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A Comparative Analysis of a Nonlinear Phase Space Evolution of SU(2) and SU(1,1) Coherent States

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Abstract

We carried out a comparative study of the phase space evolution of SU(2) and SU(1,1) coherent states generated by the same nonlinear two-mode Hamiltonian. We analyze the dynamics of the Wigner functions in the respective phase spaces and discuss the principal associated physical effects: the squeezing of the appropriate observables and the Schrödinger's cat state generation characteristic of both the considered symmetry groups.

Keywords: phase space; Wigner function; SU(2); SU(1,1); coherent states

1. Introduction

Identifying the dynamical symmetry group of a physical system provides profound insights into both its kinematic and dynamical structures' evolution. In particular, the symmetry group associated with the Hamiltonian naturally suggests an appropriate phase space representation, wherein key aspects of quantum dynamics may be visualized and analyzed.

However, in composite systems, i.e., systems constituted by multiple interacting subsystems, the Hamiltonian can often be expressed in terms of generators belonging to different Lie algebras. In such cases, the feasibility of a phase space analysis [1] critically depends on the symmetry properties of the initial state. In general, a faithful phase space representation is only possible if the initial (pure) state (or each component of the statistical mixture) belongs to a single irreducible representation of the relevant dynamical group. If the state spans multiple irreducible components, the associated phase space formulation becomes ambiguous or ill-defined. Therefore, the phase space representation is intrinsically tied to the structure of the subspace in which the system's initial state is embedded.

Situations in which the Hamiltonian supports multiple dynamical symmetry groups commonly arise in interacting multi-mode systems. These scenarios have been extensively studied, revealing a rich interplay of physical effects governed by distinct symmetries [2–14].

In this paper, we discuss the phase space dynamics of quantum states possessing different symmetries, governed by the same nonlinear two-mode Hamiltonian, which can be equivalently expressed in terms of either SU(2) or SU(1,1) generators. We compare the time evolution of the SU(2) and SU(1,1) coherent states [15,16] in their respective classical phase spaces, namely, the two-sphere and hyperboloid phase spaces.

Our analysis focuses on the similarities and differences in their Wigner function evolution, with a particular emphasis on squeezing behavior and the emergence of Schrödinger's



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cat-like states. These quantum features manifest differently in compact (SU(2)) versus non-compact (SU(1,1)) settings, thereby providing a compelling contrast between the unitary dynamics on bounded and unbounded phase space manifolds.

In Section 2, we introduce the model Hamiltonian and define the initial states confined to the SU(2) and SU(1,1) irreducible subspaces. Section 3 presents a review of the Wigner phase space mappings corresponding to the spherical and hyperbolic geometries, along with their essential properties. In Section 4, we analyze the compact phase space dynamics for both types of symmetry. Finally, Section 5 outlines the behavior of non-compact SU(1,1) dynamics within the hyperbolic phase space.

2. The Model

2.1. The Interaction Hamiltonian

A paradigmatic scenario where the symmetry of the system is dictated by its initial state is provided by the Kerr-type interactions of two bosonic modes. The corresponding Hamiltonian in the interaction picture reads

$$\hat{H} = \gamma_1(\hat{a}^\dagger \hat{a})^2 + \gamma_2(\hat{b}^\dagger \hat{b})^2 + \gamma_3 \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b}, \quad (1)$$

where $\hat{a}$ and $\hat{b}$ are bosonic annihilation operators, i.e., the generators of the Heisenberg–Weyl algebra $H(1)$, satisfying

$$[\hat{a}, \hat{a}^\dagger] = 1, \quad [\hat{b}, \hat{b}^\dagger] = 1, \quad [\hat{a}, \hat{b}] = [\hat{a}, \hat{b}^\dagger] = 0.$$

Hamiltonian (1) comprises both self-Kerr (intramode) nonlinearities in each bosonic mode, leading to photon-number-dependent phase shifts within individual modes, and a cross-Kerr (intermode) nonlinearity that couples the two modes and produces intermode correlations, which are employed in quantum non-demolition measurements, photon blockades, cat-state stabilization, and quantum logic operations. These properties have warranted the implementation of Hamiltonian (1) across multiple physical platforms, with applications including quantum photonic gates involving photon–photon interactions [17], circuit QED platforms with superconducting cavities [18], cross-Kerr interactions in trapped-ion systems [19], photon blockade and cat-state generation in optomechanics [20], and cross-Kerr interactions plus parametric amplification [21].

The natural symmetry group of Hamiltonian (1) is $H(1) \times H(1)$, with $H(1)$ being the Heisenberg–Weyl group. However, under the condition that $\gamma_1 = \gamma_2$, the Hamiltonian admits an additional permutational symmetry, $\hat{a} \longleftrightarrow \hat{b}$, i.e.,

$$\hat{H} = \gamma_1 \left((\hat{a}^\dagger \hat{a})^2 + (\hat{b}^\dagger \hat{b})^2 \right) + \gamma_3 \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b}, \quad (2)$$

allowing us to recast it in terms of the generators of either the SU(2) or SU(1,1) Lie groups.

For SU(2), whose generators satisfy the compact Lie algebra $[\hat{S}_k, \hat{S}_l] = i\epsilon_{klm}\hat{S}_m$, we define

$$\hat{S}_1 = \frac{1}{2}(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}), \quad \hat{S}_2 = \frac{1}{2i}(\hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a}), \quad (3)$$

$$\hat{S}_3 = \frac{1}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}), \quad \hat{N} = \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}, \quad [\hat{N}, \hat{S}_k] = 0. \quad (4)$$

The Hamiltonian can then be rewritten as

$$\hat{H} = \alpha_1 \hat{S}_3^2 + \alpha_2 \hat{N}^2, \quad (5)$$

where α_1 and α_2 are constants determined by γ_1 and γ_2 .

Alternatively, using the $SU(1,1)$ generators, which satisfy the non-compact algebra

$$[\hat{K}_1, \hat{K}_2] = -i\hat{K}_0, \quad [\hat{K}_2, \hat{K}_0] = i\hat{K}_1, \quad [\hat{K}_0, \hat{K}_1] = i\hat{K}_2,$$

we define

$$\hat{K}_1 = \frac{1}{2}(\hat{a}^\dagger \hat{b}^\dagger + \hat{a} \hat{b}), \quad \hat{K}_2 = \frac{1}{2i}(\hat{a}^\dagger \hat{b}^\dagger - \hat{a} \hat{b}), \quad (6)$$

$$\hat{K}_0 = \frac{1}{2}(\hat{a}^\dagger \hat{a} + \hat{b} \hat{b}^\dagger), \quad \hat{v} = \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}, \quad [\hat{v}, \hat{K}_1] = 0, \quad (7)$$

resulting in the Hamiltonian representation

$$\hat{H} = \beta_1 \hat{K}_0^2 + \beta_2 \hat{v}^2. \quad (8)$$

The time evolution of an arbitrary initial state,

$$|\psi\rangle = \sum_{n,l} c_{nl} |n\rangle_a |l\rangle_b$$

can be analyzed in either representation by expanding the state into irreducible subspaces of $SU(2)$ or $SU(1,1)$.

For the $SU(2)$ decomposition one has

$$|\psi\rangle = \sum_{S=0,1/2,1,\dots} \sum_{m=-S}^S c_{S+m,S-m} |S, m\rangle_{SU(2)}, \quad (9)$$

where $|S, m\rangle_{SU(2)} = |S+m\rangle_a |S-m\rangle_b$, and

$$\hat{S}^2 = \hat{S}_1^2 + \hat{S}_2^2 + \hat{S}_3^2 = \frac{\hat{N}}{2} \left(\frac{\hat{N}}{2} + 1 \right), \quad (10)$$

$$\hat{N}|S, m\rangle = 2S|S, m\rangle, \quad \hat{S}_3|S, m\rangle = m|S, m\rangle. \quad (11)$$

The parameter summation index S in (9) defines the size of the effective spin in each $SU(2)$ invariant subspace (with a fixed total photon number) spanned by the vectors $\{|S, m\rangle, m = -S, \dots, S\}$.

