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Optimality of Quantum Adiabatic Search Algorithm and Its Circuit Model

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Abstract

In this paper, we study two aspects of quantum adiabatic evolution for a prototypical search problem: the optimality of the corresponding algorithm and its relation to the quantum circuit model. Firstly, we propose a general framework for proving the square-root speedup of the quantum adiabatic algorithm to be optimal over classical computation, which is readily applicable to the case of multiple targets. Through this framework, we also find that it is possible to further reduce the time complexity by increasing the physical energy of the system, encompassing results from previous works. Secondly, we find that, on the one hand, when the quantum adiabatic algorithm that achieves quadratic speedup is implemented on a quantum circuit, the time slice needed is always consistent with its time complexity, which also encompasses previous results; on the other hand, however, if a further algorithmic improvement is considered, the time slice always remains invariant. This phenomenon represents a significant observation with potential applications. We anticipate that the main results of this paper will interest the quantum adiabatic computation community and may help us to design efficient quantum algorithms for practical problems in the future.

Keywords: optimality proof; time slice invariance; quantum circuit model; quantum adiabatic search algorithm

1. Introduction

Since its introduction by Farhi and collaborators [1], quantum adiabatic computation has emerged as a computational paradigm distinct the conventional circuit models. Over the subsequent two decades, this framework has been extensively applied to a wide range of problems, particularly in the contexts of quantum search [2–6] and combinatorial optimization [7–12]. The scope of adiabatic methods has continued to expand. For example, Costa et al. [13] developed an optimal-scaling quantum linear-systems solver by formulating the problem within a discrete adiabatic framework, achieving linear complexity in the condition number and thereby matching the fundamental lower bound; Mc Keever and Lubasch [14] introduced tensor network techniques to optimize quantum circuits for adiabatic computing, incorporating counterdiabatic driving to suppress unwanted transitions and demonstrating significant improvements over conventional Trotter-based approaches.

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For readers seeking a comprehensive survey of developments in quantum adiabatic computing, the review by Albash and Lidar [15] provides an excellent starting point.

The operational mechanism underlying quantum adiabatic evolution can be described as follows: One begins with a quantum state initialized in the ground state of an initial Hamiltonian that is straightforward to prepare experimentally. This state subsequently evolves under the Schrödinger equation as the Hamiltonian is gradually deformed toward a final form, whose ground state encodes the solution to the problem under consideration. Provided that the deformation proceeds sufficiently slowly, the adiabatic theorem guarantees that the system remains in the instantaneous ground state throughout the evolution, thereby yielding the desired solution at the final time. The requisite timescale for such a process is dictated by the adiabatic theorem [16,17], which provides an estimate of the runtime based on the spectral gap and transition matrix elements.

Grover's quantum algorithm is well known for its quadratic speedup over classical search algorithms [18], and has been proven to be optimal [19]. However, when first implemented within the adiabatic evolution framework, the so-called quantum global adiabatic algorithm exhibited no performance advantage over classical search [2]. This limitation was addressed in [3,20] through the introduction of quantum local adiabatic evolution, which strategically relaxes the adiabatic condition by allowing for the evolution rate to vary with time. By enforcing adiabaticity only locally—i.e., on each infinitesimal time interval—they proved that the total running timescales as $O(\sqrt{N})$, thereby recovering Grover's quadratic speedup. Importantly, it was further established that this quadratic speedup is optimal for local adiabatic evolution, meaning that no further improvement beyond $\sqrt{N}$ is possible within this framework [3].

Subsequent developments extended these ideas in several directions. Tulsi [5] proposed a class of partial adiabatic evolutions confined to a narrow symmetric interval around the minimum gap point, thereby focusing the evolution on the most critical region of the spectrum. Zhang et al. [21,22] applied this partial adiabatic framework to quantum search and claimed a time complexity of $O(\sqrt{N/M})$, which is an $O(\sqrt{M})$ improvement over the known $O(\sqrt{N/M})$ bound for local adiabatic search, where $M \geq 1$ is the number of targets. However, Kay [23] critically examined these claims and identified an important oversight: the success probability calculations in these works require a verification condition to ensure the validity of the adiabatic approximation. While Tulsi's original argument could be corrected [5], subsequent works [24–26] have not been examined as fortunate, leaving their conclusions open to question.

Very recently, Sun et al. [27] provided a comprehensive analysis of quantum partial adiabatic evolution, establishing a clear demarcation criterion: algorithms whose time complexity aligns with the fundamental quantum limit $O(\sqrt{N/M})$ achieve a substantially high success probability, whereas those claiming to exceed this limit exhibit a vanishing success probability as the number of target elements increases. This finding implies that the $O(\sqrt{N/M})$ time complexity is also optimal for quantum partial adiabatic search algorithm.

Beyond optimality proofs, another important line of research concerns the implementation of adiabatic algorithms on quantum circuits. van Dam et al. [20] and Roland and Cerf [28] established the basic framework for approximating adiabatic evolution by quantum circuits, showing that the required time slice is consistent with the algorithm's time complexity. In [29,30], it was subsequently proven that adiabatic quantum computing is computationally equivalent to circuit-based quantum computing. In [31,32], we have shown that the invalidity of problematic quantum partial adiabatic search algorithms, e.g., those in [21,24], can be easily seen by checking their circuit model implementation. Recently, Sun et al. [33–35] investigated the circuit implementation of various adiabatic search algorithms and discovered an intriguing phenomenon: for certain "improved" adia-

batic evolutions that achieve constant-time complexity by scaling the Hamiltonian energy, the time slice required for circuit implementation remains invariant at $O(\sqrt{N/M})$, despite the reduced algorithmic runtime. This invariance reveals a fundamental trade-off between time and energy resources—a theme that resonates with the energy–time uncertainty principle and may have significant implications for practical implementations on near-term quantum devices.

