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A Derivation of the Entangled State Representation by the Slice Theorem of the Wigner Operator

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Abstract

The Wigner operator's normal ordering form is deduced by using the method of integration within the ordered product of operators, and the operator's Weyl ordering symbol is employed. The integration theory within the Weyl ordering product of operators is applied, and the Wigner operator's Weyl ordering form is deduced. Then, the Wigner operator's slice theorem is proposed, which helps project and display a new pure-state density operator. Thus, the quantization of classical tomography theory is realized. We illustrate the derivation of the bi- and tri-partite entangled state representations, respectively, which completes the argument.

Keywords: Weyl ordering; Wigner operator; tomography theory; entangled state representations

1. Introduction

In recent years, the tomographic imaging technique for reconstructing Wigner functions based on the Radon transform has attracted significant attention. As an integral transform method, the core idea of the Radon transform lies in converting two-dimensional images into one-dimensional projection data by performing line integrals at different angular orientations, thereby establishing a mathematical bridge between the original spatial distribution and the projection measurements. Since its introduction, this transform method has been widely applied in fields such as machine vision, medical imaging, geophysics, and computed tomography due to its unique advantages in solving inverse problems. In computed tomography technology, by acquiring projection data of an object from multiple angles and applying the inverse Radon transform, the internal two-dimensional or three-dimensional structural images of the object can be accurately reconstructed. This non-invasive imaging approach not only reveals the internal spatial morphology of the object but also further uncovers key attributes such as density distribution and compositional characteristics, thereby providing a powerful mathematical foundation for structural analysis and feature extraction. The application scenarios of the Radon transform extend far beyond medical imaging. In machine vision, it is employed for shape recognition and edge detection; in geophysics, it assists researchers in inverting subsurface structures from surface measurement data; in materials science, it enables quantitative analysis in three-dimensional microscopic imaging. By virtue of precise mathematical correspondence it establishes between projections and reconstructions, the Radon transform has laid the core theoretical foundation for tomographic imaging techniques. With the rapid advancement of quantum technology, the application of the Radon transform has further extended into the

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field of quantum mechanics, providing important theoretical support for the reconstruction of Wigner functions in quantum state tomography.

In quantum mechanics, the "interior" of a microphysical entity does not represent a classical structure since it diverts from certain properties in a classical sense. Alternatively, its essence characterizes a probabilistic setting, a wave function. For example, as the Heisenberg uncertainty principle suggests, the coordinates and momentum information of particles cannot be measured precisely at the same time, so all the information of the quantum state cannot be obtained concurrently. However, diverse information can be captured by making many observations on the same quantum state. Then, the characteristics of the quantum state are inferred from a running series of experiments to collect data. Physicists have developed the tomography theory of phase space using the Wigner function. Its connotation is that as long as the edge distribution of the Wigner function is measured in all directions of a light field, all the information of the light field can be attained. The quantization of phase space, mathematically represented by $\oint pdx = n\hbar$, originated from Bohr's atomic orbital quantization and consists of the parameters of p, x, $\hbar$, and n corresponding to the particle momentum, the coordinate, the Planck constant, and a positive integer, respectively. However, due to the constraint of Heisenberg's uncertainty principle, the coordinates and momentum values of particles cannot be accurately measured simultaneously. Wigner introduced a function in quantum phase space to describe the quasi-probability distribution of particles [1]. Moreover, building upon a foundational question formulated by him, Feynman distilled the conceptual essence of the Wigner function into an insightful summary, which is expressed as: whether a density function, $W(x, p)$, exists in a phase space that satisfies the relationships in Equations (1) and (2) [2]:

$$P(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dpW(x, p), \tag{1}$$

$$P(p) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dxW(x, p). \tag{2}$$

where $P(x)[P(p)]$ represents a probability of finding a particle at position x (with momentum p) in coordinate space [momentum space], which is called the two-edge distributions of the Wigner function.

The essence of quantum tomography lies in reconstructing the quantum state's Wigner function [3–6] by measuring a complete set of projection distributions. The Wigner operator serves as the integral kernel of the Wigner function, which is the quasi-probability distribution of the quantum state in phase space, and its projection measurement results correspond exactly to the mathematical form of the classical Radon transform. By applying the inverse Radon transform to the measurement data acquired across different phase directions, the pure-state density operator can be directly projected, thereby completely reconstructing the quantum state's Wigner function. This process not only enables quantum state tomography and quantum state discrimination but also provides important theoretical support for constructing multi-mode continuous-variable entangled state representations and promoting the practical application of quantum states.

The objective of the article is to construct the Wigner operator slicing theorem and provide its implementation [7–13]. To this end, we naturally introduce the normal ordering form of the Wigner operator, which follows a normal distribution, and the utilization of the operator's Weyl ordering symbol $\vdots\ \vdots$. Thus, we utilize the theory of quantum phase space and the method of integration within the Weyl ordering product of operators [14–18] and deduce the Wigner operator's Weyl ordering form, $\vdots\delta(x - X)\delta(p - P)\vdots$, where X, P

represent coordinate and momentum operators, respectively. Then, the Wigner operator slicing theorem is finally proposed, which is employed to project a new pure density of states operator and achieve the quantization of classical tomography theory. Note that the research on quantum entangled states has been rising in popularity in recent years, and the multi-body entangled state representation of continuous variables has not been well-studied yet [19–22]. Thus, the Wigner operator slicing theorem is also employed to derive the two- and three-mode entangled state representations. This paper organically integrates the integration method within the ordered product of operators, quantum phase space theory, and quantum tomography concepts, providing a theoretical foundation for operator ordering transformation and integral operations in quantum phase space. It reveals the intrinsic connection between quantum state tomography and phase-space distribution, addresses the research gap in multi-mode continuous-variable entangled state representations, and offers new approaches for quantum state reconstruction and the treatment of many-body quantum correlation problems.

