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Integral Quantization for the Discrete Cylinder

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Abstract: Covariant integral quantizations are based on the resolution of the identity by continuous or discrete families of normalized positive operator valued measures (POVM), which have appealing probabilistic content and which transform in a covariant way. One of their advantages is their ability to circumvent problems due to the presence of singularities in the classical models. In this paper, we implement covariant integral quantizations for systems whose phase space is $\mathbb{Z} \times \mathbb{S}^1$, i.e., for systems moving on the circle. The symmetry group of this phase space is the discrete & compact version of the Weyl–Heisenberg group, namely the central extension of the abelian group $\mathbb{Z} \times \mathrm{SO}(2)$. In this regard, the phase space is viewed as the right coset of the group with its center. The non-trivial unitary irreducible representation of this group, as acting on $L^2(\mathbb{S}^1)$, is square integrable on the phase space. We show how to derive corresponding covariant integral quantizations from (weight) functions on the phase space and resulting resolution of the identity. As particular cases of the latter we recover quantizations with de Bièvre-del Olmo–Gonzales and Kowalski–Rembielevski–Papaloucas coherent states on the circle. Another straightforward outcome of our approach is the Mukunda Wigner transform. We also look at the specific cases of coherent states built from shifted gaussians, Von Mises, Poisson, and Fejér kernels. Applications to stellar representations are in progress.

Keywords: covariant Weyl–Heisenberg integral quantization; discrete cylinder; coherent states; angle operator; quantum mechanics on the circle; Wigner function

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1. Introduction

Quantum physics for systems whose configuration space is the circle $\mathbb{S}^1$ has been considered by many authors over the last six decades [1–20]. These studies include work by Carruthers (1968) [1], Lévy-Leblond (1976) [2], Kowalski et al. (1996 and 2021) [9,17], De Bièvre [10], Gazeau et al. [11,16], and the most recent one by Mista et al. [20]. The phase space associated with this configuration space is the cylinder, whether continuous or discrete. These works included questions such as, for a given quantization scheme of the phase space,

(i) What is the appropriate quantum position operator?
(ii) What is the appropriate quantum momentum operator?
(iii) What are the states (coherent states) that saturate some uncertainty relations?
(iv) Do these operators explicitly display the topology of the circle or the fact that the circle is a curved manifold or a group?

The latter question raises interest in the extension of these investigations to higher-dimensional configuration spaces, such as the group of rotation SO(3) in three dimensions. Note that this group can be viewed as the configuration space for the motion of a rigid body about one of its points.

Covariant integral quantizations [21] linearly transform functions ("classical observables") on phase spaces (in a wider sense) into operators "quantum observables") on some

Hilbert spaces of "quantum states". They are based on the resolution of the identity by continuous or discrete families of normalized positive operator valued measures (POVM), which transform in a covariant way under some symmetry group actions. In the simplest cases these symmetries are described by the Weyl–Heisenberg group (projective representations of translations in 2D), or by the affine groups (translations of a subset of variables combined with dilations of the remnant subset of bounded below variables). These quantization methods, whose origin can be traced back to Klauder, Berezin, Toepiltz, are relatively easy to manipulate compared with geometric or deformation or other quantizations. Beyond their appealing probabilistic content, they allow us to circumvent ordering problems, or those caused by the presence of singularities in the classical models.

In this work, we implement the covariant integral quantizations for systems whose phase space or "discrete cylinder" is $\mathbb{Z} \times \mathbb{S}^1$, i.e., for systems moving on the circle and with integral momentum. The symmetry group of this phase space is the discrete and compact version of the Weyl–Heisenberg group H_1. More precisely, it is a central extension of the abelian group $\mathbb{Z} \times \mathrm{SO}(2)$, homeomorphic to $\mathbb{R} \times \mathbb{Z} \times \mathrm{SO}(2)$, and here denoted by $\mathrm{H}_1^{\mathrm{dc}}$. The phase space can be viewed as the right coset $\Gamma := \mathbb{R} \backslash \mathrm{H}_1^{\mathrm{dc}}$.

In Section 2, we briefly review the covariant integral quantization procedure. We particularly emphasize the role of the resolution of the identity provided by the integration of operator-valued functions over a measure space. In Section 3, we derive the non-trivial unitary irreducible representation U of $\mathrm{H}_1^{\mathrm{dc}}$, acting on the Hilbert space $L^2(\mathbb{S}^1, \mathrm{d}\gamma)$ of functions that are square integrable on the circle. This representation is square integrable on the discrete cylinder. We give an overview of the properties of the unitary Weyl operator acting on $L^2(\mathbb{S}^1, \mathrm{d}\gamma)$. We define the related Gabor transform associated with a resolution of the identity and the resulting family of coherent states on the circle. In Section 4, we first define our quantization tools: a weight function ϖ defined on the discrete cylinder and the related integral operator M^{ϖ} acting on the representation space. In Section 5, we define the quantization map, which associates a function f on the discrete cylinder with an operator A_f^{ϖ} acting in $L^2(\mathbb{S}^1, \mathrm{d}\gamma)$. We compute A_f^{ϖ} for separable functions in position and momentum, in momentum only, and in position only. Finally, we examine smooth de-quantizations under the form of the so-called semi-classical portraits $\check{f}$ of A_f^{ϖ}. In Section 6, we give some concrete examples of quantum operators built through coherent states for $\mathrm{H}_1^{\mathrm{dc}}$, and those yielded by the weight $\varpi = 1$ corresponding to the parity operator. In Section 7, we examine quantizations resulting from various choices of weight function ϖ. We conclude in Section 8 with a few comments on the content of our work and on new directions to be explored. In Appendix A, some fiducial vectors and their corresponding reproducing kernels are presented in a table.

