

Article

Is Quantum Field Theory Necessarily “Quantum”?

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Abstract

The mathematical universe of the quantum topos, which is formulated on the basis of classical Boolean snapshots, delivers a neo-realist description of quantum mechanics that preserves realism. The main contribution of this article is developing formal objectivity in physical theories beyond quantum mechanics in the topos-theory approach. It will be shown that neo-realist responses to non-perturbative structures of quantum field theory do not preserve realism. In this regard, the method of Feynman graphons is applied to reframe the task of describing objectivity in quantum field theory in terms of replacing the standard Hilbert-space/operator-algebra ontology with a new context category built from a certain family of topological Hopf subalgebras of the topological Hopf algebra of renormalization as algebraic/combinatorial data tied to non-perturbative structures. This topological-Hopf-algebra ontology, which is independent of instrumentalist probabilities, enables us to reconstruct gauge field theories on the basis of the mathematical universe of the non-perturbative topos. The non-Boolean logic of the non-perturbative topos cannot be recovered by classical Boolean snapshots, which is in contrast to the quantum-topos reformulation of quantum mechanics. The article formulates a universal version of the non-perturbative topos to show that quantum field theory is a globally and locally neo-realist theory which can be reconstructed independent of the standard Hilbert-space/operator-algebra ontology. Formal objectivity of the universal non-perturbative topos offers a new route to build objective semantics for non-perturbative structures.

Keywords: quantum field theory; logical foundation of quantum theories; topological Hopf algebra of renormalization; non-perturbative topos and its representations



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1. Introduction

The concept of *objective reality*, as a metaphysical notion, aims to explain the existence of a world of objects governed by definite structures or rules without dependence on any particular perception, namely, mind-independent entities. Scientific methods are based on the working assumption that the universe is an objective reality which can be discovered by instruments and explained in terms of models. Scientific methods apply mathematical structures, as theoretical knowledge, to optimize designed models. This procedure improves the knowledge of prediction of a target objective reality in terms of designed models. The most important foundational challenge of modern mathematical and theoretical physics is to determine their position with respect to the universe as the target objective reality experienced and observed by instruments underlying scientific methods. The concept of objective reality has been challenged by three basic philosophical stances, namely, realism, anti-realism and neo-realism. Realism considers the existence of a mind-independent objective reality such that unobservables could be real entities and truth is located in the correspondence between theory and objective reality. Anti-realism

considers observables and empirically accurate models where truth is located in practical conditions. Anti-realism argues the appearance of underdetermination issue when ontologically different theories generate equivalent empirical data. Neo-realism supports achieving a mind-independent world based on structures, relations and context-relative objective features where the global classical ontology loses its meaning and unobservables could exist. There are some fundamental factors to prioritizing the neo-realist perspective rather than other stances for the description of physical theories beyond quantum mechanics [1–4].

The concept of *real objectivity* asserts a certain interpretation of the claim that “objective reality exists and that it can be described by theoretical knowledge, namely, mathematical structures”. This is the area that searches for deep interconnections between physical theories, formulated by mathematical structures, and outputs of scientific methods are extended to the foundations of mathematics and mathematical logic. Real objectivity is replaced by *formal objectivity* in this regard. Formal objectivity is definable in terms of structures, rules and relations governed by formal systems. Formal objectivity, as the mathematical logical route toward real objectivity, structurally testifies to the candidate features of reality, where real objectivity, as the metaphysical verdict, evaluates whether or not those features actually exist independent of observers. The ability to (i) select optimal candidates and (ii) test their consistency in terms of stable mathematical structures show the impact of formal objectivity in the support of real objectivity. Mathematical universes of topos models are rich enough to (i) recover real objectivity discovered by physical theories and (ii) assign neo-realist truth values to statements about them [5–24].

This article shows the capability of the topos-theory approach in assigning formal objectivity to divergences of non-perturbative structures that appear in the working platform of quantum field theory. It will be discussed how the mathematical universe of the universal non-perturbative topos can clarify the globally and locally neo-realist nature of interacting physical theories beyond quantum mechanics.

1.1. What Is a Topos?

The category theory aims to describe properties of mathematical structures in a relative setting in terms of morphisms between objects instead of membership relations. Set-theoretic notions of (sub-)sets or spaces are replaced by (sub-)objects, elements are replaced by morphisms, bijection is replaced by isomorphism, power sets are replaced by power objects and the set of truth values $\{0, 1\}$ is replaced by a subobject classifier. In set theory, a point a of a set A is described by a map $a : \{*\} \rightarrow A$ such that its generalization to category theory is based on terminal object $\mathbf{1}$ such that a global element of an object A is defined as a morphism $a : \mathbf{1} \rightarrow A$. Topos theory generalizes the notions of space and logic. It deals with the challenge of replacing or generalizing the category of sets and functions with an elementary topos as a categorical generalization of a Grothendieck topos. A Grothendieck topos is addressed as a replacement for the notion of space [12,14,18,25,26].

Definition 1. • An object A in a category $\mathcal{C}$ is called *exponentiable* if for each object C there exists the exponential C^A and the evaluation morphism $\text{ev} : C^A \times A \rightarrow C$ such that for each map $h : B \times A \rightarrow C$ there exists the unique exponential transpose $\hat{h} : B \rightarrow C^A$ which contributes to the commutative diagram $h = \text{ev} \circ \text{id}_A \times \hat{h}$. The category $\mathcal{C}$ is called *Cartesian closed* if all of its objects are exponentiable and has finite products.

- A subobject classifier or a generalized truth-value object is an object $\Omega_{\mathcal{C}}$ together with the true morphism $\top : \mathbf{1} \rightarrow \Omega_{\mathcal{C}}$ in the category $\mathcal{C}$ such that for each monic morphism $f : A \rightarrow B$ there exists the unique characteristic morphism $\chi : B \rightarrow \Omega_{\mathcal{C}}$ which contributes to the commutative diagram $\chi \circ f = \top \circ g_A$ for some morphism $g_A : A \rightarrow \mathbf{1}$. The subobject classifier is unique up to isomorphism.

- If $\mathcal{C}$ is a Cartesian closed category with the subobject classifier $\Omega_{\mathcal{C}}$, then for each object A its power object $P(A)$ is the object $\Omega_{\mathcal{C}}^A$.
- A sieve on an object B in the category $\mathcal{C}^{\text{op}}$ is a collection S of morphisms such that

$$\forall f \in S \Rightarrow \text{cod}(f) = B, (\forall f \in S : \text{cod}(g) = \text{dom}(f)) \Rightarrow f \circ g \in S. \quad (1)$$

- The principal sieve on an object B , presented by $\uparrow B$, is the set of all morphisms such that $\text{cod}(f) = B$ [14,18,26].

Definition 2. A topos is a category which has a terminal object, exponential objects, pullbacks, equalizers, a subobject classifier and all limits. In other words, a topos is a category which satisfies one of the following equivalent conditions: (i) it is a complete category with a subobject classifier and its power object, (ii) it is a complete category with exponentials and a subobject classifier or (iii) it is a Cartesian closed category with equalizers and a subobject classifier [14,18,25].

The category **Set** of sets and functions is a topos, and in a more general setting, for any set I , the category $\mathbf{Set}^I$ of I -indexed families of sets is a topos such that the constant family $(\mathbf{2})_{i \in I}$ is its subobject classifier where $\mathbf{2}$ is a two-element set. For any group G , the category $\mathbf{Set}^G$ of left G -sets is a topos such that the set $\mathbf{2}$ with trivial G -action is its subobject classifier. For any topological space S , the category $\mathbf{Sh}(S)$ of sheaves on S is a topos, and in a more general setting, the category $\mathbf{Sh}_v(\mathcal{C})$ of sheaves over a base category $\mathcal{C}$ equipped with a Grothendieck topology v is a topos. For any topological space A with the corresponding poset of open subsets $\text{Open}(A)$, which can be seen as a category, a *presheaf* on A is a contravariant functor F from $\text{Open}(A)$ to **Set**. It assigns to each open subset $U \subseteq A$ a set $F(U)$ of sections over U .

Fundamental Example. The category $\hat{\mathcal{C}} := \mathbf{Set}^{\mathcal{C}^{\text{op}}}$ of contravariant functors from any small category $\mathcal{C}$ to the category **Set** is a topos. Its subobject classifier is given by

$$\Omega_{\hat{\mathcal{C}}}(a) \simeq \text{Sub}(\mathcal{C}(-, a)) \quad (2)$$

for any object a in $\mathcal{C}$.

Structures in a theory, modeled as objects in $\hat{\mathcal{C}}$, are representable by set-valued functors over $\mathcal{C}$. Geometric morphisms are basic tools to formulate bridges between topoi to recognize differences of the interpretation of a theory in different topoi [8,14,18,25–27].

In a topos $\mathcal{C}$ with subobject classifier $\Omega_{\mathcal{C}}$, the subobjects of each object A is determined by

$$\text{Sub}(A) \simeq \text{hom}_{\mathcal{C}}(A, \Omega_{\mathcal{C}}). \quad (3)$$

The poset $\text{Sub}(A)$ is a distributive Cartesian closed lattice such that the exponential $b^a := (a \Rightarrow b)$ satisfies in the formula

$$a \leq (b \Rightarrow c) \text{ iff } a \wedge b \leq c. \quad (4)$$

It is actually a *Heyting algebra*. (A Heyting algebra is a bounded lattice equipped with an implication operator $\Rightarrow$ and a negation operator $\neg x := (x \Rightarrow 0)$ such that (i) the formula $(x \wedge a) \leq b$ is equivalent to the formula $x \leq (a \Rightarrow b)$ for all of its elements, and (ii) $\neg x$ is the largest element which obeys the formula $x \wedge \neg x = 0$, while the formula $x \vee \neg x = 1$ is not valid in general.) The subobject classifier in a topos carries a *bi-Heyting algebra* structure. (A bounded lattice equipped with an implication operator, Heyting and co-Heyting structures

is called a bi-Heyting algebra. The dual of the implication operator determines a second negation.) A Heyting algebra H is Boolean iff for any $x \in H$, we have

$$x \vee \neg x = 1, \neg \neg x = x. \quad (5)$$

(A Boolean algebra is a certain bi-Heyting algebra such that first and second negations are the same.) In the topos of presheaves $\hat{\mathcal{C}}$, if $\mathcal{C} = \mathbf{Set}$, then its subobject classifier $\Omega_{\mathbf{Set}}$ is a Boolean algebra [18,25,27–30].

1.2. Previous Tasks to Resolve the Challenge of Formal Objectivity of Physical Theories

Mathematical logic enables us to study the interface between mathematical and logical foundations of physical theories. While the Boolean algebra $\{0, 1\}$ is the approved background logic of classical mechanics, there are alternative approaches to describe the logical foundation of quantum mechanics. The one approach formulates propositional calculi on the space of propositions obtained from orthomodular lattices and Boolean algebras to describe quantum logics from the perspective of the classical logic. (An ortholattice is a bounded lattice equipped with a complementation operator $\sim$ which satisfies De Morgan's laws. An orthomodular lattice is an ortholattice which obeys the law

$$x \leq y \Rightarrow x \vee (\tilde{x} \wedge y) = y$$

for all of its elements.) The other approaches replace the mathematical universe of the classical logic with a new mathematical universe of non-Boolean logics. This replacement can be performed on the basis of the topos theory, where the Boolean topos $\mathbf{Set}$ is replaced by a certain non-Boolean topos known as the *quantum topos*. (A Boolean topos is a topos whose internal logic is classical Boolean logic. A non-Boolean topos is a topos whose internal logic is a non-Boolean Heyting algebra. In other words, the law of excluded middle is not valid in the internal logic of non-Boolean topoi.) Elements of the quantum topos are contravariant functors from the base category to the category $\mathbf{Set}$. They encode propositions or statements about quantum systems. The internal logic of the quantum topos, which is locally Boolean but globally Heyting, determines required truth values for the formulation of a propositional calculus on the space of propositions [7–11,13–15,18,20–23,31].

The topos theory as a sub-discipline of the category theory [18,25–27,32] has been developed to initiate a reconstruction program for physical theories and their interfaces [5,8–17,33]. It was a foundational step toward formulating formal objectivity for physical theories with respect to the observer-independent truth. In other words, the toposification program initiated the origin of a neo-realist perspective to posit objects of the universe at the quantum scale independently of observers such that, by changing the mathematical setting, propositions find their truth values in an internal logic. The quantum topos argues for the challenge of instrumentalism in terms of replacing the standard Hilbert-space ontology with a certain topos of presheaves on the base category of classical contexts. The foundational difference between classical mechanics and quantum mechanics is represented by their separate topos models. Classical mechanics is reconstructed in the topos $\mathbf{Set}$, which supports full classical global Boolean realism. Quantum mechanics is reconstructed in the quantum topos built from a certain family of commutative subalgebras as the contexts, which supports a context-dependent non-Boolean realism.

According to formal objectivity of topos models, mathematical universes of Boolean topoi are equivalent to the stance of realism, and mathematical universes of non-Boolean topoi are equivalent to the stance of neo-realism. The mathematical universe of any Boolean topos is based on the working assumption that the axiom of choice and the law of excluded middle are undoubtedly valid. The mathematical universe of any non-Boolean topos does

not have full axiom of choice, and the law of excluded middle fails. Therefore a physical theory cannot be recovered by classical mechanics if its corresponding topos model is non-Boolean. These physical theories are quantum theories and their beyond theories such as non-perturbative gauge field theories formulated by the working platform of quantum field theory.

Global Neo-Realism but Local Realism

On the one hand, instrumentalism handles physical theories as tools for the study of observations such that theoretical structures do not necessarily represent some real entities. Measurement devices generate information to predict observations in this regard. On the other hand, realism seeks to assign meaningful values to physical quantities such that truth values of propositions can be recovered by classical Boolean logic. It makes sense to apply instrumentalism to achieve more accurate predictability in realism [1–5,8,14,15]. The challenge for both perspectives is using non-computable real or complex numbers to present the required realistic meaningful values and instrumentalist probabilities. The neo-realist nature of non-Boolean topoi addresses a new route to deal with this challenge of non-computability.

Classical mechanics is mathematically formulated on the basis of a phase space presented by a symplectic manifold M with the Poisson algebra $(C^\infty(M), \{.,.\})$ and the Euler–Lagrange equations of motion extracted from the principle of least action. (A symplectic manifold is a smooth manifold equipped with a closed non-degenerate differential 2-form which governs the dynamics of particles in a classical physical system. The phase space M is the cotangent bundle of a fixed configuration manifold.) On the one hand, canonical quantization replaces the position and momentum as phase-space coordinates with some operators $\hat{p}, \hat{q}$ as elements of the C^* -algebra of bounded operators on the complex Hilbert space $\mathcal{H} = L^2(X \subseteq [0, \infty), m)$ of states of quantum mechanics. (Here $(X \subseteq [0, \infty), m)$ is the Lebesgue measure space defined on the Borel σ -algebra. A C^* -algebra is a Banach $*$ -algebra over the field of complex numbers equipped with the involution map which is compatible with the norm structure, namely, the C^* -identity is valid.) These operators, which obey the canonical commutation relation $[\hat{p}, \hat{q}] = i\hbar$, represent observables as operator-valued functionals acting on states in $\mathcal{H}$. On the other hand, deformation quantization replaces the pointwise product of $C^\infty(M)$ with a noncommutative star product. It encapsulates a geometric description for the route to quantum mechanics from classical mechanics in terms of deforming the Poisson bracket $\{.,.\}$ with respect to the deformation parameter $\hbar$. The value $O(\hbar^2)$ encodes quantum corrections generated as the result of a deformation of the classical algebra. Quantum mechanics works for physical systems with a finite number of quantum particles [34].