For the $SU(1,1)$ decomposition,

$$|\psi\rangle = \sum_{k=\pm\frac{1}{2}, \pm 1, \dots} \sum_{m=0}^{\infty} c_{|2k+m-1|, m} |k, k+m\rangle_{SU(1,1)},$$

with $|k, k+m\rangle = ||2k+m-1\rangle_a \otimes |m\rangle_b$, and

$$\hat{\mathbf{K}}^2 = \hat{K}_0^2 - \hat{K}_1^2 - \hat{K}_2^2 = \frac{1}{4}(\hat{v}^2 - 1), \quad (12)$$

$$\hat{\mathbf{K}}^2 |k, k+m\rangle = k(k-1) |k, k+m\rangle, \quad (13)$$

$$\hat{K}_0 |k, k+m\rangle = (k+m) |k, k+m\rangle. \quad (14)$$

The index k (the Bargmann index), setting the difference between the number of photons in both modes according to (7), labels the irreducible representations of the $SU(1,1)$ group.

2.2. Group-Theoretical Equivalence and Structural Asymmetry

Clearly, any two-mode state can be represented as a distribution in the manifold constructed by a direct product of two complex planes, corresponding to the $H(1) \times H(1)$ symmetry. However, the analysis of the quantum evolution generated by Hamiltonian (2) in such a four-dimensional composite phase space is far from being visualizable and interpretable from a classical perspective.

A suitable phase space representation of the state dynamics is well-defined only when the state initially belongs to a single irreducible representation of either SU(2) or SU(1,1). Accordingly, the group symmetry of the state [22] dictates whether it is mapped onto a distribution on the two-sphere or hyperbolic planes, which are the classical phase spaces of the SU(2) and SU(1,1) groups, respectively.

Due to the covariance of the phase space maps under symmetry group actions, the phase space evolution generated by Hamiltonians related to group transformations remains equivalent. However, a crucial structural distinction emerges between the SU(2) and SU(1,1) groups, i.e., between spin and pseudo-spin systems.

In SU(2), all the generators are compact, and group transformations (i.e., inner automorphisms) can map any generator onto an arbitrary linear combination of the others. In particular, Hamiltonians such as

$$\hat{H} = \alpha_1 \hat{S}_2^2 + \alpha_2 \hat{N}^2 \quad (15)$$

are SU(2)-equivalent to (5) and yield identical dynamics in the phase space, modulo a rotation of the coordinate system.

In contrast, SU(1,1) is a non-compact group with a fundamental asymmetry among its generators. Only $\hat{K}_0$ is compact and possesses a discrete spectrum, while $\hat{K}_1$ and $\hat{K}_2$ are non-compact and generate continuous spectra. Consequently, it is not possible to transform $\hat{K}_0$ into $\hat{K}_1$ or $\hat{K}_2$ through SU(1,1) operations. This means that, for example, the Hamiltonian

$$\hat{H} = \beta_1 \hat{K}_2^2 + \beta_2 \hat{v}^2 \quad (16)$$

is not equivalent to (8) under SU(1,1) transformations. Therefore, the dynamics generated by (8) and (16) are fundamentally distinct, even though they may share formal similarities in their structures. This distinction addresses the observation raised regarding the interpretation of Equations (15) and (16): while (15) and (5) are group-equivalent within SU(2), the same cannot be said of (8) and (16) within SU(1,1). This lack of internal equivalence among the generators in non-compact groups such as SU(1,1) must be taken into account when interpreting Hamiltonian symmetries and their corresponding phase space dynamics.

Accordingly, in what follows, we will focus on comparing the time evolution of SU(1,1) coherent states under Hamiltonians (8) and (16), as these give rise to qualitatively different behaviors that are not connected by symmetry transformations [15].

2.3. Initial States

The SU(2) coherent states in an invariant subspace spanned by the basis vectors in (11) are defined as [15]

$$|\xi\rangle = (1 + |\xi|^2)^{-S} \sum_{m=0}^{2S} \left(\frac{(2S)!}{m!(2S-m)!} \right)^{1/2} \xi^m |S, m\rangle, \quad (17)$$

where

$$\xi = e^{-i\varphi} \tan \frac{\theta}{2}, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi, \quad (18)$$

and $\hat{S}_{\pm} = \hat{S}_1 \pm i\hat{S}_2$. The states in (17) are in the orbit of the state $|S, S\rangle_{SU(2)} = |2S\rangle_a |0\rangle_b$ generated by Hamiltonians linear in (3) and (4), i.e., preserving the sum of the excitations, $N = 2S$, in both physical modes.

In the case of the SU(1,1) group, we will consider discrete positive series defined by the Bargmann index $k = \frac{1}{2}, 1, \frac{3}{2}, \dots$ (physically fixing the most and least excited modes), where the SU(1,1) coherent states are [15]

$$|\zeta\rangle = (1 - |\zeta|^2)^k \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+2k)}{m!\Gamma(2k)} \right)^{1/2} \zeta^m |k, k+m\rangle, \quad (19)$$

where

$$\zeta = e^{-i\phi} \tanh \frac{\tau}{2}, \quad 0 \leq \tau < \infty, \quad 0 \leq \phi \leq 2\pi,$$

and $\hat{K}_{\pm} = \hat{K}_1 \pm i\hat{K}_2$, with $|k, k+m\rangle$ used as a basis in (13) and (14). The states in (19) are obtained by shifting the state $|k, k\rangle_{SU(1,1)} = |2k-1\rangle_a |0\rangle_b$ using a unitary operator generated by Hamiltonians linear in (6) and (7), i.e., preserving the difference in the number of excitations between the modes.

3. Phase Space Representations

We will employ the so-called Wigner map for the representation of quantum states in their respective classical phase spaces. The Wigner mapping establishes a correspondence between quantum operators acting within a fixed invariant subspace of the dynamical symmetry group associated with the Hamiltonian and functions (symbols) defined over the corresponding classical phase space. $\mathcal{M}$. This correspondence is implemented using a quantization kernel, $\hat{w}(\Omega)$, which encodes the following mapping rule:

$$\hat{f} \iff W_f(\Omega) = \text{Tr}(\hat{f} \hat{w}(\Omega)), \quad \Omega \in \mathcal{M}, \quad (20)$$

where $W_f(\Omega)$ is the phase space symbol associated with the operator $\hat{f}$, and $\hat{w}(\Omega)$ is a fixed operator-valued distribution defined over $\mathcal{M}$, often referred to as the Stratonovich–Weyl kernel. This kernel is required to satisfy the standard set of Stratonovich–Weyl axioms: normalization, covariance under the symmetry group actions, and self-duality [23–26].

In the special case where $\hat{f} = \hat{\rho}$ is the density operator of the system, the symbol $W_{\rho}(\Omega)$ becomes the Wigner function. The expectation value of any observable $\hat{f}$ can then be expressed as a phase space integral:

$$\langle \hat{f} \rangle = \int_{\mathcal{M}} d\mu(\Omega) W_f(\Omega) W_{\rho}(\Omega),$$

where $d\mu(\Omega)$ denotes the invariant measure on the phase space $\mathcal{M}$, ensuring consistency with the underlying group-theoretical structure.

3.1. The SU(2) Wigner Function

The SU(2) Wigner map of operators acting in an invariant subspace and distributions on the two-dimensional sphere S_2 [23,26,27] is defined using the quantization kernel

$$\hat{w}(\theta, \phi) = \sqrt{\frac{4\pi}{2S+1}} \sum_{L=0}^{2S} \sum_{M=-L}^L \hat{T}_{LM}^{(S)\dagger} Y_{LM}(\theta, \phi), \quad (\theta, \phi) \in S_2, \quad (21)$$

where $\hat{T}_{LM}^{(S)}$ are irreducible tensor operators,

$$\hat{T}_{LM}^{(S)} = \sqrt{\frac{2L+1}{2S+1}} \sum_{m, m'=-S}^S C_{Sm, LM}^{Sm'} |S, m\rangle \langle S, m'|, \quad (22)$$

and $Y_{LM}(\theta, \phi)$ are spherical harmonics, with $C_{Sm, LM}^{Sm'}$ being the Clebsch–Gordan coefficients [28].