2. (Covariant) Integral Quantization: A Survey

Let (X, μ) be a measure space and $\mathcal{H}$ be a (separable) Hilbert space. An operator-valued function

$$X \ni x \mapsto \mathsf{M}(x) \text{ acting in } \mathcal{H}\,, \tag{1}$$

resolves the identity operator $\mathbb{1}$ in $\mathcal{H}$ with respect to the measure μ if

$$\int_X \mathsf{M}(x)\, \mathrm{d}\mu(x) = \mathbb{1} \tag{2}$$

holds in a weak sense.

Integral quantization based on Equation (2) is the linear map of a function on X to an operator in $\mathcal{H}$, which is defined by

$$f(x) \mapsto \int_X f(x)\mathsf{M}(x)\, \mathrm{d}\mu(x) := A_f\,, \quad 1 \mapsto \mathbb{1}\,. \tag{3}$$

If the operators $\mathsf{M}(x)$ in Equation (2) are non-negative and bounded, one says that they form a (normalized) positive operator-valued measure (POVM) on X. If they are assigned a unit trace class for all $x \in X$, i.e., if the $\mathsf{M}(x)$'s are density operators, then the map

$$f(x) \mapsto \check{f}(x) := \mathrm{tr}(\mathsf{M}(x)A_f) = \int_X f(x')\,\mathrm{tr}(\mathsf{M}(x)\mathsf{M}(x'))\,\mathrm{d}\mu(x') \tag{4}$$

is a local averaging of the original $f(x)$ (which can be very singular, similar to a Dirac) with respect to the probability distribution on X,

$$x' \mapsto \mathrm{tr}(\mathsf{M}(x)\mathsf{M}(x'))\,. \tag{5}$$

This averaging, or semi-classical portrait of the operator A_f, is in general a regularization —depending, of course, on the topological nature of the measure space (X, μ) and the functional properties of the $\mathsf{M}(x)$'s.

Now, consider a set of parameters $\boldsymbol{\kappa}$ and corresponding families of POVM $\mathsf{M}_{\boldsymbol{\kappa}}(x)$ solving the identity

$$\int_X \mathsf{M}_{\boldsymbol{\kappa}}(x)\,\mathrm{d}\mu(x) = \mathbb{1}\,, \tag{6}$$

One says that the *classical limit* $f(x)$ holds at $\boldsymbol{\kappa}_0$ if

$$\check{f}_{\boldsymbol{\kappa}}(x) := \int_X f(x')\,\mathrm{tr}(\mathsf{M}_{\boldsymbol{\kappa}}(x)\mathsf{M}_{\boldsymbol{\kappa}}(x'))\,\mathrm{d}\mu(x') \to f(x) \quad \text{as} \quad \boldsymbol{\kappa} \to \boldsymbol{\kappa}_0\,, \tag{7}$$

where the convergence $\check{f} \to f$ is defined in the sense of a certain topology.

Otherwise said, $\mathrm{tr}(\mathsf{M}_{\boldsymbol{\kappa}}(x)\mathsf{M}_{\boldsymbol{\kappa}}(x'))$ tends to

$$\mathrm{tr}(\mathsf{M}_{\boldsymbol{\kappa}}(x)\mathsf{M}_{\boldsymbol{\kappa}}(x')) \to \delta_x(x') \tag{8}$$

where δ_x is a Dirac measure with respect to μ,

$$\int_X f(x')\,\delta_x(x')\,\mathrm{d}\mu(x') = f(x)\,. \tag{9}$$

Nothing guarantees the existence of such a limit on a general level. Nevertheless, if the semi-classical $\check{f}_{\boldsymbol{\kappa}}$ might appear more realistic and more easily manageable than the original f, the next step is to evaluate the range of acceptability of the parameters $\boldsymbol{\kappa}$.

Let us now assume that $X = G$ is a Lie group with left Haar measure $\mathrm{d}\mu(g)$, and let $g \mapsto U(g)$ be a unitary irreducible representation (UIR) of G in a Hilbert space $\mathcal{H}$. Let M be a bounded self-adjoint operator on $\mathcal{H}$, and let us define g-translations of M as

$$\mathrm{M}(g) = U(g)\mathrm{M}U(g)^\dagger\,. \tag{10}$$

Suppose that the operator

$$R := \int_G \mathrm{M}(g)\,\mathrm{d}\mu(g)\,, \tag{11}$$

is defined in a weak sense. From the left invariance of $\mathrm{d}\mu(g)$, the operator R commutes with all operators $U(g)$, $g \in G$, and so, from Schur's Lemma, we have the resolution of the unity up to a constant,

$$R = c_{\mathrm{M}}\mathbb{1}\,. \tag{12}$$

The constant c_{M} can be found from the formula

$$c_{\mathrm{M}} = \int_G \mathrm{tr}(\rho_0\,\mathrm{M}(g))\,\mathrm{d}\mu(g)\,, \tag{13}$$

where ρ_0 is a given unit trace positive operator. ρ_0 is chosen, if manageable, in order to make the integral convergent. Of course, it is possible that no such finite constant exists for a given M; or worse, it cannot exist for any M (which is not the case for square integrable representations). Now, if c_{M} is finite and positive, the true resolution of the identity follows:

$$\int_G \mathrm{M}(g)\,\mathrm{d}\nu(g) = \mathbb{1}\,, \quad \mathrm{d}\nu(g) := \mathrm{d}\mu(g)/c_{\mathrm{M}}\,. \tag{14}$$

For instance, in the case of a square-integrable unitary irreducible representation $U : g \mapsto U(g)$, let us pick a unit vector $|\psi\rangle$ for which $c_{\mathrm{M}} = \int_G \mathrm{d}\mu(g)|\langle\psi|U(g)\psi\rangle|^2 < \infty$, i.e., $|\psi\rangle$ is an admissible unit vector for U. With $\mathrm{M} = |\psi\rangle\langle\psi|$, the resolution of the identity (14) provided by the family of states $|\psi_g\rangle = U(g)|\psi\rangle$ reads

$$\int_G |\psi_g\rangle\langle\psi_g| \frac{\mathrm{d}\mu(g)}{c_{\mathrm{M}}} = \mathbb{1}\,. \tag{15}$$

Vectors $|\psi_g\rangle$ are named (generalized) coherent states for the group G.