Suppose M is the phase space of states of a classical physical system C . Any physical quantity A about C is determined in terms of a unique real valued function $f_A : M \rightarrow \mathbb{R}$. If C is at a state x , then its corresponding physical quantity A has the value $f_A(x)$. Propositions about C , which are determined by Lebesgue measurable subsets, can be encoded in terms of sentences such as “ $A \in \mathcal{A}$ ” for some Borel subsets $\mathcal{A} \subseteq \mathbb{R}$. The set $\text{Prop}(C)$ of propositions about C together with inclusion forms a Boolean algebra. There exists a homomorphism of Boolean algebras between $\text{Prop}(C)$ and $\{0, 1\}$, which assigns truth values to all propositions about C . This setting assigns a topos model to C such that (i) there is a state object, (ii) properties of the system and values of physical quantities are meaningful and (iii) propositions about the system are representable by a Boolean algebra. This topos model, which is represented by the topos **Set**, provides the realist description for the system C . In this realist description, (i) any physical quantity is represented by

a morphism from the state object to the quantity-value object, and (ii) propositions are represented by subobjects of the state object. [8,25,28,31]

Suppose $\mathcal{H}_{\mathbf{Q}}$ is the complex Hilbert space of states of a quantum physical system $\mathbf{Q}$. A space-time background should be fixed to assign definite location and time values to our observations via measurement devices. Real objectivity of $\mathbf{Q}$ is controlled by the chosen measurement devices. It means that characterizing elements of $\mathcal{H}_{\mathbf{Q}}$ is highly related to agent-dependent devices as observers. This instrumentalist characteristic has caused several unsolved contextual and conceptual problems and paradoxes. Any physical quantity B about $\mathbf{Q}$ is encoded in terms of a self-adjoint bounded operator $\hat{B} : \mathcal{H}_{\mathbf{Q}} \rightarrow \mathcal{H}_{\mathbf{Q}}$ such that $\mathbf{Q}$ at a state $|\psi\rangle$ means that $|\psi\rangle$ is an eigenvector for $\hat{B}$. A proposition about a particle in $\mathbf{Q}$ with an assigned momentum and up spin or down spin is not equivalent to the proposition about the particle with an assigned momentum and up spin or an assigned momentum and down spin. Formal objectivity of this experiment is encoded by a certain orthomodular lattice structure on the space of propositions about $\mathbf{Q}$ such that the distribution of meet and join might fail with respect to each other, but the law of excluded middle is valid. The set $\text{Prop}(\mathbf{Q})$ of propositions about $\mathbf{Q}$ corresponds to the set of closed linear subspaces of $\mathcal{H}_{\mathbf{Q}}$ or their related projection operators as elements of $\mathbf{B}(\mathcal{H}_{\mathbf{Q}})$. The Kochen–Specker theorem shows that for $\dim(\mathcal{H}_{\mathbf{Q}}) \geq 3$, there exist some elements of $\mathbf{B}(\mathcal{H}_{\mathbf{Q}})$ recovering non-commutative observables such that it is impossible to simultaneously assign definite values to their corresponding propositions in $\text{Prop}(\mathbf{Q})$. In other words, any effort to formulate quantum mechanics as a fully realist theory fails [16,17,19]. If we proceed toward formal objectivity of the quantum topos, then commutative von Neumann subalgebras of $\mathbf{B}(\mathcal{H}_{\mathbf{Q}})$, as objects of the base category of contexts, generate classical snapshots of $\mathbf{Q}$. (For a given complex Hilbert space $\mathcal{H}$ and the algebra $\mathbf{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$, a von Neumann subalgebra on $\mathcal{H}$ is a unital $*$ -subalgebra of $\mathbf{B}(\mathcal{H})$ that is closed with respect to the weak operator topology. They form a certain family of C^* -algebras on $\mathcal{H}$.) Each commutative von Neumann subalgebra $\mathcal{V}$ of $\mathbf{B}(\mathcal{H}_{\mathbf{Q}})$ is equipped with a Boolean algebraic structure. Therefore propositions in $\text{Prop}(\mathbf{Q})$ corresponding to projection operators in $\mathcal{V}$ can be evaluated from the classical perspective of $\mathcal{V}$. The spectral presheaf, as the analog of the classical phase space, avoids assigning definite values to all observables at once. Physical quantities and propositions about the system find a geometric/categorical nature. Measurement outcomes are interpreted by contextual truth assignments rather than the sole source of reality in this regard. The process of replacing the orthomodular lattice method with a Heyting structure is encoded by the daseinization mapping. However, the Heyting algebra of the quantum topos contains some elements that do not correspond to any object of the orthomodular lattice method [8–17,32].

In short, formal objectivity of the quantum topos has the following properties:

- It makes observers and measurement devices secondary concepts.
- It makes the standard Hilbert-space ontology a secondary concept.
- It is dependent on the standard operator-algebra ontology.
- It assigns real numbers, either computable or non-computable, to quantities, which is in contrast to the definiteness of classical observables.
- It is globally non-Boolean, but it is locally recovered by formal objectivity of the Boolean mathematical universe.
- It is blind to the differences in the nature of physical theories when degrees of freedom tend to infinity.
- It is blind to divergences with the origin of interacting physical theories.

1.3. Essential Progress of This Research from Previous Tasks: Global and Local Neo-Realism

The method of toposification of physical theories addresses a highly precise route to real objectivity of physical theories in the context of alternative mathematical universes where internal truth values of propositions are independent of instrumentalist probabilities.

Thanks to canonical quantization and Feynman path integral, physical theories beyond quantum mechanics, as special relativistic extensions of quantum mechanics, are mathematically formulated by the platform of quantum field theory. Moving from quantum mechanics to quantum field theory changes particles to fields and increases degrees of freedom from a finite number to an infinite number where elementary particles, as excitations of fields, can be created or annihilated. This working platform led us to develop gauge field theories for the study of the physics of sub-atomic particles at high-energy scales [34,35]. Gauge field theories can be smoothly studied by perturbative techniques whenever coupling constants are small enough such that Green's functions generate only convergent perturbative series of higher-loop-order Feynman diagrams. These perturbative series drastically diverge when coupling constants become strong, and this is the area of using non-perturbative techniques such as Dyson–Schwinger equations, large N limits and numerical approximations.

The Hilbert space of states in quantum field theory becomes an infinite tensor product of Hilbert spaces corresponding to an infinite number of particles. Commutative von Neumann subalgebras of the C^* -algebra of bounded operators are much more complicated than quantum mechanics. In addition, when we pass from the perturbation domain to the non-perturbation domain of the physical theory, all standard computational algorithms fail because perturbative series of Green's functions diverge to infinity. In fact, the mathematical universe of the quantum topos is totally blind to the impact of the strength of coupling constants of physical theories in generating non-perturbative structures. In other words, the Heyting algebra of the quantum topos is almost surely powerless to assign truth values to propositions about non-perturbative structures. As this stage, it is unavoidable to search for an alternative mathematical universe where alternative techniques must be devised to efficiently assign truth values to propositions about these types of divergences extracted from the physics of high-energy scales such as the problems of confinement and triviality or the “zero charge problem”. In addition, a quantum theory for gravity is a space-time background-independent theory which cannot be fully formulated under instrumentalism or local realism conditions. These evidences must be considered as the red flag in using the mathematical universe of the quantum topos for the description of real objectivity of physical theories beyond quantum mechanics.

Thanks to the recent progress in the topological enrichment of the renormalization Hopf algebra via the method of Feynman graphons, von Neumann subalgebras of $\mathbf{B}(\mathcal{H})$ can be replaced by a certain family of topological Hopf subalgebras of the topological Hopf algebra of renormalization associated with quantum motions [24,36–38]. They are considered as objects of the base category of contexts of a new non-Boolean topos model called *non-perturbative topos*. This article shows that the mathematical universe of this new topos model is rich enough to recover formal objectivity of physical phenomena in perturbative and non-perturbative structures of interacting physical theories beyond quantum mechanics. The universal version of the non-perturbative topos, formulated in this work, provides a theory-independent formal objectivity which recovers mathematical universes of gauge field theories in a neo-realist setting.

In short, formal objectivity of the universal non-perturbative topos has the following properties:

- It makes observers and measurement devices secondary concepts.
- It makes the standard Hilbert-space ontology a secondary concept.

- It makes the standard operator-algebra ontology a secondary concept.
- It is independent of the formal objectivity of the standard Hilbert-space ontology.
- It is independent of physical theories because its skeletons are defined on a certain combinatorial Hopf algebra of non-planar rooted trees as context-independent mathematical structures without the need for any measurement process.
- It recovers non-perturbative topoi of physical theories as sub-topoi.
- It makes assigning real numbers to quantities a secondary concept. In fact, stretched Feynman graphons and their renormalized values are applied to present quantities.
- It is locally non-Boolean because each topological Hopf subalgebra in the base context category can be equipped with a bi-Heyting algebra structure.
- It is globally non-Boolean because topological Hopf algebra of renormalization can be equipped with a bi-Heyting algebra structure.
- It recognizes the differences in the nature of physical theories when degrees of freedom tend to infinity.
- It handles divergences with the origin of interacting physical theories.

The immediate consequence of formal objectivity of the universal non-perturbative topos is to show that quantum field theory is describable as a globally and locally neo-realist theory. It formally supports the notion that quantum field theory, as a “neo-realist” theory, is necessarily “quantum”. In addition, this progress makes it possible to find answers to a series of fundamental questions addressed by Doring and Isham in [39] about the nature of physical theories beyond quantum mechanics. These questions, which are listed here, will be answered in Section 5.1 of Conclusions.

- Is it possible to interpret a theory in a neo-realist manner because instrumentalist theories are problematic?
- Since Hilbert-space formalism almost inevitably forces an instrumentalist interpretation, is it possible to formulate a theory of quantum gravity independent of the Hilbert-space formalism?
- Is it possible to deal with conceptual issues that any approach to quantum gravity has to confront?
- Is quantum gravity an instrumentalist theory? Or it is a realist theory?
- Is it accurate to expand concepts of quantum ideas to a theory of quantum gravity with the usual mathematical apparatus of quantum theory?
- If we aim at a realist form of theories of beyond quantum mechanics, what kind of logic should be used to evaluate no-go theorems such as the Kochen–Specker theorem?
- In an encompassing theory of the whole universe, which roles do a mathematician mathematician physicist has? Is he or she necessarily part of the description?

1.4. Achievements

This article presents the flexibility of formal objectivity derived from topo-theory approach to pass from quantum mechanics to quantum theories beyond quantum mechanics built by the working platform of quantum field theory. The appearance of a certain family of analytic graphs, called stretched Feynman graphons, in the backbone of the non-perturbative topos together with replacing commutative von Neumann subalgebras of $\mathbf{B}(\mathcal{H})$ with a certain family of topological Hopf subalgebras of the topological Hopf algebra of renormalization [24,37] furnish an alternative working candidate for the construction of formal objectivity of gauge field theories and their non-perturbative structures. The non-perturbative topos and its universal version are globally and locally non-Boolean in contrast to the quantum topos, which is globally non-Boolean but locally Boolean. In fact, the incapability of the method of the classical Boolean snapshots in the quantum topos is clarified in terms of the deviation of the Heyting algebra of the non-perturbative topos from

the Heyting algebra of the quantum topos. The Heyting algebra of the non-perturbative topos cannot be locally described by classical Boolean algebra because each topological Hopf subalgebra in the base category of contexts is equipped with a certain bi-Heyting algebra structure [36].

The aim of this paper is to formulate a universal model of the non-perturbative topos independent of physical theories.

- We study the formulation of stretched Feynman graphons in Section 2.1 to explain the topological enrichment of the renormalization Hopf algebra. See Definitions 6–8, Theorem 1 and Remark 2.
- Feynman graph limits, which contribute to (1PI) Green's functions of a gauge field theory and (non-)perturbative solutions of their fixed point equations, are represented by large Feynman diagrams. The graphon representations of these large Feynman diagrams provide stochastic models for the description of solutions of quantum motions. See Lemma 1, Remark 3, Theorem 2, Definition 10 and Corollary 2.
- Examples A, B, C, D, E, F and G present the process of passing from Feynman integrals to random graph representations of (non-)perturbative series which contribute to (1PI) Green's functions and their fixed point equations.
- The topological Hopf algebra of renormalization of an interacting gauge field theory and its topological Hopf subalgebras associated with solutions of quantum motions are applied as the building blocks of the non-perturbative topos. This particular topos (i) recovers propositions about the physical theory and (ii) determines required truth values to evaluate propositions in a neo-realist setting. See Section 3.1.
- It will be discussed that how the internal logic of the non-perturbative topos provides the background logic of a gauge field theory where its dimensional computable Heyting algebra determines truth values for a neo-realist evaluation of propositions about quantum motions and the interface between perturbative and non-perturbative structures of their solutions. See [24] and Theorem 3.
- It will be shown that random graph representations of Feynman graph limits can be characterized by their expectation values. See Corollary 4.
- A new representation of the non-perturbative topos over a category algebra of random graphs is formulated. See Corollary 5.
- The Connes–Kreimer renormalization Hopf algebra of non-planar rooted trees provides the combinatorial representation of Feynman diagrams and a universal platform for the BPHZ renormalization process [40–45]. The topological enrichment of this particular combinatorial Hopf algebras and a certain class of its topological Hopf subalgebras are building blocks of the universal non-perturbative topos. It will be shown that the universal non-perturbative topos is non-boolean, and it includes non-perturbative topoi of gauge field theories as sub-topoi. This means that the internal logic of the universal non-perturbative topos provides the background logics of gauge field theories. See Corollary 6.

2. Topological Hopf Algebra of Renormalization

Random graphs are useful tools for assigning graph limits to sequences of finite weighted dense or sparse graphs. Fix a sequence $\{G_n\}_{n \geq 1}$ of finite graphs with an increasing vertex number $|G_n| = n$ and any integer $k \geq 1$. Let $\{G_n[k]\}$ be a sequence of distributions of random graphs such that for each n , $G_n[k]$ is a labeled subgraph in G_n with k vertices built by uniformly selecting distinct vertices $v_1, \dots, v_k$ from vertices of G_n . When $|G_n|$ tends to infinity, the behavior of $\{G_n\}_{n \geq 1}$ is controlled by the random graph limit of the sequence $\{G_n[k]\}$ for any fixed k . There are many choices of random graph models in terms of different normalizing functions. For the normalizing function $1/n$, $\{G_n\}_{n \geq 1}$

behaves like a Cauchy sequence whenever $|E(G_n)| = o(n)$. There is an alternative approach to assigning a metric structure on the space of graphs. The cut norm, as a semi-norm, is defined on the space of finite matrices. For any $n \times n$ matrix A , its cut norm is given by

$$\|A\|_{\text{cut}} := \frac{1}{n^2} \max_{I, J \subseteq [n]} \left| \sum_{i \in I, j \in J} a_{ij} \right| \quad (6)$$

such that $\|A\|_{\text{cut}} \leq \|A\|_1 \leq \|A\|_2 \leq \|A\|_\infty < \infty$. For any finite weighted graph G with the weight collection $\{\alpha_{ij}\}_{e_{ij} \in E(G)}$ for the edge set and the collection $\{\beta_s\}_{v_s \in V(G)}$ for the vertex set, its adjacency matrix defines a bounded Lebesgue measurable symmetric function $W_G^{\sigma|G|} : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ given by

$$W_G^{\sigma|G|}(x, y) = \sum_{i=1}^{|G|} \sum_{j=1}^{|G|} \chi_{I_i}(x) \chi_{I_j}(y) \alpha_{ij}. \quad (7)$$

If there is no edge between vertices v_i, v_j , then $W_G^{\sigma|G|}(x, y) = 0$. Here $\sigma|G| := (I_1, \dots, I_{|G|})$ is a partition of $[0, 1]$ such that $m(I_j) = \frac{|\beta_j|}{\sum_{s=1}^{|G|} |\beta_s|}$. The function $W_G^{\sigma|G|}$ is called a labeled graphon representation of the graph G . Its cut norm is given by

$$\|G\|_{\text{cut}} := \|W_G^{\sigma|G|}\|_{\text{cut}} = \sup_{S, T \subseteq [0, 1]} \left| \int_{S \times T} W_G^{\sigma|G|}(x, y) dx dy \right| \quad (8)$$

such that the supremum is with respect to Lebesgue measurable subsets of $[0, 1]$. This norm defines the pseudo-distance

$$d_{\text{cut}}(G_1, G_2) := \inf_{\rho_1, \rho_2} \|W_{G_1}^{\sigma|G_1|, \rho_1} - W_{G_2}^{\sigma|G_2|, \rho_2}\|_{\text{cut}} \quad (9)$$

on the space of finite graphs. The infimum is with respect to the invertible Lebesgue measure-preserving transformations on $[0, 1]$ such that $W_{G_k}^{\sigma|G_k|, \rho_k}(x, y) = W_{G_k}^{\sigma|G_k|}(\rho_k(x), \rho_k(y))$ for $k = 1, 2$. The weakly isomorphic relation identifies those graphons which will be equal almost everywhere after applying a suitable Lebesgue measure-preserving transformations. Up to this particular equivalence relation, d_{cut} defines a complete metric structure on the space of finite graphs. Graph limits of Cauchy sequences in this space are representable by graphons. The analysis of the space of graphs had been extended to spaces of stretched graphons, defined on arbitrary σ -finite measure spaces, with respect to L^p -modifications of the cut-distance metric. Rescaling and stretching techniques are applied to assign non-trivial meaningful graph limits to sequences of sparse graphs [46–50].