2. The Normal Distribution of the Wigner Operator

In 1932, Eugene Wigner first introduced the Wigner function as a quasi-probability distribution description of quantum states in phase space. Through the Wigner operator, the density operator can be mapped to a phase-space distribution, transforming all the information of a quantum state into a functional form in phase space. This provides the possibility for intuitive description and quantitative analysis of quantum states, not only laying the theoretical foundation for quantum state tomography, but also playing an indispensable role in characterizing non-classical properties of quantum states, studying quantum dissipation laws, analyzing many-body entangled systems, and establishing classical–quantum correspondences.

To define the Wigner operator from the physical meaning of the Wigner distribution function, Equations (1) and (2) can be re-expressed by the Dirac symbol as follows:

$$P(x) = \langle \psi \left| \int_{-\infty}^{+\infty} dp \Delta(x,p) \right| \psi \rangle, \tag{3}$$

$$P(p) = \langle \psi \left| \int_{-\infty}^{+\infty} dp \Delta(x,p) \right| \psi \rangle. \tag{4}$$

where $\Delta(x,p)$ represents the Wigner operator. By combining the method of integration within the ordered product (IWOP) of operators [23–27], $|x\rangle\langle x|$ and $|p\rangle\langle p|$ can be transformed into the normal distribution representations in Equations (5) and (6):

$$|x\rangle\langle x| = \frac{1}{\sqrt{\pi}} : e^{-(x-\hat{X})^2} :, \tag{5}$$

$$|p\rangle\langle p| = \frac{1}{\sqrt{\pi}} : e^{-(p-\hat{P})^2} :, \tag{6}$$

where $|x\rangle\langle x|$ denotes the coordinate projection operator, and $|p\rangle\langle p|$ designates the momentum projection operator; the symbol : : denotes the normal ordering of the operator [28–30], $\hat{X}$ represents the coordinate operator, and $\hat{P}$ designates the momentum operator. Equation (7) presents them separately:

$$\hat{X} = \frac{a + a^\dagger}{\sqrt{2}}, \hat{P} = \frac{a - a^\dagger}{\sqrt{2}i}, \tag{7}$$

where $a, a^\dagger$ represent the Bose annihilator and creator operators, respectively, where $\left[a, a^\dagger\right] = 1$. The solution of the Wigner operator can be obtained by using Equations (5) and (6), and the IWOP method, expressed by Equation (8):

$$\Delta(x,p) = \frac{1}{\pi} : e^{-(x-\hat{X})^2-(p-\hat{P})^2} : . \tag{8}$$

The Wigner operator gives, $\int_{-\infty}^{+\infty} dp\Delta(x,p) = \frac{1}{\sqrt{\pi}} : e^{-(x-\hat{X})^2} := |x\rangle\langle x|$ and $\int_{-\infty}^{+\infty} dx\Delta(x,p) = \frac{1}{\sqrt{\pi}} : e^{-(p-\hat{P})^2} := |p\rangle\langle p|$, so Equation (8) satisfies the normal distribution form. The Wigner function of a quantum state ρ is $tr[\rho\Delta(x,p)]$, and all the information of the quantum state can be understood indirectly by measuring it.

3. Weyl Ordering Form of Wigner Operator: Delta Operator Form

The most fundamental role of Weyl ordering lies in providing a canonical mapping method from classical mechanics to quantum mechanics. It consistently plays an important role in foundational research in quantum theory, applications in quantum information science, and studies in quantum–classical interdisciplinary fields.

By using the IWOP method, the $\Delta(x,p)$ in Equation (8) can be expanded to an integration within a normal ordering as follows:

$$\Delta(x,p) = \frac{1}{\pi} : e^{-(x-\hat{X})^2-(p-\hat{P})^2} := \int\int_{-\infty}^{\infty} \frac{dudv}{4\pi^2} : \exp\left[i(x-\hat{X})u + i(p-\hat{P})v - \frac{u^2+v^2}{4}\right] : . \tag{9}$$

Then, using the Baker–Hausdorff expression, Equation (9) can be transformed into:

$$\Delta(x,p) = \int\int_{-\infty}^{\infty} \frac{dudv}{4\pi^2} \exp\left[i(x-\hat{X})u + i(p-\hat{P})v\right], \tag{10}$$