For instance, the Wigner symbol of a (non-Hermitian) operator, $|\xi_1\rangle \langle \xi_2|$, where $|\xi\rangle$ is a coherent state (17), has the form

$$W_{\xi_1, \xi_2}(\zeta) = \left(\frac{(1 + \xi_1 \bar{\xi}_2^*)^2}{(1 + |\xi_1|^2)(1 + |\xi_2|^2)} \right)^S \sum_{l=0}^{2S} \frac{2l+1}{2S+1} C_{SS, l0}^{SS} P_l \left(\frac{\bar{\xi}_1 \bar{\xi}_2^* - 1}{\bar{\xi}_1 \bar{\xi}_2^* + 1} \right), \quad (23)$$

where $\bar{\xi}_i = (\xi_i - \zeta)/(1 + \xi_i^* \zeta)$ and the phase space points are labeled by the complex parameter in (18). Thus, the Wigner function of a coherent state, $|\xi_0\rangle$, is

$$W_{\zeta_0}(\zeta) = \frac{(2S)!}{\sqrt{2S+1}} \sum_{l=0}^{2S} \frac{2l+1}{\sqrt{(2S-l)!(2S+l+1)!}} P_l(\cos \omega), \quad (24)$$

where $\cos \omega = \cos \theta \cos \theta_0 + \sin \theta \sin \theta_0 \cos(\phi - \phi_0)$.

3.2. The SU(1,1) Wigner Function

The SU(1,1) quantization kernel, determining the Wigner map (20) of the operators acting in an invariant subspace spanning from elements of the basis in (13) and (14), with $k = \frac{1}{2}, 1, \frac{3}{2}, \dots$, to the upper sheet of a two-sheet hyperboloid, $\mathbb{H}^2$, is convenient to define as an integral transformation of an operator, $\hat{w}^{(+)}(\zeta)$, $\zeta \in \mathbb{H}^2$, fixed by its expectation value in the coherent states (19) [29]:

$$\langle \zeta' | \hat{w}^{(+)}(\zeta) | \zeta' \rangle = \delta(\zeta, \zeta') = \delta(\cosh \tau - \cosh \tau') \delta(\phi - \phi'),$$

as follows:

$$\hat{w}(\zeta) = \frac{2}{2k-1} \int d\mu(\zeta') g_k(\zeta'^{-1}\zeta) \hat{w}^{(+)}(\zeta'), \quad (25)$$

$$g_k(\zeta) = \int_0^\infty d\lambda \lambda \tanh(\pi\lambda) \Phi_k^{1/2}(\lambda) P_{-\frac{1}{2}+i\lambda}(\zeta), \quad (26)$$

where $\zeta'^{-1}\zeta = \cosh \tau \cosh \tau' - \cos(\phi - \phi') \sinh \tau \sinh \tau'$.

$$\Phi_k(\lambda) = \frac{(2k-1)|\Gamma(2k-\frac{1}{2}+i\lambda)|^2}{\Gamma^2(2k)}, \quad (27)$$

where $P_{-\frac{1}{2}+i\lambda}(\zeta)$ is the conic function [30], and the invariant measure on $\mathbb{H}^2$ is $d\mu(\zeta) = (1 - |\zeta|^2)^{-2} d^2\zeta$.

In a way similar to that in the SU(2) case, we obtain the symbol of $|\zeta_1\rangle\langle\zeta_2|$ in the form

$$W_{\zeta_1\zeta_2}(\zeta) = \frac{2}{2k-1} \left(\frac{(1-|\zeta_1|^2)(1-|\zeta_2|^2)}{(1-\zeta_1\zeta_2^*)^2} \right)^k g_k \left(\frac{2(1-\zeta^*\zeta_1)(1-\zeta\zeta_2^*)}{(1-|\zeta|^2)(1-\zeta_1\zeta_2^*)} - 1 \right), \quad (28)$$

and thus, the Wigner function of a coherent state, $|\zeta_0\rangle$ (19), is

$$W_{\zeta_0}(\zeta) = \frac{2}{2k-1} g_k(\zeta_0^{-1}\zeta). \quad (29)$$

4. Nonlinear Evolution of Coherent States (Compact Case)

Hamiltonian (1) in the form (5) (where the commutative term $\sim \hat{N}^2$ is omitted) describes a nonlinear medium of identical two-level atoms interacting pair-wise [31–33], characterized by the generation of (angular momentum) squeezing with the subsequent production of Schrödinger's cat states, starting with a coherent state initially located at $\theta = \pi/2$, $\phi = 0$ [34–36].

The Hamiltonian in the form (8) (the term $\hat{v}^2$ is also omitted) governs the so-called compact evolution of the SU(1,1) coherent states (19) and leads to similar physical effects. To compare the phase space dynamics on the sphere and the hyperboloid, it is convenient to represent the evolved state in the integral form

$$|\psi(t)\rangle = e^{-it\hat{H}}|\mu_0\rangle = \frac{1}{\sqrt{i\pi}} \int dx e^{i(x^2-2\eta x\sqrt{t})} |\mu_0(x)\rangle, \quad \mu_0(x) = \mu_0 e^{2ix\sqrt{t}}, \quad (30)$$

where $\mu = (\zeta, \bar{\zeta})$ defines the SU(2) or SU(1,1) coherent states, and the parameter η takes the value S or k correspondingly. This allows us to obtain the general expression for the Wigner functions:

$$W_{\mu_0}(\mu, t) = \frac{1}{\pi} \int dx dy e^{i(x^2 - y^2)} e^{-2i\eta\sqrt{t}(x-y)} W_{\mu_0(x)\mu_0(y)}(\mu), \quad (31)$$

where $W_{\mu_0(x)\mu_0(y)}(\mu)$ is given in (23) and (28) with the form $\mu_1 = \mu_0(x)$, $\mu_2 = \mu_0(y)$.

In Figure 1a–f the short-time evolution of the Wigner function of the initial SU(2) and SU(1,1) coherent states on the sphere and the hyperboloid, respectively, is plotted at times scaled by the best squeezing times in each case. The well-known spin squeezing effect is observed as a compression of the Wigner distribution on the sphere and quantified by the normalized fluctuation in the spin components $(\Delta \mathbf{s}_\perp(t))^2$ on the tangent plane, orthogonal to the mean spin vector $\mathbf{n}(t) = \langle \hat{\mathbf{S}}(t) \rangle$, $\hat{\mathbf{s}}_\perp \cdot \mathbf{n}(t) = 0$, and the so-called Kitagawa–Ueda parameter, $\zeta_s^2(t) = 2(\Delta \mathbf{s}_\perp(t))^2_{\min}/S$ [34], with $\zeta_s^2(0) = 1$.

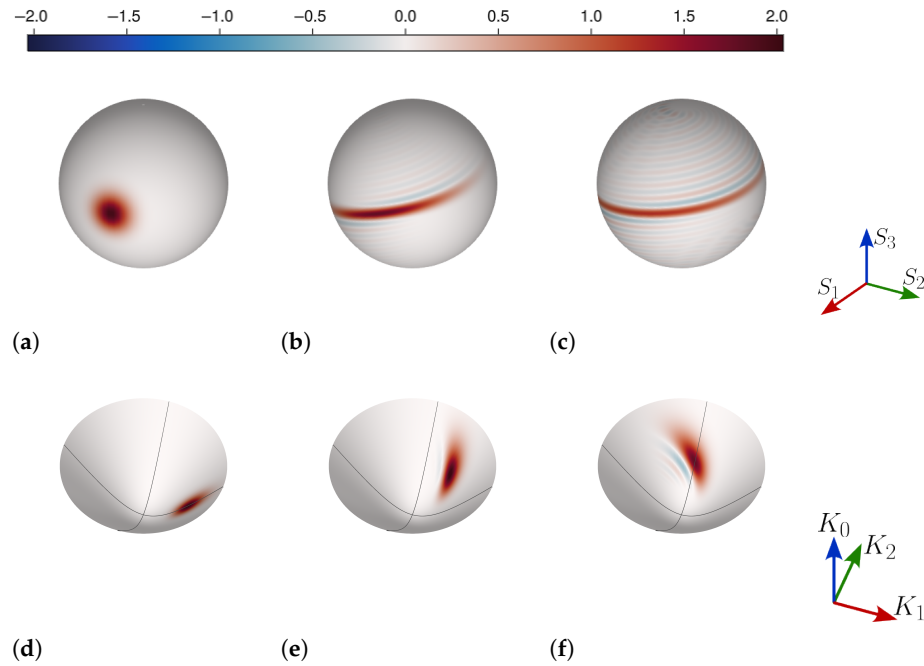


Figure 1. The Wigner functions of the evolution of the SU(2) coherent state $|\zeta = 1\rangle$ (a–c) and SU(1,1) coherent state $|\zeta = 0.5\rangle$ (d–f) according to Hamiltonian (1) at the initial moment, $t = 0$ (a,d); at the best squeezing time, $t = t_{\text{sqz}}$ (b,e); and at two times the best squeezing time, $t = 2t_{\text{sqz}}$ (c,f). The functions were plotted on the sphere and hyperboloid, with $S = 20$ and $k = 15$, respectively.

