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Towards a Theory of Quantum Hyperstructures

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Abstract

Quantum logic combines two sharp propositions by one join, which exists whether or not the propositions can be tested together and which keeps no record of that difference. We define a quantum hyperstructure to be a multivalued composition on the orthogonal projectors of a complex Hilbert space. Given two projectors, the composition returns every projector that commutes with both and whose range lies in the sum of their ranges. Its value is the family of sharp propositions still jointly compatible with the pair and supported on the subspace the pair generates. Associativity fails in general, already for a commuting triple, and holds on every Boolean algebra of mutually commuting projectors. In finite dimension, the family is classified by the joint eigenspaces of the two projectors and by the principal angles between their ranges. The same composition determines the projector order and the compatibility relation. It is not a hyper effect sum, and the poset of its commuting joins is not a quantum suplattice. The construction makes the indeterminacy of incompatible measurement a law of composition, rather than only a failure of distributivity.

Keywords: quantum hyperstructure; hyperoperation; orthogonal projector; orthomodular lattice; quantum logic; compatibility; effect algebra; quantum suplattice; foundations of quantum mechanics; quantum axiomatisation

MSC: 81P10; 20N20; 06C15; 03G12; 81P16

1. Introduction

Since the pioneering work of Birkhoff and von Neumann [1], the mathematical foundations of quantum mechanics have been understood to diverge fundamentally from those of classical physics. One of the central observations is that the propositions associated with a quantum system do not form a Boolean algebra but an *orthomodular lattice*: the lattice of closed subspaces, equivalently of orthogonal projectors, of a complex Hilbert space. This departure from Boolean logic is the algebraic signature of two intertwined physical phenomena. The first is the *non-commutativity* of observables, which forbids the simultaneous sharp definition of conjugate quantities; the second is *contextuality*, made precise by the Kochen–Specker and Bell theorems [2,3], which rules out global non-contextual hidden-variable models.

Standard quantum logic captures these features at the level of individual propositions, but it composes incompatible propositions by a single, deterministic join $P \vee Q$, which exists in $\mathcal{L}(\mathcal{H})$ for any two projectors. Although well-defined, this join loses the intuitive meaning of "P or Q" when P and Q do not commute: It returns the closure of the algebraic

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sum of the two ranges, a unique object that records nothing of the multiplicity of outcomes intrinsic to the joint consideration of two non-commuting observables. It is precisely this multiplicity that our construction sets out to make algebraic.

Why should this multiplicity be modelled by a genuinely multivalued operation rather than by a cleverer single-valued one? The reason lies in the operational meaning of an incompatible pair. When P and Q cannot be tested jointly, the request to combine them into a single proposition has no sharp answer; what the theory does determine is the *family* of sharp propositions that remain simultaneously testable with both. In the commuting case, this family is the residual-compatibility filter below the join; in the non-commuting case, it shrinks, but it does not in general collapse to a single element. A deterministic connective, returning one proposition, cannot carry this family, whereas an operation whose values are *sets* of propositions can. Set-valuedness is therefore not a technical convenience but a faithful reflection of how incompatible quantum questions compose, and it is this structural fact that we take as our point of departure.

The present paper proposes a remedy grounded in the theory of *hyperstructures*. Classical hyperstructure theory, initiated by Marty [4] and developed extensively by Krasner, Corsini, Massouros, Vougiouklis, and others [5–8], generalises a binary operation $\star : G \times G \to G$ to a *hyperoperation* $\star : G \times G \to \mathcal{P}(G) \setminus \{\emptyset\}$ assigning to each pair of elements a non-empty set of results. The novelty we exploit is that this multivaluedness is exactly the tool needed to represent, at the level of composition, the indeterminacy inherent in the interaction of non-commuting observables: Instead of collapsing two incompatible propositions to one join, the hyperoperation returns the whole family of sharp propositions that remain jointly meaningful. Multivalued (hyper)operations have already entered the theory of quantum structures at the *unsharp* level: Dvurečenskij and Hyčko [9] introduced hyper effect algebras, and Ji [10] and Ji and Xie [11] studied hyper-join, hyper-meet and hyper-commutators on effect algebras. Our construction is of a different kind, in both its object and its source of multivaluedness. It acts on *sharp* projectors rather than on unsharp effects, and it remains genuinely multivalued on the projection lattice $\mathcal{L}(\mathcal{H})$. That lattice is complete, so the hyper-join of [10], the set of minimal common upper bounds, collapses there to the singleton $\{P \vee Q\}$: Our multivaluedness comes from gating the sub-projectors of an existing join by compatibility, not from the non-uniqueness of that join. The principal novelty is accordingly a single, geometrically defined operation on the sharp projectors of a Hilbert space, whose values remain genuine sets even on the complete lattice $\mathcal{L}(\mathcal{H})$, together with its exact finite-dimensional classification; Section 5 locates it against hyper effect algebras, the hyperoperations on effect algebras, and quantum suplattices.

At first sight, the two subjects have little to say to one another. Hyperstructure theory arose within algebra and number theory, in the study of quotients of groups by non-normal subgroups and of multivalued analogues of rings and fields, and has developed largely independently of analysis or physics. Quantum logic, on the other side, has been pursued almost entirely with single-valued connectives, inherited from the Boolean template it seeks to weaken. What allows the two to meet is a change of emphasis, from the propositions themselves to the way they *compose*: The lattice of projectors supplies the objects and the compatibility relation, while the multivalued composition supplies exactly the ingredient that quantum logic lacks and that hyperstructure theory is designed to provide. The bridge runs in both directions. It endows the lattice of quantum propositions with a compositional calculus that respects incompatibility, and it furnishes hyperstructure theory with a concrete, physically motivated class of hyperoperations whose associativity, order theory, and representation problems are governed by the geometry of subspaces of a Hilbert space.

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This meeting is where the contribution of the paper lies, and it is why the foundations of quantum mechanics are its natural setting. Quantum logic and the theory of hyperstructures have grown up in separate communities with little traffic between them: the first within the foundations and interpretation of physics and the second within algebra. We approach their junction from the quantum-logic side, taking Piron's lattice of projectors as given and asking how its incompatible elements compose. The hyperoperation answers that question, and it yields structural results that neither tradition records on its own, among them the finite-dimensional classification of $P \circ Q$ by joint eigenspaces and principal angles developed in Section 3. The two theories thus have definite content to exchange, and a foundations venue is where the exchange belongs: The construction supplies an algebraic account of how incompatibility shapes the composition of quantum propositions, in terms that quantum logic by itself does not provide.

Read in this light, the construction belongs to the long effort, renewed in what is now called the *second quantum century* (the phrase marks the centenary of the 1925–1926 formulation of quantum mechanics; the specifically algebraic reading of the theory, which the present work continues, is almost as old, dating to the 1936 analysis of Birkhoff and von Neumann [1]), to find the algebraic language that makes the distinctively quantum features of nature explicit. The quantum-logical tradition already locates one such feature, the incompatibility of sharp questions, in the non-distributivity of the projector lattice, where it surfaces only negatively, as something the Boolean calculus cannot reproduce. The hyperoperation turns this negative datum into a positive one: For each incompatible pair, it names the family of propositions that survive their joint consideration, recording incompatibility in the law of composition rather than merely in the shape of the lattice. Our aim is accordingly structural. We propose neither a dynamics nor a reading of the quantum state, and we return in the conclusions to the sense in which such a calculus bears on the foundational debate.

The construction. To a pair of orthogonal projectors, one assigns the set of all projectors compatible with both and supported on the subspace the pair generates. When the two projectors can be tested together, this set is the filter of residual compatibility below their join; when they cannot, it recedes from its factors. The formal definition is given in Section 3, after the notation of Section 2. The same formula, read in an arbitrary orthomodular lattice through the theory of blocks and commutants of orthomodular lattices developed by Greechie and by Bruns and Greechie [12–15] (the commutant of an element, that is, the set of elements compatible with it, and the covering of a lattice by its maximal Boolean subalgebras, its blocks), extends the construction beyond Hilbert space; we record that extension together with the definition.

Organisation of the paper. Section 2 collects the necessary background on projectors, orthomodular lattices (including the commutant and the blocks of Bruns and Greechie), Piron's axiomatic framework, and classical hyperstructures. Section 3 gives the formal definition, records its form on an arbitrary orthomodular lattice, develops the compatible regime, classifies the hyperproduct in finite dimension, analyses associativity on Boolean subalgebras, and isolates a compact abstract axiomatisation. Section 4 works through explicit examples in $\mathbb{C}^2$, $\mathbb{C}^3$, and the two-qubit space $\mathbb{C}^4$. Section 5 is a brief comparison with neighbouring constructions. Section 6 draws conclusions and discusses directions for future work.

2. Preliminaries

Before introducing quantum hyperstructures, we fix notation and collect the background results that will be needed throughout. The reader familiar with orthomodular lattices and Piron's axiomatic approach may skim this section and return to it as needed;

the reader whose primary background is in hyperstructure theory will find here a concise account of the quantum-logical setting.

2.1. Projectors and the Orthomodular Lattice

Physically, an orthogonal projector is the mathematical counterpart of a sharp *yes/no experiment*. To a projector P, one associates the question "Does the system pass the test represented by $\mathrm{ran}(P)$?", an observable with spectrum $\{0,1\}$ whose two eigenspaces are the states answering "no" and "yes" with certainty. This is why projectors, and not states, are the primitive objects of quantum logic: They are the elementary propositions that a laboratory can actually decide, and the whole probabilistic content of the theory is recovered from the Born rule applied to them. Two propositions can be tested together precisely when their projectors commute, so the algebra of projectors already encodes which questions are jointly answerable and which are irreducibly incompatible. It is this structure of sharp, partially incompatible questions that our hyperoperation is built upon, which is why we take $\mathcal{L}(\mathcal{H})$ rather than states or effects as our starting point.

Let $\mathcal{H}$ be a separable complex Hilbert space. A bounded operator $P : \mathcal{H} \to \mathcal{H}$ is an *orthogonal projector* if $P^2 = P$ and $P^* = P$. We write $\mathcal{L}(\mathcal{H})$ for the set of all orthogonal projectors on $\mathcal{H}$. There is a canonical bijection between $\mathcal{L}(\mathcal{H})$ and the set of closed subspaces of $\mathcal{H}$: To $P \in \mathcal{L}(\mathcal{H})$, we associate $\mathrm{ran}(P)$, and to a closed subspace M, we associate the orthogonal projection P_M onto M.