Motivated by these developments, this paper presents a unified framework for proving the optimality of quantum adiabatic search algorithms in the multi-target case and investigates their circuit-level implementation. In fact, it was shown in [3] that the quantum local adiabatic algorithm is optimal, but when there are multiple targets, it seems difficult to prove that the square-root speedup is also optimal by directly adopting the method from [3]. To dealing with this dilemma, in this paper, we show a general framework for proving the optimality of quantum adiabatic search algorithms, and through it, we can easily see that the algorithm presented in [3] is optimal for both the single- and multiple-target cases. In fact, what we show is stronger: our result reveals a fundamental energy-time trade-off in quantum adiabatic computation, showing that constant-time evolution is achievable if the spectral weight on the final Hamiltonian scales as $\Omega(\sqrt{N/M})$.

In the early works of [36,37], it was shown that the quantum local adiabatic computations could achieve an O(1) time complexity if the interpolating functions within the system Hamiltonian were endowed with an enlarged factor which was proportional to square-root of the problem size. Note that these do not contradict the optimality result of the quantum algorithm in [3] because there the interpolating functions are standard, which means that there is no extra enlarged factor within them. By the method of proof of optimality here, it is easy to see that to further get algorithmic performance improvement is possible using many other ways.

When implementing standard quantum adiabatic evolution within the quantum circuit model, we find that the derived time slice is always consistent with the time complexity, regardless of the algorithm's optimality. This implies that the optimality of a quantum adiabatic algorithm cannot be inferred solely from its circuit approximation. However, this conclusion does not hold for other quantum adiabatic algorithm, such as those exhibiting O(1) time complexity. For these peculiar cases, we observe that the required time slice remains invariant, scaling inversely with the square root of the ratio of the number of target states to the total number of elements. To our knowledge, this phenomenon has been reported exclusively in [33–35]. It may provide insight into such quantum adiabatic evolutions themselves and their connection to the quantum circuit model.

The main contributions and novel aspects of this work are as follows:

1. **A unified optimality proof for multi-target adiabatic search**. While the optimality of quantum adiabatic search for a single marked item was established by Roland and Cerf [3], extending this proof to the multi-target case is not straightforward using their method. We develop a general framework that naturally accommodates multiple targets, proving that the quadratic speedup $O(\sqrt{N/M})$ is optimal for any adiabatic evolution governed by the Hamiltonian in Equations (1) and (2) with interpolating functions satisfying Equation (3). This framework unifies and generalizes previous results such as [3,36,38], providing a rigorous foundation for multi-target adiabatic search.
2. **A fundamental energy-time trade-off theorem**. Our proof yields a stronger result than a simple lower bound: we establish that $\int_0^T g(s)dt \geq \Omega(\sqrt{N/M})$, which reveals a direct trade-off between the time complexity T and the energy scale encoded in the interpolating function $g(s)$. This theorem encompasses and extends the early observations of Das et al. [36] from the single-target to the multi-target case and pro-

vides a unified explanation for various "improved" adiabatic algorithms [36,37]; their efficiency gains come from effectively increasing the Hamiltonian energy, not from circumventing fundamental quantum limits.

3. **Discovery of time slice invariance in circuit implementation**. When analyzing the circuit-level implementation of adiabatic algorithms, we uncover a striking phenomenon: for certain "improved" evolutions that achieve constant-time complexity by scaling the Hamiltonian energy, the time slice required for circuit implementation remains invariant at $O(|\langle\varphi|\psi\rangle|^{-1})$, in which $|\psi\rangle$ and $|\varphi\rangle$ are respectively the initial and final quantum states in the search problem, despite the reduced algorithmic runtime. To our knowledge, this invariance has been reported exclusively in our recent works [33–35], and it reveals a deep connection between the adiabatic and circuit models that goes beyond their established computational equivalence [29,30]. This finding has significant implications for practical implementations, as it suggests that reducing algorithmic time through energy scaling does not reduce the circuit resources required for simulation.
4. **A resolution to the partial adiabatic search controversy**. Building on Kay's critical insight [23] and recent developments by Sun et al. [27], our framework clarifies why certain "improved" partial adiabatic search algorithms [21,24] cannot achieve their claimed speedup: the algorithmic performance limit imposes constraints that these algorithms fail to satisfy. This provides a rigorous criterion for distinguishing valid adiabatic algorithms from invalid ones—a contribution that addresses an ongoing debate in the literature.

In the concluding section, we will substantiate these claims through a systematic comparison with prior work. The remainder of this paper is organized as follows: Section 2 presents the prototype of the quantum adiabatic search problem. Section 3 provides a framework for proving the optimality of the quantum adiabatic search algorithm and discusses its implications. In Section 4, we show how to simulate the adiabatic computation using quantum circuits and explore possible generalizations. Section 5 concludes the paper with a summary and final remarks.

2. The Quantum Adiabatic Search Prototype Problem

In this section, we introduce the quantum adiabatic search prototype central to our work. Specifically, consider a system governed by the following Hamiltonian:

$$H(s) = f(s)H_i + g(s)H_f, \tag{1}$$

in which

$$H_i = I - |\psi\rangle\langle\psi|, H_f = I - |\varphi\rangle\langle\varphi| \tag{2}$$

are the initial and final Hamiltonians whose ground state are $|\psi\rangle$ and $|\varphi\rangle$, respectively. A common choice for the interpolation functions is the linear form

$$f(s) = 1 - s, g(s) = s, s = s(t). \tag{3}$$

More generally, the interpolation need only satisfy the boundary conditions

$$f(0) = g(1) = 1, f(1) = g(0) = 0. \tag{4}$$

The quantum adiabatic evolution commences from the initial state $|\psi\rangle$ at $t = 0$. At $t = T$, when the quantum adiabatic evolution is completed, the system will be in the target state $|\varphi\rangle$.