The main purpose of the operator Weyl ordering symbol $\vdots\vdots$ is to quantize the classical function $e^{\lambda p}e^{\sigma x}$ into the operator $e^{\lambda\hat{P}}e^{\sigma\hat{X}}$, represented as follows:

$$e^{\lambda p}e^{\sigma x} \to e^{\lambda\hat{P}}e^{\sigma\hat{X}}, \tag{11}$$

$e^{\lambda\hat{P}+\sigma\hat{X}}$ is different from $e^{\lambda\hat{P}}e^{\sigma\hat{X}}$ and $e^{\sigma\hat{X}}e^{\lambda\hat{P}}$, so it is given a symbol $\vdots\vdots$, and then Equation (11) can be rewritten as follows:

$$e^{\lambda\hat{P}+\sigma\hat{X}} = \vdots e^{\lambda\hat{P}+\sigma\hat{X}} \vdots, \tag{12}$$

Just as the Bose creation operator and annihilation operators can be exchanged within a normal ordering symbol : :, so do the operators $\hat{P}$ and $\hat{X}$ within a Weyl ordering symbol $\vdots\vdots$. Therefore, according to Equation (12), Equation (10) can be rewritten as follows:

$$\begin{aligned} \Delta(x,p) &= \int\int_{-\infty}^{\infty} \frac{dudv}{4\pi^2} : \exp\left[i(x-\hat{X})u + i(p-\hat{P})v\right] : \\ &= \vdots \ \delta(x-\hat{X})\delta\ (p-\hat{P}) \ \vdots , \end{aligned} \tag{13}$$

Equation (13) represents the Weyl ordering form of the Wigner operator, a Delta operator form. Then, Equation (14) is attained as follows:

$$\vdots f(\hat{X},\hat{P}) \vdots = \int\int_{-\infty}^{\infty} f(x,p)\Delta(x,p)dxdp. \tag{14}$$

Equation (14) implies that the classical correspondence of a Weyl ordering operator $\vdots f(\hat{X},\hat{P}) \vdots$ turns out to be exactly $f(x,p)$.

4. The Slicing Projection of the Wigner Operator to Derive a New Pure-State Density Operator

Assume that $f(p,x) = \delta(u - \lambda x - \tau p)$, according to Equation (14), where λ, τ denote real numbers, and Equation (15) is attained by using the IWOP method and performing a projection transformation on the Wigner operator in phase space:

$$\int\int_{-\infty}^{\infty} \delta(u - \lambda x - \tau p)\Delta(x,p)dxdp = \vdots \delta\left(u - \lambda\hat{X} - \tau\hat{P}\right) \vdots, \tag{15}$$

Then, $\lambda = \cos\theta, \tau = \sin\theta$ are set and $f(p,x)$ represents the image function, and $\delta(u - \lambda x - \tau p)$ in Equation (15) represents a ray equation on the (x,p) plane, expressed by:

$$x\cos\theta - p\sin\theta = u, \tag{16}$$

Equation (16) is the Radon Transform, an integral transform proposed by the Austrian mathematician Johann Radon in 1917, which integrates a two-dimensional function (such as an image) along all possible lines to obtain a set of projection data. The core application of the Radon transform is to reconstruct images or distributions from projection data, and it is widely used in fields such as medical CT imaging, quantum state tomography, atmospheric remote sensing, and biological tissue structure analysis. The projection $P_\theta(u)$ formed by integrating the image function $f(p,x)$ along a set of rays determined by θ is commonly called parallel projection, which satisfies Equation (17):

$$P_\theta(u) = \int\int_{-\infty}^{\infty} f(x,p)\delta(u - x\cos\theta - p\sin\theta)dxdp, \tag{17}$$

Equation (17) is known as the Fourier slice-projection theorem, which is widely implemented in image processing, speech signal analysis, and medical imaging. Hence, Equation (15) represents the sliced projection of the Wigner operator.

On the other hand, combining Equation (8) and the method of within normal ordering operator, Equation (15) can be rewritten as follows:

$$\int\int_{-\infty}^{\infty} \delta(u - \lambda x - \tau p)\Delta(x,p)dxdp = \frac{1}{\sqrt{\pi(\tau^2+\lambda^2)}} : \exp\left[-\frac{1}{\tau^2+\lambda^2}\left(u - \lambda\hat{X} - \tau\hat{P}\right)^2\right] : . \tag{18}$$

Substituting $\hat{X} = \frac{a+a^\dagger}{\sqrt{2}}, \hat{P} = \frac{a-a^\dagger}{\sqrt{2}i}$ into Equation (18) with the utilization of the Weyl ordering form of the vacuum projection operator $|0\rangle\langle 0|$ leads to Equation (19):

$$\vdots e^{-a^\dagger a} \vdots = |0\rangle\langle 0|, \tag{19}$$

Then, Equation (20) is attained as follows:

$$:\exp\left[-\frac{1}{\lambda^2+\tau^2}\left(u-\lambda\hat{X}-\tau\hat{P}\right)^2\right]:=e^{\frac{\sqrt{2}ua^\dagger}{\lambda-i\tau}-\frac{u^2}{2(\lambda^2+\tau^2)}-\frac{\lambda+i\tau}{2(\lambda-i\tau)}a^{\dagger 2}}|0\rangle\langle 0|e^{\frac{\sqrt{2}ua}{\lambda+i\tau}-\frac{u^2}{2(\lambda^2+\tau^2)}-\frac{\lambda-i\tau}{2(\lambda+i\tau)}a^2}, \tag{20}$$