The set $\mathcal{L}(\mathcal{H})$ carries a natural partial order:

$$P \leq Q \iff \mathrm{ran}(P) \subseteq \mathrm{ran}(Q) \iff PQ = QP = P.$$

Under this order, $\mathcal{L}(\mathcal{H})$ is a complete orthomodular lattice with

$$\begin{aligned} P \wedge Q &= P_{\mathrm{ran}(P)\cap\mathrm{ran}(Q)}, \\ P \vee Q &= P_{\overline{\mathrm{ran}(P)+\mathrm{ran}(Q)}}, \\ P^{\perp} &= \mathbf{1} - P, \end{aligned}$$

where $\overline{V}$ denotes the closure of V, $\mathbf{1}$ is the identity operator, and $\mathbf{0}$ will denote the zero operator. For $P \in \mathcal{L}(\mathcal{H})$, we write $\downarrow P := \{R \in \mathcal{L}(\mathcal{H}) : R \leq P\}$ for the order interval below P. The same arrow denotes principal down-sets in abstract posets later. The *orthomodular law* states that if $P \leq Q$, then $Q = P \vee (Q \wedge P^{\perp})$.

Two projectors $P, Q \in \mathcal{L}(\mathcal{H})$ are *orthogonal*, written $P \perp Q$, if $PQ = QP = 0$, equivalently, $\mathrm{ran}(P) \perp \mathrm{ran}(Q)$ as subspaces. They *commute*, written $[P, Q] = 0$, if $PQ = QP$, where $[A, B] := AB - BA$ denotes the commutator of operators. Commutativity is strictly weaker than orthogonality: $P \perp Q$ implies $[P, Q] = 0$, but the converse fails. When $[P, Q] = 0$, the pair is simultaneously diagonalizable: $\mathcal{H}$ splits into *joint eigenspaces* for the eigenvalue pairs $(p, q) \in \{0,1\}^2$ (the joint spectral decomposition of $\{P, Q\}$), and $W^*(P, Q)$ denotes the abelian von Neumann algebra they generate.

These operator-theoretic facts have a lattice-theoretic counterpart. In an orthomodular lattice L, two elements a, b are *compatible*, written $a \mathrel{\mathsf{C}} b$, when they lie in a common Boolean subalgebra of L; equivalently, $a = (a \wedge b) \vee (a \wedge b')$ [16]. The *commutant*

$$\mathsf{C}(a) := \{x \in L : x \mathrel{\mathsf{C}} a\}$$

is a subalgebra of L [13,14]. Every orthomodular lattice is the union of its *blocks*, the maximal Boolean subalgebras [12,14,15]. For projectors, $P \mathrel{\mathsf{C}} Q$ if and only if $[P, Q] = 0$, and $\mathsf{C}(P)$ is the complete orthomodular sublattice of projectors commuting with P. The operator

commutant $\{P, Q\}' \subseteq \mathcal{B}(\mathcal{H})$ will appear in Section 3; the orthocomplement of P is written $P' = \mathbf{1} - P$ in the abstract axioms of that section.

2.2. Piron's Axiomatic Framework

The orthomodular lattice $\mathcal{L}(\mathcal{H})$ is essentially, under mild operational axioms, the only structure that a reasonable propositional system for a quantum theory can produce. Piron [17,18] took the empirical propositions of a physical system, ordered by the relation "whenever the first is certain, so is the second", and isolated the axioms this ordered set must satisfy: It is a complete, atomistic, orthocomplemented, orthomodular lattice with the *covering property*. His representation theorem then recovers the Hilbert-space model from these purely logical data. Any irreducible lattice satisfying these axioms, of length at least four, is isomorphic to the lattice of closed subspaces of a Hermitian space over a division ring, and under the standard operational restrictions, this collapses to $\mathcal{L}(\mathcal{H})$ for a Hilbert space over $\mathbb{R}$, $\mathbb{C}$, or the quaternions $\mathbb{H}$. Two elements of Piron's lattice are compatible exactly when they generate a Boolean sublattice, the abstract shadow of the commutation $[P, Q] = 0$; incompatibility is thus written into the failure of distributivity itself.

Piron's construction fixes the objects of quantum logic and their order, but it says nothing about how two incompatible propositions should be *composed*. The lattice offers only the join $P \vee Q$, a single deterministic element, whereas the physics of incompatible questions is irreducibly non-deterministic. It is exactly at this point, downstream of Piron's reconstruction, that we take his projector lattice as given and introduce our hyperoperation: It keeps Piron's objects and order and adds the multivalued composition law they lack. Asking whether an abstract quantum hyperstructure admits a Piron-type representation (Section 6) is then the natural hyperoperational counterpart of Piron's theorem.

Remark 1. *The reconstruction just sketched is the outcome of a long effort to express quantum theory in intrinsically algebraic terms. It opens with the 1936 proposal of Birkhoff and von Neumann [1] to read the experimental propositions of a system as an orthocomplemented, non-distributive lattice, in deliberate contrast with the Boolean algebra of classical logic, and runs parallel to von Neumann's own study of rings of operators. Mackey [19] recast the subject operationally, as a small set of axioms on the pairing of states and observables, and the lattice-theoretic axioms isolated in that tradition by Jauch and Piron led to the representation theorem recalled above [17,18]. One gap remained, the a priori admissibility of division rings other than $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$; it was closed much later by Solèr's theorem [20]: An orthomodular space containing an infinite orthonormal sequence is a Hilbert space over one of those three fields. The lattice $\mathcal{L}(\mathcal{H})$ on which we work is the object this chain of results singles out, and the hyperoperation is a construction internal to it.*

2.3. Classical Hyperstructures

The notion of a hyperoperation is a natural generalisation of that of a binary operation: Instead of assigning a single element to each pair, one assigns a non-empty subset. This multivaluedness will be the key feature exploited in the quantum setting.

Definition 1. *A* hypergroupoid *is a pair $(G, \star)$ where G is a non-empty set and $\star : G \times G \to \mathcal{P}(G) \setminus \{\varnothing\}$ is a* hyperoperation. *A hypergroupoid is a* semihypergroup *if $\star$ is* associative*: For all $a, b, c \in G$,*

$$(a \star b) \star c \;=\; a \star (b \star c),$$

where $A \star c := \bigcup_{x \in A} (x \star c)$ for $A \subseteq G$. It is a hypergroup *if, in addition, the* reproduction axiom *holds: $a \star G = G \star a = G$ for all $a \in G$.*

The canonical example relevant to us is the following. Given a group $(G, \cdot)$ and a subgroup $H \leq G$, the *coset hypergroup* on G/H is defined by $aH \star bH = \{cH \mid c \in aHbH\}$; this is a hypergroup. The analogy with the quantum setting is instructive: Just as the coset hypergroup assigns to a pair of cosets all cosets reachable by combining their representatives, the quantum hyperoperation assigns to a pair of projectors all projectors compatible with both.

3. Quantum Hyperstructures

We now introduce the central object of the paper: a multivalued composition of projectors designed to encode the indeterminacy that arises when incompatible quantum propositions are combined. After the definition, we record its form on an arbitrary orthomodular lattice, via the commutant of Section 2. We then analyse the compatible regime, classify the hyperproduct in finite dimension, and study associativity on Boolean subalgebras. A compact abstract axiomatisation closes the section.

3.1. Definition and First Properties

Throughout this section, $\mathcal{H}$ denotes a complex Hilbert space (finite- or infinite-dimensional) and $\mathcal{F} \subseteq \mathcal{L}(\mathcal{H})$ a fixed collection of orthogonal projectors containing $\mathbf{0}$ and $\mathbf{1}$.

Definition 2. *The* quantum hyperoperation *associated with $\mathcal{F}$ is the map $\circ : \mathcal{F} \times \mathcal{F} \to \mathcal{P}(\mathcal{F}) \setminus \{\varnothing\}$ defined by*

$$P \circ Q := \big\{ R \in \mathcal{F} \;\big|\; \operatorname{ran}(R) \subseteq \operatorname{ran}(P) + \operatorname{ran}(Q),\ [R, P] = [R, Q] = 0 \big\}.$$

The pair $(\mathcal{F}, \circ)$ is called a quantum hyperstructure.

Proposition 1. *Let L be an orthomodular lattice, and write $\mathrm{C}(a)$ for the commutant of a in L. The formula*

$$a \circ b := \downarrow (a \vee b) \cap \mathrm{C}(a) \cap \mathrm{C}(b)$$

defines a hyperoperation on L. If $L = \mathcal{L}(\mathcal{H})$ and $\operatorname{ran}(P) + \operatorname{ran}(Q)$ is closed, this agrees with Definition 2. In particular, the two agree in finite dimension and whenever $[P, Q] = 0$.

Proof. The value $a \circ b$ is non-empty because 0 commutes with every element and $0 \leq a \vee b$. In $\mathcal{L}(\mathcal{H})$, the range condition $\operatorname{ran}(R) \subseteq \operatorname{ran}(P) + \operatorname{ran}(Q)$ is equivalent to $R \leq P \vee Q$ precisely when the algebraic sum of ranges is closed, which holds in finite dimension and, in any dimension, when P and Q commute. The remaining conditions $[R, P] = [R, Q] = 0$ say $R \in \mathrm{C}(P) \cap \mathrm{C}(Q)$. □

Consequently, every identity below that uses only the order, the orthocomplement and the compatibility relation holds, with the same proof, in an arbitrary orthomodular lattice. The finite-dimensional classification does not: It uses the geometry of subspaces of $\mathcal{H}$.