In a quantum search, the initial state is usually designed as follows:

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle, \tag{5}$$

and the final state is an equal superposition over a set S of M marked items,

$$|\varphi\rangle = \frac{1}{\sqrt{M}} \sum_{j\in S, |S|=M} |j\rangle. \tag{6}$$

Unlike the Hamiltonian in [3], which projects onto the entire marked subspace,

$$H_f = I - \sum_{j\in S, |S|=M} |j\rangle\langle j|, \tag{7}$$

our final Hamiltonian $H_f = I - |\varphi\rangle\langle\varphi|$ has a unique ground state $|\varphi\rangle$. Nevertheless, both yield the same optimal $O(\sqrt{N/M})$ scaling, as the adiabatic evolution only needs to end in the subspace of marked states. So, for convenience, in the following discussion, we concentrate on the final Hamiltonian within our proposed model. Typically, we assume that $N \gg M$.

It is straightforward to show that, within the framework of quantum local adiabatic evolution, the corresponding algorithm achieves a time complexity of $T = O(\sqrt{N/M})$. Likewise, for the same problem, an $O(\sqrt{N/M})$ complexity can also be attained via quantum partial adiabatic evolution, as demonstrated in [25]. These results highlight the versatility of the adiabatic approach in reaching the fundamental quadratic speedup. Our proposed model, based on the Hamiltonian forms given in Equations (1) and (2), is compatible with both interpretations and offers a unified starting point for analyzing their complexity. In what follows, we demonstrate that this $O(\sqrt{N/M})$ scaling is indeed optimal, thereby establishing that the quantum adiabatic search saturates the lower bound for unstructured quantum search.

3. Proof of Optimality of Quantum Adiabatic Search

In this section, we show that the quantum adiabatic evolution of the problem in Section 2 should have at least a time complexity of $T = O(\sqrt{N/M})$. In fact, if there is a single target, which implies the corresponding time complexity is given by $T = O(\sqrt{N})$, the optimality of the quantum adiabatic algorithm has already been proven in some of the related literature, for example, [3,36]. However, it is not difficult to verify whether, if there are multi-target quantum states for the problem, the optimality can be proven by directly adopting the method in this literature. As far as we know, this remains unclear. This is also our main motivation for finding a unified framework for dealing with this dilemma.

We define $S \neq S'$ as the two target states sets and suppose that $|S| = |S'| = M$. Also, we can define the following two orthonormal bases:

$$|\alpha\rangle := \frac{1}{\sqrt{M}} \sum_{k\in S} |k\rangle, \ |\beta\rangle := \frac{1}{\sqrt{N-M}} \sum_{k\notin S} |k\rangle.$$

$$|\alpha'\rangle := \frac{1}{\sqrt{M}} \sum_{k\in S'} |k\rangle, \ |\beta'\rangle := \frac{1}{\sqrt{N-M}} \sum_{k\notin S'} |k\rangle. \tag{8}$$

It is easy to verify that

$$\langle\alpha|\beta'\rangle = \langle\beta|\alpha'\rangle = \sqrt{\frac{M}{N-M}}. \tag{9}$$

Let $|\varphi_\alpha(t)\rangle$ be the instantaneous quantum state during the adiabatic evolution when the target state is $|\alpha\rangle$, which is also the ground-energy state $|E_0(t)\rangle$ for the system Hamiltonian $H(s) = (1-s)H_i + sH_e$ for $H_i = I - |\Phi\rangle\langle\Phi|$ and $H_e = I - |\alpha\rangle\langle\alpha|$. Also, $|\varphi_{\alpha'}(t)\rangle$ and $|E_0'(t)\rangle$ can be defined analogously in the case that the system Hamiltonian is given as $H'(s) = (1-s)H_i + sH_e'$ for $H_e' = I - |\alpha'\rangle\langle\alpha'|$. At the end of the adiabatic evolution, different target states should have sufficiently small overlap, which means that

$$1 - |\langle\varphi_\alpha(T)|\varphi_{\alpha'}(T)\rangle|^2 \geq \varepsilon, \alpha \neq \alpha'. \tag{10}$$

Expanding $|\varphi_\alpha(t)\rangle$ and $|\varphi_{\alpha'}(t)\rangle$ with the above two orthonormal bases, respectively, we have

$$\begin{aligned} |\varphi_\alpha(t)\rangle := |E_0(t)\rangle &= \langle\alpha|E_0(t)\rangle|\alpha\rangle + \langle\beta|E_0(t)\rangle|\beta\rangle, \\ |\varphi_{\alpha'}(t)\rangle := |E_0'(t)\rangle &= \langle\alpha'|E_0'(t)\rangle|\alpha'\rangle + \langle\beta'|E_0'(t)\rangle|\beta'\rangle. \end{aligned} \tag{11}$$

At the beginning of the adiabatic evolution, we obviously have

$$|\varphi_\alpha(0)\rangle := |\varphi_{\alpha'}(0)\rangle = |\Phi\rangle. \tag{12}$$

The quantum system evolves according to the Schrödinger equation, which means that

$$\begin{aligned} i\frac{d}{dt}|\varphi_\alpha(t)\rangle &= H(s)|\varphi_\alpha(t)\rangle, \\ i\frac{d}{dt}|\varphi_{\alpha'}(t)\rangle &= H'(s)|\varphi_{\alpha'}(t)\rangle. \end{aligned} \tag{13}$$

The following equality is easy to verify:

$$\begin{aligned} &\left|\frac{d}{dt}[1 - |\langle\varphi_\alpha(t)|\varphi_{\alpha'}(t)\rangle|^2]\right| = \left|\frac{d}{dt}[\langle\varphi_\alpha(t)|\varphi_{\alpha'}(t)\rangle\langle\varphi_{\alpha'}(t)|\varphi_\alpha(t)\rangle]\right| \\ &= \big|\langle\dot{\varphi}_\alpha(t)|\varphi_{\alpha'}(t)\rangle\langle\varphi_{\alpha'}(t)|\varphi_\alpha(t)\rangle + \langle\varphi_\alpha(t)|\dot{\varphi}_{\alpha'}(t)\rangle\langle\varphi_{\alpha'}(t)|\varphi_\alpha(t)\rangle \\ &+\langle\varphi_\alpha(t)|\varphi_{\alpha'}(t)\rangle\langle\dot{\varphi}_{\alpha'}(t)|\varphi_\alpha(t)\rangle + \langle\varphi_\alpha(t)|\varphi_{\alpha'}(t)\rangle\langle\varphi_{\alpha'}(t)|\dot{\varphi}_\alpha(t)\rangle\big|. \end{aligned} \tag{14}$$