So,

$$\int\int_{-\infty}^{\infty}dxdp\delta(u-\lambda x-\tau p)\Delta(x,p)=|u\rangle_{\lambda,\tau\ \lambda,\tau}\langle u|, \tag{21}$$

where

$$|u\rangle_{\lambda,\tau}=\left[\pi\left(\lambda^2+\tau^2\right)\right]^{-1/4}e^{\frac{\sqrt{2}ua^\dagger}{\lambda-i\tau}-\frac{u^2}{2(\lambda^2+\tau^2)}-\frac{\lambda+i\tau}{2(\lambda-i\tau)}a^{\dagger 2}}|0\rangle. \tag{22}$$

represents a new state vector. Equation (21) shows that the classical Weyl corresponding function of the pure-state operator $|u\rangle_{\lambda,\tau\ \lambda,\tau}\langle u|$ is represented by $\delta(u-\lambda x-\tau p)$, and also indicates that the Weyl ordering form of $|u\rangle_{\lambda,\tau\ \lambda,\tau}\langle u|$ is given by:

$$|u\rangle_{\lambda,\tau\ \lambda,\tau}\langle u|=\vdots\delta\left(u-\lambda\hat{X}-\tau\hat{P}\right)\vdots, \tag{23}$$

where $|u\rangle_{\lambda,\tau}$ exactly represents the eigenstate of the operator, $\lambda\hat{X}+\tau\hat{P}$, that is:

$$\left(\lambda\hat{X}+\tau\hat{P}\right)|u\rangle_{\lambda,\tau}=u|u\rangle_{\lambda,\tau}. \tag{24}$$

Therefore, $|u\rangle_{\lambda,\tau}$ denotes the coordinate–momentum intermediary representation between momentum and coordinate representations. Moreover, the previous calculations help the slice projection transformation of the single-mode Wigner operator derive the coordinate–momentum intermediate representation. In addition, by using the IWOP method, we found that $|u\rangle_{\lambda,\tau}$ satisfies the completeness relation:

$$\int_{-\infty}^{\infty}du|u\rangle_{\lambda,\tau\ \lambda,\tau}\langle u|=\frac{1}{\sqrt{\pi(\lambda^2+\tau^2)}}\int_{-\infty}^{\infty}du:e^{-\frac{1}{\lambda^2+\tau^2}\left(u-\lambda\hat{X}-\tau\hat{P}\right)^2}:=1, \tag{25}$$

In addition, the orthogonality is represented by:

$${}_{\lambda,\tau}\left\langle u'\middle|u\right\rangle_{\lambda,\tau}=\delta\left(u-u'\right), \tag{26}$$

So, $|u\rangle_{\lambda,\tau\ \lambda,\tau}\langle u|$ represents the projection result of the Wigner operator on a ray. Finally, we summarize a slicing theorem of the Wigner operator. If a Hermitian operator $R\left(\hat{X},\hat{P}\right)$ possesses orthogonal eigenstates $|r\rangle$ with corresponding eigenvalues r, Equation (27) can be obtained:

$$\begin{aligned}r|r\rangle&=R\left(\hat{X},\hat{P}\right)|r\rangle,\\ \langle r'|r\rangle&=\delta(r'-r),\end{aligned} \tag{27}$$

and the Weyl ordering form of $R\left(\hat{X},\hat{P}\right)$ is represented by Equation (28):

$$R\left(\hat{X},\hat{P}\right)=\vdots R\left(\hat{X},\hat{P}\right)\vdots, \tag{28}$$

Then, there exists a projection transformation relationship:

$$|r\rangle\langle r|=\int\int_{-\infty}^{\infty}\delta[r-R(x,p)]\Delta(x,p)dxdp=\vdots\delta\left(r-\hat{R}\right)\vdots. \tag{29}$$

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$R(x,p)$ in Equation (29) is obtained by directly replacing $\hat{X}, \hat{P}$ in $\vdots R(\hat{X},\hat{P}) \vdots$, with x, p. From Equation (29), we can find that the classical Weyl correspondence function of the operator $|r\rangle\langle r|$ is represented by $\delta[r - R(x,p)]$. The equality in Equation (29) is proven as follows.

From the property of the delta function and Equation (27), we have:

$$\begin{aligned} &|r\rangle = \int_{-\infty}^{\infty} dr' \delta(r - r')|r'\rangle \\ &= \int_{-\infty}^{\infty} dr' \delta(r - G)|r'\rangle \\ &= \int_{-\infty}^{\infty} dr' \delta\left[r - \begin{array}{c}\vdots\\ R(\hat{X},\hat{P})\\ \vdots\end{array}\right]|r'\rangle \\ &= \int_{-\infty}^{\infty} dr' \int \int_{-\infty}^{\infty} dx dp \delta[r - R(x,p)] \begin{array}{c}\vdots\\ \delta(x-\hat{X})\delta(p-\hat{P})\\ \vdots\end{array}|r'\rangle \\ &= \int_{-\infty}^{\infty} dr' \int \int_{-\infty}^{\infty} dx dp \delta[r - R(x,p)]\Delta(x,p)|r'\rangle, \end{aligned} \tag{30}$$

So, we obtain:

$$\int \int_{-\infty}^{\infty} \delta[r - R(x,p)]\Delta(x,p) dx dp = |r\rangle\langle r|. \tag{31}$$