Remark 2. *We fix the conventions used throughout. Unless stated otherwise, $\mathcal{F} = \mathcal{L}(\mathcal{H})$, the full lattice of projectors. For a proper subfamily $\mathcal{F} \subseteq \mathcal{L}(\mathcal{H})$, the hyperproduct is, by Definition 2, the set of those of its members meeting the two conditions, so every identity below between $P \circ Q$ and a set of projectors is to be read relative to $\mathcal{F}$, that is, intersected with $\mathcal{F}$. The elementary properties and the compatible-case description require only $\mathbf{0}, \mathbf{1} \in \mathcal{F}$; the statements asserting that a join is attained, for instance, that $P \vee Q$ is the greatest element of $P \circ Q$ when $[P, Q] = 0$, require in addition that $\mathcal{F}$ be closed under the join of commuting pairs, and this hypothesis is stated where it is used.*

The classification of Section 3.3 (Proposition 3, Theorem 2 and Corollary 4) is finite-dimensional and takes $\mathcal{F} = \mathcal{L}(\mathcal{H})$: The principal-angle decomposition it rests on is a finite-dimensional tool, and the obstruction in infinite dimension is the one recorded in Remark 3, namely that $\mathrm{ran}(P) + \mathrm{ran}(Q)$ *need not be closed. Finally, $\circ$ is extended to subsets as usual: For $X \subseteq \mathcal{F}$ and $R \in \mathcal{F}$, we set $X \circ R := \bigcup_{A \in X} A \circ R$ and $R \circ X := \bigcup_{A \in X} R \circ A$, so that iterated expressions such as $(P \circ Q) \circ R$ are unambiguous.*

Remark 3. *The zero projector satisfies* $\mathrm{ran}(\mathbf{0}) = \{0\} \subseteq \mathrm{ran}(P) + \mathrm{ran}(Q)$ *and $[\mathbf{0}, P] = [\mathbf{0}, Q] = 0$ for every P, Q, so $\mathbf{0} \in P \circ Q$ always and the hyperoperation is well-defined. The sum* $\mathrm{ran}(P) + \mathrm{ran}(Q)$ *is the algebraic sum of ranges; it need not be closed in infinite dimension, so the range condition is strictly stronger than inclusion in the closed sum. In finite dimension, the algebraic sum is closed and equals* $\mathrm{ran}(P \vee Q)$ *whether or not the factors commute, and the range condition is then equivalent to $R \leq P \vee Q$. When $[P, Q] = 0$, the sum is already closed and equals* $\mathrm{ran}(P \vee Q)$*, in any dimension.*

Remark 4. *Several elementary properties are immediate from the definition. The conditions defining $P \circ Q$ are symmetric in the two arguments, so $P \circ Q = Q \circ P$. The identity is weakly absorbing: $\mathbf{1} \in P \circ \mathbf{1}$ for every $P \in \mathcal{F}$, since* $\mathrm{ran}(\mathbf{1}) = \mathcal{H}$ *and $[\mathbf{1}, P] = 0$. Finally,* $\mathrm{ran}(P) + \mathrm{ran}(\mathbf{0}) = \mathrm{ran}(P)$*, so $R \in P \circ \mathbf{0}$ means $R \leq P$ with $[R, P] = 0$; taking $R = P$ yields the local neutrality $P \in P \circ \mathbf{0}$.*

Both the projector order and commutativity are internal to $\circ$.

Proposition 2. *Let $(\mathcal{F}, \circ)$ be a quantum hyperstructure. For $P, Q \in \mathcal{F}$, the following hold:*

(i) $P \leq Q$ *if and only if $P \in \mathbf{0} \circ Q$.*
(ii) $[P, Q] = 0$ *if and only if $P \in P \circ Q$ (equivalently, $Q \in P \circ Q$).*
(iii) *If $[P, Q] = 0$ and $P \vee Q \in \mathcal{F}$, then $P \vee Q$ is the unique $\leq$-maximal element of $P \circ Q$.*

Proof. (i) $\mathbf{0} \circ Q = \{R \in \mathcal{F} \mid R \leq Q\}$, since $R \leq Q$ implies $[R, Q] = 0$ for projectors.

(ii) $P \in P \circ Q$ requires $\mathrm{ran}(P) \subseteq \mathrm{ran}(P) + \mathrm{ran}(Q)$ (always) and $[P, Q] = 0$. The converse is identical; symmetry of the defining conditions of $\circ$ yields the parenthetical claim.

(iii) By Theorem 1 below, $P \circ Q = \{R \in \mathcal{F} : R \leq P \vee Q,\ [R, P] = [R, Q] = 0\}$. If $P \vee Q \in \mathcal{F}$, then $P \vee Q$ belongs to this set, and every member is $\leq P \vee Q$. □

3.2. The Compatible Case

When P and Q commute, $P \circ Q$ reduces to the order interval below $P \vee Q$ filtered by *residual compatibility* with both factors (the requirement $[R, P] = [R, Q] = 0$ on the outcome R), in line with the fact that compatible measurements can be jointly performed.

Theorem 1. *Let $P, Q \in \mathcal{F}$ with $[P, Q] = 0$. Then,*

$$P \circ Q \;=\; \mathcal{F} \cap {\downarrow}(P \vee Q) \cap \mathrm{C}(P) \cap \mathrm{C}(Q).$$

Proof. This is the case $[P, Q] = 0$ of Proposition 1, intersected with $\mathcal{F}$. □

Remark 5. *In $\mathcal{L}(\mathcal{H})$, the identity of Theorem 1 says that $R \in P \circ Q$ if and only if R is block-diagonal for the joint spectral decomposition of $\{P, Q\}$, vanishes on* $\ker(P \vee Q)$*, and has an arbitrary orthogonal projector on each joint eigenspace inside* $\mathrm{ran}(P \vee Q)$*. The same set contains every spectral projection of $W^*(P, Q)$ below $P \vee Q$, and the containment is strict precisely when some such eigenspace has dimension greater than one.*

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Corollary 1. *If* $[P, Q] = 0$ *and* $\mathcal{F} = \mathcal{L}(\mathcal{H})$*, then* $P \circ Q = {\downarrow}(P \vee Q)$ *if and only if* $P = Q$ *or one of* P, Q *is* $\mathbf{0}$*.*

Proof. The joint eigenspaces inside $\mathrm{ran}(P \vee Q)$ are those of types $(1,1)$, $(1,0)$ and $(0,1)$. Whenever two of them are nonzero, the rank-one projector onto the sum of a unit vector from each lies under $P \vee Q$ and fails to commute with at least one of P and Q. This is precisely the case when both factors are nonzero and $P \neq Q$. If $P = Q$ or one of P, Q is $\mathbf{0}$, every projector under the join commutes with both factors. □

Corollary 2. *If* $P \perp Q$*, then*

$$P \circ Q = \{R \in \mathcal{F} \mid R \leq P + Q,\ [R, P] = [R, Q] = 0\}.$$

In $\mathcal{L}(\mathcal{H})$*, these are the projectors* $R_P \oplus R_Q \oplus 0$ *on* $\mathrm{ran}(P) \oplus \mathrm{ran}(Q) \oplus \mathrm{ran}(P+Q)^{\perp}$*. When* P *and* Q *are both nonzero, the inclusion in the order interval* ${\downarrow}(P+Q)$ *is proper.*

Proof. Orthogonality implies commutativity and $P \vee Q = P + Q$; apply Theorem 1. If both are nonzero, unit vectors $e_P \in \mathrm{ran}(P)$ and $e_Q \in \mathrm{ran}(Q)$ give a rank-one projector onto $\mathbb{C}(e_P + e_Q)$ which lies below $P + Q$ and does not commute with P. □

Remark 6. *If* $[P, Q] \neq 0$*, then neither* P *nor* Q *lies in* $P \circ Q$ *(Proposition 2(ii)). Thus, incompatibility forces the hyperproduct away from its factors: The multivalued result retains only projectors compatible with both. In finite dimension, that set is the block lattice of Theorem 2. It equals* $\{\mathbf{0}, \mathbf{1}\}$ *when the two ranges span* $\mathcal{H}$ *and the only block is one simple principal angle, as for a pair of non-orthogonal rank-one projectors on a qubit.*

3.3. The Hyperproduct as a Projection Lattice

Throughout this subsection, $\mathcal{H}$ is finite-dimensional and $\mathcal{F} = \mathcal{L}(\mathcal{H})$. Write

$$e := P \vee Q, \qquad \mathcal{W} := \{P, Q\}' = \{X \in \mathcal{B}(\mathcal{H}) : [X, P] = [X, Q] = 0\}$$

for the join and the operator commutant of the pair; $\mathcal{W}$ is a von Neumann algebra and $e \in \mathcal{W}$. In finite dimension, the algebraic sum of ranges is always closed, so Proposition 1 gives $P \circ Q = {\downarrow} e \cap \mathrm{C}(P) \cap \mathrm{C}(Q)$ for every pair, compatible or not. The lattice commutant of Section 2 is the projection lattice of $\mathcal{W}$: $\mathrm{C}(P) \cap \mathrm{C}(Q) = \mathrm{Proj}(\mathcal{W})$.

Proposition 3. *For* $P, Q \in \mathcal{L}(\mathcal{H})$ *with* $\mathcal{H}$ *finite-dimensional,*

$$P \circ Q = \{R \in \mathcal{L}(\mathcal{H}) :\ R \leq e,\ [R, P] = [R, Q] = 0\} = \mathrm{Proj}(e\mathcal{W}e),$$

the projection lattice of the von Neumann algebra $e\mathcal{W}e$ *acting on* $\mathrm{ran}(e)$*. In particular,* $P \circ Q$ *is a complete orthomodular lattice under the projector order, with least element* $\mathbf{0}$ *and greatest element* $e = P \vee Q$*.*

Proof. Since $\mathcal{H}$ is finite-dimensional, $\mathrm{ran}(P) + \mathrm{ran}(Q)$ is closed and equals $\mathrm{ran}(e)$; hence, the range condition $\mathrm{ran}(R) \subseteq \mathrm{ran}(P) + \mathrm{ran}(Q)$ of Definition 2 is equivalent to $R \leq e$, *whether or not* P and Q commute. The remaining conditions $[R, P] = [R, Q] = 0$ say exactly that $R \in \mathcal{W}$. As $e \in \mathcal{W}$, the compression $e\mathcal{W}e$ is a von Neumann subalgebra of $\mathcal{B}(\mathrm{ran}\, e)$, and a projection $R \leq e$ lies in $\mathcal{W}$ if and only if it lies in $e\mathcal{W}e$. Therefore, $P \circ Q = \mathrm{Proj}(e\mathcal{W}e)$, and the projection lattice of a von Neumann algebra is a complete orthomodular lattice. □

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Remark 7. *The hyperoperation is unitarily covariant. For every unitary U on $\mathcal{H}$ and all $P, Q \in \mathcal{L}(\mathcal{H})$,*

$$(UPU^*) \circ (UQU^*) = U\,(P \circ Q)\,U^*$$

since conjugation by U preserves ranges, the inclusion of ranges, and commutators, and maps $\mathcal{L}(\mathcal{H})$ onto itself. The representation above transforms accordingly: $UeU^* = (UPU^*) \vee (UQU^*)$ and $U\mathcal{W}U^* = \{UPU^*, UQU^*\}'$, so Proj$(e\mathcal{W}e)$ *is carried to the projection lattice of the conjugated pair. Consequently, $P \circ Q$ depends on (P, Q) only up to unitary equivalence, and every isomorphism invariant of $P \circ Q$, including its Boolean-or-continuum type, the principal angles between the two ranges, and the block dimensions of Theorem 2, is a unitary invariant of the pair.*

Corollary 3. *With the notation above, $P \circ Q$ is a Boolean algebra (equivalently, a finite set) if and only if $e\mathcal{W}e$ is abelian; otherwise, $P \circ Q$ contains a continuum of elements.*

Proof. By Proposition 3, $P \circ Q$ is the projection lattice of the finite-dimensional von Neumann algebra $e\mathcal{W}e \cong \bigoplus_i M_{n_i}(\mathbb{C})$. Its projection lattice is distributive (a Boolean algebra with 2^k elements, for k the number of factors) if and only if every $n_i = 1$, that is, if and only if $e\mathcal{W}e$ is abelian. A factor M_n with $n \geq 2$ contributes the continuum of rank-one projections of $\mathbb{C}^n$. □

The factors of $e\mathcal{W}e$ are identified by the two-subspace decomposition of Halmos [21].