From (13), we know that

$$\langle\dot{\varphi}_\alpha(t)| = i\langle\varphi_\alpha(t)|H(s), \langle\dot{\varphi}_{\alpha'}(t)| = i\langle\varphi_\alpha(t)|H'(s). \tag{15}$$

Substituting (13) and (15) into (14), and doing some simple calculations, we have

$$\left|\frac{d}{dt}[1 - |\langle\varphi_\alpha(t)|\varphi_{\alpha'}(t)\rangle|^2]\right| \leq 2s(t)|\langle\varphi_\alpha(t)|(|\alpha'\rangle\langle\alpha'| - |\alpha\rangle\langle\alpha|)|\varphi_{\alpha'}(t)\rangle|. \tag{16}$$

By (9) and (11), the above inequality can be further simplified. To see this, note that the term $\langle\varphi_\alpha(t)|(|\alpha'\rangle\langle\alpha'| - |\alpha\rangle\langle\alpha|)|\varphi_{\alpha'}(t)\rangle$ can be expanded by substituting the expressions for $|\varphi_\alpha(t)\rangle$ and $|\varphi_{\alpha'}(t)\rangle$ from Equation (11). Using the orthonormal bases defined in Equation (8), we obtain cross terms involving $\langle\alpha|\beta'\rangle$ and $\langle\beta|\alpha'\rangle$, which, from Equation (9), are both equal to $\sqrt{M/(N-M)}$. The absolute value then yields the factor $\sqrt{M/(N-M)}$, multiplying the difference $|\langle E_0(t)|\alpha\rangle\langle\beta'|E_0'(t)\rangle - \langle E_0(t)|\beta\rangle\langle\alpha'|E_0'(t)\rangle|$. This step is crucial because it isolates the dependence on the problem size N and the number of targets M, setting the stage for the subsequent inequality. Therefore, (16) becomes

$$\left|\frac{d}{dt}[1 - |\langle\varphi_\alpha(t)|\varphi_{\alpha'}(t)\rangle|^2]\right|$$

$$\leq 2s(t)\sqrt{\frac{M}{N-M}}\left|\langle E_0(t)|\alpha\rangle\langle\beta'|E_0'(t)\rangle - \langle E_0(t)|\beta\rangle\langle\alpha'|E_0'(t)\rangle\right|. \tag{17}$$

Since the following chain of inequalities holds:

$$\begin{aligned}
&\left|\langle E_0(t)|\alpha\rangle\langle\beta'|E_0'(t)\rangle - \langle E_0(t)|\beta\rangle\langle\alpha'|E_0'(t)\rangle\right| \\
&\leq \left|\langle E_0(t)|\alpha\rangle\langle\beta'|E_0'(t)\rangle\right| + \left|\langle E_0(t)|\beta\rangle\langle\alpha'|E_0'(t)\rangle\right| \\
&\leq \frac{(|\langle E_0(t)|\alpha\rangle|^2 + |\langle E_0(t)|\beta\rangle|^2) + (|\langle\beta'|E_0'(t)\rangle|^2 + |\langle\alpha'|E_0'(t)\rangle|^2)}{2} = 1,
\end{aligned} \tag{18}$$

where the AM-GM inequality is applied, and the following relations between the coefficients are adopted from (11):

$$|\langle\alpha|E_0(t)\rangle|^2 + |\langle\beta|E_0(t)\rangle|^2 = 1, \qquad |\langle\alpha'|E_0'(t)\rangle|^2 + |\langle\beta'|E_0'(t)\rangle|^2 = 1. \tag{19}$$

Then, (17) now has a compact form given by

$$\left|\frac{d}{dt}[1 - |\langle\varphi_\alpha(t)|\varphi_{\alpha'}(t)\rangle|^2]\right| \leq 2s(t)\sqrt{\frac{M}{N-M}}. \tag{20}$$

Integrating this inequality from 0 to T and applying (12) yields

$$\left|1 - \langle\varphi_\alpha(T)|\varphi_{\alpha'}(T)\rangle\right|^2 \leq 2\sqrt{\frac{M}{N-M}}\int_0^T s(t)dt. \tag{21}$$

Using the fact that $s(t) \in [0,1]$, together with (10), we finally obtain

$$T \geq \frac{\varepsilon}{2}\sqrt{\frac{N-M}{M}} = O\left(\sqrt{\frac{N}{M}}\right). \tag{22}$$

So we have finished the proof of optimality of quantum adiabatic computation of the search type problem in the case of multiple targets. The single target state case is followed trivially.

From the proof procedure, we observe that the adiabatic time lower bound scales as $T \geq O\left(\frac{\sqrt{N/M}}{A}\right)$ when a global factor $A \gg 1$ is introduced. This scaling occurs in two typical cases: (i) when the Hamiltonian is uniformly scaled, $H(s) = A[(1-s)H_i + sH_f]$; or (ii) when a common term proportional to A is added to the interpolation functions, e.g., $H(s) = [1 - s + As(1-s)]H_i + [s + As(1-s)]H_f$. Mathematically, setting $A = O(\sqrt{N/M})$ would yield a constant lower bound. However, it is crucial to recognize that this scaling directly amplifies the physical energy range of the implemented Hamiltonian. This represents a trade-off between time and energy resources, not a genuine algorithmic enhancement. In physical implementations, the maximum achievable energy scale is severely constrained by the experimental platform (e.g., qubit coherence times, coupling strengths). Therefore, the scaling factor A cannot be made arbitrarily large, and the constant-time limit is physically inaccessible for large problem sizes. This trade-off does not violate the established $\Omega(\sqrt{N/M})$ quantum lower bound for search, as that bound assumes a bounded Hamiltonian norm. This energy–time relationship offers a perspective to reinterpret some results in the literature. For instance, works like [36,37] explore optimized interpolation paths. Their efficiency gains can be viewed, in part, as effectively maximizing the use of a fixed, limited energy budget to minimize time, rather than proposing unbounded energy scaling.

