

Article

Proportional Stationarity and Structural Stability in Perturbative Field Theories

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Abstract

We formulate a structural stability criterion for dimensionless physical constants within standard perturbative field frameworks. The analysis introduces a response-ratio functional $\Gamma = \kappa/\tau$, defined from second-order sensitivity and first-order deformation measures associated with admissible variations in a field configuration. Stability is characterized by proportional stationarity of Γ , expressed as a first-order operator condition along transformation flows. The framework characterizes, within a declared variational model, when invariance of fixed constants can be represented as a stationarity condition. Under compactness and convexity assumptions typical of variational systems, stationary response ratios arise as isolated solutions of the associated operator equation; more general settings permit continuous spectra. Explicit functional definitions are provided within a conventional analytic setting, and the criterion is illustrated in representative classical field models. The results position proportional stationarity as a model-relative structural consistency condition for perturbative stability; isolation is conditional on compactness and non-degeneracy hypotheses, and continuous families may occur outside that regime. Limitations and possible extensions, including discretized spacetime formulations, are discussed.

Keywords: fundamental constants; curvature–strain invariants; operator geometry; proportional stability; equilibrium ratios; cosmological structure; variational systems

1. Introduction

The appearance of fixed dimensionless parameters in physical theory has long motivated structural inquiry. In classical and quantum field frameworks, parameters such as coupling strengths, mass ratios, and scale factors are introduced as fixed inputs to the theory. Their numerical values are determined empirically, yet within the formalism they are treated as invariant under admissible transformations of the underlying fields. Discussions of the status and interpretation of such constants span effective field theory, quantum gravity, and cosmological contexts [1–8].

The stability of such parameters is usually understood in dynamical or renormalization-group terms. In perturbative quantum field theory, effective couplings evolve with scale while maintaining structural coherence within the renormalized framework [1,2]. In classical field theory, fixed parameters enter Lagrangians as coefficients preserved under symmetry transformations and boundary conditions [9]. In both settings the mathematical formalism presumes that certain ratios or constants remain stable under admissible deformations of the fields.

This paper does not attempt to derive numerical values of physical parameters. Nor does it modify established field equations or introduce new dynamical principles. Instead,



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it isolates a variational structure that clarifies when fixed quantities may arise as stationary ratios of first- and second-order responses of a functional under admissible perturbations.

The relevance of the ratio construction to dimensionless constants is structural rather than ontological. In perturbative field frameworks, fixed dimensionless parameters enter through variational functionals and are required to remain invariant under admissible transformations of the configuration space. The present analysis does not assert that such constants must arise as response ratios. It identifies the analytic conditions under which invariance of a ratio constructed from first- and second-order response is consistent with the operator structure of the model. The plausibility of the construction therefore rests on a compatibility question: whether stability of a dimensionless quantity under admissible deformations can be expressed as proportional balance within the same variational framework that defines the theory.

The analysis is model-relative: proportional stationarity is evaluated within a specified variational functional and admissible transformation class, and its conclusions depend entirely on the analytic properties of that model.

The key construction is a ratio functional defined from the first and second variations in a twice-differentiable functional F on an admissible configuration space. Given a generator X of an admissible transformation group, define

$$\tau_X(\psi) = DF(\psi)[X\psi], \kappa_X(\psi) = D^2F(\psi)[X\psi, X\psi],$$

and consider the ratio

$$\Gamma_X(\psi) = \frac{\kappa_X(\psi)}{\tau_X(\psi)}.$$

Proportional stationarity occurs when the second variation along the generator direction is proportional to the first variation. In such cases, the induced ratio Γ_X is stationary under admissible perturbations.

The question addressed here is structural: under what analytic conditions can such stationary ratios arise as isolated elements within the admissible class?

This question can be framed entirely within standard variational and operator-theoretic settings. When the underlying functional is coercive and twice Fréchet differentiable on a Sobolev space, and when admissible perturbations preserve regularity and boundary conditions, the proportional-stationarity condition reduces to a constrained operator equation. Under compact-resolvent hypotheses and a non-degenerate constrained extremum, equilibrium ratios may emerge as isolated spectral quantities of the linearized operator [10–15]. This is a conditional operator-theoretic statement within a specified model; no claim is made that physical constants are eigenvalues of any universal operator.

The framework therefore connects three elements: first- and second-order variational response; generator-induced perturbative flows; and conditional spectral isolation of equilibrium ratios. This connection does not rely on specific physical models. It is compatible with classical field functionals, quadratic action integrals, and perturbative quantum field settings in which functionals of field configurations determine observable structure [1,2,9].

Previous discussions of parameter stability often proceed either through renormalization-group flow analysis [1,2] or through symmetry arguments rooted in Lagrangian invariance [9]. The present approach instead focuses on proportional balance between first- and second-order response under admissible transformations. In quadratic and elliptic settings, this balance reduces to spectral relations familiar from Sturm–Liouville and compact-operator theory [13–15]. In more general nonlinear settings, it defines a constrained stationarity problem whose solutions depend on coercivity and compactness properties of the admissible class.

The technical development proceeds as follows. Section 2 formalizes the configuration space, admissible transformations, and ratio functional. Section 3 analyzes proportional

stationarity and establishes conditional isolation results under standard variational assumptions. Appendices A and B provide detailed analytic support, including compactness hypotheses, differentiability of the ratio functional, and invariance under admissible rescaling. Section 4 and Appendix C illustrate the construction in a quadratic scalar-field functional on a bounded domain, where proportional stationarity reduces explicitly to a spectral condition.

The aim is to provide a structural criterion that clarifies how fixed ratios may arise as stationary quantities within perturbative field-theoretic models. The analysis remains entirely within established functional-analytic frameworks and does not introduce additional physical hypotheses beyond those standard in variational treatments of field theory [1,2,9,10].

2. Operational Framework

We work in a conventional analytic setting and specify the admissible configurations and variations under which proportional stationarity is evaluated. The definitions below introduce no new geometric objects; they only fix the functional-analytic environment in which the ratio is well-defined. We consider a configuration space $\mathcal{X}$ of admissible field profiles defined over a smooth manifold M . In typical applications, $\mathcal{X}$ is taken as a Banach or Hilbert subspace of $H^k(M)$ with $k \geq 2$, so that first and second variations are well-defined in the Fréchet sense. This setting is classical in variational treatments of classical field theory and operator analysis [10–12]. The construction does not depend on the specific choice of field; it applies equally to scalar, gauge, or metric perturbations provided the functional derivatives exist in the admissible topology.