Theorem 2. *Let $\mathcal{H}$ be finite-dimensional and let $P, Q \in \mathcal{L}(\mathcal{H})$. Write*

$$\mathcal{H}_{11} = \mathrm{ran}(P) \cap \mathrm{ran}(Q), \qquad \mathcal{H}_{10} = \mathrm{ran}(P) \cap \ker Q,$$
$$\mathcal{H}_{01} = \ker P \cap \mathrm{ran}(Q), \qquad \mathcal{H}_{00} = \ker P \cap \ker Q,$$

and let $\mathcal{H}_g$ be the orthogonal complement of $\mathcal{H}_{00} \oplus \mathcal{H}_{01} \oplus \mathcal{H}_{10} \oplus \mathcal{H}_{11}$. There is an orthogonal decomposition

$$\mathcal{H}_g = \bigoplus_\theta \mathcal{H}(\theta),$$

indexed by the principal angles $\theta \in (0, \pi/2)$ between ran(P) *and* ran(Q)*. Each block has dimension $2m_\theta$ for a multiplicity $m_\theta \geq 1$, and $\mathcal{H}(\theta)$ admits an orthonormal basis in which*

$$P = \begin{pmatrix} I_{m_\theta} & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} \cos^2\theta\, I & \cos\theta \sin\theta\, I \\ \cos\theta \sin\theta\, I & \sin^2\theta\, I \end{pmatrix}.$$

On $\mathcal{H}_{ij}$, both P and Q act as the scalars i and j. A projector R lies in $P \circ Q$ if and only if R vanishes on $\mathcal{H}_{00}$, the compression of R to each of $\mathcal{H}_{11}$, $\mathcal{H}_{10}$ and $\mathcal{H}_{01}$ is an arbitrary orthogonal projector, and, in the basis just chosen, the compression of R to $\mathcal{H}(\theta)$ equals diag(π, π) *for an arbitrary orthogonal projector π on $\mathbb{C}^{m_\theta}$.*

Proof. The four subspaces $\mathcal{H}_{ij}$ are pairwise orthogonal and reduce both P and Q; their orthogonal complement $\mathcal{H}_g$ therefore reduces both projections. By the two-subspace theorem of Halmos [21], $\mathcal{H}_g$ decomposes as an orthogonal sum of blocks $\mathcal{H}(\theta)$ indexed by the principal angles $\theta \in (0, \pi/2)$ between ran(P) and ran(Q), each of dimension $2m_\theta$, in which P and Q take the stated matrix form. (In finite dimension, the direct integral of [21] reduces to a finite sum.) On each $\mathcal{H}_{ij}$, both operators act as the scalars i and j.

Now, let $R \in P \circ Q$. By Proposition 3, $R \leq e = P \vee Q$ and $R \in \mathrm{C}(P) \cap \mathrm{C}(Q)$. Thus, R vanishes on $\mathcal{H}_{00} = \ker e$ and preserves every joint eigenspace of $\{P, Q\}$, hence each $\mathcal{H}_{ij}$ and each $\mathcal{H}(\theta)$. On $\mathcal{H}_{ij}$ with $(i, j) \neq (0, 0)$, the operators P and Q are scalars, so the compression

of R is an arbitrary projector. On $\mathcal{H}(\theta)$, commutation with the stated matrices forces the compression to be $\mathrm{diag}(\pi, \pi)$ for a projector π on $\mathbb{C}^{m_\theta}$.

Conversely, any projector of this block form is supported on $\mathrm{ran}(e)$ and commutes with P and with Q, hence lies in $P \circ Q$. The same computation yields

$$e\mathcal{W}e \;\cong\; \mathcal{B}(\mathcal{H}_{11}) \oplus \mathcal{B}(\mathcal{H}_{10}) \oplus \mathcal{B}(\mathcal{H}_{01}) \oplus \bigoplus_{\theta} M_{m_\theta}(\mathbb{C}),$$

omitting zero summands. □

Corollary 4. *In the notation of Theorem 2, $e\mathcal{W}e$ is abelian if and only if* $\dim \mathcal{H}_{11} \leq 1$, $\dim \mathcal{H}_{10} \leq 1$, $\dim \mathcal{H}_{01} \leq 1$, *and $m_\theta = 1$ for every principal angle θ. In that case, $P \circ Q$ is a Boolean algebra with 2^k elements, where N is the number of principal angles and*

$$k = \dim \mathcal{H}_{11} + \dim \mathcal{H}_{10} + \dim \mathcal{H}_{01} + N;$$

otherwise, $P \circ Q$ contains a continuum of elements. If $[P, Q] = 0$, then no angle θ occurs, and Theorem 2 reduces to the residual block description of Remark 5.

Proof. The isomorphism at the end of the proof of Theorem 2 exhibits the factors of $e\mathcal{W}e$. A full matrix algebra $M_n(\mathbb{C})$ is abelian if and only if $n \leq 1$. Corollary 3 converts that condition into the Boolean-or-continuum alternative for $P \circ Q$. When $[P, Q] = 0$, the two ranges are reducing for each other, so every principal angle is 0 or $\pi/2$ and $\mathcal{H}_g = \{0\}$. □

Remark 8. *The orthogonal decomposition of two subspaces extends to infinite dimension, with a direct integral over the angle in place of the sum over θ [21]. The membership criterion of Definition 2 does not extend in the same way. Remark 3 records that* $\mathrm{ran}(P) + \mathrm{ran}(Q)$ *need not be closed, in which case it is a proper dense subspace of* $\mathrm{ran}(P \vee Q)$. *The join $P \vee Q$ itself then fails the range condition in Definition 2, even though it commutes with both factors, and Proposition 3 is special to finite dimension.*

Figure 1 records the block structure produced by Theorem 2. The space splits into the four joint eigenspaces and the principal-angle summands, and the labels record the freedom left to a member R of $P \circ Q$ on each of them. The summand $\mathcal{H}_{00}$ is the kernel of the join, and every such R vanishes there.

3.4. Associativity

Whether $(\mathcal{F}, \circ)$ is a semihypergroup depends critically on compatibility *and* on the ambient collection: Associativity fails in general (even for some mutually commuting triples in $\mathcal{L}(\mathcal{H})$), but is restored on classical fragments. The two regimes are made precise below: Proposition 4 exhibits a triple for which the two associations differ, while Theorem 3 shows that associativity is restored on the Boolean cores. Throughout, the iterated products are formed with the set-valued extension of $\circ$ fixed in Remark 2, so that $(P \circ Q) \circ R$ denotes $\bigcup_{A \in P \circ Q} A \circ R$. The first counterexample is minimal, using three projectors in $\mathbb{C}^3$ with a non-commuting pair.

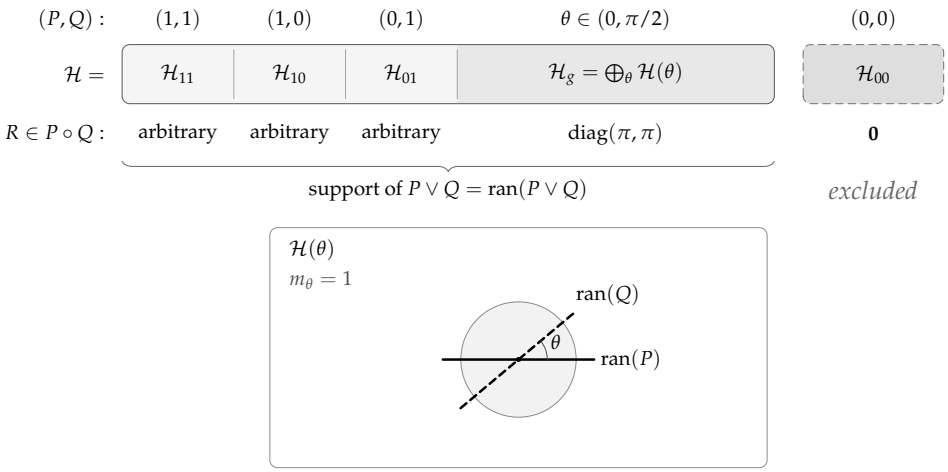


Figure 1. The finite-dimensional classification of Theorem 2. Widths in the bar are schematic. The space $\mathcal{H}$ splits into the joint eigenspaces $\mathcal{H}_{ij}$, on which P and Q act by the scalars (i,j), and the generic summand $\mathcal{H}_g = \bigoplus_\theta \mathcal{H}(\theta)$, where each block $\mathcal{H}(\theta)$ has dimension $2m_\theta$. Every $R \in P \circ Q$ vanishes on $\mathcal{H}_{00} = \ker(P \vee Q)$ and is supported on the complementary sum, which equals $\mathrm{ran}(P \vee Q)$. Its compression is an arbitrary projector on each of $\mathcal{H}_{11}$, $\mathcal{H}_{10}$ and $\mathcal{H}_{01}$, and equals $\mathrm{diag}(\pi,\pi)$ on $\mathcal{H}(\theta)$ for a projector π on $\mathbb{C}^{m_\theta}$. The inset is the simple case $m_\theta = 1$: $\mathcal{H}(\theta)$ is a two-plane, the lines $\mathrm{ran}(P)$ and $\mathrm{ran}(Q)$ meet at the principal angle θ, and R is there either $\mathbf{0}$ or the whole plane. The hyperproduct is a Boolean algebra if and only if each of $\mathcal{H}_{11}$, $\mathcal{H}_{10}$ and $\mathcal{H}_{01}$ has dimension at most one and every angle is simple, and it contains a continuum of elements otherwise (Corollary 4).