Before concluding this section, we emphasize that our current work provides a generalization of the optimality proof presented in our previous study in [38], which addressed the quantum partial adiabatic evolution in [25]. While the core proof strategy shares sim-

ilarities, the key advancement here is that we establish optimality for a broader class of quantum adiabatic computations. Specifically, we consider a generalized Hamiltonian form $H(s)$ in (1), where the target Hamiltonian is a ground-state projector and the interpolation function associated with it takes the simple form $0 \leq g(s) = s \leq 1$. We prove that any quantum adiabatic computation within this defined class that achieves a quadratic speedup over classical algorithms is optimal. Consequently, the result in [38] naturally emerges as a special instance of this more general framework. In fact, it is not difficult to check that the preceding proof yields the following theorem, which establishes a fundamental trade-off between the energy scale and time complexity of quantum adiabatic computation.

Theorem 1 (**Energy-Time Trade-off for Adiabatic Search**). *For a quantum adiabatic evolution governed by the Hamiltonian in Equation (1) with the initial and final Hamiltonians given as in Equation (2) and interpolating functions satisfying* $f(0) = g(1) = 1$ *and* $f(1) = g(0) = 0$*, the following lower bound holds:*

$$\int_0^T g(s(t))\, dt = \Omega\left(\sqrt{\frac{N}{M}}\right). \tag{23}$$

The key to proving the above theorem lies in decomposing the Hamiltonian into two parts:

$$H(s) = H_1(s) + H_2(s), \tag{24}$$

where

$$H_2(s) = -g(s)|\varphi\rangle\langle\varphi|, \tag{25}$$

encapsulates the dependence on the target-state superposition, while $H_1(s)$ is a generic term independent of $|\varphi\rangle$. For the Hamiltonian given in (1), one finds that $H_1(s) = f(s) + g(s) - f(s)|\psi\rangle\langle\psi|$.

Interpretation. This theorem reveals a fundamental trade-off between the time complexity T and the "energy resource" encoded in the interpolating function $g(s)$. Intuitively, $g(s)$ determines how much weight is placed on the target Hamiltonian H_f during the evolution; a larger $g(s)$ means the system Hamiltonian has a larger component in the direction of the final problem encoding, which typically corresponds to a larger energy scale. The theorem states that the time-integrated energy resource—i.e., the total "budget" of $g(s)$ accumulated over the evolution—must be at least $\Omega(\sqrt{N/M})$. Consequently,

- If $g(s) = O(1)$ (the standard case with bounded energy), then $T = \Omega(\sqrt{N/M})$. This recovers the well-known quadratic speedup limit.
- If $g(s) = \Omega(\sqrt{N/M})$ (i.e., the energy scale grows with problem size), then $T = \Omega(1)$: constant-time evolution is theoretically possible, but at the cost of proportionally increased physical energy.

This trade-off is not merely a mathematical curiosity; it has profound implications for practical implementations. In any real quantum system, the achievable energy scale is bounded by experimental constraints such as qubit coherence times, coupling strengths, and control electronics limitations. Therefore, while the theorem shows that constant-time evolution is possible in principle, it is physically inaccessible for large problem sizes under realistic conditions. This perspective reconciles the apparent tension between theoretical "improvements" [36,37] and the established quantum lower bound; the former achieve speedup by effectively increasing energy, not by beating fundamental limits. Also, based on our optimal result, it is evident that the "improved" quantum partial adiabatic search algorithms [21,24] are invalid because their claimed time complexity $O(\sqrt{N}/M)$ violates the bound in (23).

We should point out that a similar theorem was obtained for the special case of $M = 1$ in [36]. However, the extension of that theorem to $M > 1$ is not straightforward, whereas our present result naturally holds for the general multi-target case.

It should be emphasized that if both interpolating functions are constant, i.e., $f(s) = g(s) = E$ for a constant E, the theorem above yields a time lower bound of $\Omega\left(\frac{\sqrt{N/M}}{E}\right)$ for the analog quantum search proposed by Farhi et al. [39], in which $H_i = |\psi\rangle\langle\psi|, H_f = |\varphi\rangle\langle\varphi|$ in this case. This shows that, even without resorting to complex mathematical induction, the performance limit of quantum computation can still be readily derived.

Physical Feasibility of Energy Scaling. The energy scaling parameter A introduced in our analysis raises a crucial question: to what extent can A be increased in a realistic physical implementation? While the theorem shows that setting $A = \Omega(\sqrt{N/M})$ would theoretically yield constant-time evolution, this comes at the cost of proportionally increasing the Hamiltonian norm. In practice, several fundamental and technological constraints limit the achievable energy scale:

1. **Coherence time limitations**. Adiabatic evolution requires the total evolution time T to satisfy $T \ll T_{\text{deph}}$, where T_{deph} is the system's decoherence time [40]. Increasing A reduces T proportionally, which helps satisfy this condition. However, as noted in Ref. [41], this benefit saturates when T approaches the minimum time required for coherent control; beyond this point, further energy scaling yields no additional advantage.
2. **Maximum achievable energy scales**. The norm of the Hamiltonian is ultimately bounded by the maximum coupling strengths available in the physical platform. As shown in Ref. [42], physically realizable control Hamiltonians exist only within certain quantized energy resources. These limits arise from material properties, fabrication constraints, and the need to avoid coupling to states outside the computational subspace.
3. **Leakage to higher energy levels**. A large Hamiltonian norm may bring the computational states closer in energy to higher excited states, increasing the risk of leakage due to non-adiabatic transitions or control imperfections. This is particularly problematic in systems with dense spectra, where the assumption of an isolated two-level subspace is no longer valid (a rigorous analysis of leakage in the large-norm regime remains an open problem; for general discussions of level crossings; see standard references on quantum adiabatic computation).
4. **Control precision requirements**. This includes the sensitivity to control noise scales with the Hamiltonian norm. Recent work on robust quantum control shows that the fidelity error under parametric uncertainties is bounded by the control duration and the uncertainty magnitude [43]. Moreover, Ref. [44] derives a fundamental performance limit for coherent quantum control in the presence of uncertainties, establishing that the worst-case gate fidelity obeys a lower bound that depends on the product of gate duration and an aggregate uncertainty measure. Any relative error in the interpolation functions $f(s)$ and $g(s)$ thus translates into an absolute energy error proportional to A, imposing increasingly stringent demands on calibration and pulse fidelity as A grows.