The theory of graphons has been also developed for quantum mathematics and the space of Feynman diagrams as combinatorial weighted graphs which contribute to gauge field theories [36,38,51]. Since stretched Feynman graphons are fundamental elements of our new formal objectivity, here we explain in detail the procedure of formulating large Feynman diagrams as graph limits of sequences of Feynman diagrams and their graphon representations.

Physical systems of elementary particles underlying special relativity are studied in the context of interacting gauge field theories as a certain class of quantum field theories with gauge symmetries defined on $D + 1$ -dimensional Minkowski space-time or other models of space-time. Quantum fields are operator-valued distributions which cannot be evaluated at a single point and make sense over regions of space-time. Localized measurements in regions of space-time give information of quantum fields. An interacting gauge field theory is formulated on the basis of a number of gauge fields for the presentation of elementary particles (i.e., fermions), a number of gauge fields for the presentation of force carriers

(i.e., bosons) and a number of interactions among those fields which is encoded by the interaction part of the Lagrangian. For the rest of this paper, suppose Φ is a quantum field theory on a $D + 1$ -dimensional Minkowski space-time background with at least a strong coupling constant ≥ 1 such that its Lagrangian has interaction terms which generate non-perturbative structures.

Definition 3. A Feynman diagram $\Gamma = (\Gamma^{[0]}, \Gamma_{\text{int}}^{[1]}, \Gamma_{\text{ext}}^{[1]})$ in Φ is a decorated, oriented space-time graph which encodes interactions of a limited number of particles. (i) The collection $\Gamma^{[0]}$ of vertices represents interactions. (ii) The collection $\Gamma_{\text{int}}^{[1]}$ of internal edges (i.e., edges with beginning and ending vertices) represents virtual particles. (iii) The collection $\Gamma_{\text{ext}}^{[1]}$ of external edges, (i.e., edges with beginning or ending vertices) represents elementary particles which can be created or annihilated. (iv) The sum of input momenta is the same as output momenta at each vertex and for the whole graph [44,52].

A Feynman diagram Γ is called 1PI if it remains a connected graph after removing one arbitrary internal edge. The residue graph $\text{res}(\Gamma)$ is a graph generated by shrinking all internal edges of Γ into a single vertex such that external edges of Γ are attached to it. A subgraph γ in Γ is called a Feynman subdiagram if $\gamma_{\text{int}}^{[1]} \subseteq \Gamma_{\text{int}}^{[1]}$ and $\gamma^{[0]} \subseteq \Gamma^{[0]}$ such that for each $v \in \gamma^{[0]}$, $\text{res}_{\gamma}(v) = \text{res}_{\Gamma}(v)$ [40,52].

The interaction part of the Lagrangian of Φ generates (1PI) Green's functions, which are perturbative series of powers of running coupling constants such that their coefficients are given by Feynman integrals as a certain class of iterated momentum integrals. The convergence or divergence of Feynman integrals depends on the domain of the momentum parameter, which changes between zero and infinity, and the dimension of space-time. Feynman rules in Φ introduce a dictionary between Feynman integrals and Feynman diagrams such that sub-divergences in integrals are represented by nested loops in graphs. Therefore coefficients of those perturbative series are represented in terms of higher-loop-order Feynman diagrams [53–56].

Example A. In a toy model, for $r > 0$, $1 \gg \epsilon > 0$, the integrand of the functional

$$I_{((j_2)(j_3)j_1)}(r) = \int_0^\infty \int_0^\infty \int_0^\infty \frac{x_1^{-j_1\epsilon}}{(x_1 + r)} \frac{x_2^{-j_2\epsilon}}{(x_1 + x_2 + r)} \frac{x_3^{-j_3\epsilon}}{(x_1 + x_3 + r)} d^{D+1}x_1 d^{D+1}x_2 d^{D+1}x_3 \quad (10)$$

is represented by a j_1 -loop-order 1PI Feynman diagram which has two independent nested j_2 and j_3 -loop-order divergent 1PI Feynman subgraphs. The integrand of the functional

$$I_{(((j_3)j_2)j_1)}(r) = \int_0^\infty \int_0^\infty \int_0^\infty \frac{x_1^{-j_1\epsilon}}{(x_1 + r)} \frac{x_2^{-j_2\epsilon}}{(x_1 + x_2 + r)} \frac{x_3^{-j_3\epsilon}}{(x_1 + x_2 + x_3 + r)} d^{D+1}x_1 d^{D+1}x_2 d^{D+1}x_3 \quad (11)$$

is represented by a j_1 -loop-order 1PI Feynman diagram which has a nested j_2 -loop-order divergent 1PI Feynman subgraph such that this subgraph has also a nested j_3 -loop-order divergent 1PI Feynman sub-subgraph. Regularization techniques, such as momentum cut-off or dimensional regularization, allow us to replace these divergent integrals with some Laurent series with finite pole parts where the minimal subtraction map projects these series onto their pole parts to extract some renormalized values [44].

Let $\mathcal{A}_\Phi = \{v_i, e_j\}_{i,j}$ be the collection of all types of particles and interactions in Φ . The combinatorial presentations of (1PI) Green's functions are given by

$$G^r = \mathbb{I} \pm \sum_{\Gamma, \text{res}(\Gamma)=r} g_\lambda^{|\Gamma|} \frac{\Gamma}{\text{Sym}(\Gamma)}, \quad r \in \mathcal{A}_\Phi \quad (12)$$

such that $|\Gamma|$ is the loop number of Γ and g_λ , for $0 < \lambda \leq 1$, is a function of the bare coupling constant g generated by regularization techniques. If $g_\lambda \ll 1$ is small enough, then these perturbative series are convergent, but if $c_\lambda \geq 1$, then these perturbative series are divergent, and we need to apply non-perturbative techniques such as Dyson–Schwinger equations to extract some finite values [36,38,40,43,52,53,55–57]. The method of Feynman graphons has been introduced and developed to assign a certain family of graph functions to graph limits of sequences of Feynman diagrams which contribute to solutions of fixed point equations of (1PI) Green's functions [36–38].

2.1. Stretched Feynman Graphons

The Connes–Kreimer theory formulates a graded connected commutative non-cocommutative Hopf algebra $H_{\text{FG}}(\Phi) = H^{(n)}(\Phi)$ over the field $\mathbb{Q}$ of rational numbers (or any field with characteristic zero) on the space of Feynman diagrams of Φ . It is called *renormalization Hopf algebra*. For $n = 0$, $H^{(0)}(\Phi) = \mathbb{Q}$ and for each $n \geq 1$, $H^{(n)}(\Phi)$ is the vector space generated by 1PI Feynman diagrams with the loop number n and disjoint unions of 1PI Feynman diagrams with the overall loop number n . The coproduct of this Hopf algebra, which encodes Zimmermann's forest formula in perturbative renormalization, is given by

$$\Delta(\Gamma) = \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma + \sum_{\gamma} \gamma \otimes \Gamma/\gamma \quad (13)$$

such that the sum is taken over all disjoint unions of superficial divergent 1PI Feynman subgraphs [40–42,44].

The combinatorial version of the renormalization Hopf algebra is formulated on the polynomial algebra of non-planar rooted trees where the coproduct is reformulated by admissible cuts on rooted trees. An admissible cut c of t is a subset of edges of t such that any path in t from its root r_t to each leaf meets at most one edge from c . Each admissible cut divides t into two separate parts $R_c(t)$, a subtree of t which contains the root, and $P_c(t)$ a forest of remaining subtrees. The renormalization coproduct (13) is translated to the formula

$$\Delta_{\text{CK}}(t) = t \otimes 1 + 1 \otimes t + \sum_c R_c(t) \otimes P_c(t) \quad (14)$$

such that the sum is taken over all non-trivial admissible cuts in t . This combinatorial Hopf algebra is called the Connes–Kreimer Hopf algebra and is presented by H_{CK} [40,42,43,52].

The combinatorial representation of a Feynman diagram Γ with nested loops is a decorated non-planar rooted tree t_Γ with the following properties. (i) The loop number of Γ , as the number of independent loops in Γ , is the cardinal of the vertex set of t_Γ . (ii) Each vertex in t_Γ is decorated by a primitive divergent (1PI) subgraph in Γ . (iii) There exists an edge between vertices $v_1, v_2 \in t_\Gamma$ if their corresponding loops in Γ are nested with respect to each other. (iv) There exists no edge between vertices $v_1, v_2 \in t_\Gamma$ if their corresponding loops in Γ are independent with respect to each other, which means that loops do not have any mutual internal edge, (v) The rooted tree t_Γ together with an embedding into the plane is called planar; otherwise, it is called non-planar [40–42,44].

Example B. Rooted tree representations of some 3-loop-order Feynman diagrams, which are applicable to encode Feynman integrals such as (10) and (11) in Example A, are given by Figure 1.

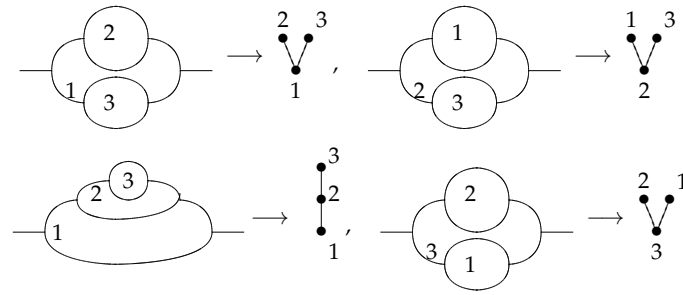


Figure 1. Replacing some 3-loop order Feynman diagrams with their rooted tree representations underlying the renormalization Hopf algebra.

Example C. Rooted tree representations of some 4-loop-order Feynman diagrams are given by Figure 2.

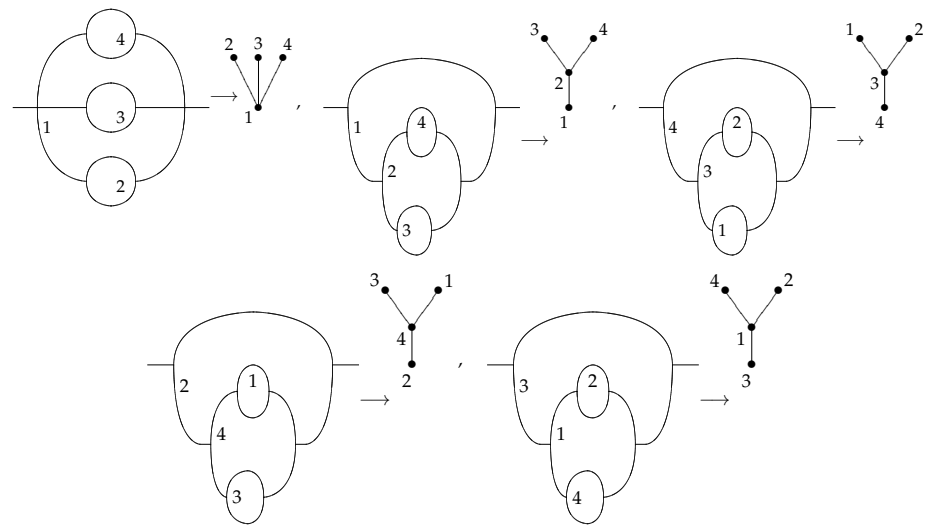


Figure 2. Replacing some 4-loop-order Feynman diagrams with their rooted tree representations underlying the renormalization Hopf algebra.

Remark 1. If Γ has some overlapping loops, then t_Γ , as a forest, is a linear combination of decorated non-planar rooted trees [52]. Figure 3 presents a simple example.

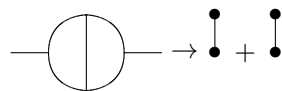


Figure 3. Rooted forest representation of a Feynman diagram with an overlapping loop underlying the renormalization Hopf algebra.

For combinatorial representations t_Γ, t'_Γ of Γ , there exists an isomorphism of non-planar rooted trees between them. There exists an injective Hopf algebra homomorphism $\Gamma \mapsto t_\Gamma$ from $H_{FG}(\Phi)$ to $H_{CK}(\Phi)$ given by the Formula (70). It allows us to reconstruct a Feynman diagram from its combinatorial representation where vertices of non-planar rooted trees are decorated by primitive (1PI) Feynman diagrams in Φ [44,52].

Definition 4. Consider a σ -finite measure space $(\Omega \subseteq [0, \infty), \mu)$. A pixel picture presentation of Γ is a symmetric bounded μ -measurable function $P_\Gamma^{g_n} : \Omega \times \Omega \rightarrow [0, 1]$ with $|t_\Gamma| = n$ and $\sigma_n := (I_1, \dots, I_n)$ which satisfies the following conditions.

- $I_1 \sqcup \dots \sqcup I_n \subseteq \Omega$, $I_k \cap I_s = \emptyset$, $k \neq s$, $\mu(I_i) = \frac{1}{n}$, $1 \leq i \leq n$;

- For any pair $(x_k, x_l) \in I_k \times I_l$, $P_{\Gamma}^{\sigma_n}(x_k, x_l) = \text{ad}_{t_{\Gamma}}(v_k v_l)$ with respect to the adjacency matrix $\text{Ad}_{t_{\Gamma}} = (\text{ad}_{t_{\Gamma}}(v_i v_j))_{i,j}$ of t_{Γ} ;
- For any invertible μ -measure-preserving transformation ρ on Ω together with the partition $\sigma_n^{\rho} := (\rho(I_1), \dots, \rho(I_n))$, the function $P_{\Gamma}^{\sigma_n^{\rho}}$ is called the labeled pixel picture presentation of Γ .

Example D. Pixel picture presentations corresponding to 3-loop-order Feynman diagrams in Figure 1 are determined by graph functions in Figure 4. The permutation matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad (15)$$

can be applied to translate three pixel pictures from right to left to the first one from the left in Figure 4.

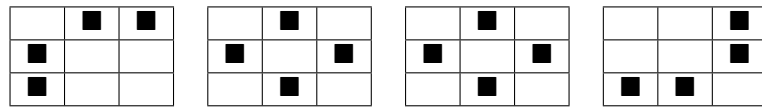


Figure 4. Required graph functions to build pixel picture presentations for finite graphs of order 3.

Example E. Pixel picture presentations corresponding to 4-loop-order Feynman diagrams in Figure 2 in Example C are determined by graph functions in Figure 5. The permutation matrices

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad (16)$$

can be applied to translate four pixel pictures from right to left to the first one from the left in Figure 5.

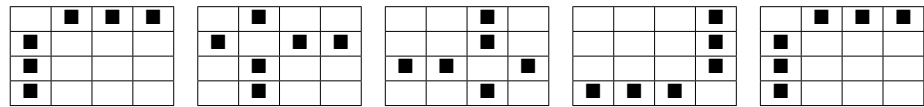


Figure 5. Required graph functions to build pixel picture presentations for finite graphs of order 4.

Definition 5. Consider a σ -finite measure space $(\Omega \subseteq [0, \infty), \mu)$.

- A bounded symmetric μ -measurable function $W : \Omega \times \Omega \rightarrow \mathbb{R}$ is weakly isomorphic to $P_{\Gamma}^{\sigma_n}$ iff there exist μ -measure-preserving transformations τ_1, τ_2 on Ω such that

$$P_{\Gamma}^{\sigma_n^{\tau_1}} = W^{\tau_2}, \mu - \text{almost everywhere} \quad (17)$$

where $W^{\tau_2}(x, y) = W(\tau_2(x), \tau_2(y))$.

- The equivalence class

$$W_{\Gamma} := [P_{\Gamma}^{\sigma_n}]_{\approx} \quad (18)$$

with respect to the weakly isomorphic relation $\approx$ is called a stretched Feynman graphon associated with Γ .

- For any invertible μ -measure-preserving transformation ρ on Ω , W_{Γ}^{ρ} is called a labeled stretched Feynman graphon.