Proposition 4. *There exist projectors $P, Q, R \in \mathcal{L}(\mathcal{H})$ such that $(P \circ Q) \circ R \neq P \circ (Q \circ R)$.*

Proof. Let $\mathcal{H} = \mathbb{C}^3$ with canonical orthonormal basis $\{e_1, e_2, e_3\}$ and set $u = \frac{1}{\sqrt{2}}(e_2 + e_3)$ and $v = \frac{1}{\sqrt{2}}(e_2 - e_3)$. Define

$$A = |e_1\rangle\langle e_1|, \qquad B = |u\rangle\langle u|, \qquad C = |e_2\rangle\langle e_2|.$$

Then, $\{e_1, u, v\}$ is an orthonormal basis, $A \perp B$, $A \perp C$, and $[B, C] \neq 0$ (since $BC = \frac{1}{\sqrt{2}}|u\rangle\langle e_2|$, while $CB = \frac{1}{\sqrt{2}}|e_2\rangle\langle u|$).

Step 1. Since $A \perp B$, Corollary 2 gives

$$A \circ B = \{R \in \mathcal{L}(\mathcal{H}) \mid R \leq A + B,\ [R, A] = [R, B] = 0\}.$$

Any such R is supported in $\mathrm{span}\{e_1, u\}$ and must have both e_1 and u as eigenvectors, so $A \circ B = \{\mathbf{0}, A, B, A + B\}$.

Step 2. Here, $\mathrm{ran}(B) + \mathrm{ran}(C) = \mathrm{span}\{e_2, e_3\}$, and $\langle u, e_2\rangle = 1/\sqrt{2}$, so the two lines are neither equal nor orthogonal. Both projectors vanish on e_1, and neither line meets the kernel of the other inside that plane. Thus, $\mathcal{H}_{11} = \mathcal{H}_{10} = \mathcal{H}_{01} = \{0\}$, $\mathcal{H}_{00} = \mathrm{span}\{e_1\}$, and $\mathcal{H}_g = \mathrm{span}\{e_2, e_3\}$ is a single simple principal-angle block. Theorem 2 gives $B \circ C = \{\mathbf{0}, D\}$ with $D := |e_2\rangle\langle e_2| + |e_3\rangle\langle e_3| = \mathbf{1} - A$.

Step 3. We compute each factor of $(A \circ B) \circ C = \bigcup_{S \in \{\mathbf{0}, A, B, A+B\}} (S \circ C)$:

$$\begin{aligned}
\mathbf{0} \circ C &= \{\mathbf{0}, C\}, \\
A \circ C &= \{\mathbf{0}, A, C, A + C\} \quad \text{(by } A \perp C \text{ and Corollary 2)}, \\
B \circ C &= \{\mathbf{0}, D\}, \\
(A + B) \circ C &= \{\mathbf{0}, A, D, \mathbf{1}\}.
\end{aligned}$$

For the last identity, pass to the orthonormal basis $\{e_1, u, v\}$. There, $A + B = \mathrm{diag}(1,1,0)$ and

$$C = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

while $\mathrm{ran}(A+B) + \mathrm{ran}(C) = \mathbb{C}^3$, so the range condition is vacuous and $(A+B) \circ C$ is the set of projections commuting with both. Any such projection has the form $\begin{pmatrix} a & b & 0 \\ \bar{b} & c & 0 \\ 0 & 0 & \lambda \end{pmatrix}$ with $\lambda \in \{0,1\}$ and with $\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}$ itself a projection. Commutation with C forces $b = 0$ and $c = \lambda$, so $a \in \{0,1\}$. The four choices are $\mathbf{0}$, A, D and $\mathbf{1}$. Hence,

$$(A \circ B) \circ C = \{\mathbf{0},\ A,\ C,\ A + C,\ D,\ \mathbf{1}\}, \tag{1}$$

so in particular $|e_3\rangle\langle e_3| \notin (A \circ B) \circ C$.

Step 4. On the other side, $A \circ (B \circ C) = (A \circ \mathbf{0}) \cup (A \circ D)$. The first factor is $\{\mathbf{0}, A\}$. Since $D = \mathbf{1} - A$, one has $[A,\ D] = 0$ and $\mathrm{ran}(A) + \mathrm{ran}(D) = \mathbb{C}^3$, so $A \circ D = \{R \in \mathcal{L}(\mathcal{H}) \mid [R,\ A] = 0\}$, which contains $|e_3\rangle\langle e_3|$. Thus, $|e_3\rangle\langle e_3| \in A \circ (B \circ C)$. Combined with (1), this shows that the two associations differ. □

Remark 9. *Mutual commutativity of a triple is not by itself enough to restore associativity on the full lattice $\mathcal{L}(\mathcal{H})$. In $\mathbb{C}^3$, the pairwise orthogonal (hence mutually commuting) projectors $P = |e_1\rangle\langle e_1|$, $Q = |e_2\rangle\langle e_2|$, and $R = |e_3\rangle\langle e_3|$ satisfy $P \circ Q = \{\mathbf{0}, P, Q, P+Q\}$ by Corollary 2. The rank-one projector $M = |\frac{1}{\sqrt{2}}(e_1 + e_2)\rangle\langle\frac{1}{\sqrt{2}}(e_1 + e_2)|$ then obeys $M \leq (P+Q) + R = \mathbf{1}$ and $[M,\ P+Q] = [M,\ R] = 0$, so $M \in (P+Q) \circ R \subseteq (P \circ Q) \circ R$; yet $Q \circ R = \{\mathbf{0}, Q, R, Q+R\}$, and membership of M in $P \circ S$ for any $S \in Q \circ R$ would require $[M,\ P] = 0$, which fails. Thus, $M \notin P \circ (Q \circ R)$.*

Associativity is restored in a classical ambient, where residual compatibility below the join is automatic, that is, precisely on classical fragments of the hyperstructure.

Theorem 3. *Let $P, Q, R \in \mathcal{F}$ with $[P,\ Q] = [Q,\ R] = [P,\ R] = 0$, and assume $P \vee Q$, $Q \vee R$, and $P \vee Q \vee R$ lie in $\mathcal{F}$. Suppose moreover that every $S \in \mathcal{F}$ with $S \leq P \vee Q \vee R$ satisfies $[S,\ P] = [S,\ Q] = [S,\ R] = 0$ (as holds whenever $\mathcal{F}$ itself is a Boolean algebra of projectors). Then,*

$$(P \circ Q) \circ R = P \circ (Q \circ R) = \{S \in \mathcal{F} \mid S \leq P \vee Q \vee R\}.$$

Proof. Write $J := P \vee Q \vee R$. By Theorem 1 and the residual-compatibility hypothesis, $P \circ Q = \{S \in \mathcal{F} : S \leq P \vee Q\}$, and in particular $P \vee Q \in P \circ Q$.

Inclusion $\subseteq$. Any element of $(P \circ Q) \circ R$ lies in $S \circ R$ for some $S \in P \circ Q$. That S lies below $P \vee Q$, hence below J, so the residual-compatibility hypothesis gives $[S,\ R] = 0$. Theorem 1 therefore bounds the element by $S \vee R$. Since $S \vee R \leq (P \vee Q) \vee R = J$, one has $(P \circ Q) \circ R \subseteq \{T \in \mathcal{F} : T \leq J\}$.

Inclusion $\supseteq$. Let $T \in \mathcal{F}$ with $T \leq J$. By the residual-compatibility hypothesis, T commutes with P, Q and R, hence with $P \vee Q \in W^*(P, Q)$; since $[P \vee Q,\ R] = 0$ and $(P \vee Q) \vee R = J \in \mathcal{F}$, Theorem 1 gives $(P \vee Q) \circ R = \{U \in \mathcal{F} : U \leq J\} \ni T$. As $P \vee Q \in P \circ Q$, this places T in $(P \circ Q) \circ R$.

Both inclusions give $(P \circ Q) \circ R = \{T \in \mathcal{F} : T \leq J\}$, and the same argument with the factors reversed yields $P \circ (Q \circ R) = \{T \in \mathcal{F} : T \leq J\}$. □

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The maximal domains of global associativity are the classical fragments of the hyperstructure.

Definition 3. *A subset $B \subseteq \mathcal{L}(\mathcal{H})$ containing* $\mathbf{0}$ *and* $\mathbf{1}$ *is a* Boolean core *if every pair of elements of B is compatible, B is closed under $\vee$ and under $P \mapsto \mathbf{1} - P$, and $(B, \leq)$ is a Boolean algebra under these operations. In the language of Bruns and Greechie [12,14,15], a Boolean core is a Boolean subalgebra of $\mathcal{L}(\mathcal{H})$, and a maximal Boolean core is a block. Equipping B with the quantum hyperoperation of Definition 2 (i.e., taking the ambient collection to be B itself) yields a quantum hyperstructure $(B, \circ)$.*

Theorem 4. *If $B \subseteq \mathcal{L}(\mathcal{H})$ is a Boolean core, then $(B, \circ)$ is a commutative semihypergroup, and for all $P, R, S \in B$,*

$$P \circ R = \{T \in B \mid T \leq P \vee R\},$$

$$(P \circ R) \circ S = P \circ (R \circ S) = \{T \in B \mid T \leq P \vee R \vee S\}.$$

In particular, every Boolean subalgebra of $\mathcal{L}(\mathcal{H})$ is a Boolean core and is associative as a quantum hyperstructure on itself (not as a subset of $(\mathcal{L}(\mathcal{H}), \circ)$ under the unrestricted ambient hyperoperation).