5. Using the Slicing Theorem of the Two-Mode Wigner Operator to Establish the Two-Mode Entangled State Representation

Equation (29) is utilized to investigate the relative momentum $\hat{P}_1 - \hat{P}_2$ and centroid coordinate $(\hat{X}_1 + \hat{X}_2)/2$ of two particles. Since $[\hat{X}_1 + \hat{X}_2,\ \hat{P}_1 - \hat{P}_2] = 0$, therefore, the relative momentum and centroid coordinates have a common eigenstate, which is denoted as $|\beta_1, \beta_2\rangle$. Then, the corresponding eigen-equations are presented as follows:

$$\begin{aligned} &(\hat{X}_1 + \hat{X}_2)|\beta_1,\beta_2\rangle = \beta_1|\beta_1,\beta_2\rangle, \\ &(\hat{P}_1 - \hat{P}_2)|\beta_1,\beta_2\rangle = \beta_2|\beta_1,\beta_2\rangle, \end{aligned} \tag{32}$$

where the eigenvalues β_1, β_2 correspond to the values in the classical phase space, namely, $(\hat{X}_1 + \hat{X}_2) \to (x_2 + x_1)$ and $(\hat{P}_1 - \hat{P}_2) \to (p_1 - p_2)$. The Weyl ordering of $(\hat{X}_2 + \hat{X}_1)(\hat{P}_1 - \hat{P}_2)$ is represented by:

$$(\hat{X}_2 + \hat{X}_1)(\hat{P}_1 - \hat{P}_2) = \vdots(\hat{X}_2 + \hat{X}_1)(\hat{P}_1 - \hat{P}_2)\vdots = \vdots(\hat{P}_1 - \hat{P}_2)(\hat{X}_2 + \hat{X}_1)\vdots, \tag{33}$$

Then, according to Equations (31) and (32), Equation (34) is attained as follows:

$$\begin{aligned} &|\beta_1,\beta_2\rangle\langle\beta_1,\beta_2| \\ &= \int_{-\infty}^{\infty} dp_1 dx_1 dp_2 dx_2 \delta[\beta_1 - (x_1 + x_2)]\delta[\beta_2 - (p_1 - p_2)]\Delta(p_1,x_1)\Delta(p_2,x_2). \end{aligned} \tag{34}$$

Equation (34) shows that the Weyl correspondence of the operator $|\beta_1,\beta_2\rangle\langle\beta_1,\beta_2|$ is represented by $\delta[\beta_1 - (x_1 + x_2)]\delta[\beta_2 - (p_1 - p_2)]$.

Next, the IWOP method and the two-mode Wigner operator are employed to derive the specific form of $|\beta_1,\beta_2\rangle\langle\beta_1,\beta_2|$. The normal ordering of the two-mode Wigner operator is represented by:

$$\Delta(p_1,x_1)\Delta(p_2,x_2) = \frac{1}{\pi^2} : e^{-(x_1-\hat{X}_1)^2 - (p_1-\hat{P}_1)^2 - (x_2-\hat{X}_2)^2 - (p_2-\hat{P}_2)^2} :, \tag{35}$$

Substituting Equation (35) into Equation (34), the Radon transformation is obtained as follows:

$$|\beta_1,\beta_2\rangle\langle\beta_1,\beta_2| = \int_{-\infty}^{\infty} dp_1 dx_1 dp_2 dx_2 \delta[\beta_1-(x_1+x_2)]\delta[\beta_2-(p_1-p_2)]\frac{1}{\pi^2} : e^{-(x_1-\hat{X}_1)^2-(p_1-\hat{P}_1)^2-(x_2-\hat{X}_2)^2-(p_2-\hat{P}_2)^2} :, \tag{36}$$

Operating x_2 on $\int_{-\infty}^{\infty} dx_2$ within the normal ordering symbol : : yields:

$$\int_{-\infty}^{\infty} dx_2 \delta[\beta_1-(x_1+x_2)] e^{-(x_2-\hat{X}_2)^2} = e^{-(\beta_1-x_1-\hat{X}_2)^2}, \tag{37}$$

Then, using the convolution formula leads to Equation (38):

$$\frac{1}{2\pi\sigma\tau}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2\sigma^2}} e^{-\frac{x^2}{2\tau^2}} dx = \frac{1}{\sqrt{2\pi(\sigma^2+\tau^2)}} e^{-\frac{y^2}{2(\sigma^2+\tau^2)}}. \tag{38}$$

Operating x_1 on $\int_{-\infty}^{\infty} dx_1$ yields:

$$\begin{aligned} &\int_{-\infty}^{\infty} dx_1 e^{-(x_1-\hat{X}_1)^2-(-x_1+\beta_1-\hat{X}_2)^2} \\ &= \sqrt{\frac{\pi}{2}} \exp\left\{-\frac{1}{2}\left[\beta_1-(\hat{X}_1+\hat{X}_2)\right]^2\right\}, \end{aligned} \tag{39}$$

So, Equation (40) is attained:

$$|\beta_1,\beta_2\rangle\langle\beta_1,\beta_2| = \frac{1}{2\pi} : e^{-\frac{1}{2}[\beta_1-(\hat{X}_1+\hat{X}_2)]^2-\frac{1}{2}[\beta_2-(\hat{P}_1-\hat{P}_2)]^2} : . \tag{40}$$