An admissible transformation group G acts continuously on $\mathcal{X}$. The group may represent diffeomorphisms, gauge transformations, or other perturbative flows consistent with the underlying model. Its Lie algebra is denoted $\mathfrak{g}$, and generators $X \in \mathfrak{g}$ induce infinitesimal variations in the form

$$\delta_X \psi = \left. \frac{d}{dt} \right|_{t=0} \Phi_t^X(\psi),$$

where Φ_t^X is the flow generated by X . Throughout, variations are restricted to this admissible class. When X acts linearly on $\mathcal{X}$ (as in Sections 3 and 4), we use the shorthand $\delta_X \psi = X\psi$.

Let $F : \mathcal{X} \rightarrow \mathbb{R}$ be a twice Fréchet differentiable functional associated with the model under consideration. In classical field settings, F may be taken as an action functional or energy density integrated over M ; in operator-theoretic contexts, it may represent a response amplitude or expectation value. No specific interpretation is imposed at this stage.

Two response measures are defined along an admissible direction X . Define the first- and second-order responses along the generator direction by

$$\tau_X(\psi) = DF(\psi)[\delta_X \psi], \kappa_X(\psi) = D^2F(\psi)[\delta_X \psi, \delta_X \psi].$$

where $\tau_X(\psi) \neq 0$, define the response-ratio functional by

$$\Gamma_X(\psi) = \frac{\kappa_X(\psi)}{\tau_X(\psi)}.$$

We write

$$A_X := \{\psi \in \mathcal{X} : \tau_X(\psi) \neq 0\}$$

for the admissible subset on which Γ_X is defined. When a nonnegative “magnitude” version is desired, one may replace τ_X, κ_X by smooth norms induced by a positive-definite metric on

the response space; all differentiability statements below are understood on the admissible domain where $\tau_X \neq 0$.

Stability in the present sense is characterized by proportional stationarity of this ratio along admissible flows. Explicitly, proportional stationarity requires that

$$D\Gamma_X(\psi)[\delta_X\psi] = 0,$$

or equivalently,

$$\tau_X(\psi) D\kappa_X(\psi)[\delta_X\psi] = \kappa_X(\psi) D\tau_X(\psi)[\delta_X\psi].$$

This condition expresses a first-order balance between quadratic sensitivity and linear deformation along the chosen perturbative direction. It does not impose extremality of either functional individually.

The proportional-stationarity condition can be interpreted variationally. For fixed X , consider the auxiliary functional

$$\mathcal{S}_\lambda(\psi) = \kappa_X(\psi) - \lambda\tau_X(\psi).$$

Stationarity of $\mathcal{S}_\lambda$ with respect to admissible variations yields a relation between first and second derivatives of F . Identifying λ with $\Gamma_X(\psi)$ recovers the proportional balance condition. This formulation situates the construction within familiar constrained-variation settings [16,17], without introducing additional dynamical assumptions.

The character of stationary solutions depends on structural properties of the admissible class. The existence and isolation claims are conditional on compactness/coercivity hypotheses stated explicitly in Appendix A. The precise hypotheses under which isolation holds are stated in Appendix A, where the conditional discreteness argument is given in operator-theoretic form.

It is important to note that the framework does not require global spectral completeness. The proportional-stationarity condition is local along admissible flows. In contexts where F is derived from a Lagrangian density,

$$F(\psi) = \int_M \mathcal{L}(\psi, \partial\psi) d\mu,$$

the deformation measure corresponds to first variation in the action, while the sensitivity measure corresponds to its second variation. This connects the construction directly to perturbative treatments of classical and effective field theories [1,2,9,18–20]. No modification of field equations is introduced; the proportional condition operates at the level of response structure.

Technical details (admissibility, differentiability of Γ_X , and the isolation mechanism under compact-resolvent structure) are recorded in Appendices A and B to keep the main argument continuous. The central analytic question is therefore whether the proportionality condition defining Γ_X can arise as a stationary constraint within the admissible class and, if so, whether such stationary ratios are structurally isolated. Section 3 addresses these questions under standard compactness and coercivity assumptions.

In Sections 3 and 4 we assume: (i) $F : \mathcal{X} \rightarrow \mathbb{R}$ is twice Fréchet differentiable on the admissible subset; (ii) admissible flows preserve $\mathcal{X}$ and the relevant boundary/regularity constraints; (iii) $\tau_X(\psi) \neq 0$ on the subset where Γ_X is evaluated; and (iv) when isolation is claimed, the second-variation operator satisfies the compactness/non-degeneracy hypotheses recorded in Appendix A. These are standard in coercive elliptic variational settings and are invoked only when needed.

3. Proportional Stationarity and Structural Isolation

The ratio functional introduced in Section 2,

$$\Gamma_X(\psi) = \frac{\kappa_X(\psi)}{\tau_X(\psi)},$$

encodes the relative magnitude of second-order to first-order response along admissible generator directions. Throughout, Γ_X is considered on the admissible subset where $\tau_X(\psi) \neq 0$; degenerate directions (vanishing first variation along X) are excluded as standard for ratio-type variational functionals. We identify conditions under which stationary ratios occur and the operator hypotheses under which such values are isolated in the admissible class.

Let $\mathcal{X} \subset H^1(\Omega)$ denote the configuration space described in Section 2 and assume that the functional $F : \mathcal{X} \rightarrow \mathbb{R}$ is twice Fréchet differentiable. Generators X are taken to preserve admissibility and regularity. These assumptions place the framework within the standard variational setting for coercive functionals on Sobolev spaces, in the usual Sobolev/elliptic variational setting [10–15].

3.1. Proportional Stationarity as a Constrained Variational Condition

For a fixed generator X , proportional stationarity occurs at configurations $\psi^* \in \mathcal{A}_X$ satisfying

$$D^2F(\psi^*)[X\psi^*, X\psi^*] = \Gamma^* DF(\psi^*)[X\psi^*],$$

where $\Gamma^* = \Gamma_X(\psi^*)$, and $\psi^* \in \mathcal{A}_X$.

This condition can be interpreted as a constrained extremality requirement in which the second variation along the generator direction is proportional to the first variation. The proportionality constant Γ^* emerges from the structure of the functional and the transformation direction; it is not externally imposed.

When F is quadratic in the admissible variables, the relation reduces to a spectral balance between bilinear forms. In more general nonlinear settings, the condition corresponds to a nonlinear eigenvalue-type problem. Problems of this form arise naturally in variational treatments of stability and bifurcation, where spectral parameters emerge from constrained optimization in infinite-dimensional spaces [10,12,14].

The analytic consistency of the ratio functional under differentiation and perturbation is established in Appendix B, where differentiability of Γ_X on the admissible set is verified under standard regularity hypotheses. This permits the proportional-stationarity condition to be studied using the tools of Fréchet calculus and operator linearization.

3.2. Existence Under Coercivity and Compactness

The existence of stationary ratios depends on the structure of the functional and the admissible class. When F is coercive and sequentially weakly lower semicontinuous on $\mathcal{X}$, minimizing sequences admit convergent subsequences under compact embedding assumptions. Such conditions are classical in the direct method of the calculus of variations [10,11].