The Equation (14) provides an integral quantization of complex-valued functions on group G, as follows

$$f \mapsto A_f = \int_G \mathrm{M}(g) f(g) \frac{\mathrm{d}\mu(g)}{c_{\mathrm{M}}}\,. \tag{16}$$

Furthermore, this quantization is *covariant* in the sense that:

$$U(g) A_f U(g)^\dagger = A_F \quad \text{where} \quad F(g') = (\mathfrak{U}(g)f)(g') = f(g^{-1}g')\,, \tag{17}$$

i.e., $\mathfrak{U}(g) : f \mapsto F$ is the left regular representation if $f \in L^2(G, \mathrm{d}\mu(g))$.

For the group $\mathrm{H}_1^{\mathrm{dc}}$ considered in this paper, an adaption of this material is necessary in the sense that we have to replace the group $\mathrm{H}_1^{\mathrm{dc}}$ with its right coset $\Gamma = \mathbb{R}\backslash\mathrm{H}_1^{\mathrm{dc}} \sim \mathbb{Z}\times\mathrm{SO}(2)$, which amounts to replacing its UIR with its projective version.

3. Overview: Weyl Operator Acting on $L^2(\mathbb{S}^1, \mathrm{d}\gamma)$

The group $\mathrm{H}_1^{\mathrm{dc}}$ is the set of triplets $(s, m, \theta) \in \mathbb{R}\times\mathbb{Z}\times\mathbb{S}^1$ equipped with the following Weyl–Heisenberg-like multiplication:

$$(s, m, \theta)\,(s', m', \theta') = \left(s + s' + \frac{m\theta' - m'\theta}{2}, m + m', \theta + \theta' \bmod 2\pi\right). \tag{18}$$

The neutral element is $e = (0, 0, 0)$ and the inverse is given by

$$(s, m, \theta)^{-1} = (-s, -m, -\theta \bmod 2\pi)\,.$$

Let us define the representation of $\mathrm{H}_1^{\mathrm{dc}}$ acting on $L^2(\mathbb{S}^1, \mathrm{d}\gamma)$ as:

$$(V(s, m, \theta)\psi)(\gamma) = e^{\mathrm{i}\,s} e^{-\mathrm{i}\,\frac{m\theta}{2}} e^{\mathrm{i}\,m\gamma} \psi(\gamma - \theta)\,. \tag{19}$$

This representation is unitary and irreducible. Due to the central parameter $s \in \mathbb{R}$, it is not square integrable on the group equipped with its semi-discrete Haar measure $\mathrm{d}(s, m, g)$, i.e., there is no $\psi \in L^2(\mathbb{S}^1, \mathrm{d}\gamma)$, such that $\sum_{m\in\mathbb{Z}} \int_{\mathbb{R}\times\mathbb{S}^1} \mathrm{d}s\,\mathrm{d}\theta\,|(\psi, (V(s, m, \theta)\psi)|^2 < \infty$. On the other hand, it is square integrable on the right coset $\mathbb{R}\backslash\mathrm{H}_1^{\mathrm{dc}} =$ of $\mathrm{H}_1^{\mathrm{dc}}$ with its center. This right coset is identified with the phase space or discrete cylinder $\Gamma = \mathbb{Z}\times\mathbb{S}^1$. The representation (19) induces the unitary ($\sim$ Weyl) operator U defined on Γ by the section $s = 0$ on the group, i.e., $U(m, \theta) := V(0, m, \theta)$:

$$(U(m, \theta)\psi)(\gamma) = e^{-\mathrm{i}\,\frac{m\theta}{2}}\, e^{\mathrm{i}\,m\gamma} \psi(\gamma - \theta)\,. \tag{20}$$

One can show, through the use of Schur's Lemma, that for any ϕ in $L^2(\mathbb{S}^1, \mathrm{d}\gamma)$ the family $\{U(m, \theta)\phi\}_{(m,\theta)\in\Gamma}$ constitutes an overcomplete family resolving the identity in the sense of Equations (2) or (14) or (15):

$$\sum_{m\in\mathbb{Z}}\int_{\mathbb{S}^1} \mathrm{d}\theta\, |U(m,\theta)\phi\rangle\langle U(m,\theta)\phi| = \mathbb{1}\,. \tag{21}$$

This implies that for any ϕ, ψ in $L^2(\mathbb{S}^1, \mathrm{d}\gamma)$, we have:

$$\sum_{m\in\mathbb{Z}}\int_{\mathbb{S}^1} \mathrm{d}\theta\, |\langle\psi|U(m,\theta)\phi\rangle|^2 = ||\psi||^2||\phi||^2\,. \tag{22}$$

The function $\Gamma \ni (m,\theta) \mapsto \phi_{(m,\theta)} := U(m,\theta)\phi$ is a coherent state (CS) for the group $\mathrm{H}_1^{\mathrm{dc}}$ in the sense given by Perelomov [18], and the function ϕ is then called a fiducial vector. The interpretation of the projection of ψ on $\phi_{(m,\theta)}$, namely

$$\langle\phi_{(m,\theta)}|\psi\rangle = \int_{\mathbb{S}^1}\mathrm{d}\gamma\,\overline{(U(m,\theta)\phi)(\gamma)}\psi(\gamma) = \int_{\mathbb{S}^1}\mathrm{d}\gamma\, e^{\mathrm{i}\frac{m\theta}{2}} e^{-\mathrm{i}\,m\gamma}\overline{\phi(\gamma-\theta)}\psi(\gamma)\,, \tag{23}$$

is the phase space or momentum-angular position representation of ψ with respect to the family of coherent states $\phi_{(m,\theta)}$.