The visible deformations of the distribution in the hyperbolic phase space, which is a non-compact manifold with a negative curvature, are slightly misleading, since the fluctuations proper to the SU(1,1) coherent states are only isotropic in the tangent plane at the origin ($\zeta = 1$), i.e., only for the fundamental state $|k, k\rangle$. Thus, for an analytical computation of the squeezing parameter one should translate the center of the distribution, determined by the average pseudo-vector $\mathbf{n}(t) = \langle \hat{\mathbf{K}}(t) \rangle$, to the origin, $\mathbf{n}_0 = (0, 0, 1)$, by applying the required group transformation to the generators in (6) and (7), i.e., $\hat{\mathbf{K}}(t) \rightarrow \hat{\mathbf{k}}(t) = \hat{V}^\dagger \hat{\mathbf{K}}(t) \hat{V}$ and $\hat{V} \in SU(1,1)$, so that $\mathbf{n}_0 \cdot \langle \hat{\mathbf{k}}(t) \rangle = 0$. Then, the Kitagawa–Ueda criterion consisting of a minimization of the fluctuations in the tangent plane (see Appendix A),

$$\zeta_k^2(t) = 2(\Delta \hat{\mathbf{k}}(t))^2_{\min}/k \leq 1, \quad (32)$$

detects the SU(1,1) squeezing [37]. In Figure 2a,b the evolution of the squeezing parameter for both the SU(2) and SU(1,1) cases is plotted in the unit of the best squeezing time, where the squeezing parameter reaches its minimum value (A12). Observe the symmetric form of the SU(1,1) squeezing parameters, $\zeta_k^2(t)$ in contrast to the asymmetry of $\zeta_s^2(t)$.

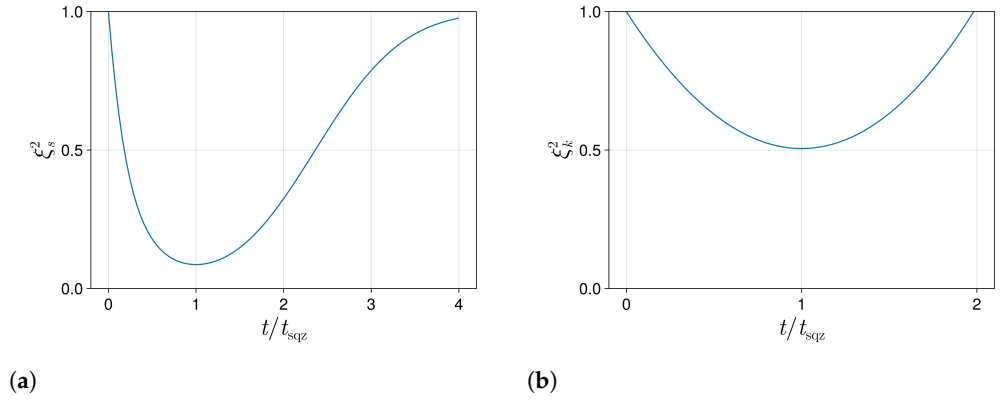


Figure 2. Evolution of (a) the fluctuations in ζ_s^2 for SU(2) and (b) ζ_k^2 for SU(1,1) in the unit of the best squeezing time.

It is instructive to compare the stereographic projections of the sphere and the hyperboloid on the plane and the Poincare disk, correspondingly, which are shown in Figure 3a,b. One can observe that the interference pattern is significantly less pronounced and has a more localized form in the case of the hyperbolic evolution.

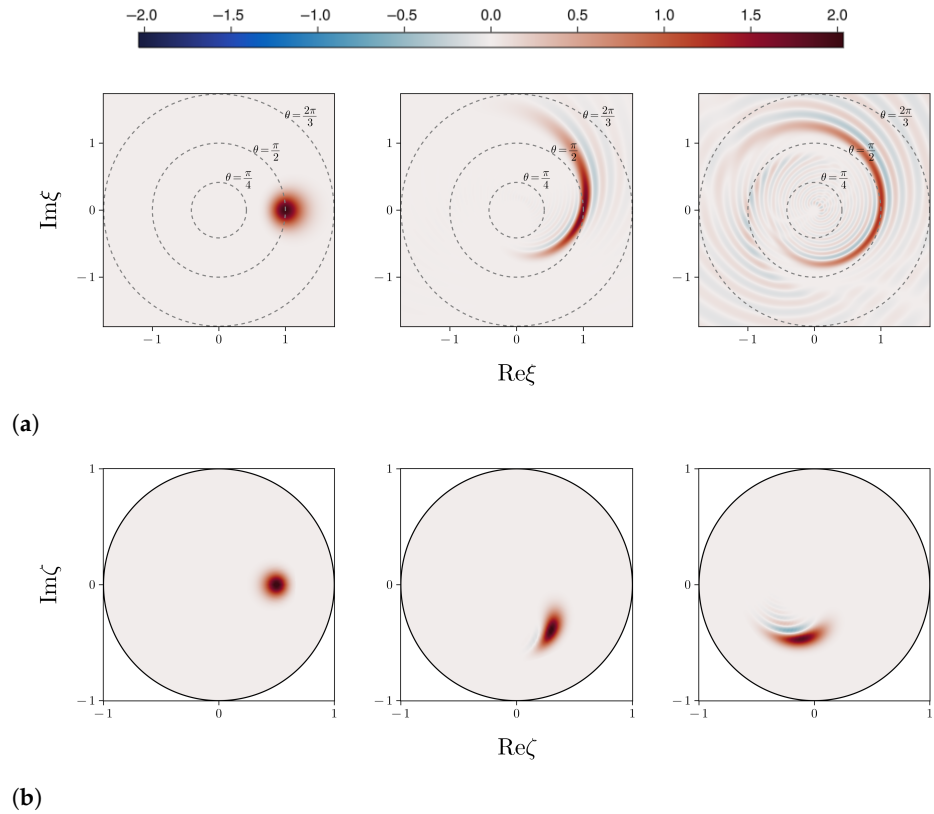


Figure 3. (a) The stereographic projection on the SU(2) Wigner function onto the 2-dim plane and (b) the stereographic projection on the SU(1,1) Wigner function onto the Poincare plane, at the same times as in Figure 1.

Over longer durations both distributions start exhibiting self-interference effects.

The discrete character of the spectra of both Hamiltonians (5) and (8) results in periodic dynamics, allowing us to analytically assess the Wigner function at $t_M = \pi/M$ and $M \in \mathbb{N}$. Making use of the finite Fourier transformation method, we arrive at the following expressions for an even M :

$$W_{\mu_0}(\mu, t_M) = M^{-1} \sum_{p=0}^{M-1} W_{\mu_p}(\mu) + 2M^{-1} \sum_{\substack{p,q=0 \\ p < q}}^{M-1} \operatorname{Re} \left(e^{i \frac{\pi}{M} (p-q)(2\eta+p+q)} W_{\mu_p, \mu_q}(\mu) \right),$$

where $\mu_p = \mu_0 e^{i\pi \frac{2p}{M}}$, and for an odd M ,

$$W_{\mu_0}(\mu, t_M) = M^{-1} \sum_{p=0}^{M-1} W_{\mu_p}(\mu) + 2M^{-1} \sum_{\substack{p,q=0 \\ p < q}}^{M-1} \operatorname{Re} \left(e^{i \frac{\pi}{M} (p-q)(2\eta+1+p+q)} W_{\mu_p, \mu_q}(\mu) \right),$$

where $\mu_p = \mu_0 e^{i\pi \frac{2p+1}{M}}$ and we have used the same notations as in (31). In other words, M copies of the initial state (described by $W_{\mu_p}(\mu)$ in the above equations) along with the corresponding interferences (calculated using the second sums in the above equations) are generated on both the sphere and the hyperboloid. In the particular case that $M = 2$, we obtain the Wigner function of a two-component Schrödinger's cat state:

$$W_{\mu_0}(\mu, t_M) = \frac{1}{2} \left[W_{\mu_0}(\mu) + W_{-\mu_0}(\mu) + 2\operatorname{Re} \left(e^{-i \frac{\pi}{2} (2\eta+1)} W_{\mu_0, -\mu_0}(\mu) \right) \right], \quad (33)$$

where $W_{\mu_p, \mu_q}(\mu)$ are defined in (23) and (28).