Proof. On a mutually commuting family, every residual commutator among elements of B vanishes, so Theorem 1 (applied to the quantum hyperstructure $(B, \circ)$) gives $P \circ R = \{T \in B : T \leq P \vee R\}$. Theorem 3, whose residual-compatibility hypothesis holds automatically on B, yields associativity. Every Boolean subalgebra of $\mathcal{L}(\mathcal{H})$ satisfies Definition 3 by the spectral theorem. □

Remark 10. *Thus, associativity of $\circ$ is restored on every Boolean core. Whether every maximal associative subcollection arises in this way is discussed in Section 6; one direction (Boolean core $\Rightarrow$ associative) is Theorem 4, while the converse remains open.*

Remark 11. *It is useful to record where $\circ$ meets the standard hyperstructure axioms. Closure is immediate from Definition 2, since every value is a non-empty subset of the ambient family (Remark 3), and $\circ$ is commutative (Remark 4). Associativity is the delicate axiom: $(\mathcal{F}, \circ)$ is a semihypergroup exactly when $\circ$ is associative throughout $\mathcal{F}$, which fails in general (Proposition 4) and holds whenever $\mathcal{F}$ is a Boolean core (Theorem 4); the sharp necessary and sufficient description, whether the maximal associative subcollections are exactly the Boolean cores, is the open problem of Section 6.*

On a Boolean core B, the reproduction axiom holds as well: Since $\mathbf{1} \in B$, for every $P \in B$, one has $P \circ \mathbf{1} = \{T \in B : T \leq P \vee \mathbf{1}\} = B$, so $\bigcup_{R \in B} P \circ R = B$. With Theorem 4, this shows that every Boolean core is a commutative hypergroup under $\circ$. *The pairwise orthogonal projectors $|e_1\rangle\langle e_1|, \ldots, |e_n\rangle\langle e_n|$ of $\mathbb{C}^n$ generate such a core, the 2^n-element Boolean algebra of coordinate projections, which gives a family of non-trivial commutative hypergroups; the three-dimensional case appears in Remark 9.*

The identity $P \circ R = \{T \in B : T \leq P \vee R\}$ also fixes the interaction of $\circ$ with the lattice operations on a Boolean core: It gives the absorption relations $P \in P \circ (P \wedge R)$ and $P \circ (P \vee R) = P \circ R$, and makes $\circ$ monotone in each argument. Off the Boolean cores, no distributive law can be expected, since associativity itself fails there; a systematic account of weak modular or distributive identities relating $\circ$ to $\wedge$ and $\vee$ is left open.

3.5. Abstract Form and Related Structures

The properties above depend on ranges and commutators, but their algebraic content can be isolated. We record a compact axiomatisation in which order and compatibility are *derived* from a single hyperoperation.

Definition 4. *An* abstract quantum hyperstructure *(AQH) is a structure* $(H, \circ, 0, 1)$ *with* $\circ : H \times H \to \mathcal{P}(H) \setminus \{\emptyset\}$, *where one writes* $a \leq b :\Leftrightarrow a \in 0 \circ b$ *(the* derived order*) and* $a \mathsf{C} b :\Leftrightarrow a \in a \circ b$ *(the* derived compatibility *relation). Write* $\downarrow a := \{x \in H : x \leq a\}$ *for the principal down-set of a in* $(H, \leq)$. *The structure is subject to the following axioms:*

(AQ1) $a \circ b = b \circ a$.

(AQ2) $0 \in a \circ b$ *for all* a, b.

(AQ3) $\leq$ *is a bounded partial order with bounds* $0, 1$.

(AQ4) $a \leq b$ *implies* $a \mathsf{C} b$, *and* C *is symmetric* ($a \mathsf{C} b$ *iff* $b \mathsf{C} a$).

(AQ5) $a \circ a = 0 \circ a$.

(AQ6) $1 \in a \circ 1$.

(AQ7) *If* $a \mathsf{C} b$, *then* $a \circ b$ *has a* $\leq$*-greatest element, written* $a \sqcup b$, *with* $a, b \leq a \sqcup b$ *(the* compatible join*).*

(AQ8) *If* $a \mathsf{C} b$, *then* $a \circ b = \{x \mid x \leq a \sqcup b,\ x \mathsf{C} a,\ x \mathsf{C} b\}$.

An AQH is orthocomplemented *if it carries a map* $(-)'$ *(an* orthocomplement*) such that* $a'' = a$, $a \mathsf{C} a'$, $a \sqcup a' = 1$, $\downarrow a \cap \downarrow a' = \{0\}$, $a \leq b \Rightarrow b' \leq a'$, *and the orthomodular law holds whenever the relevant meets exist.*

Theorem 5. *Let* $\mathcal{F} \subseteq \mathcal{L}(\mathcal{H})$ *contain* $\mathbf{0}$ *and* $\mathbf{1}$ *and be closed under* $P \vee Q$ *whenever* $[P, Q] = 0$ *and* $P, Q \in \mathcal{F}$. *Then,* $(\mathcal{F}, \circ, \mathbf{0}, \mathbf{1})$ *is an AQH. If* $\mathcal{F}$ *is also closed under* $P \mapsto \mathbf{1} - P$, *it is orthocomplemented with* $P' = \mathbf{1} - P$. *In particular,* $(\mathcal{L}(\mathcal{H}), \circ, \mathbf{0}, \mathbf{1})$ *is an orthocomplemented AQH.*

Proof. (AQ1) is the symmetry of Remark 4; (AQ2) is Remark 3. For (AQ3), the relation $P \leq Q \Leftrightarrow P \in \mathbf{0} \circ Q$ is the restriction of the projector order to $\mathcal{F}$ (Proposition 2(i)), hence a partial order; bounds $\mathbf{0} \leq P \leq \mathbf{1}$ hold because $\mathbf{0}, \mathbf{1} \in \mathcal{F}$. (AQ4) combines the standard fact that $P \leq Q$ implies $[P, Q] = 0$, rewritten via Proposition 2(ii), with the symmetry of commutativity of projectors. (AQ5): $P \circ P = \{R \in \mathcal{F} : R \leq P\} = \mathbf{0} \circ P$. (AQ6) is the weak absorption of $\mathbf{1}$ in Remark 4. (AQ7)–(AQ8): If $[P, Q] = 0$, join-closure puts $P \vee Q \in \mathcal{F}$, and Theorem 1 identifies $P \circ Q$ with the residual-compatibility filter below $P \vee Q$, whose greatest element is $P \vee Q$. Orthocomplementation by $P \mapsto \mathbf{1} - P$ is standard in $\mathcal{L}(\mathcal{H})$ and restricts to any $\mathcal{F}$ closed under that map. □

Remark 12. *The quantum hyperoperation is not a mosaic in the sense of Nakamura and Reyes [22]: Neither* $\mathbf{0}$ *nor* $\mathbf{1}$ *is a mosaic identity, since* $\mathbf{0} \circ P = \downarrow P \neq \{P\}$ *in general. The order recovery* $R \leq P \Leftrightarrow R \in P \circ P$ *matches that of L-mosaics [23], while global joins exist for* $\circ$ *only on compatible pairs. Comparison with L-mosaics, quantum suplattices and hyper effect algebras is given in Section 5.*

4. Examples

The abstract theory of Sections 2 and 3 is best illuminated by working out explicit cases. We proceed in order of increasing complexity: the qubit ($\mathbb{C}^2$), where the hyperoperation is completely classified; three-dimensional space ($\mathbb{C}^3$), which already exhibits the full range of compatible and incompatible behaviours; and the two-qubit system ($\mathbb{C}^4$), where the hyperoperation of two incompatible propositions can return a projector onto an entangled state. Each of these hyperproducts is the case, in dimension two, three or four, of the block description in Theorem 2. A recurring theme is that $P \circ Q$ tracks the degree of compatibility between P and Q: For rank-one projectors, the more incompatible the pair, the fewer non-trivial elements survive, while for higher-rank projectors, the surviving elements record exactly the subspaces on which the two propositions agree.

4.1. The Qubit

Let $\mathcal{H} = \mathbb{C}^2$ with canonical basis $\{|0\rangle, |1\rangle\}$. The projectors on $\mathbb{C}^2$ are $\mathbf{0}$, $\mathbf{1}$, and the rank-one projectors $|v\rangle\langle v|$ for unit vectors $v \in \mathbb{C}^2$. For rank-one P, Q, the hyperoperation is completely determined by the results of Section 3. If $P = Q$, then $P \circ P = \{\mathbf{0}, P\}$. If $P \perp Q$ (a complementary pair of rank-one projectors), then $P + Q = \mathbf{1}$ and Corollary 2 gives $P \circ Q = \{\mathbf{0}, P, Q, \mathbf{1}\}$. If $[P, Q] \neq 0$, then the two lines are distinct and not orthogonal, so $\mathrm{ran}(P) + \mathrm{ran}(Q) = \mathbb{C}^2$. The only projectors commuting with the rank-one projector P are $\mathbf{0}$, P, $\mathbf{1} - P$ and $\mathbf{1}$, and neither P nor $\mathbf{1} - P$ commutes with Q. Hence, $P \circ Q = \{\mathbf{0}, \mathbf{1}\}$.

Example 1. *Let* $P_z = |0\rangle\langle 0|$ *and* $P_x = |+\rangle\langle +|$, *where* $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. *These are incompatible, since*

$$[P_z, P_x] = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \neq 0.$$

Hence, $P_z \circ P_x = \{\mathbf{0}, \mathbf{1}\}$. *This reflects maximal indeterminacy: The only projectors compatible with both a z-measurement and an x-measurement are the trivial ones.*

4.2. Three Dimensions

The richer structure of $\mathbb{C}^3$ allows non-trivial intermediate projectors to appear in the hyperoperation. In particular, the hyperproduct of two incompatible rank-two projectors can contain a non-trivial rank-one projector: physically, the unique direction common to both, the sharp proposition they still agree on despite being otherwise incompatible.

Example 2. *Let* $\mathcal{H} = \mathbb{C}^3$, *and define*

$$P = |e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|, \qquad Q = |e_1\rangle\langle e_1| + |v\rangle\langle v|, \quad v = \tfrac{1}{\sqrt{2}}(e_2 + e_3).$$

In matrix form,

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \qquad [P, Q] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix} \neq 0.$$

The sum $\mathrm{ran}(P) + \mathrm{ran}(Q) = \mathrm{span}\{e_1, e_2, v\} = \mathbb{C}^3$ *(as* e_1, e_2, v *are linearly independent).*

A projector R commutes with P iff R is block-diagonal with blocks on $\mathrm{ran}(P) = \mathrm{span}\{e_1, e_2\}$ *and its orthogonal complement* $\mathrm{span}\{e_3\}$. *A projector R commutes with Q iff R preserves the eigenspaces of Q, which are* $\mathrm{span}\{e_1\}$, $\mathrm{span}\{v\} = \mathrm{span}\{e_2 + e_3\}$, *and* $\mathrm{span}\{e_2 - e_3\}$ *(with eigenvalues* $1, 1, 0$*). The only common eigendirection of P and Q is* $\mathrm{span}\{e_1\}$*; on the orthogonal plane* $\mathrm{span}\{e_2, e_3\}$, *the operators P and Q act as two distinct rank-one projectors, so their joint commutant there reduces to the scalars. Hence, a projector commuting with both is a diagonal block* $a|e_1\rangle\langle e_1| \oplus b\,\mathbf{1}_{\mathrm{span}\{e_2,e_3\}}$ *with* $a, b \in \{0, 1\}$, *and the image constraint* $\mathrm{ran}(R) \subseteq \mathbb{C}^3$ *is automatic. Therefore,*

$$P \circ Q = \{\mathbf{0},\ |e_1\rangle\langle e_1|,\ \mathbf{1} - |e_1\rangle\langle e_1|,\ \mathbf{1}\},$$

a four-element Boolean algebra. The rank-one element $|e_1\rangle\langle e_1|$ *is the unique atom: It projects onto the line* $\mathrm{span}\{e_1\}$ *that P and Q share as a common eigenspace, the single sharp proposition on which the two incompatible rank-two projectors still agree. Its complement* $\mathbf{1} - |e_1\rangle\langle e_1| = |e_2\rangle\langle e_2| + |e_3\rangle\langle e_3|$ *is the projector onto the plane where they disagree (Figure 2).*