- Feynman diagrams Γ_1, Γ_2 are called fractionally isomorphic if their corresponding pixel picture presentations $P_{\Gamma_1}^{\sigma_{n_1}}$ and $P_{\Gamma_2}^{\sigma_{n_2}}$ are fractionally isomorphic. This means that there exists a doubly stochastic matrix $S_{\Gamma_1\Gamma_2}$, such as permutation matrices, such that $\text{Ad}_{t_{\Gamma_1}} S_{\Gamma_1\Gamma_2} = S_{\Gamma_1\Gamma_2} \text{Ad}_{t_{\Gamma_2}}$.
- Labeled stretched Feynman graphons $W_{\Gamma_1}^{\rho_1} \in [P_{\Gamma_1}^{\sigma_{n_1}}]_{\approx}$ and $W_{\Gamma_2}^{\rho_2} \in [P_{\Gamma_2}^{\sigma_{n_2}}]_{\approx}$ are called fractionally isomorphic iff pixel picture presentations $P_{\Gamma_1}^{\sigma_{n_1}}$ and $P_{\Gamma_2}^{\sigma_{n_2}}$ are fractionally isomorphic.
- For the Lebesgue measure space $([0, 1], m)$, as the ground measure space, W_{Γ} is called a Feynman graphon.

Thanks to [58], it is observed that isomorphic Feynman diagrams are fractionally isomorphic.

2.2. Topological Enrichment of the Renormalization Hopf Algebra

Following Definition 5, the quotient space

$$\mathcal{S}_{\text{gr}, \approx}^{\Phi, \Omega} := \left\{ W_{\Gamma} : \Gamma \in H_{\text{FG}}(\Phi) \right\} \quad (19)$$

is equipped and topologically completed with the distance function

$$d_{\text{cut}}(W_{\Gamma_1}, W_{\Gamma_2}) = \inf_{\psi_1, \psi_2} \sup_{A, B \subseteq \Omega} \left| \int_{A \times B} \left(W_{\Gamma_1}^{\psi_1}(x, y) - W_{\Gamma_2}^{\psi_2}(x, y) \right) d\mu(x) d\mu(y) \right| \quad (20)$$

to obtain a complete Hausdorff metric space such that inf is taken over (invertible) μ -measure-preserving transformations on Ω and sup is taken over μ -measurable subsets of Ω . The space $\mathcal{S}_{\text{gr}, \approx}^{\Phi, \Omega}$ is a closed subspace of the space $\mathcal{W}_{\approx}^{\Omega}$ of real valued stretched graphons. For the Lebesgue measure space $([0, 1], m)$, the subspace $\mathbb{R}_{\geq 0}$ -valued Feynman graphons in $\mathcal{S}_{\text{gr}, \approx}^{\Phi, [0, 1]}$ is compact.

Definition 6. • The space of Feynman diagrams of Φ can be equipped with a new pseudo-metric given by

$$d_{\text{cut}}(\Gamma_1, \Gamma_2) := d_{\text{cut}}(W_{\Gamma_1}, W_{\Gamma_2}). \quad (21)$$

- Following the equivalence class (18), Feynman diagrams $\Gamma_1, \Gamma_2 \in H_{\text{FG}}(\Phi)$ are called weakly isomorphic iff $W_{\Gamma_1} = W_{\Gamma_2}$.
- Feynman diagrams Γ_1, Γ_2 are called weakly isomorphic iff $d_{\text{cut}}(\Gamma_1, \Gamma_2) = 0$. This means that d_{cut} is a metric on the quotient space of weakly isomorphic classes of Feynman diagrams.

Definitions 5 and 6 show that isomorphic Feynman diagrams are weakly isomorphic and weakly isomorphic Feynman diagrams are fractionally isomorphic. This new metric space is used to define the notions of the Feynman graph limit and large Feynman diagrams.

Definition 7. • A sequence $\{\Gamma_n\}_{n \geq 1}$ of Feynman diagrams is called convergent when n tends to infinity iff its corresponding sequence $\left\{ \frac{W_{\Gamma_n}}{\|W_{\Gamma_n}\|_{\text{cut}}} \right\}_{n \geq 1}$ of stretched Feynman graphons in $\mathcal{S}_{\text{gr}, \approx}^{\Phi, \Omega}$ converges with respect to the metric (20). Up to the weakly isomorphic relation, the limit of $\{\Gamma_n\}_{n \geq 1}$, whenever it exists, is unique.

- For

$$\lim_{n \rightarrow \infty} \frac{W_{\Gamma_n}}{\|W_{\Gamma_n}\|_{\text{cut}}} = W \in \mathcal{S}_{\text{gr}, \approx}^{\Phi, \Omega}, \quad (22)$$

the stretched Feynman graphon W represents the Feynman graph limit of the sequence $\{\Gamma_n\}_{n \geq 1}$. Following Definition 5, we have $W = [P_X^{\sigma_n}]_{\approx}$ for an infinite graph X with the rooted tree or forest representation t_{∞} . Here X is called the large Feynman diagram or Feynman graph limit generated by the sequence $\{\Gamma_n\}_{n \geq 1}$ such that $W_X \in \mathcal{S}_{\text{gr}, \approx}^{\Phi, \Omega}$ and $W_X = W$.

Modulo weakly isomorphic equivalence relation, the stretched Feynman graphon representation of a large Feynman diagram is unique. A certain class of continuous functionals on $\mathcal{S}_{\text{gr},\approx}^{\Phi,\Omega}$, known as homomorphism densities, are useful tools for the characterization of large Feynman diagrams.

Definition 8. Consider the Lebesgue measure space $([0, 1), \mathfrak{m})$ and the space $\mathcal{S}_{\text{gr},\approx}^{\Phi,[0,1]}$. For a finite Feynman diagram Γ and a large Feynman diagram X with $W_X \in \mathcal{S}_{\text{gr},\approx}^{\Phi,[0,1]}$, the probability that a random map of vertices of t_Γ to the vertices of t_X is a homomorphism is computed in terms of the homomorphism density

$$t(\Gamma, W_X) = \int_{[0,1]^{|t_\Gamma|}} \prod_{e_i e_j \in E(t_\Gamma)} \frac{1}{|W_X|} W_X(x_{e_i}, x_{e_j}) dx_1 \dots dx_{|t_\Gamma|} \times \int_{[0,1]^{|t_\Gamma|}} \prod_{e_i e_j \notin E(t_\Gamma)} 1 - \frac{1}{|W_X|} W_X(x_{e_i}, x_{e_j}) dx_1 \dots dx_{|t_\Gamma|}. \quad (23)$$

Definitions 5, 6 and 8 show that stretched Feynman graphons $W_X, W_Y \in \mathcal{S}_{\text{gr},\approx}^{\Phi,[0,1]}$ are fractionally isomorphic if $t(\Gamma, W_X) = t(\Gamma, W_Y)$ for any finite Feynman diagram Γ , and they are weakly isomorphic if $t(\Gamma, W_X) = t(\Gamma, W_Y)$ for any finite and infinite Feynman diagram Γ .

Theorem 1. Consider the space of Feynman diagrams, large Feynman diagrams in Φ together with their stretched Feynman graphons as elements in $\mathcal{S}_{\text{gr},\approx}^{\Phi,\Omega}$. We present this new space as $H_{\text{FG}}^{\text{cut}}(\Phi)$. Up to the weakly isomorphic equivalence relation, $H_{\text{FG}}^{\text{cut}}(\Phi)$, which can be equipped with the renormalization Hopf algebra, is a separable Banach space with respect to the metric (21).

Proof. For $\Gamma = \alpha_1 \Gamma_1 + \dots + \alpha_n \Gamma_n$, its corresponding stretched Feynman graphon is given by the direct sum $W_\Gamma = W_{\Gamma_1} + \dots + W_{\Gamma_n}$ such that for each $1 \leq i \leq n$, W_{Γ_i} is the stretched Feynman graphon associated with Γ_i defined on a μ -measurable subset $I_i \subset \Omega$ such that $\mu(I_i) = \frac{|\alpha_i|}{\sum_{i=1}^n |\alpha_i|}$, $I_r \cap I_s = \emptyset$ for $r \neq s$ and

$$W_\Gamma|_{I_i} = \begin{cases} W_{\Gamma_i}, & \alpha_i > 0 \\ -W_{\Gamma_i}, & \alpha_i < 0 \end{cases}. \quad (24)$$

Thanks to [46,47], any sequence $\{U_n\}_{n \geq 1}$ in $\mathcal{S}_{\text{gr},\approx}^{\Phi,\Omega}$ has a Cauchy subsequence $(U_{n_i})_{i \geq 1}$. It determines a Cauchy subsequence $(\Gamma_{n_i})_{i \geq 1}$ in $\{\Gamma_n\}_{n \geq 1}$ such that its large Feynman diagram X , with respect to the metric (21), is represented by the stretched Feynman graphon W_X in $\mathcal{S}_{\text{gr},\approx}^{\Phi,\Omega}$. In addition, the countable collection of stretched Feynman graphons W_Γ corresponding to 1PI Feynman diagrams with loop numbers n , $n \geq 1$, is dense in $H_{\text{FG}}^{\text{cut}}(\Phi)$ such that as the Schauder basis, its linear combinations might be infinite. Therefore $H_{\text{FG}}^{\text{cut}}(\Phi)$ is a separable complete metric space.

Consider the Banach space $H_{\text{FG}}^{\text{cut}}(\Phi) \otimes H_{\text{FG}}^{\text{cut}}(\Phi)$ with respect to the cross norm given by

$$\|x\| := \inf \left\{ \sum_{i=1}^n \|\gamma'_i\|_{\text{cut}} \|\gamma''_i\|_{\text{cut}} : x = \sum_{i=1}^n \gamma'_i \otimes \gamma''_i \right\}. \quad (25)$$

The renormalization coproduct $\Delta(X) \in H_{\text{FG}}^{\text{cut}}(\Phi) \otimes H_{\text{FG}}^{\text{cut}}(\Phi)$ and its related antipode $S(X) \in H_{\text{FG}}^{\text{cut}}(\Phi)$ are defined as the Feynman graph limits of the Cauchy subsequences $\{\Delta(\Gamma_{n_i})\}_{i \geq 1}$ and $\{S(\Gamma_{n_i})\}_{i \geq 1}$ with respect to the metric (21) such that for each n ,

$$S(\Gamma_{n_i}) = -\Gamma_{n_i} - \sum_{\gamma_{n_i}} S(\gamma_{n_i}) \Gamma_{n_i} / \gamma_{n_i}, \quad S(\mathbb{I}) = \mathbb{I}, \quad (26)$$

with

$$\Delta(\Gamma_{n_i}) = \mathbb{I} \otimes \Gamma_{n_i} + \Gamma_{n_i} \otimes \mathbb{I} + \sum_{\gamma_{n_i}} \gamma_{n_i} \otimes \Gamma_{n_i} / \gamma_{n_i} \quad (27)$$

such that the sum is taken over all disjoint unions of superficial divergent 1PI subgraphs of Γ_{n_i} .

For the Feynman graph limit X , stretched Feynman graphons $W_{\Delta(X)}, W_{S(X)} \in \mathcal{S}_{\text{gr}, \approx}^{\Phi, \Omega}$ are determined as the limits of some Cauchy sequences $\{W_{\Delta(\Gamma_{n_i})}\}_{i \geq 1}$ and $\{W_{S(\Gamma_{n_i})}\}_{i \geq 1}$ with respect to the metric (20). The inequalities

$$\begin{aligned} \|W_{\Delta(X)}\|_{\text{cut}} &\leq \|W_{\Delta(X)}\|_1 \leq \|W_{\Delta(X)}\|_{\infty} < \infty, \\ \|W_{S(X)}\|_{\text{cut}} &\leq \|W_{S(X)}\|_1 \leq \|W_{S(X)}\|_{\infty} < \infty, \end{aligned} \quad (28)$$

show that linear operators Δ and S are bounded. Therefore they are continuous maps.

Let $\{\Gamma'_{m_i}\}_{i \geq 1}$ be another Cauchy subsequence of $\{\Gamma_n\}_{n \geq 1}$ in $H_{\text{FG}}^{\text{cut}}(\Phi)$ which converges to the Feynman graph limit X . For each $\epsilon > 0$, there exist orders N, N' such that for each $n_i \geq N$ and $m_i \geq N'$,

$$d_{\text{cut}}(\Gamma_{n_i}, X) < \epsilon/2, \quad d_{\text{cut}}(\Gamma'_{m_i}, X) < \epsilon/2. \quad (29)$$

For $s \geq N^* := \max\{N, N'\}$, we get

$$d_{\text{cut}}(\Gamma_s, \Gamma'_s) \leq d_{\text{cut}}(\Gamma_s, X) + d_{\text{cut}}(X, \Gamma'_s) < \epsilon. \quad (30)$$

In other words, for $s \geq N^*$, $d_{\text{cut}}(W_{\Gamma_s}, W_{\Gamma'_s}) < \epsilon$ such that according to Definition 5, we get

$$P_{\Gamma_s}^{\sigma_{|\Gamma_s|}^{\tau_1}} = P_{\Gamma'_s}^{\sigma_{|\Gamma'_s|}^{\tau_2}}, \quad \mu - \text{almost everywhere} \quad (31)$$

for some μ -measure-preserving transformations τ_1, τ_2 on Ω . This means that $W_{\Gamma_s} = W_{\Gamma'_s}$. In other words, the subsequences $\{\Gamma_s\}_{s \geq N^*}$ and $\{\Gamma'_s\}_{s \geq N^*}$ are weakly isomorphic. Therefore, up to the weakly isomorphic relation, Feynman graph limits $\Delta(X), S(X)$ and their corresponding stretched Feynman graphons are independent of the chosen subsequences. $\square$

Remark 2. Consider the topological Hopf algebra of renormalization $H_{\text{FG}}^{\text{cut}}(\Phi)$.

- The renormalization coproduct and its related antipode can be continuously extended to $\mathcal{S}_{\text{gr}, \approx}^{\Phi, \Omega}$. We have

$$\Delta(W_X) = W_{\mathbb{I}} \otimes W_X + W_X \otimes W_{\mathbb{I}} + \sum_{W_{\Gamma}} W_{\Gamma} \otimes W_{X/\Gamma} \quad (32)$$

such that the sum is taken over all stretched Feynman graphons corresponding to Feynman diagrams $X \in H_{\text{FG}}^{\text{cut}}(\Phi)$ which contribute to the renormalization coproduct $\Delta(X)$. In addition,

$$S(W_X) = -W_X - \sum_{W_{\Gamma}} S(W_{\Gamma}) W_{X/\Gamma}. \quad (33)$$

- There exists a grading structure on $\mathcal{S}_{\text{gr}, \approx}^{\Phi, \Omega}$. For each n , define $\mathcal{S}_{\text{gr}, \approx}^{\Phi, \Omega, (n)}$ as the vector space generated by stretched Feynman graphons W_Γ corresponding to 1PI Feynman diagrams Γ with the loop number n .

2.3. Applications of Feynman Graph Limits

Fixed point equations of (1PI) Green's functions, known as *Dyson–Schwinger equations*, provide a quantized version of Euler–Lagrange equations of motion in classic mechanics, and because of that they are addressed as quantum motions in quantum field theories [35,52,53,55–57]. According to the Hochschild cohomology of the renormalization Hopf algebra, a combinatorial version of these equations is reformulated which enables us to apply combinatorial and topological tools in dealing with their solutions. Solutions of these fundamental equations can be interpreted by stretched Feynman graphons which represent limits of a certain family of Cauchy sequences of stretched Feynman graphons [36–38,40,43,51,57].