In Figure 4a–c and Figure 4d–f, the Wigner functions corresponding to four-, three-, and two-component cat states are plotted on the sphere and the hyperboloid, respectively. The differences in the interference structures can be clearly observed in Figure 5a,b, in which the stereographic projections of the Wigner distributions on the plane and Poincaré disk are plotted. It is especially interesting to observe the presence of a fine superstructure localized at the bottom of the hyperboloid (at the center of the Poincaré disk) and the poles of the sphere in the cases with an even number of cat states. Such an interference pattern, consisting of a grid of alternative rings where the size of the cells is below the standard quantum limits of $\sim S^{-1/2}$ (sphere) and $\sim k^{-1/2}$ (hyperboloid), was recently investigated in [38] by employing a different (non-self) dual map [39,40] to the hyperbolic phase space. The sub-Planck structures [41,42] of extensions $\sim S^{-1}$ (sphere) and $\sim k^{-1}$ (hyperboloid) are characterized by a supersensitivity to phase space displacements.

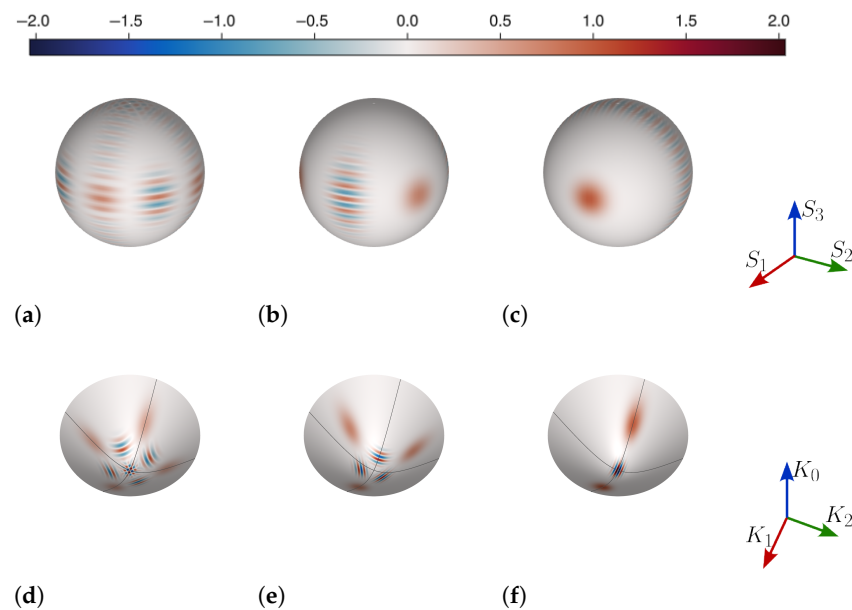


Figure 4. The Wigner functions of the SU(2) (a–c) and SU(1,1) (d–f) Schrödinger's cat states plotted on the sphere and hyperboloid, with $S = 20$ and $k = 15$, respectively: (a,d) four components at $t = \pi/4$; (b,e) three components at $t = \pi/3$; (c,f) two components at $t = \pi/2$.

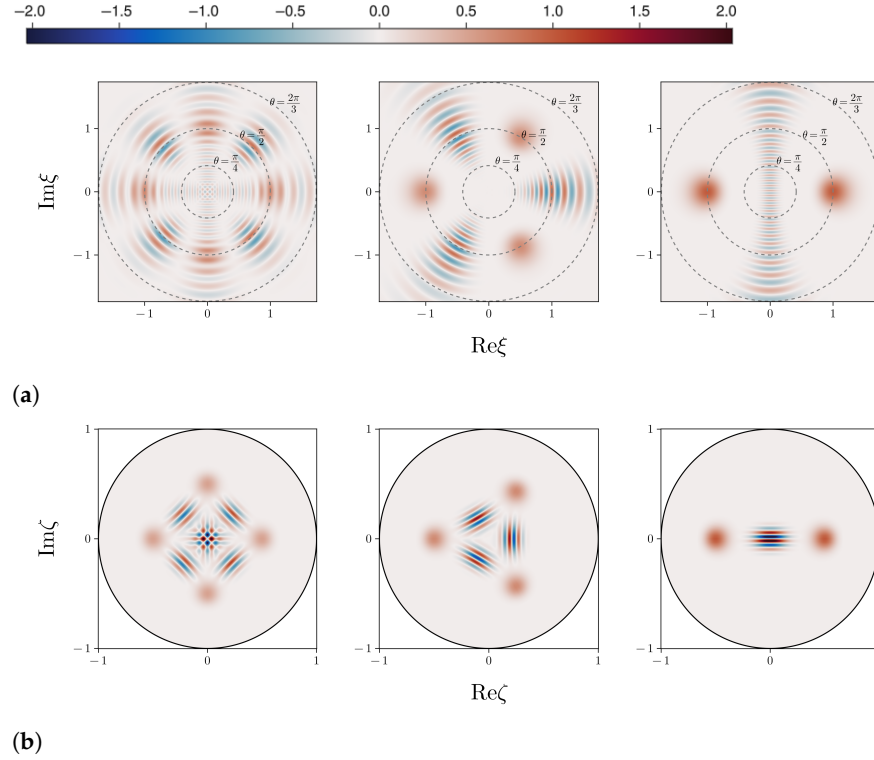


Figure 5. (a) The stereographic projection on the SU(2) Wigner function onto the 2-dim plane and (b) the stereographic projection on the SU(1,1) Wigner function onto the Poincare plane, at the same times as in Figure 4.

5. SU(1,1) Non-Compact Nonlinear Evolution

As was discussed in Section 2, in the case of SU(2) symmetry the phase space dynamics of mutually $\pi/2$ -rotated coherent states generated by the Hamiltonians in (5) and (16) are indistinguishable due to the covariance property of the Wigner map (21). The situation is entirely distinct in the case of SU(1,1) symmetry, where the Hamiltonian in (15) results in qualitatively different evolution to that resulting from (8). The principal difference consists in the continuous character of the spectrum of the Hamiltonian in (15), which is not SU(1,1) group-equivalent to (8).

Representing the evolved SU(1,1) coherent state in a similar manner to (30),

$$|\psi(t)\rangle = e^{-it\hat{K}_2^2}|\zeta_0\rangle = \frac{1}{\sqrt{i\pi}} \int dx e^{i(x^2+2ix\sqrt{t}\hat{K}_2)}|\zeta_0\rangle,$$

and we obtain the Wigner function in a form convenient for numerical calculation:

$$W_{\zeta_0}(\zeta, t) = \frac{2}{\pi(2k-1)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy e^{i(x^2-y^2)} f(\zeta_x, \zeta_y; \zeta, \zeta_0) \quad (34)$$

$$\times g_k \left(\frac{2(1-\zeta^* \gamma_x)(1-\zeta \gamma_y^*)}{(1-|\zeta|^2)(1-\gamma_x \gamma_y^*)} - 1 \right), \quad (35)$$

where g_k is defined in (26).

$$f(u, v; \zeta, \zeta_0) = (1-u^2)^k (1-v^2)^k \left(1 - uv + \frac{\zeta_0 - \zeta_0^*}{1-|\zeta_0|^2} (v-u) \right)^{-2k}, \quad (36)$$

and

$$\gamma_z = \frac{\zeta_z + \zeta_0}{1 + \zeta_z \zeta_0^*}, \quad \zeta_z = \tanh(z\sqrt{t}), \quad z = x, y. \quad (37)$$

Function (34) can be normalized at any time through construction. However, as we can see below, this leads to non-physical effects starting to occur at a particular time. In Figure 6a–d, the evolution of the initial coherent state $|\zeta = 0\rangle = |k, k\rangle$ is plotted, corresponding to the vacuum states of both field modes. The initial distribution undergoes a squeezing-type deformation, which is always centered at the bottom of the hyperboloid. The distribution continues to line up with the axis K_1 and compress over longer evolution times, twisting slightly in the orthogonal direction. A similar (except for its natural asymmetry) dynamic is observed for a coherent state located outside of the origin. No Schrödinger's cat states arise in the course of this evolution.