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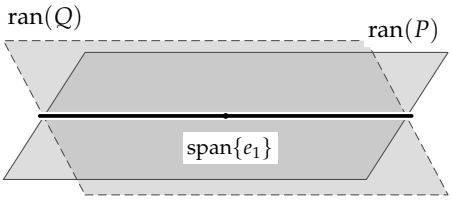


Figure 2. The two incompatible rank-two projectors of Example 2 in $\mathbb{C}^3$. The planes $\mathrm{ran}(P) = \mathrm{span}\{e_1, e_2\}$ and $\mathrm{ran}(Q) = \mathrm{span}\{e_1, v\}$, $v = \frac{1}{\sqrt{2}}(e_2 + e_3)$, meet along their full intersection, the line $\mathrm{span}\{e_1\}$ (bold). This common eigenline is the single sharp proposition the two incompatible questions still share: the rank-one atom $|e_1\rangle\langle e_1|$ of $P \circ Q = \{\mathbf{0}, |e_1\rangle\langle e_1|, \mathbf{1} - |e_1\rangle\langle e_1|, \mathbf{1}\}$.

Example 3. *Take* $P = |e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|$ *and* $Q = |e_1\rangle\langle e_1|$. *These commute since* $\mathrm{ran}(Q) \subseteq \mathrm{ran}(P)$, *so* $QP = Q = PQ$. *We have* $P \vee Q = P$. *By Theorem 1,*

$$P \circ Q = \{R \in \mathcal{L}(\mathcal{H}) \mid R \leq P,\ [R, P] = 0,\ [R, Q] = 0\} = \{\mathbf{0}, |e_1\rangle\langle e_1|, |e_2\rangle\langle e_2|, P\}.$$

The hyperproduct is the Boolean interval of subprojections of P *that commute with both* P *and* Q *(here* $\{\mathbf{0}, |e_1\rangle\langle e_1|, |e_2\rangle\langle e_2|, P\}$*), not the set of all projectors* $R \leq P$.

4.3. Two-Qubit System

Let $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$, with computational basis $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$.

Example 4. *Write the Bell vectors* $|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ *and* $|\Psi^{+}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, *and define the rank-two projectors*

$$P = |\Phi^{+}\rangle\langle\Phi^{+}| + |01\rangle\langle 01|, \qquad Q = |\Phi^{+}\rangle\langle\Phi^{+}| + |\Psi^{+}\rangle\langle\Psi^{+}|.$$

Both contain the entangled direction $|\Phi^{+}\rangle$ *as an eigenvector of eigenvalue* 1, *but they are incompatible:* $|01\rangle$ *is an eigenvector of* P, *while* $Q|01\rangle = \frac{1}{2}(|01\rangle + |10\rangle)$ *is not proportional to it, so* $[P, Q] \neq 0$. *Their ranges span the three-dimensional subspace* $\mathrm{ran}(P) + \mathrm{ran}(Q) = \mathrm{span}\{|\Phi^{+}\rangle, |01\rangle, |10\rangle\}$.

A projector commutes with both P *and* Q *exactly when it preserves their common eigendirections. These are* $|\Phi^{+}\rangle$ *(eigenvalue* $(1, 1)$*) and* $|\Phi^{-}\rangle$ *(eigenvalue* $(0, 0)$*); on the remaining plane* $\mathrm{span}\{|01\rangle, |10\rangle\}$, *the projectors* P *and* Q *act as the distinct rank-one projectors* $|01\rangle\langle 01|$ *and* $|\Psi^{+}\rangle\langle\Psi^{+}|$, *whose joint commutant is scalar. Imposing in addition* $\mathrm{ran}(R) \subseteq \mathrm{ran}(P) + \mathrm{ran}(Q)$ *removes the* $|\Phi^{-}\rangle$ *block, and one obtains*

$$P \circ Q = \big\{\mathbf{0},\ |\Phi^{+}\rangle\langle\Phi^{+}|,\ |01\rangle\langle 01| + |10\rangle\langle 10|,\ |\Phi^{+}\rangle\langle\Phi^{+}| + |01\rangle\langle 01| + |10\rangle\langle 10|\big\}.$$

The non-trivial atom $|\Phi^{+}\rangle\langle\Phi^{+}|$ *projects onto an entangled state: The hyperoperation singles out, among the propositions jointly compatible with two incompatible rank-two questions, the Bell proposition on which both agree. This is the two-qubit analogue of the common-eigenline phenomenon of Example 2, now carried by a genuinely non-separable vector, and shows that* $\circ$ *can return entangled propositions.*

Remark 13 (Companion interactive tool)**.** *The examples above, and arbitrary instances of the hyperoperation in dimension up to twelve, can be explored interactively in the browser companion* HyperOn, *at the address https://canjioh.github.io/hyperon/ (accessed on 25 November 2026). Given two projectors* P, Q *entered either as spanning vectors of their ranges or as matrices, the tool computes the joint eigenspaces and the principal-angle blocks of Theorem 2, and the resulting elements of* $P \circ Q$, *flagging the cases in which a continuum of outcomes survives. A separate view tests associativity by comparing* $(P \circ Q) \circ R$ *with* $P \circ (Q \circ R)$ *on a chosen triple. It is a single self-contained web page, with no external dependencies. Since the hyperproduct is a unitary*

invariant of the pair (Remark 7), the tool also generates families of worked examples in higher dimension from a single representative, and every computation in Section 4 was checked with it.

5. Connections with Related Structures

The neighbouring constructions are recorded in Table 1. We keep here only the two comparisons that yield a statement about $\circ$, and we omit restating the source definitions.

Table 1 locates $\circ$ against the neighbouring frameworks.

Table 1. The quantum hyperoperation in relation to neighbouring algebraic frameworks.

Framework	Objects	Operation	Relation to $\circ$
Quantum hyperstructure $(\mathcal{F}, \circ)$	sharp projectors of $\mathcal{H}$	set-valued: range condition and commutation with both arguments	reference construction; multivalued, associative only on Boolean cores
Classical hyperstructures [4,5]	a set G with $\star : G \times G \to \mathcal{P}(G) \setminus \{\emptyset\}$	total; closure only	$(\mathcal{L}(\mathcal{H}), \circ)$ is a concrete geometric instance and a commutative hypergroup on each Boolean core (Remark 11)
Quantum suplattices [24]	quantum posets, from quantised sets	supremum-preserving maps	a different category; the shadow functor of $\circ$ lands in compatible-join posets
Hyper effect algebras [9]	effects, including unsharp ones	partial sum $+_h$, associative by axiom	on sharp projectors; fails associativity, complement, top and antisymmetry
Hyper-join, hyper-meet, and hyper-commutator [10,11]	elements of an effect algebra	minimal or maximal common bounds	single-valued on $\mathcal{L}(\mathcal{H})$ (hyper-join $= \{P \vee Q\}$); $\circ$ stays multivalued
Mosaics and L-mosaics [22,23]	hypermagmas; mosaics are unital and reversible	total hyperoperation; associativity not assumed	commutative hypermagma, not a mosaic ($\mathbf{0}, \mathbf{1}$ are not identities); see (AQ1)–(AQ8)

Note: $\mathbf{0}$ and $\mathbf{1}$ denote the zero operator and the identity operator on $\mathcal{H}$ (Section 2).

5.1. The Compatible-Join Shadow

A quantum suplattice in the sense of Jenča and Lindenhovius [24] is a quantum poset with a lower Galois adjoint for the down-set embedding, not an orthomodular lattice restricted to commuting joins. Ordinary Boolean algebras need not be quantum suplattices [24] (Section 12) , and we claim no functor into that category. What $\circ$ determines is elementary.

Definition 5. *A* compatible-join poset *is a bounded poset $(L, \leq)$ with least element 0 and a symmetric binary relation C such that $a \leq b$ implies $a \mathsf{C} b$, and such that every C-related pair has a least upper bound in L. A morphism of compatible-join posets is an order-preserving map sending 0 to 0, preserving C, and preserving least upper bounds of C-related pairs.*

By a *morphism* of quantum hyperstructures $f : (\mathcal{F}_1, \circ) \to (\mathcal{F}_2, \circ)$, we mean a map with $f(\mathbf{0}) = \mathbf{0}$ and $f(P \circ Q) \subseteq f(P) \circ f(Q)$ for all P, Q.

Proposition 5. *Let $(\mathcal{F}, \circ)$ be a quantum hyperstructure whose collection $\mathcal{F}$ is closed under the join of any commuting pair. Define*

$$A \preceq B \iff A \in \mathbf{0} \circ B, \qquad A \mathsf{C} B \iff [A, B] = 0.$$

Then, $(\mathcal{F}, \preceq, \mathsf{C})$ is a compatible-join poset, and $(\mathcal{F}, \circ) \mapsto (\mathcal{F}, \preceq, \mathsf{C})$ is a functor into the category of compatible-join posets. Explicitly:

(i) $\preceq$ *is the projector order* $\leq$ *restricted to* $\mathcal{F}$*, and* $A \preceq B$ *implies* $A \mathrel{\mathsf{C}} B$*.*
(ii) *If* $A \mathrel{\mathsf{C}} B$*, the join* $A \vee B$ *of* $\mathcal{L}(\mathcal{H})$ *lies in* $\mathcal{F}$ *and is the least upper bound of* $\{A, B\}$ *in* $(\mathcal{F}, \preceq)$*.*
(iii) *A morphism of join-closed quantum hyperstructures preserves* $\preceq$*, preserves* C*, and preserves joins of* C*-related pairs.*

Proof. **(i)** $\mathbf{0} \circ B = \{R \in \mathcal{F} \mid R \leq B\}$ by Proposition 2(i), so $A \preceq B \iff A \leq B$; in particular, $\preceq$ is a partial order. For projectors, $A \leq B$ implies $[A, B] = 0$, hence $A \mathrel{\mathsf{C}} B$.