For (1PI) Green's function G^r given by (12), $r \in \mathcal{A}_\Phi$, and a family $\{\gamma_n\}_{n \geq 1}$ of primitive (1PI) Feynman diagrams with $\text{res}(\gamma) = r$, the fixed point equation of G^r is given by the recursive equation

$$\text{DSE} : X = \mathbb{I} + \sum_{n \geq 1} g_\lambda^n \omega_n B_{\gamma_n}^+(X^{n+1}) \quad (34)$$

in $H_{\text{FG}}(\Phi)[[g_\lambda]]$. It is called a *combinatorial Dyson–Schwinger equation*. In the formula (34), for each $n \geq 1$, $B_{\gamma_n}^+ : H_{\text{FG}}(\Phi) \rightarrow H_{\text{FG}}(\Phi)$ is a Hochschild 1-cocycle and it sends Γ to a linear combination of Feynman diagrams generated by the insertion of Γ into γ_n in terms of types of vertices of γ_n and types of external edges of Γ . Its solution $X = \sum_{n \geq 0} g_\lambda^n X_n$ is given by

$$X_n = \sum_{j=1}^n \omega_j B_{\gamma_j}^+ \left(\sum_{k_1 + \dots + k_{j+1} = n-j, k_i \geq 0} X_{k_1} \dots X_{k_{j+1}} \right), \quad X_0 = \mathbb{I} \quad (35)$$

such that $\{X_n\}_{n \geq 1}$ is a collection of generators for a connected graded free commutative Hopf subalgebra $H_{\text{DSE}}(\Phi)$ of $H_{\text{FG}}(\Phi)$ [40,43,44,53].

Lemma 1. The space $\mathcal{S}_{\text{gr}, \approx}^{\Phi, \Omega}$ recovers the solution X of the equation DSE.

Proof. According to Theorem 1, the solution X can be described as the Feynman graph limit of the Cauchy sequence $\{Y_m\}_{m \geq 1}$ of partial sums,

$$Y_m = \mathbb{I} + g_\lambda X_1 + \dots + g_\lambda^m X_m \quad (36)$$

with respect to the metric (21). For each $m \geq 1$, the stretched Feynman graphon

$$\tilde{W}_{Y_m} = \tilde{W}_{X_1} + \dots + \tilde{W}_{X_m} : \Omega \times \Omega \rightarrow \mathbb{R} \quad (37)$$

is the direct sum of stretched Feynman graphons $\tilde{W}_{X_k} : \tilde{J}_k \times \tilde{J}_k \rightarrow \mathbb{R}$ with $\mu(\tilde{J}_k) = g_\lambda^k$ such that $\tilde{J}_r \cap \tilde{J}_s = \emptyset$, $r \neq s$. When m tends to infinity, the sequence $\{\tilde{W}_{Y_m}\}_{m \geq 1}$ converges to the stretched Feynman graphon

$$\tilde{W}_{\text{DSE}} = \tilde{W}_{X_1} + \dots + \tilde{W}_{X_m} + \dots : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R} \quad (38)$$

as the infinite direct sum of the stretched Feynman graphons $\tilde{W}_{X_k} : \tilde{J}_k \times \tilde{J}_k \rightarrow \mathbb{R}$ with $\mu(\tilde{J}_k) = g_\lambda^k$ such that $\tilde{J}_r \cap \tilde{J}_s = \emptyset$, $r \neq s$. For each subinterval $\tilde{J}_k$, $\tilde{W}_{\text{DSE}}|_{\tilde{J}_k \times \tilde{J}_k} = \tilde{W}_{X_k}$. Therefore the Feynman graph limit X is represented by elements of the class

$$W_{\text{DSE}} := [\tilde{W}_{\text{DSE}}]_{\approx} = \left\{ \tilde{W}_{\text{DSE}}^\rho, \rho : \mu - \text{measure preserving transformation on } \Omega \right\} \quad (39)$$

such that $W_{\text{DSE}} \in \mathcal{S}_{\text{gr}, \approx}^{\Phi, \Omega}$. $\square$

Remark 3. • It is possible to apply μ -measure-preserving and affine monotone transformations on Ω to project the stretched Feynman graphon $\tilde{W}_{\text{DSE}}$ onto its corresponding Feynman graphon on the ground measure space $([0, 1], \mu)$.

- Combinatorial Dyson–Schwinger equations $\text{DSE}_1, \text{DSE}_2$ with the solutions $X_{\text{DSE}_1}, X_{\text{DSE}_2}$ are called weakly isomorphic iff X_{DSE_1} is weakly isomorphic to X_{DSE_2} iff $W_{\text{DSE}_1} = W_{\text{DSE}_2}$.

According to Lemma 1, it is possible to extend the metric (21) to the space of Feynman graph limits which contribute to solutions of combinatorial Dyson–Schwinger equations of Φ . Up to the weakly isomorphic relation given by Remark 3, the distance between combinatorial Dyson–Schwinger equations $\text{DSE}_1, \text{DSE}_2$ with the solutions $X_{\text{DSE}_1}, X_{\text{DSE}_2}$ and sequences of partial sums $\{Y_m^{(1)}\}_{m \geq 1}$ and $\{Y_m^{(2)}\}_{m \geq 1}$ is given by

$$d_{\text{cut}}(X_{\text{DSE}_1}, X_{\text{DSE}_2}) = \lim_{m \rightarrow \infty} d_{\text{cut}}(Y_m^{(1)}, Y_m^{(2)}) = \lim_{m \rightarrow \infty} d_{\text{cut}}(W_{Y_m^{(1)}}, W_{Y_m^{(2)}}). \quad (40)$$

Therefore, up to the weakly isomorphic relation given by Remark 3, we have

$$\text{DSE}_1 \approx \text{DSE}_2 \Leftrightarrow d_{\text{cut}}(X_{\text{DSE}_1}, X_{\text{DSE}_2}) = 0. \quad (41)$$

Therefore we can apply the metric (40) to complete the quotient space of Feynman graph limits which contribute to solutions of combinatorial Dyson–Schwinger equations of Φ for running coupling constants g_λ as functions of the bare coupling constant. The resulting space $\mathcal{S}_{\approx}^{\Phi, \mathcal{G}}$ is a separable complete Banach subspace of $\mathcal{S}_{\text{gr}, \approx}^{\Phi, \Omega}$. It was shown in other works of the author that the geometry of this particular Banach space has led to the development of some advanced analytic tools for the study of solutions of combinatorial Dyson–Schwinger equations.

Example F. Consider a divergent perturbative series and its rooted forest representation given by Figure 6. Thanks to [36], it likely cut-distance converges to some labeled Feynman graphon models with domains such as the one presented by Figure 7. Its non-zero Feynman graph limit is determined by some $W_{G_3^-} \in \mathcal{S}_{\text{gr}, \approx}^{\Phi, \Omega}$ with a stretched domain such as the one given by Figure 8. Weakly isomorphic stretched Feynman graphons corresponding to $W_{G_3^-}$ defined on stretched domains are obtained by applying μ -measure-preserving transformations on Ω . For instance, transformations $x \mapsto nx$ for the case of the Lebesgue measure spaces and fractional transformations $x \mapsto \frac{1}{n}x$ for the case of Gaussian measure spaces can be applied to generate non-zero stretched Feynman graphons on stretched domains.

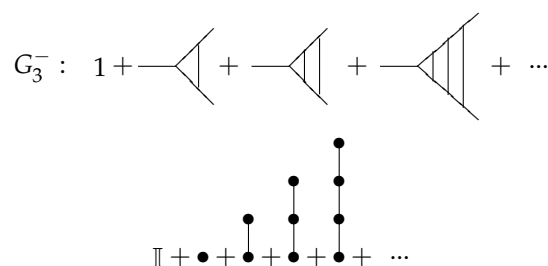


Figure 6. A (1PI) Green's function together with its rooted forest representation underlying the renormalization Hopf algebra.

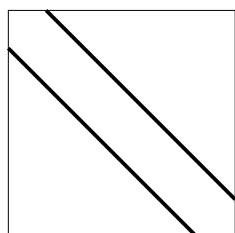


Figure 7. The domain of a labeled Feynman graphon model for the Feynman graph limit of the divergent perturbative series presented in Figure 6.

Corollary 1. Let X_{DSE} be the solution of a combinatorial Dyson–Schwinger equation DSE with the sequence $\{Y_m\}_{m \geq 1}$ of partial sums. For any Feynman diagram Γ , the sequence $\{t(\Gamma, W_{Y_m})\}_{m \geq 1}$ of homomorphism densities converges to $t(\Gamma, W_{\text{DSE}})$ when m tends to infinity.



Figure 8. The stretched version of the domain presented in Figure 7.

Proof. It is a consequence of Definition 8, Theorem 1, Lemma 1 and Theorem 11.21 in [36,37,50]. $\square$

2.4. Stochastic Models for Feynman Graph Limits

A new application of the method of (Feynman) graphons has been found in describing the asymptotics of (divergent) perturbative series of higher-loop-order Feynman diagrams and also beta functions of the physical theories. The notion of triviality for interacting quantum theories is remodeled in terms of a certain family of discrete-time Markov chains of random operators [51]. It opens a new stochastic platform for the study of non-perturbative structures.

Definition 9. • A random graph $R(n, p)$ is a non-oriented graph with n vertices such that every two vertices are independently connected by an edge with the probability p . It is called sparse if $p = c/n$ for $c > 0$ with sufficiently large n .

- For a given probability measure space, a stochastic process is defined by a time-dependent family of random variables, such as random graphs on the probability measure, which takes values in a state space of the process. If the family of random variables is a countable set, then the process is called a discrete time stochastic process; otherwise, it is called a continuous time stochastic process.

Theorem 2. *Feynman graph limits of Cauchy sequences of Feynman diagrams in $\mathcal{S}_{\text{gr},\approx}^{\Phi,\Omega}$ are generated by stochastic processes.*

Proof. This follows from [36,38], Definition 4 and Theorem 1. Let X be the Feynman graph limit of a Cauchy sequence $\{\Gamma_n\}_{n \geq 1}$ with respect to the metric (21). For each $n \geq 1$, let $\tilde{W}_{\Gamma_n} : \Omega \times \Omega \rightarrow \mathbb{R}$ be stretched Feynman graphons corresponding to Feynman diagrams Γ_n such that

$$X = \lim_{n \rightarrow \infty} \Gamma_n \Leftrightarrow \tilde{W}_X = \lim_{n \rightarrow \infty} \tilde{W}_{\Gamma_n}. \quad (42)$$

According to the model given in [59], for each n , we explain a stochastic process $S_n := \{S_{n,t} : t = 0, 1, 2, \dots\}$ to build sparse random graphs R_{n,p_n} with the vertex set $\{x_1, \dots, x_n\}$ uniformly chosen from Ω such that with the probability

$$p_n = |\tilde{W}_{\Gamma_n}|_1^{-1} \left| \tilde{W}_{\Gamma_n}(x_r, x_s) \right| \quad (43)$$

there exists an edge between x_r and x_s in R_{n,p_n} .

At $t = 0$, the process chooses no vertex. At $t = 1$, an arbitrary vertex is selected from $\{x_1, \dots, x_n\}$ such that it is independently connected by some edges to other vertices with the probability p_n . The selected vertex is called first generated and then saturated. Define $S_{n,1}$ as the number of vertices in the resulting connected component. All connected vertices at this stage are called generated. At $t = 2$, the process chooses one of the generated but non-saturated vertices, if any, and then saturates it by independently connecting it with the probability p_n to the vertices that either have not been generated yet or have been generated but not saturated. If there are no generated non-saturated vertices, then an arbitrary non-generated vertex is selected and then saturated by independently connecting it to other non-generated vertices. Define $S_{n,2}$ as the total number of vertices generated at steps $t = 1, 2$. The process runs by saturating vertices until the step $t = n$ to generate the random graph R_{n,p_n} .

When n tends to infinity, the sequence $\{R_{n,p_n}\}_{n \geq 1}$ of random graphs, generated by a stochastic process, converges to an infinite random graph R_X with the vertex set $\{x_1, \dots, x_n, \dots\}$ uniformly chosen from Ω such that with the probability $|\tilde{W}_X|_1^{-1} \left| \tilde{W}_X(x_k, x_l) \right|$, there exists an edge between x_k and x_l in R_X . $\square$

Definition 10. *Let X be the Feynman graph limit of a Cauchy sequence $\{\Gamma_n\}_{n \geq 1}$ with respect to the metric (21). For each $n \geq 1$, let t_{Γ_n} be the combinatorial representation of Γ_n . For any fixed integer k , define $t_{\Gamma_n}[k]$ as a subtree of t_{Γ_n} labeled with vertices $1, \dots, k$ obtained by uniformly choosing k distinct vertices $v_1, \dots, v_k \in t_{\Gamma_n}$. For each n , $\Gamma_n[k]$ is called a random Feynman subgraph of rank n in Γ_n corresponding to $t_{\Gamma_n}[k]$.*

According to Theorem 2 and Definition 10, when n tends to infinity, the distribution $\{\Gamma_n[k]\}_{n \geq 1}$ is a discrete time stochastic process which converges to $X[k]$.

Corollary 2. *(Non-)perturbative solutions of quantum motions can be described by stochastic processes.*

Proof. This follows from [36,38], Lemma 1, Theorem 2 and Definition 10. Consider a combinatorial Dyson–Schwinger equation DSE with the solution X_{DSE} and the corresponding sequence $\{Y_m\}_{m \geq 1}$ of partial sums. We have

$$X_{\text{DSE}} = \lim_{m \rightarrow \infty} Y_m \Leftrightarrow \tilde{W}_{\text{DSE}} = \lim_{m \rightarrow \infty} \tilde{W}_{Y_m}. \quad (44)$$

According to the model given in [59] and modified in the proof of Theorem 2, for each m , there exists a stochastic process $S'_m := \{S'_{m,t} : t = 0, 1, 2, \dots\}$ to build sparse random graphs R_{m,q_m}^{DSE} with the vertex set $V_m \subset \Omega$ with cardinal $|t_{Y_m}|$ uniformly chosen from Ω such that with the probability

$$q_m := |\tilde{W}_{Y_m}|_1^{-1} \left| \tilde{W}_{Y_m}(x_k, x_l) \right| \quad (45)$$

there exists an edge between x_k and x_l in R_{m,q_m}^{DSE} .

When m tends to infinity, the sequence $\{R_{m,q_m}^{\text{DSE}}\}_{m \geq 1}$ of random graphs, generated by a stochastic process, converges to an infinite random graph R_∞^{DSE} with the vertex set $V(\text{DSE}) = \{x_1, \dots, x_m, \dots\}$, uniformly chosen from Ω , such that with the probability $|\tilde{W}_{\text{DSE}}|_1^{-1} \left| \tilde{W}_{\text{DSE}}(x_r, x_s) \right|$, there exists an edge between x_r and x_s in R_∞^{DSE} .

For any fixed integer k , let $t_{Y_m}[k]$ be a sub-forest of t_{Y_m} labeled with vertices $1, \dots, k$ obtained by uniformly choosing k distinct vertices $u_1, \dots, u_k \in t_{Y_m}$. For each $m \geq 1$, random Feynman subgraphs $Y_m[k]$ of rank m in Y_m corresponding to $t_{Y_m}[k]$ can be generated by a stochastic process, such as the one given in [59] and modified here, such that the sequence $\{Y_m[k]\}_{m \geq 1}$ converges to $X_{\text{DSE}}[k]$ when m tends to infinity. $\square$

Example G. The stretched Feynman graphon, with the stretched domain presented in Figure 8, corresponding to the divergent perturbative series given in Figure 6 determines a certain family of random graph representations such as those given by Figure 9. These random graph representations generate processes of random graphs for the description of solutions of the fixed point equations of the divergent perturbative series given by Figure 6. One important note is that we can formulate other random graph processes for the description of a (1PI) Green's function in terms of changing the ground measure space of stretched Feynman graphon models.

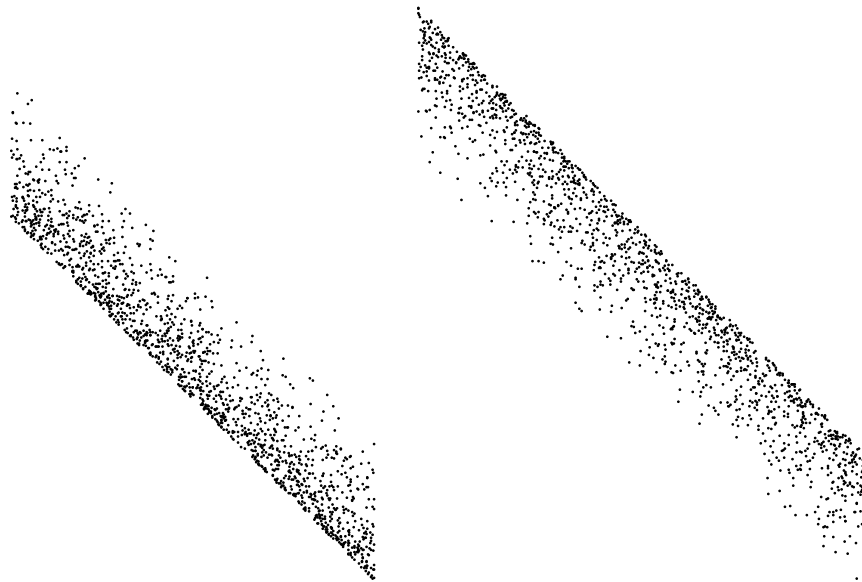


Figure 9. Samples of random graph representations assigned to the divergent perturbative series given by Figure 6. These representations are generated by Feynman graphon models.