Let us now summarize the most important features of the above CS analysis of the elements of $L^2(\mathbb{S}^1, \mathrm{d}\gamma)$ resulting from the resolution of the identity (21).

Proposition 1. *Let $L^2(\Gamma, \mathrm{d}(m,\theta))$ be the Hilbert space of square integrable functions on Γ equipped with its semi-discrete measure. For any function $\phi, \psi \in L^2(\mathbb{S}^1, \mathrm{d}\gamma)$, such that $||\phi|| = 1$, the map*

$$L^2(\Gamma, \mathrm{d}\gamma) \ni \psi \to \langle\phi_{(m,\theta)}|\psi\rangle \in L^2(\Gamma, \mathrm{d}(m,\theta)) \tag{24}$$

satisfies the following properties:

- *It is an isometry:*

$$\|\psi\|^2 = \int_{\mathbb{S}^1}\mathrm{d}\gamma\,|\psi(\gamma)|^2 = \sum_{m\in\mathbb{Z}}\int_{\mathbb{S}^1}\mathrm{d}\theta\,|\langle\phi_{(m,\theta)}|\psi\rangle|^2\,, \tag{25}$$

- *It can be inverted on its range:*

$$\psi(\gamma) = \sum_{m\in\mathbb{Z}}\int_{\mathbb{S}^1}\mathrm{d}\theta\,\langle\phi_{(m,\theta)}|\psi\rangle\phi_{(m,\theta)}(\gamma)\,, \tag{26}$$

- *Its range is a reproducing kernel space:*

$$\Psi(m,\theta) := \langle\phi_{(m,\theta)}|\psi\rangle = \sum_{m'\in\mathbb{Z}}\int_{\mathbb{S}^1}\mathrm{d}\theta'\,\langle\phi_{(m,\theta)}|\phi_{(m',\theta')}\rangle\Psi(m',\theta')\,, \tag{27}$$

where the reproducing kernel K_ϕ is the two-point function on the phase space:

$$K_\phi((m,\theta),(m',\theta')) = \langle\phi_{(m,\theta)}|\phi_{(m',\theta')}\rangle\,. \tag{28}$$

One can show that:

$$K_\phi((m,\theta),(m',\theta')) = A(m,m',\theta,\theta')\Phi(m-m',\theta-\theta') \tag{29}$$

$$A(m,m',\theta,\theta') = e^{\mathrm{i}\,\frac{m(\theta-\theta')-(m-m')\theta'}{2}} \tag{30}$$

$$\Phi(m-m',\theta-\theta') = \int_{\mathbb{S}^1}\mathrm{d}\gamma\, e^{-\mathrm{i}(m-m')}\overline{\phi(\gamma-(\theta-\theta'))}\phi(\gamma) \tag{31}$$

Let us now present a list of useful formulae for the sequel.

$$\begin{aligned} U^\dagger(m,\theta)\psi(\gamma) &= e^{\mathrm{i}\frac{m\theta}{2}} e^{-\mathrm{i}\, m(\gamma+\theta)}\psi(\gamma+\theta) = e^{-\mathrm{i}\frac{m\theta}{2}} e^{-\mathrm{i}\, m\gamma}\psi(\gamma+\theta) \\ &= U(-m,-\theta)\psi(\gamma)\,. \end{aligned} \tag{32}$$

$$U(m,\theta)\,U(m',\theta') = e^{-\mathrm{i}\frac{m\theta'-m'\theta}{2}}\, U(m+m',\theta+\theta')\,. \tag{33}$$

$$U(m',\theta')U(m,\theta)\,U^\dagger(m',\theta') = e^{\mathrm{i}(m\theta'-m'\theta)}U(m,\theta)\,. \tag{34}$$

$$U^\dagger(m,\theta)\,U^\dagger(m',\theta') = e^{-\mathrm{i}\frac{m\theta'-m'\theta}{2}}\, U^\dagger(m+m',\theta+\theta')\,. \tag{35}$$

$$\mathrm{Tr}[U(m,\theta)] = \delta_{m\,0}\delta(\theta)\,. \tag{36}$$

As was stated, any square integrable function on the circle can be used as a starting function, i.e., a fiducial vector to build a coherent state. However, one is mostly interested in considering fiducial vectors that have some "good" localization both in angular position and momentum. One way of achieving this is to take the periodized version of a square integrable function that has good localization properties on the real line (see [22], page 146, for the equivalent of the uncertainty principle on the circle). For this purpose, we use periodization combined with the Poisson summation formula (see [22], pages 137–138). So, let us pick a function ϕ that is integrable on the real line. The periodization operator Per is defined by

$$(\mathrm{Per}\,\phi)(x) = 2\pi \sum_{n\in\mathbb{Z}} \phi(x+2\pi\, n)\,, \tag{37}$$

and the application of the Poisson summation formula reads:

$$2\pi \sum_{n\in\mathbb{Z}} \phi(x+2\pi\, n) = \sum_{n\in\mathbb{Z}} \mathcal{F}[\phi](n)e^{\mathrm{i}\, n\, x} \tag{38}$$

where $\mathcal{F}[\phi]$ is the Fourier transform of ϕ, which is defined with its inverse by

$$\mathcal{F}[\phi](\omega) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}\, x\, e^{-\mathrm{i}\,\omega\, x}\phi(x)\,, \quad \phi(x) = \int_{\mathbb{R}} \mathrm{d}\, x\, e^{\mathrm{i}\, x\,\omega}\mathcal{F}[\phi](\omega)\,. \tag{39}$$

Let us present some examples of such periodizations [22] and other examples of fiducial vectors.