Therefore, while the energy–time trade-off identified in the theorem is theoretically valid, it does not imply that constant-time quantum search is practically achievable for large problem sizes. Instead, it should be understood as a fundamental bound that delineates the limits of what is possible given finite energy resources—a manifestation of the quantum energy-time uncertainty principle [42]. This perspective aligns with the resource-theoretic view of quantum computation, where time and energy are both considered valuable resources that must be optimally allocated. In the following, we describe how to simulate the

above adiabatic algorithm using quantum circuits, thereby providing a concrete framework for its implementation on gate-based quantum computers.

4. The Circuit Model of Optimal Quantum Adiabatic Evolution

Now, we turn to approximating the optimal quantum adiabatic evolution by a quantum circuit. There are two steps for this, namely, estimating the time slice from the difference of the resulting unitary operations from that of two supposed system Hamiltonians and implementing the whole unitary transformations by two elementary operations, respectively.

In the first stage, we will need the following lemma, the proof of which can be found in [28].

Lemma 1 (**Evolutionary Distance under Hamiltonian Perturbation**). *Consider two time-dependent Hamiltonians, $H(t)$ and $H'(t)$, defined for $t \in [0, T]$, which generate unitary evolutions $U(t)$ and $U'(t)$, respectively. Suppose the difference between the Hamiltonians is bounded pointwise in the operator norm by $|||H(t) - H'(t)|||_2 \leq \delta(t)$ for all t, where for a linear operator M, $|||M|||_2 := \sup_{||x||_2=1} ||Mx||_2$. Then the distance between the final evolution operators satisfies $|||U(T) - U'(T)|||_2 \leq \sqrt{2\int_0^T \delta(t)dt}$.*

Firstly, by defining the Hamiltonian $H'(s) = (1 - s'(t))H_i + s'(t)H_f$, we have the following equality:

$$\begin{aligned} |||H(t) - H'(t)|||_2 &= |||H(s) - H'(s)|||_2 \\ &= ||| [s'(t) - s(t)]H_i + [s(t) - s'(t)]H_f |||_2 \\ &= |s'(t) - s(t)| \cdot |||H_i - H_f|||_2 \end{aligned} \tag{26}$$

Let $\Delta T = T/R$, where R is the number of time slices in the approximation. Define $s'(t) = s(t + \Delta T)$. Then we have the following inequality:

$$|||H(t) - H'(t)|||_2 \leq s(t + \Delta T) - s(t), \tag{27}$$

in which $|||H_i - H_f|||_2 = \sqrt{1 - M/N} < 1$ is used. Since $s(t) = 1$ for all $t \geq T$, we have the following estimate:

$$\begin{aligned} \int_0^T [s(t + \Delta T) - s(t)]\, dt &= \int_0^T s(t + \Delta T)\, dt - \int_0^T s(t)\, dt \\ &= \int_{\Delta T}^{T+\Delta T} s(u)\, du - \int_0^T s(t)\, dt \qquad (u = t + \Delta T) \\ &= \left(\int_{\Delta T}^{T} s(u)\, du + \int_{T}^{T+\Delta T} s(u)\, du \right) - \left(\int_0^{\Delta T} s(t)\, dt + \int_{\Delta T}^{T} s(t)\, dt \right) \\ &= \int_{T}^{T+\Delta T} s(u)\, du - \int_0^{\Delta T} s(t)\, dt \\ &= \Delta T - \int_0^{\Delta T} s(t)\, dt \leq \Delta T, \end{aligned} \tag{28}$$

which, together with the lemma above, implies

$$|||U(T) - U'(T)|||_2 \leq \sqrt{2T/R}. \tag{29}$$

Therefore, the number of time slices in the first stage of the circuit-based approximation of the optimal adiabatic evolution is $R_{\text{ad1}} = O(\sqrt{N/M})$, consistent with the algorithm's overall time complexity.

We now proceed to the second stage of the approximation. For this, consider the following unitary transformations:

$$U_k^{'} = e^{-iH_k\Delta T} = e^{-i[(1-s_k)H_i+s_kH_f]\Delta T}, \tag{30}$$

where $H_k = (1-s_k)H_i + s_kH_f$. This evolution can be approximated via a first-order Trotter splitting, yielding

$$U_k^{''} = e^{-i(1-s_k)H_i\Delta T}\, e^{-is_kH_f\Delta T}, \tag{31}$$

which is realized by elementary quantum gates.The Campbell–Baker–Hausdorff theorem [45] provides an estimate for the error incurred when approximating the unitary evolution in (30) by that in (31). This leads to the following bound:

$$|||U_k^{'} - U_k^{''}|||_2 \in O((1-s_k)s_k\Delta T^2||||[H_i, H_f]||||_2). \tag{32}$$

Since $||||[H_i, H_f]||||_2 = \sqrt{\frac{M}{N}}\sqrt{1-\frac{M}{N}}$, the estimate above simplifies to

$$|||U_k^{'} - U_k^{''}|||_2 \in O(\sqrt{N/M}/R_{\mathrm{ad2}}^2). \tag{33}$$

Moreover, by induction one can show the following inequality:

$$|||\prod_k U_k^{'} - \prod_k U_k^{''}|||_2 \leq \sum_k |||U_k^{'} - U_k^{''}|||_2. \tag{34}$$

For R_{ad2} steps,we have $U'(T) = \prod_k U_k^{'}$, which combined with (34) yields

$$|||U'(T) - \prod_k U_k^{''}|||_2 \in O(\sqrt{N/M}/R_{\mathrm{ad2}}). \tag{35}$$

Thus, we obtain $R_{\mathrm{ad2}} = O(\sqrt{N/M})$, and together with $R_{\mathrm{ad1}} = O(\sqrt{N/M})$, the total number of time slices required to implement the optimal adiabatic evolution on a quantum circuit is also $R = O(\sqrt{N/M})$, which matches the algorithm's time complexity $T = O(\sqrt{N/M})$.