According to $\hat{X}_j = \frac{1}{\sqrt{2}}\left(a_j + a_j^\dagger\right)$, $\hat{P}_j = \frac{1}{i\sqrt{2}}\left(a_j - a_j^\dagger\right)$ and using the normal ordering form of the two-mode vacuum state [31–33] leads to Equation (41):

$$: \exp\left(-a_1^\dagger a_1 - a_2^\dagger a_2\right) := |00\rangle\langle 00|, \tag{41}$$

The right-hand side of Equation (40) is expanded to obtain:

$$|\beta_1,\beta_2\rangle = \frac{1}{\sqrt{2\pi}} e^{-\frac{\beta_1^2+\beta_2^2}{4}+\frac{\beta_1+i\beta_2}{\sqrt{2}}a_1^\dagger+\frac{\beta_1-i\beta_2}{\sqrt{2}}a_2^\dagger-a_1^\dagger a_2^\dagger}|00\rangle, \tag{42}$$

Assume $\xi = \frac{1}{\sqrt{2}}(\beta_1+i\beta_2)$, then Equation (42) leads to:

$$|\xi\rangle = \sqrt{2\pi}|\beta_1,\beta_2\rangle = e^{\xi a_1^\dagger - \frac{|\xi|^2}{2} - a_1^\dagger a_2^\dagger + \xi^* a_2^\dagger}|00\rangle, \tag{43}$$

where $|\xi\rangle$ represents a class of entangled states, integrating the entangled state $|\xi\rangle$ to obtain:

$$\int_{-\infty}^{\infty} \frac{d^2\xi}{\pi}|\xi\rangle\langle\xi| = 1. \tag{44}$$

Equation (44) indicates the completeness of the entangled state $|\xi\rangle$. So, we find that the entangled state representation can be derived by the Radon transform of the two-mode Wigner operator. Note that:

$$a_1|\xi\rangle = \left(\xi - a_2^\dagger\right)|\xi\rangle, a_2|\xi\rangle = \left(\xi^* - a_1^\dagger\right)|\xi\rangle. \tag{45}$$

Therefore, the entangled state $|\xi\rangle$ obeys eigenvector equations.

6. Using the Slicing Projection of the Three-Mode Wigner Operator to Derive a Three-Mode Entangled State Representation

Constructing common eigenvectors for three mutually commuting operators according to the method in the previous section. The common determined eigenstates by mutually commuting operators $\hat{X}_1 - \hat{X}_2$, $\hat{X}_1 - \hat{X}_3$ and $\hat{P}_1 + \hat{P}_2 + \hat{P}_3$ satisfy the following respective eigenvalue equations:

$$\begin{aligned}
&(\hat{X}_1 - \hat{X}_2)|\rho, \chi_1, \chi_2\rangle = \chi_1|\rho, \chi_1, \chi_2\rangle, \\
&(\hat{X}_1 - \hat{X}_3)|\rho, \chi_1, \chi_2\rangle = \chi_2|\rho, \chi_1, \chi_2\rangle, \\
&(\hat{P}_1 + \hat{P}_2 + \hat{P}_3)|\rho, \chi_1, \chi_2\rangle = \rho|\rho, \chi_1, \chi_2\rangle,
\end{aligned} \tag{46}$$

The next step is to derive the explicit expression for the operator $|\rho, \chi_1, \chi_2\rangle\langle\rho, \chi_1, \chi_2|$. According to Equation (29), the Weyl ordering form of $(\hat{X}_1 - \hat{X}_2)(\hat{X}_1 - \hat{X}_3)(\hat{P}_1 + \hat{P}_2 + \hat{P}_3)$ must be derived. Utilizing the IWOP method and operator commutation relations, Equation (47) can be derived:

$$\begin{aligned}
&\vdots\ \hat{X}^2\hat{P}\ \vdots = \left(\tfrac{1}{2}\right)^2(\hat{X}^2\hat{P} + 2\hat{X}\hat{P}\hat{X} + \hat{P}\hat{X}^2) \\
&= \left(\tfrac{1}{2}\right)^2\{-2i\hat{X} - [\hat{X}^2, \hat{P}] + 4\hat{X}^2\hat{P}\} \\
&= -iX + \hat{X}^2\hat{P},
\end{aligned} \tag{47}$$

Therefore, note that the Weyl ordering form of $\hat{X}^2\hat{P}$ is represented by $\vdots\ \hat{X}^2\hat{P}\ \vdots + iX$. Then, the subsequent calculations can be done as follows:

$$\begin{aligned}
&\vdots\ (\hat{X}_1 - \hat{X}_2)(\hat{X}_1 - \hat{X}_3)(\hat{P}_1 + \hat{P}_2 + \hat{P}_3)\ \vdots \\
&= \vdots\ (\hat{X}_1^2 - \hat{X}_2\hat{X}_1 - \hat{X}_1\hat{X}_3 + \hat{X}_2\hat{X}_3)(\hat{P}_1 + \hat{P}_2 + \hat{P}_3)\ \vdots \\
&= \hat{X}_1^2\hat{P}_1 - i\hat{X}_1 + \hat{X}_1^2(\hat{P}_2 + \hat{P}_3) - \hat{X}_2(\hat{X}_1\hat{P}_1 + \hat{P}_1\hat{X}_1)/2 \\
&-\hat{X}_1(\hat{X}_2\hat{P}_2 + \hat{P}_2\hat{X}_2)/2 - \hat{X}_2\hat{X}_1\hat{P}_3 - \hat{X}_3(\hat{X}_1\hat{P}_1 + \hat{P}_1\hat{X}_1)/2 - \hat{X}_1\hat{X}_3\hat{P}_2 \\
&-\hat{X}_1(\hat{X}_3\hat{P}_3 + \hat{P}_3\hat{X}_3)/2 + \hat{X}_2\hat{X}_3\hat{P}_1 + \hat{X}_3(\hat{X}_2\hat{P}_2 + \hat{P}_2\hat{X}_2)/2 + \hat{X}_2(\hat{X}_3\hat{P}_3 + \hat{P}_3\hat{X}_3)/2 \\
&= \hat{X}_1^2\hat{P} - i\hat{X}_1 + \hat{X}_1^2(\hat{P}_2 + \hat{P}_3) - \hat{X}_2\hat{X}_1\hat{P}_1 + \tfrac{i}{2}\hat{X}_2 - \hat{X}_1\hat{X}_2\hat{P}_2 + \tfrac{i}{2}\hat{X}_1 - \hat{X}_2\hat{X}_1\hat{P}_3 \\
&-\hat{X}_3\hat{X}_1\hat{P}_1 + \tfrac{i}{2}\hat{X}_3 - \hat{X}_1\hat{X}_3\hat{P}_2 - \hat{X}_1\hat{X}_3\hat{P}_3 + \tfrac{i}{2}\hat{X}_1 \\
&+\hat{X}_2\hat{X}_3\hat{P}_1 + \hat{X}_3\hat{X}_2\hat{P}_2 - \tfrac{i}{2}\hat{X}_3 + \hat{X}_2\hat{X}_3\hat{P}_3 - \tfrac{i}{2}\hat{X}_2 \\
&= (\hat{X}_1 - \hat{X}_2)(\hat{X}_1 - \hat{X}_3)(\hat{P}_1 + \hat{P}_2 + \hat{P}_3).
\end{aligned} \tag{48}$$

Equation (48) reveals that the Weyl ordering form of the operator $(\hat{X}_1 - \hat{X}_2)(\hat{X}_1 - \hat{X}_3)(\hat{P}_1 + \hat{P}_2 + \hat{P}_3)$ is itself. Consequently, the Radon transform relation is derived from Equation (29), and the three-mode Wigner operator is defined by Equation (49):

$$\begin{aligned}
&|\rho, \chi_1, \chi_2\rangle\langle\rho, \chi_1, \chi_2| \\
&= \tfrac{1}{\pi^3}\int dp_1 dx_1 dp_2 dx_2 dp_3 dx_3 \\
&\times\delta[\chi_1 - (x_1 - x_2)]\delta[\chi_2 - (x_1 - x_3)]\delta[\rho - (p_1 + p_2 + p_3)] : \prod_{i=1}^{3} e^{-(x_i - \hat{X}_i)^2 - (p_i - \hat{P}_i)^2} : .
\end{aligned} \tag{49}$$

Next, the normal ordering form of the Wigner operator is employed to compute the operator $|\rho,\chi_1,\chi_2\rangle\langle\rho,\chi_1,\chi_2|$. According to Equation (49), we have:

$$\begin{aligned}&\int \delta[\chi_1-(x_1-x_2)]\delta[\chi_2-(x_1-x_3)]e^{-(x_1-\hat{X}_1)^2}dx_1\\&=\delta[\chi_2-\chi_1-x_2+x_3]e^{-(\chi_1+x_2-\hat{X}_1)^2},\end{aligned}\tag{50}$$

and

$$\begin{aligned}&\int \delta[\chi_2-(\chi_1+x_2-x_3)]e^{-(\chi_1-\hat{X}_1+x_2)^2}e^{-(x_2-\hat{X}_2)^2}dx_2\\&=e^{-(\chi_2+x_3-\hat{X}_1)^2}e^{-(\chi_2+x_3-\chi_1-\hat{X}_2)^2},\end{aligned}\tag{51}$$

with

$$\begin{aligned}&\int_{-\infty}^{\infty} e^{-(\chi_2-\hat{X}_1+x_3)^2}e^{-(-\chi_1+\chi_2-\hat{X}_2+x_3)^2}e^{-(x_3-\hat{X}_3)^2}dx_3\\&=\int_{-\infty}^{\infty} dx_3 e^{-3x_3^2-2x_3(\chi_2-\hat{X}_1+\hat{X}_3+\chi_2-\chi_1-\hat{X}_2+\hat{X}_3)-(\chi_2-\hat{X}_1+\hat{X}_3)^2-(\chi_2-\chi_1-\hat{X}_2+\hat{X}_3)^2}\\&=\sqrt{\frac{\pi}{3}}\exp\Big\{-\frac{1}{3}\Big[(\chi_2-\hat{X}_1+\hat{X}_3)^2+(\chi_2-\chi_1-\hat{X}_2+\hat{X}_3)^2\Big]\\&-\frac{1}{3}\big[(\chi_2-\hat{X}_1+\hat{X}_3)-(\chi_2-\chi_1-\hat{X}_2+\hat{X}_3)\big]^2\Big\}\\&=\sqrt{\frac{\pi}{3}}\exp\Big\{-\frac{1}{3}\Big[(\chi_2-\hat{X}_1+\hat{X}_3)^2+(\chi_2-\chi_1-\hat{X}_2+\hat{X}_3)^2+(\hat{X}_2-\hat{X}_1+\chi_1)^2\Big]\Big\}.\end{aligned}\tag{52}$$