(ii) Let $[A, B] = 0$. Join-closure puts $A \vee B \in \mathcal{F}$, and $A \vee B$ commutes with A and B, so $A, B \preceq A \vee B$. If $D \in \mathcal{F}$ satisfies $A, B \preceq D$, then $A, B \leq D$, hence $A \vee B \leq D$, and $A \vee B \leq D$ forces $[A \vee B, D] = 0$, i.e., $A \vee B \preceq D$. Thus, $A \vee B$ is the least upper bound of $\{A, B\}$ in $(\mathcal{F}, \preceq)$.

(iii) Let $f : (\mathcal{F}_1, \circ) \to (\mathcal{F}_2, \circ)$ be a morphism, both collections join-closed. *Order.* If $A \preceq B$, then $A \in \mathbf{0} \circ B$, so $f(A) \in f(\mathbf{0} \circ B) \subseteq f(\mathbf{0}) \circ f(B) = \mathbf{0} \circ f(B)$, that is, $f(A) \preceq f(B)$. *Compatible joins.* Suppose $[A, B] = 0$. By Proposition 2(iii), $A \vee B$ is the greatest element of $A \circ B$; applying f and the morphism inclusion gives $f(A \vee B) \in f(A) \circ f(B)$. Since $A \in A \circ B$ (Proposition 2(ii)), also $f(A) \in f(A) \circ f(B)$, so $[f(A), f(B)] = 0$, that is, $f(A) \mathrel{\mathsf{C}} f(B)$; join-closure of $\mathcal{F}_2$ then gives, again by Proposition 2(iii), that $f(A) \vee f(B)$ is the greatest element of $f(A) \circ f(B)$. As $f(A \vee B)$ belongs to $f(A) \circ f(B)$, we get $f(A \vee B) \leq f(A) \vee f(B)$, while $f(A), f(B) \preceq f(A \vee B)$ by order-preservation gives the reverse inequality. Hence, $f(A \vee B) = f(A) \vee f(B)$. □

Remark 14. *The functor retains* $\leq$ *and the joins of commuting pairs, and it discards every other element of a value* $P \circ Q$*. In finite dimension, Proposition 3 still places the lattice join* $P \vee Q$ *at the top of* $P \circ Q$ *whether or not* P *and* Q *commute. If they do not, the relation* C *does not contain the pair, and a morphism of compatible-join posets is not required to preserve that join.*

5.2. Hyper Effect Algebras

Effect algebras [25] model unsharp questions; hyper effect algebras [9] and the hyper-join of Ji [10,11] are the corresponding set-valued operations. On the lattice $\mathcal{L}(\mathcal{H})$, the hyper-join collapses to the singleton $\{P \vee Q\}$. The operation $\circ$ does not. The only comparison we need is the following, on orthogonal pairs.

Proposition 6. *Let* $P, Q \in \mathcal{L}(\mathcal{H})$ *satisfy* $P \perp Q$ *and* $P, Q \neq \mathbf{0}$*. Then,*

$$\{P + Q\} \subsetneq P \circ Q \subsetneq \{R \in \mathcal{L}(\mathcal{H}) : R \leq P + Q\}.$$

Thus, $P \circ Q$ *coincides neither with the singleton hyper-sum nor with the interval hyper-sum of the effect algebra* $\mathcal{L}(\mathcal{H})$*.*

Proof. Orthogonality gives $[P, Q] = 0$ and $P \vee Q = P + Q$. Corollary 2 identifies $P \circ Q$ with the set of projectors $R \leq P + Q$ that commute with both P and Q. That set contains $\mathbf{0}$, P, Q and $P + Q$. Since $Q \neq \mathbf{0}$, one has $P \neq P + Q$, and the singleton $\{P + Q\}$ is a proper subset. The same corollary records that for unit vectors $e_P \in \mathrm{ran}(P)$ and $e_Q \in \mathrm{ran}(Q)$, the rank-one projector on $\mathbb{C}(e_P + e_Q)$ lies under $P + Q$ and does not commute with P, so the inclusion in the order interval is proper as well. □

Remark 15. *Restricting* $\circ$ *to commuting pairs yields a partial hyperoperation which fails the associativity, unique-complement, top and antisymmetry axioms of a hyper effect algebra [9]. Associativity already fails for the pairwise orthogonal triple of Remark 9. A hyper effect sum on the full effect interval* $E(\mathcal{H}) = \{A : 0 \leq A \leq \mathbf{1}\}$*, agreeing with the operator sum on orthogonal effects, is a different construction and is left open in Section 6.*

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6. Conclusions

We have proposed the quantum hyperoperation as an algebraic device for the one thing standard quantum logic cannot express: the way two incompatible sharp propositions combine. Starting from Piron's projector lattice, we replaced the single deterministic join $P \vee Q$ by the multivalued $P \circ Q$, the set of sharp propositions compatible with both factors and supported in the sum of their ranges. This turns the indeterminacy of incompatible measurement, usually visible only through the failure of distributivity, into a positive, computable object. We recorded the elementary properties of $\circ$ and showed that when the factors commute, it reduces to the residual-compatibility filter of Theorem 1. On $\mathcal{L}(\mathcal{H})$, that filter properly contains the spectral projections of $W^*(P, Q)$ supported on $P \vee Q$ precisely when a joint eigenspace inside the support has dimension greater than one, and it is strictly smaller than the order interval below the join except in the degenerate cases of Corollary 1. We also traced the exact boundary at which associativity holds: It fails already for a commuting triple in $\mathcal{L}(\mathcal{H})$, yet is restored on every Boolean core, so that the semihypergroup law marks precisely the classical fragments of the theory. In finite dimension, Theorem 2 identifies $P \circ Q$ for an arbitrary pair: The outcome is read off the joint eigenspaces $\mathcal{H}_{ij}$ and the principal-angle blocks between the two ranges. The examples in $\mathbb{C}^2$, $\mathbb{C}^3$ and $\mathbb{C}^4$, including hyperproducts that single out a shared entangled proposition, are the low-dimensional cases of that identification.

The abstract quantum hyperstructures of Section 3 isolate what is essential in the construction. A single hyperoperation generates both the order and the compatibility relation: Order is $a \in 0 \circ b$ and compatibility is $a \in a \circ b$. Proposition 5 sends a join-closed quantum hyperstructure to its compatible-join shadow. That shadow is not a quantum suplattice in the sense of Jenča and Lindenhovius: Their objects are quantum posets with a lower Galois adjoint for the down-set embedding, and ordinary Boolean algebras need not be among them. The comparison with hyper effect algebras is a separation of operations. On a pair of nonzero orthogonal projectors, $P \circ Q$ lies strictly between the singleton hyper-sum and the interval hyper-sum of Dvurečenskij and Hyčko (Proposition 6), and the partial hyperoperation induced by $\circ$ on commuting pairs fails their associativity, complement, top and antisymmetry axioms (Remark 15).

The novelties introduced here can be stated compactly. The composition $\circ$ is genuinely multivalued on the whole projection lattice, where the minimal-upper-bound hyperoperations of the effect-algebra literature reduce to a single join; its values are sets because compatibility gates the sub-projectors of an existing join. In finite dimension, the hyperproduct is classified completely by the joint eigenspaces and the principal angles and is represented as the projection lattice $\mathrm{Proj}(e\mathcal{W}e)$ of a compressed commutant, a representation covariant under unitary conjugation. Associativity is delimited exactly, being restored on the Boolean cores, each of which is a commutative hypergroup under $\circ$. The abstract axiomatisation recovers both the order and the compatibility relation from the single operation, and the comparison of Table 1 separates the construction from hyper effect algebras, from the hyperoperations on effect algebras, and from quantum suplattices.

These results open a research programme, whose principal lines we single out. The first is representational: to characterise which abstract quantum hyperstructures arise from projectors, a hyperoperational counterpart of Piron's reconstruction theorem. Separate from that is the unsharp regime. A hyper effect sum in the sense of Dvurečenskij and Hyčko [9], defined on the full effect interval $E(\mathcal{H}) = \{A : 0 \leq A \leq \mathbf{1}\}$ and reducing to the operator sum on orthogonal effects, is not supplied by $\circ$, and constructing one is open. The same formula that defines $\circ$ on $\mathcal{L}(\mathcal{H})$ makes sense in an arbitrary orthomodular lattice (Proposition 1); a Piron-type representation theorem for that abstract hyperoperation is the natural next step. The infinite-dimensional form of Theorem 2 is open for the reason given

in Remark 8: When $\mathrm{ran}(P) + \mathrm{ran}(Q)$ fails to be closed, a commutant projection below $P \vee Q$ need not lie in $P \circ Q$. The last question is the fine structure of associativity, in particular whether the maximal associative subcollections of $\mathcal{L}(\mathcal{H})$ are exactly the Boolean cores, which would tie the failure of the semihypergroup law directly to non-distributivity.

Beyond these, two longer-range directions stand out. A probabilistic enrichment would attach to $P \circ Q$ a canonical measure weighting its elements, turning the set-valued composition into a bona fide quantum-probabilistic hypergroup. And a categorical reading, interpreting $\circ$ as a multivalued morphism in the diagrammatic language of Abramsky and Coecke [26], would connect the present framework to categorical quantum mechanics and to the relation and span categories where multivalued composition is native [27]. Together, these would develop the account of incompatible composition given here into a fuller theory of quantum indeterminacy.

Seen against the broader foundational discussion, the hyperoperation is deliberately modest in its commitments. It adds no dynamics, no collapse rule and no preferred notion of state, and it does not adjudicate between ontic, epistemic and relational readings of the quantum formalism. What it supplies is structural: a precise, representation-independent record of how far two incompatible sharp propositions can still be combined, expressed in the same projector-lattice language that the operational reconstructions of quantum logic take as primitive. In this respect, the construction is closer to the reconstruction and *foil-theory* programmes, which ask what minimal algebraic data force the quantum phenomena, than to any single interpretation. Whether the compositional indeterminacy isolated here is an artefact of the lattice formalism or a genuine structural fact about quantum propositions is, we believe, a question worth posing, and the present construction is one precise way to pose it. More broadly, the construction illustrates what a dialogue between quantum logic and the theory of hyperstructures, two traditions that have so far developed apart, can offer the foundations of quantum mechanics. Approached here from the quantum-logic side, their junction yields concrete structure, such as the classification of Section 3, and opens a programme rather than a single result.

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