Corollary 3. *Weakly or fractionally isomorphic Feynman graph limits, which contribute to solutions of combinatorial Dyson–Schwinger equations, are represented by same stochastic model.*

Proof. It is a consequence of Corollaries 1 and 2 and Definitions 5 and 10. $\square$

3. Toposification of Gauge Field Theories

Topological Hopf subalgebras of $H_{\text{FG}}^{\text{cut}}$, determined by solutions of quantum motions, are assigned to space-time regions to build a new class of correlations between distinct regions. These correlations are encoded by lattices of topological Hopf subalgebras corresponding to towers of combinatorial Dyson–Schwinger equations. In fact, the standard operator-algebra ontology of commutative von Neumann subalgebras of $\mathbf{B}(\mathcal{H}_\Phi)$ is replaced by topological Hopf subalgebras of the topological Hopf algebra of renormalization in this regard [24,36,37].

3.1. Non-Perturbative Topos

Consider the topological Hopf algebra of renormalization $H_{\text{FG}}^{\text{cut}}(\Phi)$ of Φ with the bare coupling constant g . Define a small category $\mathcal{C}_\Phi^{\text{non},g}$ such that its objects are topological Hopf subalgebras of $H_{\text{FG}}^{\text{cut}}(\Phi)$ associated with solutions of combinatorial Dyson–Schwinger equations in Φ , and its morphisms are homomorphisms of Hopf algebras which are continuous with respect to cut-distance topology.

Definition 11. The non-perturbative topos $\mathbf{T}_\Phi^{\text{non},g}$ of Φ is the topos of presheaves over the category $\mathcal{C}_\Phi^{\text{non},g}$. Its objects are contravariant functors from the category $\mathcal{C}_\Phi^{\text{non},g}$ to the category **Set** of sets and functions, and its morphisms are natural transformations between these functors.

The prime spectrum Spec is a representable contravariant functor from the category of commutative rings to the category of topological spaces. For any $H_{\text{DSE}}^{\text{cut}} \in \text{obj}(\mathcal{C}_\Phi^{\text{non},g})$, its corresponding affine group scheme $\mathbb{G}_{\text{DSE}}^{\text{cut}} = \text{Spec}(H_{\text{DSE}}^{\text{cut}})$ is a representable covariant functor from the category of commutative algebras to the category of groups. For the complex Lie group

$$\mathbb{G}_{\text{DSE}}^{\text{cut}}(\mathbb{C}) = \text{Spec}(H_{\text{DSE}}^{\text{cut}})(\mathbb{C}) = \text{Hom}(H_{\text{DSE}}^{\text{cut}}, \mathbb{C}) \quad (46)$$

of characters on $H_{\text{DSE}}^{\text{cut}}$, define the spectral presheave in $\mathbf{T}_\Phi^{\text{non},g}$ given by

$$\begin{aligned} \underline{\Sigma}^\Phi : H_{\text{DSE}}^{\text{cut}} &\mapsto \text{Hom}(H_{\text{DSE}}^{\text{cut}}, \mathbb{C}) = \\ &\left\{ \phi : H_{\text{DSE}}^{\text{cut}} \rightarrow \mathbb{C} : \phi(\Gamma_1 \Gamma_2) = \phi(\Gamma_1) \phi(\Gamma_2), \phi(\mathbb{I}) = 1 \right\}. \end{aligned} \quad (47)$$

For $H_{\text{DSE},1}^{\text{cut}} \leq H_{\text{DSE},2}^{\text{cut}} \in \text{obj}(\mathcal{C}_\Phi^{\text{non},g})$, the injective morphism $i : H_{\text{DSE},1}^{\text{cut}} \rightarrow H_{\text{DSE},2}^{\text{cut}}$ is lifted onto the surjective group homomorphism $\tilde{i} : \text{Hom}(H_{\text{DSE},2}^{\text{cut}}, \mathbb{C}) \rightarrow \text{Hom}(H_{\text{DSE},1}^{\text{cut}}, \mathbb{C})$ to determine the morphism $\underline{\Sigma}^\Phi(H_{\text{DSE},2}^{\text{cut}}) \rightarrow \underline{\Sigma}^\Phi(H_{\text{DSE},1}^{\text{cut}})$ in $\text{mor}(\mathcal{C}_\Phi^{\text{non},g})$. The outer presheave in $\mathbf{T}_\Phi^{\text{non},g}$ is given by

$$\underline{Q}^\Phi : H_{\text{DSE}}^{\text{cut}} \mapsto \text{In}(H_{\text{DSE}}^{\text{cut}}) = \left\{ \psi_\Gamma : H_{\text{DSE}}^{\text{cut}} \rightarrow \mathbb{C} : \psi_\Gamma(\Gamma') = \delta_{\Gamma,\Gamma'} : \Gamma \in H_{\text{DSE}}^{\text{cut}} \right\}. \quad (48)$$

For $H_{\text{DSE},1}^{\text{cut}} \leq H_{\text{DSE},2}^{\text{cut}} \in \text{obj}(\mathcal{C}_\Phi^{\text{non},g})$, the injective morphism $i : H_{\text{DSE},1}^{\text{cut}} \rightarrow H_{\text{DSE},2}^{\text{cut}}$ is lifted onto the surjective algebra homomorphism $\hat{i} : \text{In}(H_{\text{DSE},2}^{\text{cut}}) \rightarrow \text{In}(H_{\text{DSE},1}^{\text{cut}})$ to determine the morphism $\underline{Q}^\Phi(H_{\text{DSE},2}^{\text{cut}}) \rightarrow \underline{Q}^\Phi(H_{\text{DSE},1}^{\text{cut}})$ in $\text{mor}(\mathcal{C}_\Phi^{\text{non},g})$ such that

$$Z_\Gamma \in \underline{Q}^\Phi(H_{\text{DSE},2}^{\text{cut}}) \mapsto \delta(Z_\Gamma) := \bigwedge \left\{ Z_\gamma \in \text{In}(H_{\text{DSE},2}^{\text{cut}}) : Z_\Gamma \preceq Z_\gamma \right\} \in \underline{Q}^\Phi(H_{\text{DSE},1}^{\text{cut}}) \quad (49)$$

where the partial order $\preceq$ is defined by the loop number of Feynman diagrams.

The *terminal object* in $\mathbf{T}_\Phi^{\text{non},g}$ is given by

$$\mathbf{1}^\Phi : \mathcal{C}_\Phi^{\text{non},g} \rightarrow \mathbf{Set}, \mathbf{1}^\Phi(H_{\text{DSE}}^{\text{cut}}) = \{*\}, \mathbf{1}^\Phi(H_{\text{DSE},1}^{\text{cut}} \rightarrow H_{\text{DSE},2}^{\text{cut}}) = \text{id} : \{*\} \rightarrow \{*\}. \quad (50)$$

For any object $H_{\text{DSE}}^{\text{cut}} \in \text{obj}(\mathcal{C}_\Phi^{\text{non},g})$, a *sieve* on $H_{\text{DSE}}^{\text{cut}}$ is a set $s_{H_{\text{DSE}}^{\text{cut}}}$ of morphisms $f : H_{\text{DSE},1}^{\text{cut}} \rightarrow H_{\text{DSE}}^{\text{cut}} \in \text{mor}(\mathcal{C}_\Phi^{\text{non},g})$ such that for any $h : H_{\text{DSE},2}^{\text{cut}} \rightarrow H_{\text{DSE},1}^{\text{cut}} \in \text{mor}(\mathcal{C}_\Phi^{\text{non},g})$, $f \circ h \in s_{H_{\text{DSE}}^{\text{cut}}}$. The subobject classifier $\Omega_\Phi^{\text{non},g}$ in $\mathbf{T}_\Phi^{\text{non},g}$ maps each $H_{\text{DSE}}^{\text{cut}}$ to the collection of all its sieves. For any morphism $f : H_{\text{DSE},1}^{\text{cut}} \rightarrow H_{\text{DSE}}^{\text{cut}} \in \text{mor}(\mathcal{C}_\Phi^{\text{non},g})$,

$$\begin{aligned} \Omega_\Phi^{\text{non},g}(f) : \Omega_\Phi^{\text{non},g}(H_{\text{DSE}}^{\text{cut}}) &\rightarrow \Omega_\Phi^{\text{non},g}(H_{\text{DSE},1}^{\text{cut}}), \\ \forall s_{H_{\text{DSE}}^{\text{cut}}} \in \Omega_\Phi^{\text{non},g}(H_{\text{DSE}}^{\text{cut}}) : \\ \Omega_\Phi^{\text{non},g}(s_{H_{\text{DSE}}^{\text{cut}}}) &= \left\{ k : H_{\text{DSE},2}^{\text{cut}} \rightarrow H_{\text{DSE},1}^{\text{cut}} \in \Omega_\Phi^{\text{non},g}(H_{\text{DSE},1}^{\text{cut}}) : fok \in s_{H_{\text{DSE}}^{\text{cut}}} \right\}. \end{aligned} \quad (51)$$

For any $H_{\text{DSE}}^{\text{cut}} \in \text{obj}(\mathcal{C}_\Phi^{\text{non},g})$, the inclusion introduces a poset structure on $\Omega_\Phi^{\text{non},g}(H_{\text{DSE}}^{\text{cut}})$ given by

$$s_{H_{\text{DSE}}^{\text{cut}}} \preceq t_{H_{\text{DSE}}^{\text{cut}}} \Leftrightarrow s_{H_{\text{DSE}}^{\text{cut}}} \subseteq t_{H_{\text{DSE}}^{\text{cut}}}, \quad (52)$$

such that the empty sieve is the zero element and the principal sieve

$$\uparrow H_{\text{DSE}}^{\text{cut}} := \left\{ h : H_{\text{DSE},1}^{\text{cut}} \rightarrow H_{\text{DSE}}^{\text{cut}} \in \text{mor}(\mathcal{C}_\Phi^{\text{non},g}) \right\} \quad (53)$$

is the unit element of this poset.

An algebraic structure is computable if its domain can be determined in terms of a computable set of natural numbers and a computable set of a finite number of operations together with relations [60]. The computable dimension of a computable structure is the number of isomorphic computable copies of the structure up to computable isomorphism. (Dimensionally) computable Heyting algebras have been studied to develop a theory of computation on the basis of constructive logics [61]. The subobject classifier $\Omega_\Phi^{\text{non},g}$ of $\mathbf{T}_\Phi^{\text{non},g}$, which is a Heyting algebra, provides truth values for the logical evaluation of propositions about solutions of quantum motions of Φ formulated in terms of logical connectives of this Heyting algebra. It is shown that this new Heyting algebra is dimensionally computable [24].

3.2. Propositional Language

For a combinatorial Dyson–Schwinger equation DSE with the solution X_{DSE} , let Δ_{DSE} be an open neighborhood around X_{DSE} in $H_{\text{FG}}^{\text{cut}}(\Phi)$. The Hahn–Banach theorem shows that any 1-dimensional subspace generated by any Feynman diagram $\Gamma \subset X_{\text{DSE}}$ has a closed complementary subspace in $H_{\text{DSE}}^{\text{cut}}$. In other words, there exists a bounded linear functional with value 1 at Γ which generates a projective operator p_Γ .

Definition 12. • For any $X \in H_{\text{FG}}^{\text{cut}}(\Phi)$, the formula $(X \in \Delta_{\text{DSE}})$ is called a *primitive proposition*.

- Let $\mathbf{P}_{\text{DSE}}^\Phi$ be the collection of all projections in $H_{\text{DSE}}^{\text{cut}}$. The infinitesimal characters Z_Γ , for any Feynman diagram $\Gamma \subset X_{\text{DSE}}$, define a correspondence between $\mathbf{P}_{\text{DSE}}^\Phi$ and $\text{In}(\text{DSE})$.
- Let $\mathbf{P}^\Phi$ be the collection of all projections in $H_{\text{FG}}^{\text{cut}}(\Phi)$. The orthogonal decomposition $H_{\text{FG}}^{\text{cut}}(\Phi) = H_{\text{DSE}}^{\text{cut}} \oplus H$ defines a correspondence between $\mathbf{P}^\Phi$ and $\text{In}(H_{\text{FG}}^{\text{cut}}(\Phi))$.

Definition 12 enables us to describe primitive propositions about Feynman graph limits, which contribute to solutions of quantum motions, in terms of projective operators in $H_{\text{FG}}^{\text{cut}}(\Phi)$.

Definition 13. Consider projective operators $\hat{P}_1, \hat{P}_2$ corresponding to the primitive propositions $(X \in \Delta_1)$ and $(X \in \Delta_2)$. The extension of the partial order relation given in the Formula (49) to $\mathbf{P}_{\text{DSE}}^\Phi$ is given by

$$\hat{P}_1 \succcurlyeq \hat{P}_2 \Leftrightarrow d_{\text{PH}}(\Delta_1, \Delta_{\text{DSE}}) \leq d_{\text{PH}}(\Delta_2, \Delta_{\text{DSE}}). \quad (54)$$

This new partial order relation can be extended to $\mathbf{P}^\Phi$.

Lemma 2. Let $\Gamma \underline{O}^\Phi$ be set of all global elements $\mathbf{1}^\Phi \rightarrow \underline{O}^\Phi$ of the outer presheaf $\underline{O}^\Phi$ of the non-perturbative topos. The partial order $\succcurlyeq$ defines a poset structure on $\Gamma \underline{O}^\Phi$.

Proof. We apply [8–10], Section 3.1 and Definition 13. Let

$$\delta(\hat{P}_2)_{H_{\text{DSE}}^{\text{cut}}} := \bigwedge \{ \hat{Q} \in \mathbf{P}_{\text{DSE}}^\Phi : \hat{Q} \succcurlyeq \hat{P}_2 \} \quad (55)$$

For any $\hat{P} \in \mathbf{P}^\Phi$, the assignment $H_{\text{DSE}}^{\text{cut}} \mapsto \delta(\hat{P})_{H_{\text{DSE}}^{\text{cut}}}$ determines a global element of the outer presheaf to extend δ to the injective map

$$\delta^\Phi : \mathbf{P}^\Phi \longrightarrow \Gamma \underline{O}^\Phi, \quad \hat{P} \longmapsto \left\{ \delta(\hat{P})_{H_{\text{DSE}}^{\text{cut}}} : H_{\text{DSE}}^{\text{cut}} \in \text{Obj}(\mathcal{C}_\Phi^{\text{non},g}) \right\}. \quad (56)$$

It translates the partial order $\succcurlyeq$ to a poset structure on $\Gamma \underline{O}^\Phi$. $\square$

Theorem 3. The non-perturbative topos recovers propositional language of the background logic of Φ .

Proof. We apply [8–10,25], Section 3.1 and Lemma 2. A subobject of the spectral presheaf of the non-perturbative topos is a contravariant functor $\underline{S}$ such that for each $H_{\text{DSE}}^{\text{cut}} \in \text{Obj}(\mathcal{C}_\Phi^{\text{non},g})$,

$$\underline{S}(H_{\text{DSE}}^{\text{cut}}) \subset \sum^\Phi(H_{\text{DSE}}^{\text{cut}}), \quad \underline{S}(i_{H_{\text{DSE}}^{\text{cut}}}) : \underline{S}(H_{\text{DSE}}^{\text{cut}}) \longrightarrow \underline{S}(H_{\text{DSE}}^{\text{cut}}). \quad (57)$$

The set $\text{Sub}(\sum^\Phi)$ of all subobjects of the spectral presheaf carries a Heyting algebra structure.

Let $\text{PL}(\Phi)$ be the propositional language of propositions for Feynman diagrams and Feynman graph limits which contribute to (1PI) Green's functions in Φ and solutions of their fixed point equations. We embed the lattice $\mathbf{P}_{\text{DSE}}^\Phi$ into a lattice of subobjects of $\sum^\Phi(H_{\text{DSE}}^{\text{cut}})$ for any $H_{\text{DSE}}^{\text{cut}} \in \mathcal{C}_\Phi^{\text{non},g}$ to build a representation of $\text{PL}(\Phi)$ in $\text{Sub}(\sum^\Phi)$.