- Periodized (L^2 normalized) gaussian kernel and theta function.

$$\frac{2\pi}{(2\pi)^{\frac{1}{4}}}\frac{1}{\sqrt{\sigma}}\sum_{n\in\mathbb{Z}} e^{-\frac{(x+2\pi\, n)^2}{2\sigma^2}} = 2\pi(2\pi)^{-\frac{3}{2}}\sum_{n\in\mathbb{Z}} e^{\mathrm{i}\, n\, x}e^{-\frac{\sigma^2}{2}n^2} = (2\pi)^{-\frac{1}{2}}\theta\left(\frac{x}{2\pi},\frac{\sigma}{\sqrt{2\pi}}\right), \tag{40}$$

where ϑ_3 is the third Jacobi theta function:

$$\vartheta_3(x,s) = \sum_{n\in\mathbb{Z}} e^{\mathrm{i}\, 2\pi\, n\, x}\, e^{\mathrm{i}\pi\, s\, n^2}\,. \tag{41}$$

A number of authors ([9,12]) have used variants of this function as a fiducial vector for coherent states on the circle.
The corresponding reproducing kernel for $\sigma = 1$ is:

$$\begin{aligned} &\langle\phi_{(m,\theta)}|\phi_{(m',\theta')}\rangle = \\ &= (2\pi)^2\, e^{\mathrm{i}\,\frac{m(\theta-\theta')-(m-m')\theta'}{2}}\, e^{-\frac{(m-m')^2}{2}}\sum_{n\in\mathbb{Z}} e^{-n^2}e^{\mathrm{i}\, n\,(\mathrm{i}(m-m')+(\theta-\theta'))} \\ &= (2\pi)^2\, e^{\mathrm{i}\,\frac{m(\theta-\theta')-(m-m')\theta'}{2}}\, e^{-\frac{(m-m')^2}{2}}\,\vartheta\left(\frac{\mathrm{i}(m-m')+(\theta-\theta')}{2\pi},\frac{\mathrm{i}}{\sqrt{\pi}}\right). \end{aligned} \tag{42}$$

- Periodized (L^2 normalized) Poisson kernel.

$$P_\sigma(x) = 2\sum_{n\in\mathbb{Z}} \frac{\sigma^{-1}}{(x+2\pi\,n)^2+\sigma^{-2}} = \frac{1-r^2}{1-2\,r\,\cos(x)+r^2}\,. \tag{43}$$

- Dirichlet fiducial vector (L^2 normalized).

$$D_n(\gamma) = \frac{1}{\sqrt{2\pi(n+1)}}\sum_{m=-n}^{m=+n} e^{\mathrm{i}\,n\,\varphi} = \frac{1}{\sqrt{2\pi(n+1)}} \times \frac{\sin(n+\frac{1}{2})(\gamma)}{\sin\frac{\gamma}{2}}\,. \tag{44}$$

The corresponding reproducing kernel is:

$$\langle\phi_{(m,\theta)}|\phi_{(m',\theta')}\rangle \;=\; \frac{1}{2\pi(n+1)}\,e^{\mathrm{i}\,\frac{m(\theta-\theta')-(m-m')\theta'}{2}}\,\frac{\sin((n+\frac{1}{2})(\theta-\theta'))}{\sin(\frac{\theta-\theta'}{2})}\,. \tag{45}$$

- Fejér fiducial vector.

$$\begin{aligned} F_n(\varphi) &= \frac{D_0(\varphi)+D_1(\varphi)+\ldots+D_n(\varphi)}{n+1} \\ &= \frac{1}{n+1}\left[\frac{\sin((\frac{n+1}{2})\varphi)}{\sin\frac{1}{2}\varphi}\right]^2 = \sum_{k=-n}^{k=n}(1-\frac{|\,k|}{n+1})e^{\mathrm{i}\,k\varphi}. \end{aligned} \tag{46}$$

- Von Mises fiducial vector (L^2 normalized)

$$\phi(\gamma) = \frac{e^{\lambda\cos(\gamma)}}{\sqrt{2\pi\,I_0(2\lambda)}}\,, \tag{47}$$

where I_0 is the modified Bessel of the first kind of zero order. The corresponding reproducing kernel is:

$$\langle\phi_{(m,\theta)}|\phi_{(m',\theta')}\rangle \;=\; \frac{e^{\mathrm{i}\,\frac{(m-m')\theta-m\,(\theta-\theta')}{2}}}{2\pi\,I_0(2\lambda)} \times I_{m-m'}(2\lambda)\,\cos\left(\frac{\theta-\theta'}{2}\right). \tag{48}$$

This fiducial vector is related to the Von Mises kernel, which is widely used in the area of circular data analysis. In the context of quantum mechanics on the circle, it was derived by Carruthers in 1968 [1], and several other authors, including Kowalski [17] and collaborators, have used it.

4. Quantization Operators and the Quantization Map

Following previous works [23,24], we pick a function ϖ, called weight (but not necessarily positive), on the phase space Γ. We then define the operator M^ϖ acting in $L^2(\mathbb{S}^1, \mathrm{d}\gamma)$ by

$$\mathsf{M}^\varpi = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int_{\mathbb{S}^1}\mathrm{d}\theta\,\varpi(m,\theta)\,U(m,\theta)\,. \tag{49}$$

Proposition 2. *With the assumption that the weight ϖ has been chosen such that the operator M^ϖ is bounded:*

(i) The operator M^ϖ is the integral operator:

$$(\mathsf{M}^\varpi\psi)(\gamma) = \int_{\mathbb{S}^1}\mathrm{d}\gamma\,\mathcal{M}^\varpi(\gamma,\gamma')\psi(\gamma')\,, \tag{50}$$

where the kernel $\mathcal{M}^\varpi(\gamma,\gamma')$ is given by:

$$\mathcal{M}^{\varpi}(\gamma,\gamma') = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} \varpi(m,\gamma-\gamma')\, e^{\mathrm{i}\frac{m(\gamma+\gamma')}{2}} \equiv \frac{1}{2\pi}\widetilde{\varpi}_{p_1}\left(\frac{\gamma+\gamma'}{2},\gamma-\gamma'\right). \tag{51}$$

Here, $\widetilde{\varpi}_{p_1}$ is the inverse discrete Fourier transform of ϖ with respect to the first variable.