If boundary data are fixed so that the admissible set is effectively compact modulo natural invariances, constrained extremizers exist within each admissible class. Appendix A formalizes how, under strict convexity and appropriate boundary conditions, the constrained stationarity problem reduces to a null-mode equation for an associated operator. In that setting, proportional stationarity arises from a well-posed variational problem rather than from algebraic manipulation of ratios.

The solvability of the proportionality condition therefore reflects the interaction between first and second variations under the generator flow, embedded within standard compactness and coercivity frameworks.

3.3. Spectral Reduction and Isolation

Isolation of equilibrium ratios requires additional structure. Suppose the second-variation operator

$$L_\psi = D^2F(\psi)$$

is uniformly elliptic and possesses compact resolvent on the admissible domain. Such operators have discrete spectra accumulating only at infinity, a classical result in spectral theory for elliptic operators on bounded domains [13–15]. The discreteness of the spectrum is the operator-theoretic mechanism by which proportional-stationary ratios become isolated values rather than members of continuous families.

Under these conditions, the proportional-stationarity relation induces a spectral problem of the form

$$L_\psi(X\psi) = \Gamma^* M_\psi(X\psi),$$

where M_ψ encodes the first-variation functional at ψ : in a Hilbert setting we identify $DF(\psi)$ with its Riesz representer and write $DF(\psi)[v] = \langle M_\psi, v \rangle$ for all admissible v ; in a Banach setting, M_ψ may be read as the corresponding element of $\mathcal{X}^*$. Compact-resolvent structure implies that admissible solutions correspond to discrete spectral values.

Appendix A develops this reduction explicitly. In particular, it shows that when the induced entropy-like functional is strictly convex in a neighborhood of an extremal configuration, equilibrium ratios are spectrally isolated. Isolation follows from non-degeneracy of the constrained extremum together with continuity of the ratio mapping. The spectral isolation argument relies on standard compact-operator theory and variational eigenvalue analysis [13,15].

Continuous families of ratios may arise if compactness fails or if degeneracies occur in the linearized operator. The isolation result is therefore conditional and operator-theoretic in nature.

3.4. Stability of Equilibrium Ratios

Local stability of Γ^* follows from differentiability of the ratio functional. Appendix B provides the explicit expression

$$D\Gamma_X(\psi)[\delta\psi] = \frac{D\kappa_X(\psi)[\delta\psi]}{\tau_X(\psi)} - \frac{\kappa_X(\psi)}{\tau_X(\psi)^2} D\tau_X(\psi)[\delta\psi].$$

At a proportional-stationary configuration ψ^* , first-order perturbations preserving proportionality yield vanishing first variation of Γ_X . Higher-order corrections scale quadratically in perturbation magnitude provided regularity assumptions remain valid. This behavior is consistent with classical stability analyses for constrained variational problems, where spectral gaps control the local behavior of extremal configurations [12,14].

The perturbative behavior therefore aligns with the compact-resolvent and convexity hypotheses developed in Appendices A and B.

3.5. Structural Role Within the Present Framework

Within the operator-theoretic setting adopted here, proportional stationarity expresses a balance between first-order deformation response and second-order curvature response along admissible transformation directions. The analysis characterizes when such balances may arise and under what structural conditions they are isolated.

The conclusions depend explicitly on coercivity, compactness, and non-degeneracy assumptions. These assumptions are formalized in Appendix A and analytically supported in Appendix B. Section 4 illustrates how these abstract conditions appear in a representative quadratic field-theoretic functional, where the proportional-stationarity condition reduces directly to a spectral relation.

4. Application to a Quadratic Scalar-Field Functional

To illustrate how proportional stationarity manifests within a standard field-theoretic setting, consider a real scalar field defined on a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary and Dirichlet boundary conditions. This example is chosen because it lies entirely within the Sobolev framework described in Section 2 and satisfies the coercivity and compactness assumptions used in Appendix A. Quadratic scalar functionals of this form are standard in both classical field theory and effective quantum field settings [1,2,9].

Let the configuration space be

$$\mathcal{X} = H_0^1(\Omega),$$

and consider the quadratic functional

$$F(\psi) = \frac{1}{2} \int_{\Omega} (|\nabla \psi|^2 + m^2 \psi^2) dx,$$

where $m \geq 0$ is a fixed parameter. This functional corresponds to the classical action density for a free scalar field restricted to a static domain. Its coercivity and strict convexity on $H_0^1(\Omega)$ are classical in elliptic theory [10,11]. The Fréchet differentiability properties required in Section 2 are standard consequences of the Sobolev embedding framework.

The first variation is given by

$$DF(\psi)[\phi] = \int_{\Omega} (\nabla \psi \cdot \nabla \phi + m^2 \psi \phi) dx,$$

and the associated Euler–Lagrange equation is

$$-\Delta \psi + m^2 \psi = 0,$$

which is the familiar Klein–Gordon operator in a static domain [9].

The second variation is

$$D^2F(\psi)[\phi, \phi] = \int_{\Omega} (|\nabla \phi|^2 + m^2 \phi^2) dx.$$

These expressions satisfy the differentiability and boundedness assumptions recorded in Appendix B.

4.1. Generator-Induced Variations

Let X be a linear admissible generator acting on ψ . Examples include internal symmetry directions or admissible scaling transformations preserving $H_0^1(\Omega)$. The only requirement is that $X\psi \in H_0^1(\Omega)$, consistent with the admissibility framework in Section 2.

Define

$$\tau_X(\psi) = DF(\psi)[X\psi], \kappa_X(\psi) = D^2F(\psi)[X\psi, X\psi].$$

Substituting the explicit expressions yields

$$\tau_X(\psi) = \int_{\Omega} (\nabla \psi \cdot \nabla (X\psi) + m^2 \psi (X\psi)) dx,$$

and

$$\kappa_X(\psi) = \int_{\Omega} (|\nabla(X\psi)|^2 + m^2(X\psi)^2) dx.$$

The ratio functional becomes

$$\Gamma_X(\psi) = \frac{\int_{\Omega} (|\nabla(X\psi)|^2 + m^2(X\psi)^2) dx}{\int_{\Omega} (\nabla\psi \cdot \nabla(X\psi) + m^2\psi(X\psi)) dx}.$$

The structure of this ratio parallels the abstract formulation in Section 2 and Appendix B, with numerator and denominator arising directly from second- and first-order variational response.

4.2. Reduction to a Spectral Relation

Suppose ψ is an eigenfunction of the elliptic operator

$$L = -\Delta + m^2,$$

with eigenvalue λ , satisfying

$$L\psi = \lambda\psi.$$

The operator L , with Dirichlet boundary conditions on a bounded domain, has compact resolvent and therefore a discrete spectrum accumulating only at infinity. This is a classical result in elliptic spectral theory [13–15]. The compact-resolvent structure is precisely the hypothesis invoked in Appendix A to justify isolation of admissible equilibrium ratios.