For a projection $\hat{P}_A^\Delta$ corresponding to the primitive proposition $(A \in \Delta)$ about the solution of the equation DSE, the set

$$S_{\hat{P}_A^\Delta} := \left\{ \rho \in \mathbb{G}_{\text{DSE}}^{\text{cut}}(\mathbb{C}) : \rho(\Gamma) \neq 0, \forall \Gamma \in A \cap \Delta \right\} \quad (58)$$

assigns a new morphism $\hat{P}_A^\Delta \mapsto S_{\hat{P}_A^\Delta}$ of lattices such that

$$\underline{\delta}(\hat{P}_A^\Delta) := \left\{ S_{\hat{P}_A^\Delta} \subset \sum^\Phi(H_{\text{DSE}}^{\text{cut}}) : \forall H_{\text{DSE}}^{\text{cut}} \in \mathcal{C}_\Phi^{\text{non},g} \right\} \quad (59)$$

determines a subobject of the spectral presheaf of the non-perturbative topos.

Let $\text{PL}_{\text{prim}}(\Phi)$ be the collection of primitive propositions in $\text{PL}(\Phi)$. The map π_Φ^0 which sends each primitive proposition $(A \in \Delta)$ to the subobject $\underline{\delta}(\hat{P}_A^\Delta)$ is a representation. The

extension of π_Φ^0 to a representation π_Φ for $\text{PL}(\Phi)$ provides a representation of $\text{PL}(\Phi)$ in terms of subobjects of the spectral presheaf $\underline{\Sigma}^\Phi$. $\square$

3.3. Random Representations

Here we address a new correlation between the strength of running coupling constants, which governs (non-)perturbative behavior of solutions of quantum motions, and assigned expectation values to random graphs generated by stretched Feynman graphons of Feynman graph limits.

Corollary 4. *Random graphs generated by Feynman graph limits can be characterized by their expectation values.*

Proof. For a combinatorial Dyson–Schwinger equation DSE with the solution X_{DSE} , let Δ_{DSE} be an open neighborhood around X_{DSE} in $H_{\text{FG}}^{\text{cut}}(\Phi)$. Thanks to [62], the deviation of any subset $U \subset H_{\text{FG}}^{\text{cut}}(\Phi)$ from X_{DSE} with respect to Δ_{DSE} can be defined by the Pompeiu–Hausdorff distance

$$d_{\text{PH}}(U, \Delta_{\text{DSE}}) := \max \left\{ \sup_{\Gamma \in \Delta_{\text{DSE}}} F_U(\Gamma), \sup_{\Gamma' \in U} F_{\Delta_{\text{DSE}}}(\Gamma') \right\}, \quad (60)$$

such that

$$F_U(\Gamma) := \inf \left\{ d_{\text{cut}}(\Gamma', \Gamma) : \Gamma' \in U \right\}, \quad F_{\Delta_{\text{DSE}}}(\Gamma') := \inf \left\{ d_{\text{cut}}(\Gamma', \Gamma) : \Gamma \in \Delta_{\text{DSE}} \right\} \quad (61)$$

with respect to the metric (21).

The expectation value of any primitive proposition ($X \in \Delta_{\text{DSE}}$) is defined by

$$\hat{E}[X \in \Delta_{\text{DSE}}] := 1 - \left(d_{\text{PH}}(X, \Delta_{\text{DSE}}) - \lfloor d_{\text{PH}}(X, \Delta_{\text{DSE}}) \rfloor \right) \quad (62)$$

such that $\lfloor \cdot \rfloor$ is the floor function. Therefore the expectation value of any proposition ($U \subseteq \Delta_{\text{DSE}}$) is determined by

$$\hat{E}[U \subseteq \Delta_{\text{DSE}}] := \inf_{X \in U} \hat{E}[X \in \Delta_{\text{DSE}}]. \quad (63)$$

For any Feynman graph limit X and the infinite graph R on Ω generated by W_X , the expectation value of R from the perspective X_{DSE} is given by

$$\hat{E}[R]_{\text{DSE}} := \inf_{\Delta_{\text{DSE}}} \left\{ \hat{E}[R \in \Delta_{\text{DSE}}] \right\} \quad (64)$$

such that the inf is taken over all open neighborhoods around X_{DSE} in $H_{\text{FG}}^{\text{cut}}(\Phi)$. $\square$

Given a category $\mathcal{C}$ and a commutative ring R , the category algebra RC is the free R -module such that the set of morphisms of $\mathcal{C}$ is its basis, and it is equipped with the product

$$f \bar{*} g = \begin{cases} f \circ g, & \text{if } f \text{ and } g \text{ can be composed in } \mathcal{C} \\ 0, & \text{otherwise} \end{cases}. \quad (65)$$

A representation of a category $\mathcal{C}$ over a commutative ring R is a covariant functor from $\mathcal{C}$ to the category $R\text{-mod}$ of R -modules. It was shown that if $\mathcal{C}$ is a small category with a finite number of objects, then the category of representations of $\mathcal{C}$ is equivalent to the category of unital left RC -modules. The extension of this equivalent to finite category, a

small category in which its morphisms is a finite set, and EI-category, a small category in which all endomorphisms are isomorphisms, are studied in [63].

Corollary 5. *The non-perturbative topos has a representation over a category algebra of random graphs.*

Proof. This follows from Corollary 2 and Definition 11. Consider $\mathcal{S}_{\text{gr}, \approx}^{\Phi, \Omega}$ as the commutative ring of stretched Feynman graphons with the unit $W_{\mathbb{I}}$ such that each $W_X \in \mathcal{S}_{\text{gr}, \approx}^{\Phi, \Omega}$ determines a random graph R_X with the expectation value $\hat{E}[R]_X$ given by (64). Consider the category of $\mathcal{S}_{\text{gr}, \approx}^{\Phi, \Omega}$ -modules.

Consider the covariant functor $H_{\text{DSE}}^{\text{cut}} \mapsto R_{W_{\text{DSE}}}$ from $\mathbf{T}_{\Phi}^{\text{non}, g}$ to $\mathcal{S}_{\text{gr}, \approx}^{\Phi, \Omega}$ to define a new small category $\mathcal{C}_{\Phi}^{\text{ran}}$ with

$$\text{Obj}(\mathcal{C}_{\Phi}^{\text{ran}}) = \left\{ R_{W_{\text{DSE}}} : H_{\text{DSE}}^{\text{cut}} \in \mathbf{T}_{\Phi}^{\text{non}, g} \right\} \quad (66)$$

together with the collection $\text{Mor}(\mathcal{C}_{\Phi}^{\text{ran}})$ of homomorphisms of random graphs between those objects.

Any $\text{Sub}(H_{\text{DSE}}^{\text{cut}}) \subset \text{Obj}(\mathcal{C}_{\Phi}^{\text{non}, g})$ is associated with $\text{Sub}(R_{W_{\text{DSE}}}) \subset \mathcal{C}_{\Phi}^{\text{ran}}$. Any morphism $f \in \text{mor}(H_{\text{DSE}_1}^{\text{cut}}, H_{\text{DSE}_2}^{\text{cut}}) \in \text{mor}(\mathcal{C}_{\Phi}^{\text{non}, g})$ defines a morphism $\tilde{f} \in \text{mor}(R_{W_{\text{DSE}_1}}, R_{W_{\text{DSE}_2}}) \in \text{mor}(\mathcal{C}_{\Phi}^{\text{ran}})$.

Consider the topos $\mathbf{T}_{\Phi}^{\text{ran}}$ of presheaves over the base category $\mathcal{C}_{\Phi}^{\text{ran}}$ and its corresponding category algebra $\mathcal{S}_{\text{gr}, \approx}^{\Phi, \Omega} \mathbf{T}_{\Phi}^{\text{ran}}$ with respect to the commutative ring $\mathcal{S}_{\text{gr}, \approx}^{\Phi, \Omega} \mathbf{T}_{\Phi}^{\text{ran}}$. We apply the outer presheave in $\mathbf{T}_{\Phi}^{\text{ran}}$ to define the covariant functor

$$H_{\text{DSE}}^{\text{cut}} \mapsto W_{\text{DSE}} \underline{Q}^{\Phi, \text{ran}}(R_{W_{\text{DSE}}} \rightarrow R_{W_{\text{DSE}}}) \quad (67)$$

which is a representation of $\mathbf{T}_{\Phi}^{\text{non}, g}$ over the category algebra $\mathcal{S}_{\text{gr}, \approx}^{\Phi, \Omega} \mathbf{T}_{\Phi}^{\text{ran}}$. $\square$

4. Universal Non-Perturbative Topos

A Hopf algebra is called combinatorial if it is a graded connected Hopf algebra such that as an algebra, it is isomorphic to a polynomial algebra where the variables which generate the polynomial algebra correspond to some combinatorial objects such as trees. The Connes–Kreimer Hopf algebra of non-planar rooted trees $H_{\text{CK}} = \bigoplus_{n=0}^{\infty} H_{(n)}$ is a combinatorial Hopf algebra. It is a graded connected commutative non-cocommutative Hopf algebra [40–42,44].

Remark 4. • For each $n \geq 1$, $H_{(n)}$ is the vector space generated by non-planar rooted trees of degree n or forests with the total degree n . $H_{(0)} = \mathbb{Q}$ is generated by the empty tree 1.

- The grafting operator $B^+ : H_{\text{CK}} \rightarrow H_{\text{CK}}$ is a homogeneous linear operator. It sends each forest $t_1 \dots t_n$ to a new non-planar rooted tree t by adding a new vertex r_t together with new edges $e_1, \dots, e_n$ from r_t to the roots $r_{t_1}, \dots, r_{t_n}$ of $t_1, \dots, t_n$.
- The coproduct (14) is reformulated recursively in terms of the equation

$$\Delta_{\text{CK}}(t) = \Delta_{\text{CK}} \circ B^+(t_1 \dots t_n) = ((\text{Id} \otimes B^+ + (B^+ \otimes 1\epsilon)) \circ \Delta_{\text{CK}}(t) \quad (68)$$

such that $t = B^+(t_1 \dots t_n)$, 1 is the unit and ϵ is the counit in H_{CK} [40–42,44].

The space of finite weighted graphs is a compact Hausdorff topological space completed by the space $\mathcal{W}_{\approx}^{([0,1], m)}([0,1])$ of graphons; see Chapter 8 in [50]. The collection of non-planar rooted trees, as simple graphs, is a subspace of this compact topological

space such that up to the weakly isomorphic equivalence relation, we get a metric space. According to Definition 4, elements $t, s \in H_{\text{CK}}$ are weakly isomorphic iff $W_t = W_s$ such that

$$W_t := [P_t^{\sigma|t|}]_{\sim}, W_s := [P_s^{\sigma|s|}]_{\sim}. \quad (69)$$

For $d_{\text{cut}}(t, s) := d_{\text{cut}}(W_t, W_s)$, it is seen that s, t are weakly isomorphic iff $d_{\text{cut}}(t, s) = 0$. Up to the weakly isomorphic relation, the quotient space $H_{\text{CK}}^{\text{cut}}$ of weakly isomorphic non-planar rooted trees equipped with the metric d_{cut} and completed by $\mathcal{W}_{\approx}^{([0,1],m)}([0,1])$ is a separable Banach space.

Definition 14. Define a small category $\mathcal{C}_{\text{U}}$ such that its objects are connected graded commutative Hopf subalgebras of $H_{\text{CK}}^{\text{cut}}$ completed by d_{cut} up to the weakly isomorphic equivalence relation, and its morphisms are homomorphisms of Hopf algebras which are continuous with respect to d_{cut} . The “universal non-perturbative topos” $\mathbf{T}_{\text{U}}$ is the topos of presheaves over $\mathcal{C}_{\text{U}}$. Its objects are contravariant functors from the category $\mathcal{C}_{\text{U}}$ to the category **Set** of sets and functions, and its morphisms are natural transformations between these functors.

Corollary 6. The universal non-perturbative topos describes quantum field theory as a neo-realist theory.

Proof. Step I. We show that the Banach space $H_{\text{FG}}^{\text{cut}}(\Phi)$ is embedded into the decorated version $H_{\text{CK}}^{\text{cut}}(\Phi)$ such that the vertex sets of its elements are decorated by primitive (1PI) Feynman diagrams in Φ .

According to the Hochschild cohomology of commutative graded Hopf algebras, the pair (H_{CK}, B^+) is the universal object in a certain category of pairs (H, L) of a (graded connected) commutative Hopf algebra together with a Hochschild one-cocycle L [40]. For a non-primitive 1PI Feynman diagram $\Gamma = \prod_{i=1}^{k_j} \Gamma_j \star_{j,i} \gamma_{j,i}$, with $G_{j,i}$ as the gluing information, its combinatorial representation t_{Γ} is given by

$$\Gamma \mapsto t_{\Gamma} = \Xi_{\Phi}(\Gamma) = \sum_{j=1}^r B_{\Gamma_j, G_{j,i}}^+ \left(\prod_{i=1}^{k_j} \Xi_{\Phi}(\gamma_{j,i}) \right) \quad (70)$$

such that each $B_{\Gamma_j, G_{j,i}}^+$ is a closed Hochschild one-cocycle. The map Ξ_{Φ} is an injective homomorphism of Hopf algebras which embeds $H_{\text{FG}}(\Phi)$ into $H_{\text{CK}}(\Phi)$ [52].

According to Theorem 1 and Remark 4, for any Feynman graph limit X as the limit of any sequence $\{\Gamma_n\}_{n \geq 1}$ of Feynman diagrams, $t_X = \Xi_{\Phi}(X)$ is determined by the graph limit of the sequence $\{\Xi_{\Phi}(\Gamma_n)\}_{n \geq 1}$ with respect to the metric d_{cut} . This means that the map Ξ_{Φ} can be interpreted as an injective homomorphism of Banach spaces to embed $H_{\text{FG}}^{\text{cut}}(\Phi)$ into $H_{\text{CK}}^{\text{cut}}(\Phi)$.

It is possible to project combinatorial Dyson–Schwinger equations in $H_{\text{FG}}(\Phi)[[g_{\lambda}]]$ to their corresponding equations in $H_{\text{CK}}(\Phi)[[g_{\lambda}]]$ where each Hochschild one-cocycle B_{γ}^+ is replaced by the grafting operator $B_{\bullet, \gamma}^+$ [43,53,57].

Step II. We show that combinatorial Dyson–Schwinger equations in Φ determine topological Hopf subalgebras of $H_{\text{CK}}^{\text{cut}}(\Phi)$.

On the one hand, for the running coupling constant g_{λ} and the ring $\mathbb{K}[[g_{\lambda}]]$ of one variable formal power series over the field $\mathbb{K}$ with characteristic zero, the group

$$G_{\text{FdB}} := \{g_{\lambda} + \sum_{n \geq 1} a_n g_{\lambda}^{n+1} \in \mathbb{K}[[g_{\lambda}]]\} \quad (71)$$

together with the composition of formal series determine a graded commutative non-cocommutative Hopf algebra H_{FdB} known as *Faa di Bruno Hopf algebra*. It is the Hopf algebra of functions on the opposite of the group G_{FdB} defined by the polynomial ring in variables

$$Y_i : G_{\text{FdB}} \rightarrow \mathbb{K}, \hbar + \sum_{n \geq 1} a_n g_\lambda^{n+1} \mapsto a_i \quad (72)$$

as homogeneous functionals of degree i , with the coproduct $\Delta(f)(P \otimes Q) = f(Q \circ P)$. The element $Y = 1 + \sum_{n=1}^{\infty} Y_n$ belongs to $\overline{H_{\text{FdB}}}$ completed by the n -adic metric d_{adic} given by

$$\forall u, v \in H_{\text{FdB}} : d_{\text{adic}}(u, v) = 2^{-\text{val}(u-v)}, \text{val}(w) = \max\{n \in \mathbb{N} : w \in \bigoplus_{k \geq n} H_{\text{FdB}}^{(k)}\}. \quad (73)$$

[43].

On the other hand, the terms $\{X_n\}_{n \geq 0}$ of the solution $X_{\text{DSE}} = \sum_{n \geq 0} g_\lambda^n X_n$ of any combinatorial Dyson–Schwinger equation (34) determines a connected graded commutative Hopf subalgebra H_{DSE} of $H_{\text{FG}}(\Phi)$, which is isomorphic to the Faa di Bruno Hopf algebra [40,43,57].