Similarly, the Gaussian convolution of Equation (38) yields:

$$\begin{aligned}&\int_{-\infty}^{\infty} dp_1dp_2dp_3 e^{-\sum_{i=1}^{3}(p_i-\hat{P}_i)^2}\delta[\rho-(p_1+p_2+p_3)]\\&=\int_{-\infty}^{\infty} dp_1dp_2 e^{-(\rho-p_1-p_2-\hat{P}_3)^2}e^{-(p_1-\hat{P}_1)^2}e^{-(p_2-\hat{P}_2)^2}\\&=\int_{-\infty}^{\infty} dp_1dp_2 e^{-(\rho-p_1-p_2-\hat{P}_2-\hat{P}_3)^2}e^{-(p_1-\hat{P}_1)^2}e^{-p_2^2}\\&=\sqrt{\frac{\pi}{2}}\int_{-\infty}^{\infty} dp_1 e^{-(\rho-p_1-\hat{P}_1-\hat{P}_2-\hat{P}_3)^2/2}e^{-p_1^2}\\&=\frac{\pi}{\sqrt{3}}e^{-(\rho-\hat{P}_1-\hat{P}_2-\hat{P}_3)^2/3},\end{aligned}\tag{53}$$

So, combining Equations (50)–(53), Equation (54) is attained:

$$\begin{aligned}&|\rho,\chi_1,\chi_2\rangle\langle\rho,\chi_1,\chi_2|\\&=\tfrac{1}{3\pi^{3/2}}:\exp\Big\{-\tfrac{1}{3}\Big[(\chi_2-\hat{X}_1+\hat{X}_3)^2-(\chi_2-\chi_1-\hat{X}_2+\hat{X}_3)^2\\&-(\hat{X}_2-\hat{X}_1+\chi_1)^2-(\rho-\hat{P}_1-\hat{P}_2-\hat{P}_3)^2\Big]\Big\}:.\end{aligned}\tag{54}$$

Utilizing the normal ordering form of the three-mode squeezed vacuum state leads to Equation (55), and

$$:\exp\left(-\sum_{i=1}^{3}a_i^\dagger a_i\right):=|000\rangle\langle 000|,\tag{55}$$

enables systematic disentanglement of the normal ordering on the right-hand side of Equation (54), then we have:

$$\begin{aligned}&|\rho,\chi_1,\chi_2\rangle=\tfrac{1}{\sqrt{3}\pi^{\frac{3}{4}}}\exp\Big[-\tfrac{\rho^2}{6}-\tfrac{1}{3}(\chi_1^2-\chi_1\chi_2+\chi_2^2)\\&+\tfrac{\sqrt{2}}{3}a_1^\dagger(\chi_1+\chi_2+i\rho)+\tfrac{\sqrt{2}}{3}a_2^\dagger(-2\chi_1+\chi_2+i\rho)\\&+\tfrac{\sqrt{2}}{3}a_3^\dagger(i\rho+\chi_1-2\chi_2)+\tfrac{2}{3}(a_1^\dagger a_2^\dagger+a_1^\dagger a_3^\dagger+a_2^\dagger a_3^\dagger)\\&-\tfrac{1}{6}(a_1^{\dagger 2}+a_2^{\dagger 2}+a_3^{\dagger 2})\Big]|000\rangle,\end{aligned}\tag{56}$$

Equation (56) defines the three-mode entangled state. Furthermore, we can also calculate:

$$\int_{-\infty}^{\infty} d\rho d\chi_1 d\chi_2 |\rho, \chi_1, \chi_2\rangle\langle\rho, \chi_1, \chi_2| = 1. \tag{57}$$

Therefore, the entangled state $|\rho, \chi_1, \chi_2\rangle$ satisfies the completeness relation.

7. Conclusions

Based on the introduction of the Weyl ordering symbol for operators, combined with the IWOP method and quantum phase space theory, this paper derives the normally ordered product form and the Weyl ordered form of the Wigner operator, respectively, providing a solid theoretical foundation for subsequent ordering conversion and integral operations in quantum phase space. The slice theorem of the Wigner operator is proposed, proving that the slicing operation of the Wigner operator can effectively project new pure-state density operators. This achieves the quantization of classical tomography theory, reveals the intrinsic connection between quantum state tomography and phase-space distribution, elevates classical tomographic imaging techniques to the level of quantum representation, and provides new theoretical tools for the reconstruction and characterization of quantum states. The two-mode entangled state representation and the three-mode entangled state representation are derived, respectively, and their completeness relations are proved. This not only enriches the entanglement theory of multi-mode continuous-variable systems but also provides effective representation methods for addressing many-body quantum correlation problems.

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