Using Dirichlet boundary conditions and the symmetry of $L = -\Delta + m^2$ on $H_0^1(\Omega)$ (integration by parts), we may rewrite the first variation in operator form as

$$\tau_X(\psi) = \int_{\Omega} (L\psi)(X\psi) dx,$$

while the second variation becomes

$$\kappa_X(\psi) = \int_{\Omega} (L(X\psi))(X\psi) dx.$$

If the generator commutes with the operator within the admissible class, the proportional-stationarity condition

$$\kappa_X(\psi) = \Gamma_X(\psi) \tau_X(\psi)$$

reduces to a spectral proportionality between operator actions on ψ and $X\psi$. In this case, equilibrium ratios are directly linked to spectral properties of the linearized operator, as described abstractly in Section 3.

4.3. Isolation of Equilibrium Ratios

Given the discrete spectrum of L on bounded domains [13–15], the proportional-stationarity condition selects equilibrium ratios associated with spectral modes. The proportional-stationarity condition therefore defines equilibrium ratios associated with discrete spectral modes.

Appendix A demonstrates that, when strict convexity and non-degeneracy hold in a neighborhood of an extremal configuration, the induced ratio Γ_X is locally isolated. In the present scalar-field example, strict convexity of the quadratic functional together with compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ ensures that equilibrium ratios correspond to discrete spectral quantities. This functional satisfies the coercivity, compact embedding,

and ellipticity hypotheses detailed in Appendices A and B, ensuring that the proportional-stationarity analysis applies without additional structural assumptions [10–15].

The mechanism of isolation in this setting is thus entirely operator-theoretic and rests on classical elliptic results.

4.4. Perturbative Stability

Consider a perturbation $\psi \mapsto \psi + \varepsilon\phi$, with $\phi \in H_0^1(\Omega)$. Using the differentiability of the ratio functional established in Appendix B, one obtains

$$\delta\Gamma_X = \frac{\delta\kappa_X}{\tau_X} - \frac{\kappa_X}{\tau_X^2} \delta\tau_X + O(\varepsilon^2).$$

At a proportional-stationary configuration, perturbations preserving proportionality to first-order yield vanishing first-order change in Γ_X . The quadratic scaling of higher-order terms follows from standard Taylor expansion in Banach spaces [10,11].

This stability behavior is consistent with classical analyses of constrained variational extrema and spectral stability.

4.5. Structural Implications for Field-Theoretic Models

The scalar-field example demonstrates that proportional stationarity arises naturally in a familiar Lagrangian framework. The ratio functional reorganizes the relationship between first- and second-order response along admissible transformation directions without altering the underlying field equations.

In quadratic elliptic settings, equilibrium ratios inherit the discrete spectral structure of the operator. In nonlinear field theories, the same proportional condition defines a constrained variational problem whose solvability depends on coercivity and compactness properties, as detailed in Appendices A and B.

The example therefore anchors the abstract construction in a canonical field-theoretic model widely used in both classical and quantum analyses [1,2,9], and shows how operator-theoretic isolation of ratios follows from established spectral results.

5. Structural Scope, Limitations, and Field-Theoretic Context

The preceding sections develop a conditional operator-theoretic setting in which proportional stationarity of

$$\Gamma_X(\psi) = \frac{\kappa_X(\psi)}{\tau_X(\psi)}$$

can arise as an isolated quantity. The argument rests on familiar hypotheses from variational calculus and elliptic operator theory—coercivity, compact embedding, twice Fréchet differentiability, and non-degeneracy of the constrained extremum [10–15]. Under compact-resolvent conditions, the proportionality requirement reduces to a spectral relation whose admissible solutions form a discrete set.

It is therefore appropriate to place the result within the broader landscape of field-theoretic stability analysis. In perturbative quantum field theory, stability of effective parameters is typically discussed through renormalization-group flow [1,2], with fixed points defined by scale-invariant balance (vanishing beta functions). In classical field theory, invariance of coefficients is tied to symmetry and the variational structure of the action [9]. The present paper does not replace these mechanisms. It isolates a compatibility condition that may coexist with them: proportional balance between first- and second-variation response along admissible generator directions.

In quadratic settings—exemplified by the scalar-field functional in Section 4 and Appendix C—proportional stationarity coincides with a spectral relation between bilinear

forms, and isolation of ratios follows directly from compact-operator theory [13–15]. In that regime, the mechanism is operator-theoretic rather than dynamical: it depends on the functional-analytic geometry of the admissible class and the perturbation group, not on scale flow or cosmological evolution.

For nonlinear functionals, proportional stationarity takes the form of a constrained eigenvalue-type problem. Existence and isolation are conditional on the coercivity and compactness hypotheses recorded in Appendix A, while differentiability and perturbative stability of the ratio functional are treated in Appendix B. No hypotheses beyond those routinely used in variational analysis of field equations are assumed [10–12].

It is equally important to state what is not being claimed. The framework does not propose new invariants of nature and does not determine numerical values of physical constants. It introduces no new field equations, makes no modification to Lagrangians, and adds no dynamical degrees of freedom. Its content is a variational criterion: within a specified field-theoretic model and admissible deformation class, it characterizes when ratios derived from first- and second-order response exhibit stationarity. Whether any particular parameter satisfies this condition depends entirely on the variational structure of the model under study.

The proportional-stationarity condition should therefore be understood as a consistency test internal to a declared variational model. If a dimensionless quantity is invariant under admissible configuration-space deformations within that model, and if its defining functional admits the coercivity and compactness properties analyzed above, then proportional balance between first- and second-order response is a mathematically natural candidate for expressing that invariance. The construction does not presume that physical constants universally satisfy this condition; it clarifies when such a representation is analytically coherent.

The quadratic scalar-field example of Section 4 and Appendix C is included solely to display the mechanism in a setting where all analytic hypotheses are transparent. There, the functional

$$F(\psi) = \frac{1}{2} \int_{\Omega} (|\nabla \psi|^2 + m^2 \psi^2) dx$$

satisfies coercivity and compact-resolvent conditions on bounded domains with Dirichlet boundary data [10,13]. The proportional-stationarity condition reduces to a spectral relation involving the Laplacian, and the resulting discreteness reflects classical elliptic theory rather than additional assumptions.

From a field-theoretic perspective, the criterion may be read as an internal balance condition between deformation response and curvature response within a variational model. In effective field theory terms, this concerns response structure under admissible configuration-space transformations rather than scale dependence of couplings [1,2]. In geometric terms, it specifies when second-order response scales proportionally with first-order deformation along selected directions in configuration space.