The complex Lie group $\mathbb{G}_\Phi(\mathbb{C}) = \text{Hom}(H_{\text{FG}}(\Phi), \mathbb{C})$ encodes Feynman rules of the physical theory [41,45,53,54]. The quotient Hopf algebra $H_{\text{FG}}(\Phi)/H_{\text{DSE}}$ determines a Lie subgroup of $\mathbb{G}_\Phi(\mathbb{C})$ [37]. This allows us to restrict the homomorphism Ξ_Φ given by **Step I** to a new injective homomorphism $\Xi_{\Phi, \text{DSE}}$ from H_{DSE} to $H_{\text{CK}}(\Phi)$ which is lifted onto an injective homomorphism of Banach spaces. It projects $H_{\text{DSE}}^{\text{cut}}$ to $H_{\text{DSE,CK}}^{\text{cut}} := \Xi_{\Phi, \text{DSE}}(H_{\text{DSE}}^{\text{cut}})$.

Step III. We show that the universal non-perturbative topos is non-boolean. The spectral presheave in $\mathbf{T}_U$ is given by

$$\underline{\Sigma}^U : H^{\text{cut}} \mapsto \text{Hom}(H^{\text{cut}}, \mathbb{C}) = \left\{ \phi : H^{\text{cut}} \rightarrow \mathbb{C} : \phi(\Gamma_1 \Gamma_2) = \phi(\Gamma_1) \phi(\Gamma_2), \phi(\mathbb{I}) = 1 \right\} \quad (74)$$

as the complex Lie group of characters on H^{cut} . The group structure is defined in terms of the coproduct structure of H^{cut} . For $H_1^{\text{cut}} \leq H_2^{\text{cut}} \in \mathcal{C}_U$, the injective morphism $i : H_1^{\text{cut}} \rightarrow H_2^{\text{cut}}$ can be lifted onto the surjective group homomorphism $\tilde{i} : \text{Hom}(H_2^{\text{cut}}, \mathbb{C}) \rightarrow \text{Hom}(H_1^{\text{cut}}, \mathbb{C})$. It leads us to the morphism $\underline{\Sigma}^U(H_2^{\text{cut}}) \rightarrow \underline{\Sigma}^U(H_1^{\text{cut}})$ in $\text{mor}(\mathcal{C}_U)$.

The outer presheave in $\mathbf{T}_U$ is given by

$$\underline{Q}^U : H^{\text{cut}} \mapsto \text{In}(H^{\text{cut}}) = \left\{ \psi : H^{\text{cut}} \rightarrow \mathbb{C} : \psi \text{ linear}, \psi(t_1 t_2) = \psi(t_1) \varepsilon(t_2) + \varepsilon(t_1) \psi(t_2) \right\} \quad (75)$$

as the complex algebra of infinitesimal characters. The algebra structure is defined in terms of the coproduct structure of H^{cut} . For $H_1^{\text{cut}} \leq H_2^{\text{cut}} \in \mathcal{C}_U$, the injective morphism $i : H_1^{\text{cut}} \rightarrow H_2^{\text{cut}}$ can be lifted onto the surjective group homomorphism $\hat{i} : \text{In}(H_2^{\text{cut}}) \rightarrow \text{In}(H_1^{\text{cut}})$. It leads us to the morphism $\underline{Q}^U(H_2^{\text{cut}}) \rightarrow \underline{Q}^U(H_1^{\text{cut}})$ in $\text{mor}(\mathcal{C}_U)$.

The terminal object in $\mathbf{T}_U$ is given by

$$\mathbf{1}^U : \mathcal{C}_U \rightarrow \mathbf{Set}, \mathbf{1}^U(H^{\text{cut}}) = \{*\}, \mathbf{1}^U(H_1^{\text{cut}} \rightarrow H_2^{\text{cut}}) = \text{id} : \{*\} \rightarrow \{*\}. \quad (76)$$

For any $H^{\text{cut}} \in \mathcal{C}_U$, a sieve on H^{cut} is a set $s_{H^{\text{cut}}}$ of morphisms $f : H_1^{\text{cut}} \rightarrow H^{\text{cut}} \in \text{mor}(\mathcal{C}_U)$ with this property that for any $h : H_2^{\text{cut}} \rightarrow H_1^{\text{cut}} \in \text{mor}(\mathcal{C}_U)$, $f \circ h \in s_{H^{\text{cut}}}$. The subobject classifier Ω_U in $\mathbf{T}_U$ maps each $H^{\text{cut}} \in \mathcal{C}_U$ to the collection of all its sieves. For any morphism $f : H_1^{\text{cut}} \rightarrow H^{\text{cut}} \in \text{mor}(\mathcal{C}_U)$,

$$\begin{aligned} \Omega_U(f) : \Omega_U(H^{\text{cut}}) &\rightarrow \Omega_U(H_1^{\text{cut}}), \forall s_{H^{\text{cut}}} \in \Omega_U(H^{\text{cut}}) : \\ \Omega_U(f)(s_{H^{\text{cut}}}) &= \left\{ k : H_2^{\text{cut}} \rightarrow H_1^{\text{cut}} \in \Omega_U(H_1^{\text{cut}}) : f \circ k \in s_{H^{\text{cut}}} \right\}. \end{aligned} \quad (77)$$

For any $H^{\text{cut}} \in \text{obj}(\mathcal{C}_U)$, the inclusion introduces a poset structure on $\Omega_U(H^{\text{cut}})$ given by

$$s_{H^{\text{cut}}} \preceq t_{H^{\text{cut}}} \Leftrightarrow s_{H^{\text{cut}}} \subseteq t_{H^{\text{cut}}} , \quad (78)$$

such that the empty sieve is the zero element and the principal sieve

$$\uparrow H^{\text{cut}} := \left\{ h : H_1^{\text{cut}} \rightarrow H^{\text{cut}} \in \text{mor}(\mathcal{C}_U) \right\} \quad (79)$$

is the unit element of this poset. Define the following algebraic operations on $\Omega_U(H^{\text{cut}})$.

$$s_{H^{\text{cut}}} \wedge t_{H^{\text{cut}}} := s_{H^{\text{cut}}} \cap t_{H^{\text{cut}}} , \quad s_{H^{\text{cut}}} \vee t_{H^{\text{cut}}} := s_{H^{\text{cut}}} \cup t_{H^{\text{cut}}} , \quad (80)$$

$$s_{H^{\text{cut}}} \Rightarrow t_{H^{\text{cut}}} := \quad (81)$$

$$\left\{ f : H_2^{\text{cut}} \rightarrow H^{\text{cut}} \in \text{mor}(\mathcal{C}_U) \mid \forall g : H_3^{\text{cut}} \rightarrow H_2^{\text{cut}} \in \text{mor}(\mathcal{C}_U) , \text{ if } f \circ g \in s_{H^{\text{cut}}} , \text{ then } f \circ g \in t_{H^{\text{cut}}} \right\} ,$$

$$\neg s_{H^{\text{cut}}} := \left\{ f : H_2^{\text{cut}} \rightarrow H^{\text{cut}} \in \text{mor}(\mathcal{C}_U) \mid \forall g : H_3^{\text{cut}} \rightarrow H_2^{\text{cut}} \in \text{mor}(\mathcal{C}_U) , f \circ g \notin s_{H^{\text{cut}}} \right\} . \quad (82)$$

According to [14,24,31,60,61], $\Omega_U(H^{\text{cut}})$ is equipped with a dimensionally computable Heyting algebra.

Step IV. We apply Definitions 11 and 14 together with [9,24,25,31,32] to show that non-perturbative topoi of gauge field theories are embedded as sub-topoi into the universal non-perturbative topos.

For the non-perturbative topos $\mathbf{T}_\Phi^{\text{non},g}$, we project objects in $\mathcal{C}_\Phi^{\text{non},g}$ onto some objects in $\mathcal{C}_U$. This introduces a surjective covariant functor $F_\Phi : \mathcal{C}_\Phi \rightarrow \mathcal{C}_\Phi^{\text{non},g}$. For any contravariant functor $\alpha_\Phi : \mathcal{C}_\Phi^{\text{non},g} \rightarrow \mathbf{Set} \in \text{obj}(\mathbf{T}_\Phi^{\text{non},g})$, we get the contravariant functor $\alpha_\Phi \circ F_\Phi : \mathcal{C}_U \rightarrow \mathbf{Set} \in \text{obj}(\mathbf{T}_U)$. This means that

$$\underline{\Sigma}^\Phi \circ F_\Phi \mapsto \underline{\Sigma}^U , \quad \underline{\Omega}^\Phi \circ F_\Phi \mapsto \underline{\Omega}^U , \quad \Omega_\Phi^{\text{non},g} \circ F_\Phi \mapsto \Omega_U . \quad (83)$$

Therefore we can formulate a surjective functor $\tilde{F}_\Phi : \mathbf{T}_U \rightarrow \mathbf{T}_\Phi^{\text{non},g}$ between topoi such that $\tilde{F}_\Phi^{-1}(\mathbf{T}_\Phi^{\text{non},g})$ is a sub-topos of $\mathbf{T}_U$.

Step V. We show that the universal non-perturbative topos has representations of non-perturbative topoi of gauge field theories over a category algebra of random graphs.

$\mathbf{T}_U$ provides a representation of the propositional language of the non-perturbative topos $\mathbf{T}_\Phi^{\text{non},g}$ of any gauge field theory Φ . For any $H^{\text{cut}} \in \text{obj}(\mathcal{C}_U)$, $\Omega_U(H^{\text{cut}})$ is a bi-Heyting algebra with respect to the second negation

$$\sim s_{H^{\text{cut}}} :=$$

$$\left\{ f : H_2^{\text{cut}} \rightarrow H^{\text{cut}} \in \text{mor}(\mathcal{C}_U) \mid \exists g : H_3^{\text{cut}} \rightarrow H_2^{\text{cut}} \in \text{mor}(\mathcal{C}_U) , f \circ g \notin s_{H^{\text{cut}}} \right\} \quad (84)$$

for all $s_{H^{\text{cut}}} \in \Omega_U(H^{\text{cut}})$.

Therefore the dimensionally computable Heyting algebra of $\mathbf{T}_U$ recovers truth values for the neo-realist evaluation of no-go theorems in quantum field theory. The propositional language of $\mathbf{T}_U$ governs propositional calculi for a neo-realist reconstruction of quantum field theory independent of the standard Hilbert-space/operator-algebra ontology. This reconstruction program is also theory-independent and instrument-independent because its building blocks have no need for instrumental or measurement tools to be identified. $\square$

5. Conclusions

This article discussed formal objectivity in gauge field theories and their non-perturbative structures in terms of the mathematical universe of the non-perturbative topos. This particular non-Boolean topos recovers a globally and locally neo-realist description of interacting physical theories beyond quantum mechanics. The universal version of the non-perturbative topos has been built to recover formal objectivity of quantum field theory. The universal non-perturbative topos, as a non-Boolean topos, is theory-independent and instrumentalist-probability-independent. Its formal objectivity encodes real objectivity of gauge field theories and their non-perturbative structures.

- The category of contexts of the non-perturbative topos and its universal version are defined by a certain family of topological Hopf subalgebras of the topological Hopf algebra of renormalization. Therefore formal objectivity of these non-Boolean topoi almost eliminates the implicit reliance on the standard Hilbert-space/operator-algebra ontology.
- Cut-distance topological subsets of Feynman diagrams and their Feynman graph limits represented by stretched Feynman graphons, as mathematical objects, are building blocks of the non-perturbative topos. No instrumentalist probabilities are needed to define or identify elements of this neo-realist framework.
- The non-perturbative renormalization built from the topological Hopf algebra of renormalization together with Heyting spaces corresponding to renormalization bi-Heyting algebra [36,38] show that the mathematical universe of the non-perturbative topos can assign definite computable, renormalized values to propositions about gauge field theories and their non-perturbative structures. In other words, the approach of the non-perturbative topos makes real numbers secondary mathematical tools.
- The deviation of the Heyting algebra of the universal non-perturbative topos from the Heyting algebra of a quantum topos show that the method of classical Boolean snapshots fails to recover non-perturbative structures. In other words, quantum field theory cannot be fully described by any formal model locally built from classical Boolean snapshots.
- Real objectivity of non-perturbative structures is describable only by neo-realist truth values determined by global and local neo-realist topos models.

5.1. Formal Objectivity of the Universal Non-Perturbative Topos Answers to Doring–Isham’s Questions

- The mathematical universe of the non-perturbative topos replaces instrumentalist probabilities with cut-distance topological regions of Feynman diagrams and their Feynman graph limits. Quantities in quantum field theory can be assigned to stretched Feynman graphons or their renormalized values as definite computable numbers. Therefore formal objectivity of the universal non-perturbative topos interprets quantum field theory in a neo-realist manner with no need for instrumentalist probabilities.
- The renormalization Hopf algebraic approach has already been formulated and developed for quantum gravity [64,65]. This allows us to formulate the non-perturbative topos for quantum gravity. Therefore the mathematical universe of the universal non-perturbative topos recovers a theory for quantum gravity independent of the standard Hilbert-space/operator-algebra ontology.
- Formal objectivity of the universal non-perturbative topos is theory-independent and instrument-independent. Therefore it seems its non-Boolean mathematical universe is rich enough to deal with conceptual issues of a theory for quantum gravity.
- Formal objectivity of the non-perturbative topos shows that a theory of quantum gravity is a globally neo-realist theory. It is shown that while the renormalization bi-Heyting algebra of quantum field theory is non-Boolean, the renormalization bi-

Heyting algebra of a perturbative theory of quantum gravity is Boolean [36]. This fact tells us that there is a chance that a perturbative theory of quantum gravity becomes locally realistic with no need for instrumentalism.

- The deviation of the Heyting algebra of the universal non-perturbative topos from the Heyting algebra of the quantum topos means that concepts of quantum ideas, formulated by the usual mathematical apparatus of quantum theory, fail to be expanded to interacting physical theories beyond quantum mechanics.
- Formal objectivity of the non-perturbative topos clarifies that there is no route to a realist form of theories beyond quantum mechanics. In addition, the Heyting algebra of the non-perturbative topos recovers truth values of no-go theorems.
- A mathematician mathematical physicist has no role in real objectivity generated by formal objectivity of the universal non-perturbative topos.

5.2. Applications of This New Formal Objectivity to Real Objectivity

- Triviality or the “zero charge problem” considers the analysis of the behavior of an interacting theory beyond quantum mechanics in terms of relating it to an isolated version of the theory with no interacting. Triviality opens the possibility of switching an interacting theory and its non-interacting renormalized version whenever a regularization parameter is removed by some limiting procedures. Experimental data supported the appearance of triviality in the Landau pole of quantum electrodynamics at a high-energy scale and the Landau pole quantum chromodynamics at a low-energy scale. The Gell-Mann–Low equation encapsulates triviality by a certain integral equation with respect to the beta function of a renormalization group program. Thanks to the method of stretched (Feynman) graphons, the asymptotics of the Gell-Mann–Low equation and its corrections have been modeled in terms of discrete-time Markov chains of random operators [51]. This achievement addressed a new stochastic platform for the description of non-perturbative structures in physical theories. Thanks to the formalism of the non-perturbative topos, there exists a stochastic platform to relocate formal objectivity of the non-perturbative topos to real objectivity of physical theories. In other words, it is possible to formulate computational models for the computation of the asymptotics of non-perturbative structures of physical theories. Neo-realist truth values of intermediate algorithms are determined by the Heyting algebra of the non-perturbative topos of the physical theory.
- The non-locality of interacting theories beyond quantum mechanics has been studied in the context of the topological Hopf algebra of renormalization. Space-time regions are decorated by a certain family of topological Hopf subalgebras to explain the existence of new non-trivial correlations between those space-time regions. Quantum entanglement of particles in quantum field theory is modeled in terms of lattices of topological Hopf subalgebras in this regard [37]. This approach leads us to a new theory of computation for the extraction of data controlled by quantum entanglement of particles in non-perturbative structures [37,38]. The structure of the non-perturbative topos shows that its internal logic recovers that theory of computation. In other words, the mathematical universe of the non-perturbative topos of a gauge field theory can build consistent computational algorithms to generate meaningful data from quantum entanglement in non-perturbative structures.

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