(ii) *The operator M^{ϖ} is symmetric if—and only if—the weight satisfies:*

$$\mathsf{M}^{\varpi} = \mathsf{M}^{\varpi\dagger} \Leftrightarrow \overline{\varpi(-m,\gamma)} = \varpi(m,-\gamma)\,. \tag{52}$$

(iii) *The operator M^{ϖ} is trace class, and its trace is given by*

$$\mathrm{Tr}(\mathsf{M}^{\varpi}) = \varpi(0,0)\,. \tag{53}$$

Proof. (i) The action of M^{ϖ} on ψ is given by:

$$\begin{aligned}(\mathsf{M}^{\varpi}\psi)(\gamma) &= \int_{\mathbb{S}^1}\mathrm{d}\gamma'\left(\frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\varpi(m,\gamma-\gamma')e^{\mathrm{i}\frac{m(\gamma+\gamma')}{2}}\right)\psi(\gamma')\\ &= \int_{\mathbb{S}^1}\mathrm{d}\gamma'\mathcal{M}^{\varpi}(\gamma,\gamma')\psi(\gamma')\,.\end{aligned}$$

(ii) The condition that M^{ϖ} be symmetric implies the following condition on the kernel:

$$\overline{\mathcal{M}^{\varpi}(\gamma,\gamma')} = \mathcal{M}^{\varpi}(\gamma',\gamma)$$

which gives:

$$\overline{\widetilde{\varpi}_{p_1}\left(\frac{\gamma+\gamma'}{2},\gamma-\gamma'\right)} = \widetilde{\varpi}_{p_1}\left(\frac{\gamma+\gamma'}{2},-(\gamma-\gamma')\right).$$

(iii) Therefore, the trace of M^{ϖ} corresponds the integral of the kernel over its diagonal; that is:

$$\int_{\mathbb{S}^1}\mathrm{d}\gamma\,\mathcal{M}^{\varpi}(\gamma,\gamma) = \frac{1}{2\pi}\int_{\mathbb{S}^1}\mathrm{d}\gamma\sum_{m\in\mathbb{Z}}\varpi(m,0)e^{\mathrm{i}\,m\gamma} = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\varpi(m,0)2\pi\delta_{m\,0} = \varpi(0,0).$$

□

In turn, one retrieves the weight ϖ from the quantization operator M^{ϖ} through a tracing operation.

Proposition 3. *The trace of the operator $U^{\dagger}(m,\theta)\mathsf{M}^{\varpi}$ is given by:*

$$\mathrm{Tr}[U^{\dagger}(m,\theta)\mathsf{M}^{\varpi}] = \varpi(m,\theta) \tag{54}$$

Proof. To compute this trace, one uses the expansion of the kernel in terms of the orthonormal Fourier basis $\left\{e_n(\gamma) = \frac{1}{\sqrt{2\pi}}e^{\mathrm{i}\,n\gamma}\right\}$ as follows

$$\begin{aligned}&\mathrm{Tr}[U^{\dagger}(m,\theta)\mathsf{M}^{\varpi}] = \sum_{n\in\mathbb{Z}}\langle\, e_n|\, U^{\dagger}(m,\theta)\mathsf{M}^{\varpi}e_n\rangle = \sum_{n\in\mathbb{Z}}\langle\, U(m,\theta)\, e_n|\, \mathsf{M}^{\varpi}e_n\rangle\\ &= \sum_{n\in\mathbb{Z}}\int_{\mathbb{S}^1}d\gamma\frac{1}{\sqrt{2\pi}}\,e^{\mathrm{i}\frac{m\theta}{2}}\,e^{-\mathrm{i}\,m\gamma}\,e^{-\mathrm{i}\,n\,(\gamma-\theta)}\frac{1}{2\pi}\sum_{m'\in\mathbb{Z}}\int_{\mathbb{S}^1}d\theta'\,\varpi(m',\theta')e^{-\mathrm{i}\frac{m'\theta'}{2}}e^{\mathrm{i}\,m'\gamma}\frac{1}{\sqrt{2\pi}}\,e^{\mathrm{i}\,n(\gamma-\theta')}\\ &= \frac{1}{(2\pi)^2}\sum_{m'\in\mathbb{Z}}\int_{\mathbb{S}^1}d\theta'\,\varpi(m',\theta')2\pi\delta(\theta-\theta')2\pi\delta_{m\,m'}\\ &= \varpi(m,\theta).\end{aligned}$$

□

5. Covariant Affine Integral Quantization from Weight Function

5.1. General Results

We now establish general formulae for the integral quantization issued from a weight function $\varpi(m,\theta)$ on $\Gamma = \mathbb{Z} \times \mathbb{S}^1$, yielding the bounded self-adjoint operator M^{ϖ} defined in (49). This allows us to build a family of operators obtained from the Weyl operator transport of M^{ϖ}:

$$\mathsf{M}^{\varpi}(m,\theta) = U(m,\theta)\mathsf{M}^{\varpi}U^{\dagger}(m,\theta)\,. \tag{55}$$

Then, the corresponding integral quantization is given by the linear map:

$$f \mapsto A_f^{\varpi} = \frac{1}{2\pi}\sum_{\mathbb{Z}}\int_{\mathbb{S}^1}\mathrm{d}\gamma\, f(m,\theta)\,\mathsf{M}^{\varpi}(m,\theta)\,. \tag{56}$$

We then obtain the following result.