The analysis has clear limitations. Compact-resolvent assumptions restrict attention to bounded domains or settings with an appropriate confining structure; in non-compact domains, continuous spectra may appear and equilibrium ratios need not be isolated. Degeneracies in the linearized operator can likewise produce families of proportional solutions. Finally, differentiability of the ratio functional requires non-vanishing of the denominator on the admissible set, a condition made explicit in Appendix B. These points delineate the operator regimes in which proportional stationarity behaves as a discrete quantity; outside those regimes, Γ_X may remain well-defined while losing spectral isolation.

The contribution is therefore conditional in the precise sense made explicit above. It provides a variational criterion under which ratios derived from first- and second-order response can arise as stationary and—under compactness—isolated quantities in

perturbative field-theoretic models. The criterion is formulated entirely within established functional-analytic frameworks and requires no physical hypotheses beyond those standard in the calculus of variations and elliptic operator theory [10–15].

This perspective complements existing stability analyses rather than competing with them. Renormalization-group methods address scale evolution of couplings [1,2]; symmetry arguments encode invariance under transformation groups [9]; spectral theory governs stability and eigenstructure of the linearized operator [13–15]. Proportional stationarity identifies the variational conditions under which these structures yield stationary ratios along admissible deformation directions. When coercivity, compactness, and non-degeneracy hold, equilibrium ratios are spectrally isolated and perturbatively stable; when they do not, the ratio condition remains meaningful but need not select isolated values.

6. Robustness Under Extended Perturbation Structures

Sections 2–5 formulate proportional stationarity as a constrained variational condition whose isolation depends on the operator hypotheses made explicit in Appendix A. The present section examines robustness: whether proportional stationarity is tied to a single generator choice, or whether it persists under broader admissible perturbation structures within the same analytic setting. Throughout, F is assumed twice Fréchet differentiable on the admissible class, with coercivity and compact-resolvent structure invoked whenever isolation is discussed [10–15].

6.1. Generator Families and Invariance Classes

In Section 2, Γ_X was defined relative to a fixed generator X . In many field-theoretic settings, admissible perturbations arise from a finite-dimensional Lie algebra of generators associated with symmetry or deformation groups [9]. Let $\{X_1, \dots, X_k\}$ denote such a family preserving admissibility and regularity. For each generator, define

$$\Gamma_{X_i}(\psi) = \frac{\kappa_{X_i}(\psi)}{\tau_{X_i}(\psi)}.$$

When the generators lie in a common invariance class and act smoothly on the configuration space, proportional stationarity along one generator induces compatibility constraints along the others. In quadratic regimes such as Section 4 and Appendix C, this compatibility reduces to invariance of the relevant spectral subspace under the generator family. Under the compact-resolvent hypothesis of Appendix A, eigenspaces are finite-dimensional and stable under symmetry-preserving perturbations [13–15]. In that setting, proportional stationarity reflects properties of the functional and its linearized operator rather than an arbitrary coordinate choice of deformation direction.

The differentiability statements proved in Appendix B apply generator-by-generator: for each admissible X_i , Γ_{X_i} remains Fréchet differentiable wherever $\tau_{X_i} \neq 0$, and the perturbative stability formula of Section 3.4 applies without modification.

6.2. Multi-Directional Perturbations

In many variational problems, admissible deformations span finite-dimensional subspaces rather than single directions. Let

$$X(\alpha) = \sum_{i=1}^k \alpha_i X_i$$

denote a finite linear combination of admissible generators. The induced first- and second-order responses inherit bilinearity from DF and D^2F . In quadratic functionals, this yields a

finite-dimensional reduction in which proportional stationarity corresponds to a generalized eigenvalue problem of the form

$$L_\psi v = \Gamma M_\psi v,$$

restricted to $\text{span}\{X_i\psi\}$. Under compact-resolvent assumptions, discreteness of admissible spectral values is preserved under such finite-dimensional reductions [13–15], so isolation of proportional ratios persists when perturbations are taken within a finite generator subspace.

For nonlinear settings, existence of stationary ratios under multi-directional perturbations follows from the same compactness and coercivity structure used in Section 3.2. Appendix A records the reduction in the constrained extremality condition to a null-mode equation for the linearized operator. When the admissible perturbation subspace is finite-dimensional and compactness is retained, isolation again reduces to non-degeneracy of the constrained extremum.

6.3. Weak Nonlinearity and Stability Radius

The ratio construction is not tied to quadratic structure. For nonlinear functionals satisfying the hypotheses of Section 3.2, Appendix B yields the perturbative identity

$$D\Gamma_X(\psi)[\delta\psi] = \frac{D\kappa_X(\psi)[\delta\psi]}{\tau_X(\psi)} - \frac{\kappa_X(\psi)}{\tau_X(\psi)^2} D\tau_X(\psi)[\delta\psi].$$

At a proportional-stationary configuration ψ^* , first-order variations that preserve proportionality yield vanishing first variation of Γ_X . Higher-order corrections scale quadratically in perturbation magnitude as long as the regularity hypotheses remain valid. This behavior is consistent with classical nonlinear stability analysis for constrained extrema in Banach spaces [10–12]. When the linearized operator exhibits a spectral gap (an assumption stated explicitly in Appendix A), perturbations remain controlled within a finite stability radius.

If compactness fails or the linearized operator becomes degenerate, spectral isolation may be lost and families of proportional ratios can appear. The framework therefore distinguishes isolated versus continuous behavior through operator-theoretic hypotheses, rather than through any algebraic feature of the quotient itself.

6.4. Relation to Renormalization and Fixed-Point Language

Proportional stationarity should not be identified with renormalization-group fixed points. In perturbative quantum field theory, fixed points are defined by vanishing beta functions under scale flow [1,2], describing the evolution of couplings with respect to an external scaling parameter. The present criterion instead concerns balance of first- and second-order functional response under admissible configuration-space deformations. The two perspectives therefore operate on different analytic objects: scale dependence versus directional response balance within a fixed variational model.

That said, both rely on the consistency of the operator framework. In effective field theory, renormalizability requires control of divergences and preservation of operator form under scaling [1,2]. In the proportional-stationarity setting, isolation depends on coercivity, compactness, and non-degeneracy of the induced operator [10–15]. In each case, stability conclusions are driven by the structure of the underlying operators, even though the stability questions differ.

6.5. Robustness Summary

Within the hypotheses fixed in Sections 2 and 3 and made explicit in Appendices A and B, proportional stationarity exhibits three robustness properties: it persists across symmetry-related generator families; it extends to finite-dimensional perturbation subspaces; and it remains perturbatively stable under weak nonlinear corrections. Each point is an operator-theoretic consequence of the same compact-resolvent and non-degeneracy structure used to obtain isolation in Section 3 [10–15]. No additional assumptions are introduced in this extension.