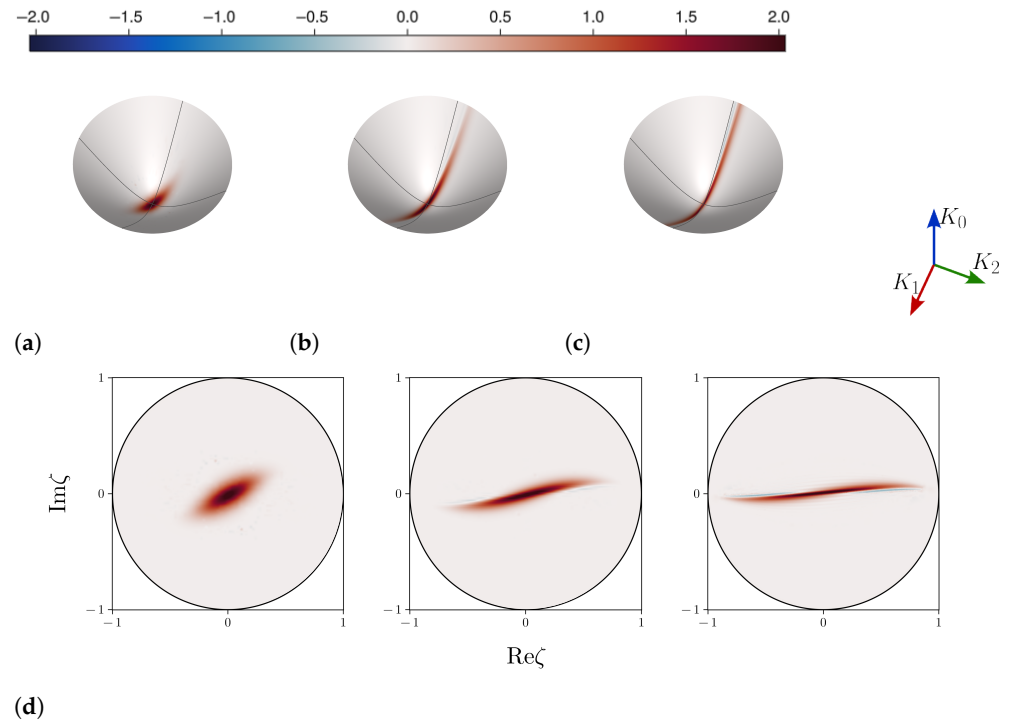


Figure 6. The SU(1,1) Wigner function of the state $e^{-it\hat{K}_2^2}|\zeta_0 = 0\rangle$ on the hyperboloid at (a) $t = 0.05$, (b) $t = 0.15$, and (c) $t = 0.3$; (d) the stereographic projection of the SU(1,1) Wigner function on the Poincaré disc at the same times. In all cases, $k = 10$.

It is worth mentioning that the evolution of the average values of some observables governed by the Hamiltonian in (16) leads to singularities at finite times. For instance, let us consider the evolution of the observable $\hat{K}_0(t) = \hat{U}^\dagger(t)\hat{K}_0\hat{U}(t)$, where the evolution operator is $\hat{U}(t) = \exp(-it\hat{K}_2^2)$, taking the average for the $|k, k\rangle$ state:

$$\langle k, k | \hat{K}_0(t) | k, k \rangle = \frac{e^{-it}}{2} \langle k, k | (\hat{J}_+ e^{-2t\hat{K}_2} + \hat{J}_- e^{2t\hat{K}_2}) | k, k \rangle = k(\cos t)^{-2k-1}, \quad (38)$$

where $\hat{J}_\pm = \hat{K}_0 \pm \hat{K}_1$ are non-compact (parabolic) operators, satisfying $[\hat{K}_2, \hat{J}_\pm] = \pm i\hat{J}_\pm$, so that $e^{it\hat{K}_2^2}\hat{J}_\pm e^{-it\hat{K}_2^2} = e^{-it}\hat{J}_\pm e^{\mp 2t\hat{K}_2}$, and the average $\langle k, k | \hat{J}_- e^{2t\hat{K}_2} | k, k \rangle$ is computed by taking into account the orbit properties of the coherent states in (19), i.e., $|\zeta = \tanh \frac{\tau}{2}\rangle = e^{i\tau\hat{K}_2} |k, k\rangle$.

The above results show a divergence at $t = \pi/2$, whose physical meaning generates a two-mode state with an infinite total number of photons in both modes. Such a pathological behavior is usually explained by a lack of self-adjointness of the operator $\hat{K}_0$ on the space of functions generated by the application of the evolution operator $\hat{U}(t)$ on the $|k, k\rangle$ state. In other words, the evolved state $\hat{U}(t \geq \pi/2) |k, k\rangle$ does not belong to the domain of $\hat{K}_0$.

Interestingly, the Wigner function of the state does not present any singularities at $t = \pi/2$ (or any other time), as shown in Figure 7. In particular, it preserves the

normalization. The divergence appears after integration over the entire phase space with the Wigner symbol of the observable:

$$\langle \hat{K}_0(t) \rangle = \int d\mu(\zeta) W_{K_0}(\zeta) W_0(\zeta, t),$$

where $W_0(\zeta, t)$ is defined in (34) at $\zeta_0 = 0$ and $W_{K_0}(\zeta) = \sqrt{k(k-1)} \cosh \tau$.

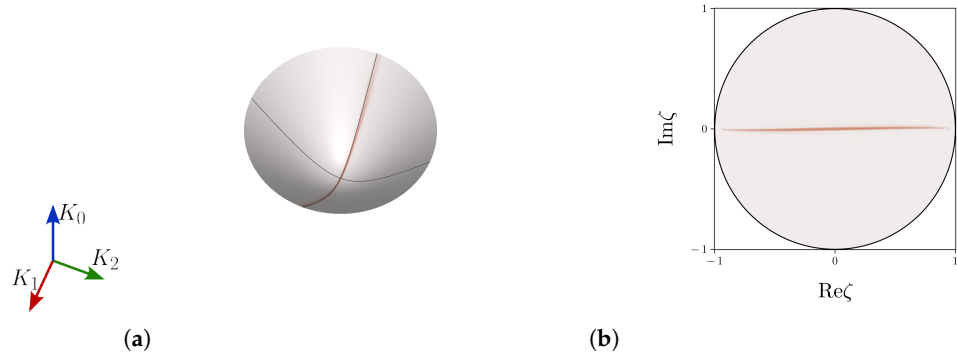


Figure 7. Representation on the hyperboloid (a) and the Poincaré disk (b) of the state $e^{-i\pi K_2^2/2}|\zeta_0\rangle$ for $\zeta_0 = 0$, with $k = 10$.

Clearly, the average values of higher-order operators become singular for shorter times, as, for instance, in

$$\hat{K}_0^2(t) = \frac{1}{4}(\hat{f}_+^2(t) + \hat{f}_-^2(t) + 2(\hat{\mathbf{K}}^2 + \hat{K}_2^2)), \quad (39)$$

where

$$\hat{f}_\pm^2(t) = e^{-4it} \hat{f}_\pm^2 e^{\mp 4it \hat{K}_2}, \quad (40)$$

and $\hat{\mathbf{K}}^2$ is the Casimir operator in (12), so that the average,

$$\langle \hat{K}_0^2(t) \rangle = \frac{k}{2} \left(\frac{2k+1}{2} \cos^{-2(k+1)}(2t) + \frac{2k-1}{2} \right), \quad (41)$$

diverges at $t = \pi/4$. A similar behavior is exhibited by other observables (not commuting with $\hat{K}_2$), for which the state $|k, k\rangle$ is not an analytic vector according to Nelson's theorem [43,44]. This in particular means that the $SU(1,1)$ squeezing parameter (32) is undefined for $t \geq \pi/4$, since it is computed as a quadratic form of the algebra generators. In Figure 8 the evolution of the squeezing parameter generated by the Hamiltonian in (16) is plotted, where the best squeezing time is significantly shorter than $\pi/4$ (in the appropriate units).

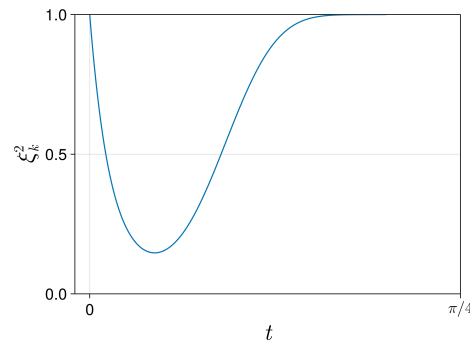


Figure 8. Evolution of the squeezing parameter generated by the Hamiltonian in (16), with $k = 10$.