From the discussion above, it is straightforward to verify that if a quantum adiabatic algorithm has time complexity $T = O(|\langle\varphi|\psi\rangle|^{-1})$, where $|\psi\rangle$ and $|\varphi\rangle$ are respectively the initial and final states in a quantum search-type problem, then the resulting number of time slices needed when the algorithm is approximated by a quantum circuit is also $R = O(|\langle\varphi|\psi\rangle|^{-1})$. Moreover, it should be noted that if a quantum adiabatic algorithm has time complexity $T = O(|\langle\varphi|\psi\rangle|^{-2})$, the resulting time slices required is $R = O(|\langle\varphi|\psi\rangle|^{-2})$, as shown in [46] and can be verified by a straightforward calculation. Consequently, this implies that solely from the time slice count obtained in approximating a quantum adiabatic evolution with a quantum circuit model, one cannot determine whether the quantum adiabatic algorithm is optimal. This observation should be distinguished from the recent works [31,32], which examine the validity of the quantum partial adiabatic search algorithm via its circuit model.

In the previous section, we discussed theoretical approaches to reducing the time complexity of quantum adiabatic computation by modifying the interpolating functions in the Hamiltonian system,as exemplified by the two principal cases considered. Interestingly, one can verify that when considering the quantum circuit models simulating these corresponding adiabatic evolutions, the required number of time slices remains invariant at $O(|\langle\varphi|\psi\rangle|^{-1})$. This result does not match the projected time complexity of the algorithm itself, which is $T = O(|\langle\varphi|\psi\rangle|^{-1}/A)$. This suggests that, for such modified evolutions, the invariant time slice reflects a trade-off between the reduced algorithmic time complex-

ity and the increased energy requirement. This interplay can be loosely analogized to a form of uncertainty principle between these computational and physical resource costs. To the best of our knowledge, this specific observation has primarily been explored in our prior works [33–35], and it may offer valuable insights for understanding such quantum adiabatic evolutions themselves and their connections to quantum circuit models. Finally, we note that the observed discrepancy between time complexity and time slice in these particular quantum adiabatic evolutions does not contradict the established equivalence proofs in [29,30] because those proofs are specifically constructed for the standard form of quantum adiabatic evolution.

Remark on the two-stage approximation. The circuit approximation of adiabatic evolution involves two conceptually distinct stages. In the first stage (time slicing), we discretize the continuous-time evolution into R steps of duration $\Delta T = T/R$, approximating the ideal evolution $U(T)$ by a product $U'(T) = \prod_k e^{-iH(s_k)\Delta T}$. The error from this discretization scales as $\sqrt{T/R}$, as shown by the lemma. In the second stage (Trotterization), each step $e^{-iH(s_k)\Delta T}$ is further approximated by a product of elementary gates $e^{-i(1-s_k)H_i\Delta T}e^{-is_kH_f\Delta T}$, introducing additional error from the non-commutativity of H_i and H_f. The total time slice R must be chosen large enough to control both sources of error. Our analysis shows that, for an optimal adiabatic search, $R = O(\sqrt{N/M})$ suffices—consistent with the algorithm's time complexity. However, for "improved" evolutions where $g(s)$ is scaled to achieve constant T, we observe the surprising time slice invariance $R = O(|\langle\varphi|\psi\rangle|^{-1})$, independent of the reduction in T. This invariance reveals that circuit resources are determined by the overlap between initial and final states, not by the algorithmic runtime—a subtle but important distinction.

5. Conclusions and Discussion

In summary, this paper investigates the optimality and circuit implementation of quantum adiabatic search algorithms, establishing a general framework for proving optimal quadratic speedups and revealing invariant properties in their circuit-level realization. While these theoretical investigations are abstract, they are ultimately motivated by the broader pursuit of optimal and efficient computational paradigms across scientific and engineering domains. This pursuit is vividly exemplified in applied research, such as the development of a MobileNet V3-based model for intelligent detection of semiconductor chip pin defects [47,48]—a direct illustration of how the principle of optimal resource allocation, central to our adiabatic optimality proofs, manifests in a concrete engineering context. That work addresses a critical industrial challenge by optimizing a classical deep learning architecture for speed, accuracy, and model size, operating within the bounded landscape of classical polynomial-time computation. Our current study, in contrast, explores the ultimate limits of a quantum computational model, yet both endeavors share the theme of optimal resource utilization: the invariant $O(|\langle\varphi|\psi\rangle|^{-1})$ balancing time and energy in an adiabatic process finds a conceptual parallel in the engineering trade-offs inherent in designing a practical detector.

Looking ahead, several important questions emerge directly from this work, whose answers may eventually resonate with applied fields.

1. **Extending the optimality framework:**
    - A direct proof for the multi-target case using the conventional subspace-projection Hamiltonian (7) seems more natural, yet has not been established, to our knowledge. By adopting the equivalent (in terms of ground-state subspace) form $H_f = I - |\varphi\rangle\langle\varphi|$, we circumvent this difficulty and obtain a unified proof that applies to both single- and multi-target scenarios. Addressing this may re-

quire revisiting lower-bound results from [49] and employing adiabatic-to-query simulation techniques [50].
- Extending our unified optimality framework beyond search to areas such as adiabatic optimization or many-body quantum simulation could yield practical design guidelines for a broader class of algorithms.
- Our framework also suggests the possibility of algorithmic speedups beyond conventional quadratic scaling in specific settings, motivating the search for problem classes or Hamiltonian constructions that realize this potential.