7. Structural Implications and Operator Regimes

The preceding sections establish proportional stationarity as a constrained variational condition defined entirely within standard functional-analytic frameworks. When a twice Fréchet differentiable functional is coercive on an admissible Sobolev class and its second-variation operator possesses compact resolvent, stationary ratios of first- and second-order response arise as discrete spectral quantities. Isolation of such ratios is not an algebraic property of the quotient construction but a consequence of operator structure, convexity, and non-degeneracy, as detailed in Sections 3–6 and formalized in Appendices A and B [1,2,9–15].

This section clarifies the operator regimes in which those conclusions hold and delineates the analytic boundaries of the framework.

7.1. Operator Regimes Supporting Isolation

The isolation results derived in Section 3 depend on compact-resolvent properties of the linearized operator $L_\psi = D^2F(\psi)$. On bounded domains with appropriate boundary conditions, uniformly elliptic operators satisfy these properties and admit discrete spectra accumulating only at infinity [13–15]. Under such circumstances, proportional stationarity reduces to a generalized eigenvalue relation, and equilibrium ratios occur as isolated spectral values.

Appendix A makes explicit that strict convexity of the induced functional near an extremal configuration ensures non-degeneracy of the constrained problem. Appendix B establishes that the ratio functional remains differentiable wherever the first variation does not vanish, permitting perturbative stability analysis.

These conditions are standard in elliptic variational theory [10–15]. The proportional-stationarity mechanism therefore operates naturally in operator regimes already familiar from stability and spectral analysis of field equations.

7.2. Regimes Where Isolation May Fail

Isolation is not automatic in arbitrary functional settings. If the configuration space is non-compact and no confining boundary conditions are imposed, the associated operator may admit continuous spectrum. In such regimes, stationary ratios—if they exist—need not be isolated. Similarly, degeneracy in the linearized operator may produce families of solutions rather than discrete values.

Loss of coercivity or failure of weak lower semicontinuity can also prevent existence of constrained extremizers. In these cases, the proportional-stationarity condition remains formally definable but may lack a well-posed variational interpretation.

These distinctions are analytic rather than conceptual. They reflect operator-theoretic structure rather than properties of the ratio functional itself. The framework therefore provides a diagnostic criterion whose behavior is determined by established compactness and spectral conditions [13–15].

7.3. Relation to Variational Field Theory

In classical field theory, stability of configurations is often analyzed through second-variation positivity and spectral properties of linearized operators [9,10]. The present construction operates within that same analytic environment. Proportional stationarity does not alter Euler–Lagrange equations or introduce additional dynamical constraints. It evaluates directional balance between first- and second-order response along admissible perturbations of an existing functional.

In perturbative quantum field theory, structural stability of effective couplings is typically analyzed through renormalization-group flow and scale dependence [1,2]. The proportional-stationarity condition addresses a different aspect of structure: it concerns response balance under configuration-space deformations rather than scale evolution of parameters. Both perspectives depend on operator consistency of the underlying theory, but they operate on distinct analytic levels.

The framework therefore complements rather than competes with established approaches. It identifies the precise operator conditions under which internal response balance yields stationary ratios, leaving dynamical and renormalization analyses unchanged.

7.4. Potential Extensions Within Analytic Scope

The analysis has been developed for functionals on Sobolev spaces with elliptic second variation. Extensions to gauge-field functionals or systems with internal symmetry structure would require verification of the same coercivity and compact-resolvent properties in those settings. Hyperbolic systems, by contrast, typically lack compact resolvent on natural domains and therefore fall outside the isolation regime described here.

Similarly, non-compact manifolds or asymptotically flat geometries introduce continuous spectral components that alter the isolation behavior. In such cases, proportional stationarity may still characterize response balance but need not produce discrete equilibrium ratios.

These extensions remain analytic questions governed by operator theory. The framework imposes no additional physical hypotheses; it requires only that the functional and admissible perturbations satisfy the regularity and compactness conditions already specified in Appendices A and B.

7.5. Concluding Structural Statement

Proportional stationarity, as developed in this paper, is a constrained variational condition linking first- and second-order functional response along admissible transformation directions. When coercivity, compactness, and non-degeneracy hold, the resulting stationary ratios are spectrally isolated and perturbatively stable. When those hypotheses fail, isolation may be lost, but the analytic structure of the condition remains well-defined.

The contribution of the present work is therefore structural. It clarifies how isolated ratios can arise from operator properties of a variational model without introducing new field equations or modifying established theoretical frameworks. The conclusions are conditional on standard analytic hypotheses and remain entirely within established operator-theoretic and variational frameworks [1,2,9–15].

Accordingly, the paper’s output is a checkable operator condition— Γ_X -stationarity along admissible flows—and a conditional isolation mechanism under standard compactness/non-degeneracy assumptions. It does not derive numerical values of constants and does not alter field equations; it specifies when a fixed ratio is structurally consistent within a declared variational model.

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Appendix A. Conditional Isolation of Proportional Stationary Ratios

This appendix records the analytic conditions under which proportional stationary ratios, defined in Section 2, occur as isolated solutions within an admissible class. The argument does not assert global discreteness in the absence of structural assumptions. It clarifies when isolation follows from standard variational and operator-theoretic hypotheses.

Let $\mathcal{X}$ be a Banach space of admissible configurations as introduced in Section 2 and let $F : \mathcal{X} \rightarrow \mathbb{R}$ be twice Fréchet differentiable. For a fixed admissible generator $X \in \mathfrak{g}$, define

$$\tau_X(\psi) = |DF(\psi)[\delta_X \psi]|, \kappa_X(\psi) = |D^2F(\psi)[\delta_X \psi, \delta_X \psi]|,$$

and the ratio

$$\Gamma_X(\psi) = \frac{\kappa_X(\psi)}{\tau_X(\psi)}, \text{ whenever } \tau_X(\psi) \neq 0.$$

The proportional stationarity condition is

$$D\Gamma_X(\psi)[\delta_X \psi] = 0.$$

We now describe conditions under which solutions of this condition are locally isolated.

Appendix A.1. Structural Hypotheses

Assume the following:

1. Regularity.
 $F \in C^2(\mathcal{X})$, and the mappings

$$\psi \mapsto \tau_X(\psi), \psi \mapsto \kappa_X(\psi)$$

are continuous on the admissible subset.

2. Admissible Compactness.

The subset $\mathcal{A} \subset \mathcal{X}$ defined by fixed boundary data or normalization constraints is weakly sequentially compact.

3. Coercivity Along Admissible Directions.

There exists a neighborhood $U \subset \mathcal{A}$ such that the second variation of F restricted to admissible directions is bounded below by a positive constant outside a finite-dimensional kernel.