6. Conclusions

We have analyzed the phase space evolution of coherent states belonging to the SU(2) and SU(1,1) groups, both driven by the same nonlinear two-mode Hamiltonian (1) and represented either on a sphere or on a hyperboloid according to the symmetry of their initial state. Despite the fundamental differences between the underlying group structures and geometries, the corresponding Wigner functions on the sphere and hyperboloid exhibit qualitatively similar dynamical features.

In both cases, the initial phase space distributions experience deformations that manifest as squeezing. However, this effect must be interpreted differently for each group. In the SU(2) case, squeezing corresponds to a reduction in fluctuations in specific components of angular momentum observables (and thus, related to photon conservation processes), defined with respect to the mean spin direction. In contrast, for SU(1,1) coherent states, which inhabit a non-compact hyperbolic phase space, squeezing involves fluctuations in pseudo-spin observables (connected to intramode photon difference preservation) around the origin and becomes meaningful only after applying an appropriate SU(1,1) transformation that centers the distribution.

Furthermore, the time evolution under both group symmetries leads to the formation of Schrödinger's cat-like states. These states emerge as superpositions of macroscopically distinguishable coherent components and are clearly visible in the Wigner representation through their interference fringes. While their structural patterns exhibit certain similarities, they are embedded in geometrically distinct manifolds: a compact sphere for SU(2) and a hyperbolic surface for SU(1,1).

A particularly remarkable case arises under the SU(1,1) dynamics generated by the Hamiltonian in (16), which is not group-equivalent to the compact case in (8). In this scenario, the long-time evolution of the Wigner function looks like a constantly narrowing distribution extending along the axis K_1 , which, however, maintains its normalization. This stretching behavior leads to a finite squeezing value being reached in relatively short times, $t_{sq} \ll \pi/4$ (actually, the distribution not only stretches but also twists slightly, which prevents the detection of the large squeezing of pseudo-spin observables). The unbound growth of the expectation values of some of the observables over a finite time (corresponding to the total energy of both physical modes) for the initial fundamental state $|k, k\rangle$ reveals its non-analyticity, which, however, does not prevent the well-behaved unitary evolution of the observables over the determined periods [2].

These results underscore the critical role of group-theoretical symmetry in shaping quantum dynamics in phase space. They also emphasize the importance of symmetry-adapted representations for capturing physically meaningful features such as squeezing, nonclassical superpositions, and dynamical singularities.

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Appendix A

In this Appendix we outline the procedure for computing the evolution of the SU(1,1) squeezing parameter $\xi_k^2(t)$ generated by the Hamiltonian $\hat{H} = K_0^2$, starting with a coherent state (19) for an arbitrary value of the Bargmann index, i.e., an invariant subspace with an arbitrary fixed photon difference in the initial state.

According to the procedure in [37], we apply an appropriate SU(1,1) transformation, $\hat{V} = \exp(i\tilde{\tau}\hat{K}_2)\exp(i\tilde{\varphi}\hat{K}_0)$, to the evolved generators, $\hat{K}_j(t) = \hat{U}^\dagger(t)\hat{K}_j\hat{U}(t)$, $\hat{U}(t) = \exp(-itK_0^2)$, to achieve $\langle\hat{k}_j(t)\rangle = 0$, where $\hat{k}_j(t) = \hat{V}^\dagger\hat{K}_j(t)\hat{V}$, $j = 1, 2$. Explicitly, one has

$$\hat{k}_1 = \cosh \tilde{\tau} \cos \tilde{\varphi} \hat{K}_1 + \cosh \tilde{\tau} \sin \tilde{\varphi} \hat{K}_2 - \sinh \tilde{\tau} \hat{K}_0, \quad (\text{A1})$$

$$\hat{k}_2 = -\sin \tilde{\varphi} \hat{K}_1 + \cos \tilde{\varphi} \hat{K}_2, \quad (\text{A2})$$

and for an initial coherent state, $|\zeta_0\rangle$, the condition $\langle\hat{k}_{1,2}(t)\rangle = 0$ is fulfilled for

$$\tilde{\tau}(t) = \tanh^{-1} \left(\frac{|\langle\hat{K}_+(t)\rangle|(1 - |\zeta_0|^2)}{k(1 + |\zeta_0|^2)} \right), \quad \tilde{\varphi}(t) = \text{Arg}(\langle\hat{K}_+(t)\rangle), \quad (\text{A3})$$

where

$$\langle\hat{K}_+(t)\rangle = 2k\zeta_0^* e^{it(2k+1)} \frac{(1 - |\zeta_0|^2)^{2k}}{(1 - e^{2it}|\zeta_0|^2)^{2k+1}}. \quad (\text{A4})$$

The minimum variance of the operator $\hat{k}(t) = \cos \nu \hat{k}_1(t) + \sin \nu \hat{k}_2(t)$ determines the SU(1,1) squeezing:

$$\xi_k^2(t) = \frac{2(\Delta k(t))_{\min}^2}{k}, \quad (\text{A5})$$

where for any SU(1,1) coherent state, $\xi_k^2 = 1$, and the state is squeezed if $\xi_k^2 < 1$. The minimum fluctuation in $\hat{k}(t)$ is given by the minimum eigenvalue of the covariance matrix

$$\Sigma(t) = \frac{1}{2} \begin{pmatrix} A_+(t) + A_-(t) & B(t) \\ B(t) & A_+(t) - A_-(t) \end{pmatrix}, \quad (\text{A6})$$

leading to

$$\xi_k^2(t) = \frac{1}{k} \left(A_+(t) - \sqrt{A_-^2(t) + B^2(t)} \right), \quad (\text{A7})$$

where $A_\pm(t) = \langle\hat{k}_1^2(t)\rangle \pm \langle\hat{k}_2^2(t)\rangle$, $B(t) = \langle\{\hat{k}_1(t), \hat{k}_2(t)\}\rangle$ and

$$\begin{aligned} A_\pm &= \frac{\cosh^2 \tilde{\tau} \pm 1}{2} (\langle\hat{K}_0^2\rangle - k(k-1)) + \sinh^2 \tilde{\tau} \langle\hat{K}_0^2\rangle \\ &+ \frac{\cosh^2 \tilde{\tau} \mp 1}{2} (\cos(2\tilde{\varphi}) \text{Re}\langle\hat{K}_+^2(t)\rangle + \sin(2\tilde{\varphi}) \text{Im}\langle\hat{K}_+^2(t)\rangle) \\ &- (2k+1) \cosh \tilde{\tau} \sinh \tilde{\tau} \text{Re} \left[e^{-i\tilde{\varphi}} \left(1 + \frac{2e^{2it}|\zeta_0|^2}{1 - e^{2it}|\zeta_0|^2} \right) \langle\hat{K}_+(t)\rangle \right], \end{aligned} \quad (\text{A8})$$

$$\begin{aligned} B &= \cosh \tilde{\tau} (\cos(2\tilde{\varphi}) \text{Im}\langle\hat{K}_+^2(t)\rangle - \sin(2\tilde{\varphi}) \text{Re}\langle\hat{K}_+^2(t)\rangle) \\ &- (2k+1) \sinh \tilde{\tau} \text{Im} \left[e^{-i\tilde{\varphi}} \left(1 + \frac{2e^{2it}|\zeta_0|^2}{1 - e^{2it}|\zeta_0|^2} \right) \langle\hat{K}_+(t)\rangle \right], \end{aligned} \quad (\text{A9})$$

with

$$\langle\hat{K}_+^2(t)\rangle = 2k(2k+1)(\zeta_0^*)^2 e^{4it(k+1)} \frac{(1 - |\zeta_0|^2)^{2k}}{(1 - e^{4it}|\zeta_0|^2)^{2k+2}}, \quad (\text{A10})$$

$$\langle\hat{K}_0^2\rangle = \frac{k(2k+1)}{2} \left(\frac{1 + |\zeta_0|^2}{1 - |\zeta_0|^2} \right)^2 - \frac{k}{2}. \quad (\text{A11})$$