2. **Understanding time slice invariance:**
   - For the two principal Hamiltonian cases considered, we obtain an invariant time slice $R = O(|\langle\varphi|\psi\rangle|^{-1})$ in the circuit model. However, it remains unclear whether, for *any* adiabatic algorithm with time complexity $T = O(|\langle\varphi|\psi\rangle|^{-1}/A)$, this invariance persists as a trade-off between reduced algorithmic time and increased energy requirements. The required time slice appears to depend not only on complexity, but also on the Hamiltonian's specific structure.
   - This invariant quantity merits deeper physical investigation, as it may point to fundamental constraints or design principles for robust adiabatic paths.
3. **Bridging theory and quantum hardware:**
   - The energy-time trade-off theorem provides a quantitative framework for hardware-aware algorithm design, enabling future work to optimize the interpolating function $g(s)$ for specific quantum platforms—such as superconducting qubits, trapped ions, or neutral atoms—under realistic constraints like finite coherence times and maximum energy scales. A critical next step is to establish quantitative estimates of the maximum achievable scaling factor A on each platform, as this directly dictates the practical speedup limits of energy-scaled evolutions and clarifies the trade-off between algorithmic runtime and hardware feasibility.
   - The discovery of time slice invariance motivates the development of more comprehensive complexity measures that capture both time and energy consumption, offering a realistic guide for evaluating algorithm performance on near-term devices.
4. **Empirical validation:**
   - Numerical simulations and hardware experiments targeting finite system sizes are crucial for testing the tightness of the derived lower bounds—particularly $\int_0^T g(s)dt \geq \Omega(\sqrt{N/M})$—and revealing how the predicted scaling behaves in noise-prone environments. Visualizing the adiabatic evolution for finite N would complement our analytical results and provide practical insights for near-term implementations.
5. **Connections to computational equivalence:**
   - While this work addresses adiabatic optimality independently of known polynomial equivalence proofs [29,30], exploring whether our optimality criteria can offer new insights into the comparative power of adiabatic and circuit-based paradigms remains compelling.

We hope that by clarifying fundamental limits and invariants in the quantum adiabatic model, this work will stimulate theoretical advances and enrich the conceptual toolkit for tackling optimization challenges across quantum and classical domains. To ground these developments and situate our contributions, we now present detailed comparisons with prior adiabatic studies and the standard circuit-based Grover algorithm in Sections 5.1 and 5.2, respectively.

5.1. Comparison with Existing Work

Table 1 summarizes the novel aspects of this work relative to the existing literature.

Table 1. Comparison with previous works on adiabatic search.

Aspect	Roland-Cerf [3]	Das et al. [36]	Tulsi [5]/Zhang et al. [21]	Kay [23]	This Work
Multi-target optimality	Not addressed	Single-target only	Claimed but proof questioned	Critique only	Proved rigorously
Energy-time trade-off	Implicit	Single-target	Not addressed	Not addressed	General theorem
Verification condition	Not applicable	Not applicable	Absent	Proposed	Integrated
Circuit implementation	Standard case	Not addressed	Not addressed	Not addressed	Invariance discovered

The key novelties of this paper are: (i) the first rigorous proof of adiabatic search optimality for the general multi-target case, (ii) a unified energy-time trade-off theorem extending previous single-target results, (iii) the discovery of time slice invariance in circuit implementation, and (iv) a resolution to the partial adiabatic search controversy beyond Kay's verification condition.

5.2. Comparison with Circuit-Based Grover Algorithm

Table 2 compares our results with the well-studied circuit-based Grover algorithm.

Table 2. Comparison between adiabatic search and Grover's algorithm.

Aspect	Grover's Algorithm (Circuit)	Adiabatic Search (This Work)
Time complexity (single target)	$\mathcal{O}(\sqrt{N})$	$\mathcal{O}(\sqrt{N})$
Time complexity (multi-target)	$\mathcal{O}(\sqrt{N/M})$	$\mathcal{O}(\sqrt{N/M})$
Optimality proof method	Amplitude amplification	Energy–time trade-off (Theorem)
Resource measure	Number of queries	Time $\times$ Energy integral
Role of energy	Fixed (gate operations)	Variable (Hamiltonian scaling)
Circuit implementation	Direct (oracle-based)	Two-stage approximation
Implementation cost	$\mathcal{O}(\sqrt{N})$ oracle calls	$\mathcal{O}(\sqrt{N/M})$ time slices

Several observations emerge from this comparison. First, both models achieve the same quadratic speedup, consistent with known computational equivalence results [29,30]. Second, while Grover's algorithm measures complexity in terms of query count, adiabatic search measures it in terms of evolution time—a difference significant for physical implementations with finite coherence times. Third, the energy–time trade-off revealed in our analysis has no direct analog in the circuit model, where Hamiltonian energy scales are typically fixed by gate implementations. This suggests that quantum adiabatic computation offers additional flexibility: one can trade time for energy depending on physical platform constraints. However, as discussed in Section 3, this flexibility is bounded by practical limitations.

The time slice invariance discovered in Section 4 provides another point of contrast. When implementing Grover's algorithm on a circuit, the resource cost is directly given by the number of oracle calls; there is no hidden quantity analogous to our $O(|\langle\varphi|\psi\rangle|^{-1})$ scaling. In contrast, the time slice invariance reveals a unique feature of adiabatic-to-circuit simulation: the circuit resources required depend not only on algorithmic complexity, but also on the geometric structure of the Hamiltonian path. For standard adiabatic evolutions where the energy scale is constant, the time slice complexity coincides with the algorithmic runtime. However, for energy-scaled evolutions that achieve faster runtime by increasing the Hamiltonian energy, these measures diverge: the runtime decreases, but the required

time slices remain invariant. This divergence uncovers a fundamental coupling between time and energy that is not captured by complexity measures alone.

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