Such hypotheses are standard in variational treatments of elliptic and Sturm–Liouville-type problems [10,13–15]. They do not require global spectral completeness, only local compactness in the admissible class.

Appendix A.2. Reduction to a Ratio Functional

Define the ratio functional restricted to admissible directions:

$$\Gamma_X : \mathcal{A} \rightarrow \mathbb{R}.$$

Because κ_X and τ_X are continuous and $\tau_X \neq 0$ on the admissible domain under consideration, Γ_X is continuous.

A proportional stationary configuration satisfies

$$D\Gamma_X(\psi) = 0$$

in the direction $\delta_X\psi$. If this condition holds for all admissible generators, it defines a constrained stationary set within $\mathcal{A}$.

Appendix A.3. Local Isolation

Under the coercivity assumption, the linearized operator associated with the second variation of F has compact resolvent on the admissible domain. Standard results from perturbation theory of linear operators [14,15] imply that non-degenerate critical points of such functionals are locally isolated.

To make this precise, suppose $\psi^* \in \mathcal{A}$ satisfies the proportional stationarity condition and that the linearization of Γ_X at ψ^* has trivial kernel in the admissible class. Then, by the implicit function theorem in Banach spaces [10,12], there exists a neighborhood $U \subset \mathcal{A}$ such that

$$\Gamma_X(\psi) = \Gamma_X(\psi^*) \Rightarrow \psi = \psi^*$$

for $\psi \in U$.

Thus, the stationary ratio $\Gamma_X(\psi^*)$ is locally isolated.

This statement is conditional. If degeneracy is present, continuous families of stationary ratios may occur. The framework does not exclude that possibility.

Appendix A.4. Relation to Spectral-Type Problems

In many classical settings, the second variation of F leads to an operator whose eigenvalue problem is of Sturm–Liouville or elliptic type. When the admissible class imposes boundary conditions that render the associated resolvent compact, the spectrum consists of isolated eigenvalues of finite multiplicity [13,14].

In such cases, proportional stationarity reduces to a condition relating first and second variations along admissible directions. If the associated linearized operator satisfies compactness conditions, the set of admissible stationary ratios inherits the discrete character of the underlying spectral problem.

The argument here does not require that the physical system itself be a Sturm–Liouville problem. It requires only that the admissible perturbative structure exhibit analogous compactness and coercivity properties.

Appendix A.5. Scope of the Result

The analysis developed above establishes that proportional stationary ratios are locally isolated under standard variational assumptions, specifically when coercivity, compactness, and non-degeneracy conditions hold for the induced operator framework. The discreteness of such ratios is therefore not automatic; it depends explicitly on compact-resolvent structure and the absence of degeneracy in the constrained extremum. In the absence of these hypotheses, no general discreteness theorem applies, and continuous families of proportional solutions may arise.

No claim is made that physical constants are eigenvalues of a universal operator. The appendix formalizes only the structural conditions under which ratio stationarity may yield isolated values within a specified admissible analytic setting. The results are conditional and operator-theoretic in nature.

Further normalization conventions and domain restrictions referenced in Section 2 are recorded in Appendix B.

Appendix B. Structural Invariance, Domain Conditions, and Perturbative Stability of the Ratio Functional

This appendix formalizes analytic properties of the ratio functional

$$\Gamma_X(\psi) = \frac{\kappa_X(\psi)}{\tau_X(\psi)},$$

as defined in Section 2. The objective is to establish that proportional stationarity possesses several structural features within the declared analytic framework. In particular, the ratio is shown to be invariant under admissible rescalings, well-defined on an appropriate functional domain, stable under small perturbations, and independent of trivial coordinate reparameterizations.

The discussion remains entirely analytic in scope and does not introduce any physical interpretation.

Appendix B.1. Functional Setting

Let $\mathcal{X} \subset H^2(\Omega)$ denote the admissible configuration space of profiles ψ , where Ω is a compact domain or a domain with boundary conditions ensuring precompactness in H^1 .

Let $F : \mathcal{X} \rightarrow \mathbb{R}$ be twice Fréchet differentiable.

For an admissible generator X of a transformation group G , define

$$\tau_X(\psi) = DF(\psi)[X\psi],$$

$$\kappa_X(\psi) = D^2F(\psi)[X\psi, X\psi].$$

Assume the following:

- $F \in C^2(\mathcal{X})$,
- $X\psi \in T_\psi\mathcal{X}$,
- $\tau_X(\psi) \neq 0$ on the admissible subset.

Then Γ_X is well-defined on

$$\mathcal{A}_X = \{\psi \in \mathcal{X} : \tau_X(\psi) > 0\}.$$

This excludes degenerate stationary directions where first variation vanishes identically.

Appendix B.2. Scaling Invariance

Let $F \mapsto \tilde{F} = \lambda F, \lambda > 0$. Then,

$$\tilde{\tau}_X = \lambda \tau_X, \tilde{\kappa}_X = \lambda \kappa_X.$$

Thus,

$$\tilde{\Gamma}_X = \Gamma_X.$$

Similarly, under generator rescaling $X \mapsto \mu X$,

$$\tau_{\mu X} = \mu \tau_X, \kappa_{\mu X} = \mu^2 \kappa_X,$$

and therefore

$$\Gamma_{\mu X} = \mu \Gamma_X.$$

Hence, proportional stationarity $L_X \Gamma_X = 0$ transforms covariantly under generator scaling. When generators are fixed by normalization (as in Section 2), Γ becomes scale-independent. This demonstrates that Γ reflects structural response rather than absolute magnitude.

Appendix B.3. Reparameterization Invariance

Let $\Psi : \mathcal{X} \rightarrow \mathcal{X}$ be a smooth bijection preserving admissibility and differentiability class. Under reparameterization, first and second variations transform via the chain rule:

$$D(F \circ \Psi) = DF \circ D\Psi,$$

$$D^2(F \circ \Psi) = D^2F[D\Psi, D\Psi] + DF[D^2\Psi].$$

Provided $D^2\Psi$ does not introduce singular terms along admissible directions, the ratio structure of κ_X/τ_X is preserved to first order.

Thus, proportional stationarity is coordinate-invariant under smooth admissible transformations.

Appendix B.4. Differentiability of the Ratio Functional

On $\mathcal{A}_X$, Γ is differentiable with derivative

$$D\Gamma_X(\psi)[\delta\psi] = \frac{D\kappa_X(\psi)[\delta\psi]}{\tau_X(\psi)} - \frac{\kappa_X(\psi)}{\tau_X(\psi)^2} D\tau_X(\psi)[\delta\psi].$$

If ψ^* satisfies proportional stationarity, i.e.,

$$\kappa_X(\psi^*) = \Gamma^* \tau_X(\psi^*),$$

then for perturbations satisfying

$$D\kappa_X(\psi^*)[\delta\psi] = \Gamma^* D\tau_X(\psi^*)[\delta\psi],$$

we obtain

$$D\Gamma_X(\psi^*)[\delta\psi] = 0.$$

Thus, Γ is first-order stable along admissible directions preserving proportionality.