The best squeezing time, t_{sq} , corresponds to the minimum value in (A7):

$$\min_t \zeta_k^2(t) = \zeta_k^2(t_{sq}). \quad (\text{A12})$$

References

1. Zachos, C.; Fairlie, D.; Curtright, T. *Quantum Mechanics in Phase Space: An Overview with Selected Papers*; World Scientific Publishing Company: Singapore, 2005.
2. Braunstein, S.L.; McLachlan, R.I. Generalized squeezing. *Phys. Rev.* **1987**, *35*, 1659. [[CrossRef](#)] [[PubMed](#)]
3. Fisher, R.A.; Nieto, M.M.; Sandberg, V.D. Impossibility of naively generalizing squeezed coherent states. *Phys. Rev. D* **1984**, *29*, 1107. [[CrossRef](#)]
4. Hillery, M.; Zubairy, M.; Wodkiewicz, K. Squeezing in higher order nonlinear optical processes. *Phys. Lett. A* **1984**, *103*, 259–261. [[CrossRef](#)]
5. Lawrie, B.J.; Lett, P.D.; Marino, A.M.; Pooser, R.C. Quantum sensing with squeezed light. *ACS Photonics* **2019**, *6*, 1307–1318. [[CrossRef](#)]
6. Schnabel, R. Squeezed states of light and their applications in laser interferometers. *Phys. Rep.* **2017**, *684*, 1–51. [[CrossRef](#)]
7. Bužek, V. SU(1,1) squeezing of SU(1,1) generalized coherent states. *J. Mod. Opt.* **1990**, *37*, 303–316. [[CrossRef](#)]
8. Gerry, C.C. Correlated two-mode SU(1,1) coherent states: Nonclassical properties. *J. Opt. Soc. Am. B* **1991**, *8*, 685–690. [[CrossRef](#)]
9. Gerry, C.C. Application of SU(1,1) coherent states to the interaction of squeezed light in an anharmonic oscillator. *Phys. Rev. A* **1987**, *35*, 2146. [[CrossRef](#)] [[PubMed](#)]
10. Solomon, A. Group theory of superfluidity. In *Dynamical Groups and Spectrum Generating Algebras*; World Scientific: Singapore, 1988; Volume 2, pp. 523–527.
11. Bassichis, W.; Foldy, L. Analysis of the Bogoliubov method applied to a simple boson model. *Phys. Rev.* **1964**, *133*, A935. [[CrossRef](#)]
12. Richardson, R. Exactly Solvable Many-Boson Model. *J. Math. Phys.* **1968**, *9*, 1327–1343. [[CrossRef](#)]
13. Lipkin, H.J.; Meshkov, N.; Glick, A. Validity of many-body approximation methods for a solvable model: (I). Exact solutions and perturbation theory. *Nucl. Phys.* **1965**, *62*, 188–198. [[CrossRef](#)]
14. Turski, L. *Model of the Josephson Effect*; Technical Report; University of Warsaw: Warsaw, Poland, 1972.
15. Perelomov, A.M. *Generalized Coherent States and Their Applications*; Springer: Berlin/Heidelberg, Germany, 1986.
16. Zhang, W.M.; Gilmore, R. Coherent states: Theory and some applications. *Rev. Mod. Phys.* **1990**, *62*, 867. [[CrossRef](#)]
17. Combes, J.; Brod, D.J. Two-photon self-Kerr nonlinearities for quantum computing and quantum optics. *Phys. Rev. A* **2018**, *98*, 062313. [[CrossRef](#)]
18. Hillmann, T.; Quijandria, F. Designing Kerr interactions for quantum information processing via counterrotating terms of asymmetric Josephson-junction loops. *Phys. Rev. Appl.* **2022**, *17*, 064018. [[CrossRef](#)]
19. Semiao, F.; Vidiella-Barranco, A. Effective cross-Kerr nonlinearity and robust phase gates with trapped ions. *Phys. Rev. A—At. Mol. Opt. Phys.* **2005**, *72*, 064305. [[CrossRef](#)]
20. Zou, F.; Fan, L.B.; Huang, J.F.; Liao, J.Q. Enhancement of few-photon optomechanical effects with cross-Kerr nonlinearity. *Phys. Rev. A* **2019**, *99*, 043837. [[CrossRef](#)]
21. Khorasani, S. Solution of cross-Kerr interaction combined with parametric amplification. *Sci. Rep.* **2019**, *9*, 1830. [[CrossRef](#)]
22. Onofri, E. A note on coherent state representations of Lie groups. *J. Math. Phys.* **1975**, *16*, 1087. [[CrossRef](#)]
23. Stratonovich, R. On the statistical interpretation of quantum theory. *Soviet Phys. JETP* **1957**, *5*.
24. Berezin, F.A. General concept of quantization. *Commun. Math. Phys.* **1975**, *40*, 153–174. [[CrossRef](#)]
25. Várilly, J.C.; Gracia-Bondía, J. The Moyal representation for spin. *Ann. Phys.* **1989**, *190*, 107–148. [[CrossRef](#)]
26. Brif, C.; Mann, A. Phase-space formulation of quantum mechanics and quantum-state reconstruction for physical systems with Lie-group symmetries. *Phys. Rev. A* **1999**, *59*, 971. [[CrossRef](#)]
27. Agarwal, G.S. Relation between atomic coherent-state representation, state multipoles, and generalized phase-space distributions. *Phys. Rev. A* **1981**, *24*, 2889. [[CrossRef](#)]
28. Varshalovich, D.A.; Moskalev, A.N.; Khersonskii, V.K. *Quantum Theory of Angular Momentum*; World Scientific: Singapore, 1988.
29. Klimov, A.B.; Seyfarth, U.; de Guise, H.; Sánchez-Soto, L.L. SU(1,1) covariant s-parametrized maps. *J. Phys. A Math. Theor.* **2021**, *54*, 065301. [[CrossRef](#)]
30. Erdélyi, A. *Higher Transcendental Functions*; McGraw Hill: New York, NY, USA, 1953; p. 59.
31. Tanas, R. Squeezed States of Light in Resonance Fluorescence. *Quantum Opt.* **1986**, *380*.
32. Kitagawa, M.; Yamamoto, Y. Number-phase minimum-uncertainty state with reduced number uncertainty in a Kerr nonlinear interferometer. *Phys. Rev. A* **1986**, *34*, 3974. [[CrossRef](#)] [[PubMed](#)]

33. Chumakov, S.; Klimov, A.; Sanchez-Mondragon, J. Collective atomic dynamics in a strong quantum field. *Opt. Commun.* **1995**, *118*, 529–536. [[CrossRef](#)]
34. Kitagawa, M.; Ueda, M. Squeezed spin states. *Phys. Rev. A* **1993**, *47*, 5138. [[CrossRef](#)]
35. Agarwal, G.S.; Puri, R.R.; Singh, R. Atomic Schrödinger cat states. *Phys. Rev. A* **1997**, *56*, 2249. [[CrossRef](#)]
36. Ma, J.; Wang, X.; Sun, C.P.; Nori, F. Quantum spin squeezing. *Phys. Rep.* **2011**, *509*, 89–165. [[CrossRef](#)]
37. Huang, Y.; Xiong, H.N.; Zhong, W.; Liu, B.; Wang, X. SU(1,1) squeezed state. *J. Phys. B At. Mol. Opt. Phys.* **2019**, *52*, 185301. [[CrossRef](#)]
38. Akhtar, N.; Sanders, B.C.; Xianlong, G. Sub-Planck phase-space structure and sensitivity for SU(1,1) compass states. *Phys. Rev. A* **2022**, *106*, 043704. [[CrossRef](#)]
39. Vourdas, A. Analytic representations in quantum mechanics. *J. Phys. Math. Gen.* **2006**, *39*, R65. [[CrossRef](#)]
40. Seyfarth, U.; Klimov, A.B.; de Guise, H.; Leuchs, G.; Sánchez-Soto, L.L. Wigner function for SU(1,1). *Quantum* **2020**, *4*, 317. [[CrossRef](#)]
41. Zurek, W.H. Sub-Planck structure in phase space and its relevance for quantum decoherence. *Nature* **2001**, *412*, 712–717. [[CrossRef](#)]
42. Zurek, W.H. Decoherence, einselection, and the quantum origins of the classical. *Rev. Mod. Phys.* **2003**, *75*, 715. [[CrossRef](#)]
43. Hillery, M. Photon number divergence in the quantum theory of n-photon down conversion. *Phys. Rev. A* **1990**, *42*, 498. [[CrossRef](#)]
44. Reed, M.; Simon, B. *II: Fourier Analysis, Self-Adjointness*; Elsevier: Amsterdam, The Netherlands, 1975; Volume 2.

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