Proposition 4. *A_f^{ϖ} is the integral operator on $L^2(\mathbb{S}^1,\mathrm{d}\gamma)$*

$$(A_f^{\varpi}\psi)(\gamma) = \int_{\mathbb{S}^1}\mathrm{d}\gamma'\,\mathcal{A}^{\varpi}(\gamma,\gamma')\psi(\gamma')\,, \tag{57}$$

and its kernel is given by

$$\mathcal{A}_f^{\varpi}(\gamma,\gamma') = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} F(\gamma-\gamma',n)\varpi(n,\gamma-\gamma')e^{\mathrm{i}\,n\frac{(\gamma+\gamma')}{2}}\,, \tag{58}$$

where $F(\varphi,k)$ is the Fourier transform of f with respect to the angle variable and the inverse (discrete) Fourier transform in the momentum variable,

$$F(\varphi,n) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_{\mathbb{S}^1}\mathrm{d}\theta\, f(k,\theta)e^{\mathrm{i}(k\varphi-n\theta)} \tag{59}$$

Proof. The calculation of the kernel of the integral operator A_f^{ϖ} goes through the following steps.

$$\begin{aligned}
(A_f^{\varpi})(\psi)(\gamma) &= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int_{\mathbb{S}^1}\mathrm{d}\theta\, f(m,\theta)\left(\mathsf{M}^{\varpi}(m,\theta)\psi\right)(\gamma)\\
&= \frac{1}{(2\pi)^2}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\int_{\mathbb{S}^1}\mathrm{d}\theta\int_{\mathbb{S}^1}\mathrm{d}\alpha\, f(m,\theta)\varpi(n,\alpha)\left(U(m,\theta)U(n,\alpha)U^{\dagger}(m,\theta)\psi\right)(\gamma)\\
&= \frac{1}{(2\pi)^2}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\int_{\mathbb{S}^1}\mathrm{d}\theta\int_{\mathbb{S}^1}\mathrm{d}\alpha\, f(m,\theta)\varpi(n,\alpha)e^{\mathrm{i}(m\alpha-n\theta)}(U(n,\alpha)\psi)(\gamma)\\
&= \frac{1}{(2\pi)^2}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\int_{\mathbb{S}^1}\mathrm{d}\theta\int_{\mathbb{S}^1}\mathrm{d}\alpha\, f(m,\theta)\varpi(n,\alpha)e^{\mathrm{i}(m\alpha-n\theta)}e^{-\mathrm{i}\frac{n\alpha}{2}}e^{\mathrm{i}\,n\gamma}\psi(\gamma-\alpha)\\
&= \frac{1}{(2\pi)^2}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\int_{\mathbb{S}^1}\mathrm{d}\theta\int_{\mathbb{S}^1}\mathrm{d}\gamma'\, f(m,\theta)\varpi(n,\gamma-\gamma')e^{\mathrm{i}(m(\gamma-\gamma')-n\theta)}e^{-\mathrm{i}\frac{n(\gamma-\gamma')}{2}}e^{\mathrm{i}\,n\gamma}\psi(\gamma')\\
&\equiv \int_{\mathbb{S}^1}\mathrm{d}\gamma'\left\{\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}F(\gamma-\gamma',n)\varpi(n,\gamma-\gamma')e^{\mathrm{i}\,n\frac{(\gamma+\gamma')}{2}}\right\}\psi(\gamma')
\end{aligned}$$

□

One easily determines that $f = 1$ gives $F(\varphi,k) = 2\pi\delta(\varphi)\delta_{k\,0}$. Inserting this in (58) gives:

$$\mathcal{A}_1^{\varpi}(\gamma,\gamma') = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}2\pi\delta(\gamma-\gamma')\delta_{n\,0}\varpi(n,\gamma-\gamma')\,e^{\mathrm{i}\,n\frac{(\gamma+\gamma')}{2}} = \delta(\gamma-\gamma')\varpi(0,0) = \delta(\gamma-\gamma') \tag{60}$$

for $\varpi(0,0) = 1$.

Covariance in the sense given by Equation (17) is easily proven in the present case.

Proposition 5. *The quantization map $f \mapsto A^{\varpi}_f$ is covariant with respect to the action of the representation V; that is,*

$$V\,A^{\varpi}_f V^{\dagger} = A^{\varpi}_{\mathcal{V}f} \tag{61}$$

where $(\mathcal{V}(s,m,\theta)f)(n,\varphi) = f(n-m,\varphi-\theta)$ is the induced action on the phase space.

Let us summarize the above set of results.

Proposition 6. *Let $\varpi(m,\theta)$ be a weight function on $\Gamma = \mathbb{Z}\times\mathbb{S}^1$ obeying*

$$\varpi(0,0) = 1\,, \quad \overline{\varpi(-m,\gamma)} = \varpi(m,-\gamma)\,, \tag{62}$$

and such that the operator M^{ϖ} defined in (49) is bounded. The operators $\mathsf{M}^{\varpi}(m,\theta) = U(m,\theta)\mathsf{M}^{\varpi}U^{\dagger}(m,\theta)$ are bounded self-adjoint, resolving the identity

$$\frac{1}{2\pi}\sum_{\mathbb{Z}}\int_{\mathbb{S}^1}\mathrm{d}\theta\,\mathsf{M}^{\varpi}(m,\theta) = \mathbb{1} \tag{63}$$

and yielding a covariant integral quantization of functions (or distributions) on the phase space Γ.

In the sequel, we suppose that the choice of a weight function complies with the conditions of Proposition 6.