Appendix B.5. Perturbative Stability

Let

$$(\kappa_X, \tau_X) \mapsto (\kappa_X + \delta\kappa_X, \tau_X + \delta\tau_X).$$

A second-order Taylor expansion gives

$$\delta\Gamma_X = \frac{\delta\kappa_X}{\tau_X} - \frac{\kappa_X}{\tau_X^2} \delta\tau_X + O(\|\delta\psi\|^2).$$

If proportionality holds to first order, deviations in Γ are quadratic in perturbation magnitude. This establishes local stability of equilibrium ratios under admissible small perturbations. No numerical tolerance assumptions are required.

Appendix B.6. Domain Restrictions and Boundary Conditions

The isolation results established in Appendix A rely on precompactness or coercivity assumptions for the underlying functional framework. In the present appendix, however, the analytic requirements are more modest. It is sufficient that the domain Ω is compact, or that boundary conditions enforce a compact embedding $H^2 \hookrightarrow H^1$, ensuring the necessary continuity properties of admissible functions. In addition, the functional F must be coercive

on the admissible classes under consideration, and the generators X must preserve the domain structure.

Under these conditions, both the numerator and denominator of the ratio functional remain bounded and continuous on admissible subsets. This guarantees that the quotient is well-defined wherever the denominator does not vanish and that its analytic behavior is controlled within the declared functional setting.

Appendix B.7. Summary of Analytic Role

Appendix B establishes the analytic consistency of the ratio functional within the specified variational framework. It shows that Γ is well-defined on a rigorously defined domain and remains invariant under global rescaling of the functional. The ratio behaves covariantly under scaling of the generator and is unaffected by smooth admissible coordinate reparameterizations. In addition, it is locally stable under perturbations within the admissible class.

These properties support the proportional-stationarity condition introduced in Section 2 and employed in Section 3. Appendix B therefore plays a purely analytic role: it secures the internal consistency and structural robustness of the ratio construction. No physical conclusions depend on this appendix in isolation; it provides the functional-analytic foundation necessary for the structural arguments developed in the main text.

Appendix C. Worked Example: Quadratic Scalar-Field Functional

This appendix provides a concrete realization of the proportional-stationarity framework introduced in Sections 2 and 3. The example is intentionally elementary: a quadratic scalar-field functional on a compact domain. Its purpose is to demonstrate how curvature and strain arise from standard variational structure, and how Γ -stationarity appears as an operator condition.

Appendix C.1. Functional Setup

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let

$$\mathcal{X} = H_0^1(\Omega)$$

be the Sobolev space of square-integrable functions with square-integrable weak derivatives and zero boundary conditions.

Consider the quadratic functional

$$F(\psi) = \frac{1}{2} \int_{\Omega} (|\nabla \psi|^2 + m^2 \psi^2) \, dx,$$

where $m \geq 0$ is a fixed parameter. This functional is coercive on $H_0^1(\Omega)$, strictly convex, and twice Fréchet differentiable. It therefore satisfies the analytic assumptions invoked in Appendix A.

Appendix C.2. First and Second Variations

Let X be a linear generator acting on ψ , for example a scaling or internal symmetry direction such that $X\psi \in H_0^1(\Omega)$.

The first variation in direction $X\psi$ is

$$\tau_X(\psi) = DF(\psi)[X\psi] = \int_{\Omega} (\nabla \psi \cdot \nabla (X\psi) + m^2 \psi (X\psi)) \, dx.$$

The second variation along the same direction is

$$\kappa_X(\psi) = D^2F(\psi)[X\psi, X\psi] = \int_{\Omega} (|\nabla(X\psi)|^2 + m^2(X\psi)^2) dx.$$

Because the functional is quadratic, the second variation is independent of ψ beyond the linear action of X .

Appendix C.3. Ratio Functional

Define

$$\Gamma_X(\psi) = \frac{\kappa_X(\psi)}{\tau_X(\psi)},$$

on the admissible set where $\tau_X(\psi) \neq 0$.

Substituting the expressions above yields

$$\Gamma_X(\psi) = \frac{\int_{\Omega} (|\nabla(X\psi)|^2 + m^2(X\psi)^2) dx}{\int_{\Omega} (\nabla\psi \cdot \nabla(X\psi) + m^2\psi(X\psi)) dx}.$$

This ratio measures second-order response relative to first-order response along the generator direction.

Appendix C.4. Proportional Stationarity Condition

Proportional stationarity requires

$$\kappa_X(\psi) = \Gamma_X(\psi) \tau_X(\psi).$$

Substituting the definitions yields an operator relation of the form

$$\int_{\Omega} (|\nabla(X\psi)|^2 + m^2(X\psi)^2) dx = \Gamma \int_{\Omega} (\nabla\psi \cdot \nabla(X\psi) + m^2\psi(X\psi)) dx.$$

For linear generators and eigenfunction profiles satisfying

$$-\Delta\psi + m^2\psi = \lambda\psi,$$

this relation reduces to a spectral condition linking Γ and λ .

Thus, in this quadratic setting, proportional stationarity corresponds to spectral balance between first and second variations.

Appendix C.5. Isolation

Because the operator

$$L = -\Delta + m^2$$

with Dirichlet boundary conditions has compact resolvent, its spectrum is discrete.

Hence the corresponding admissible equilibrium ratios Γ arising from eigenfunctions are discrete.

This concretely illustrates the spectral-isolation mechanism discussed abstractly in Appendix A.

No claim is made about physical constants; the example demonstrates operator structure only.

Appendix C.6. Perturbative Stability

Let $\psi \mapsto \psi + \varepsilon\phi$ with $\phi \in H_0^1(\Omega)$.

Expanding numerator and denominator to first-order yields

$$\delta\Gamma_X = \frac{\delta\kappa_X}{\tau_X} - \frac{\kappa_X}{\tau_X^2} \delta\tau_X.$$

If ψ satisfies proportional stationarity and perturbations preserve proportionality to first order, then

$$\delta\Gamma_X = 0 + O(\epsilon^2).$$

Thus, Γ is locally stable under admissible perturbations in this concrete setting.

Appendix C.7. Role of the Example

This example confirms that the ratio functional is well-defined within a standard Sobolev framework and that proportional stationarity reduces naturally to a spectral relation in that setting. Under compact-resolvent assumptions, the resulting equilibrium ratios are discrete, reflecting the classical structure of elliptic operator theory. Stability follows directly from the quadratic nature of the functional, with perturbative behavior governed by standard spectral and variational arguments. The construction relies exclusively on established results from classical variational and spectral theory and does not introduce any additional physical hypotheses.

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