5.2. Quantization of Separable Functions

As a first application, we consider the quantization of a separable function:

$$f(k,\theta) = g(k)\,h(\theta)\,. \tag{64}$$

In this case, the function F in (58) and the integral kernel read as

$$F(\gamma-\gamma',n) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_{\mathbb{S}^1}\mathrm{d}\theta\,g(k)h(\theta)e^{\mathrm{i}(k(\gamma-\gamma')-n\theta)} = \tilde{g}(\gamma-\gamma')\hat{h}(n)\,, \tag{65}$$

$$\mathcal{A}^{\varpi}_{gh}(\gamma,\gamma') = \tilde{g}(\gamma-\gamma')\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\hat{h}(n)\varpi(n,\gamma-\gamma')e^{\mathrm{i}\,n\frac{(\gamma+\gamma')}{2}}\,. \tag{66}$$

5.3. Quantization of Functions of Momentum Only

The restriction of (65) and (66) to a function of momentum only, $f(m,\theta) = g(m)$, yields for the function F and the corresponding integral kernel:

$$F(\gamma-\gamma',-n) = \tilde{g}(\gamma-\gamma')\delta_{n\,0}\,, \tag{67}$$

$$\begin{aligned}\mathcal{A}^{\varpi}_{g}(\gamma,\gamma') &= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\tilde{g}(\gamma-\gamma')\delta_{n\,0}\varpi(n,\gamma-\gamma')e^{-\mathrm{i}\,n\frac{(\gamma-\gamma')}{2}}\\ &= \frac{1}{2\pi}\tilde{g}(\gamma-\gamma')\varpi(0,\gamma-\gamma')\,,\end{aligned} \tag{68}$$

where $\tilde{g}(\gamma) = \sum_{m\in\mathbb{Z}} g(m)e^{\mathrm{i}\,m\gamma}$. This results in the following action of the operator A^{ϖ}_g:

$$(A^{\varpi}_g\psi)(\gamma) = \frac{1}{2\pi}\int_{\mathbb{S}^1}\mathrm{d}\gamma'\tilde{g}(\gamma'-\gamma)\varpi(0,\gamma-\gamma')\psi(\gamma')\,. \tag{69}$$

Let us give some elementary examples:

- Angular momentum $g(m) = m$, through integrating parts and appropriate derivability properties of the weight function,

$$(A_m^{\varpi}\psi)(\gamma) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int_{\mathbb{S}^1}\mathrm{d}\gamma'\left(\frac{\partial}{-\mathrm{i}\partial\gamma'}e^{\mathrm{i}\,m(\gamma-\gamma')}\right)\varpi(0,\gamma-\gamma')\psi(\gamma')$$
$$= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int_{\mathbb{S}^1}\mathrm{d}\gamma' e^{\mathrm{i}\,m(\gamma-\gamma')}\left\{-\mathrm{i}\frac{\partial}{\partial\gamma'} - \frac{\partial}{\partial\gamma'}[\mathrm{i}\varpi(0,\gamma-\gamma')]\right\}\psi(\gamma')$$
$$= \left(-\varpi(0,0)\,\mathrm{i}\frac{\partial}{\partial\gamma} - \frac{\partial}{\partial\gamma}[\mathrm{i}\varpi(0,\gamma)]_{\gamma=0}\right)\psi(\gamma) = \left(-\mathrm{i}\frac{\partial}{\partial\gamma} - \frac{\partial}{\partial\gamma}[\mathrm{i}\varpi(0,\gamma)]_{\gamma=0}\right)\psi(\gamma)\,,$$

and so, with the definition

$$\Omega(\gamma) := \varpi(0,\gamma)\,, \quad \Omega(0) = 1\,, \tag{70}$$

$$A_m^{\varpi} = -\mathrm{i}\frac{\partial}{\partial\gamma} - \frac{\partial}{\partial\gamma}[\mathrm{i}\varpi(0,\gamma)]_{\gamma=0} \equiv L - \Omega'(0)\,. \tag{71}$$

Hence, with a weight function obeying $\varpi(0,\gamma)]_{\gamma=0} = \Omega'(0) = 0$, we retrieve the usual angular momentum operator $L = -\mathrm{i}\frac{\partial}{\partial\gamma}$.

- Square angular momentum $g(m) = m^2$. Using similar methods and assumptions on the weight function, we find

$$(A_{m^2}^{\varpi}\psi)(\gamma) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int_{\mathbb{S}^1}\mathrm{d}\gamma'\left(-\frac{\partial^2}{\partial\gamma'^2}e^{\mathrm{i}\,m(\gamma-\gamma')}\right)\varpi(0,\gamma-\gamma')\psi(\gamma')$$
$$= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int_{\mathbb{S}^1} d\gamma' e^{\mathrm{i}\,m((\gamma-\gamma')}\{-\frac{\partial^2}{\partial\gamma'^2}[\varpi(0,\gamma-\gamma')\psi(\gamma')]$$
$$= \left(-\Omega(0)\frac{\partial^2}{\partial\gamma^2} + 2\Omega'(0)\frac{\partial}{\partial\gamma} - \Omega''(0)\right)\psi(\gamma)\,.$$

Finally,

$$A_{m^2}^{\varpi} = L^2 + 2\mathrm{i}\frac{\partial}{\partial\gamma}\Omega'(0)\,L - \Omega''(0)\,. \tag{72}$$

Remark 1. *At this point, it is valuable to compare the above quantization of functions defined on the discrete cylinder $\Gamma = \mathbb{Z}\times\mathbb{S}^1$ with the quantization of the same functions defined on the cylinder $\mathcal{C} := \mathbb{R}\times\mathbb{S}^1$, for instance that one using De Bièvre or similar coherent states on the circle (see [16] and references therein). Both approaches yield similar results (up to an additive constant) for the classical momentum, i.e., the angular momentum, and its square, i.e., the kinetic energy, of a particle moving on the circle. Is it a kind of elementary illustration of Hamiltonian reduction in the sense of Marsden and Weinstein [25]?*

5.4. Quantization of Function of Angular Position Only

We now turn to the quantization of a function of the position only, $f(m,\theta) = h(\theta)$. From

$$\tilde{f}(\gamma-\gamma',n) = \delta(\gamma-\gamma')\hat{h}(n)\,, \tag{73}$$

we obtain for the integral kernel:

$$\mathcal{A}_h^{\varpi}(\gamma,\gamma') = 2\pi\delta(\gamma-\gamma')\sum_{n\in\mathbb{Z}}\hat{h}(n)\,\varpi(n,\gamma-\gamma')e^{\mathrm{i}\,n\frac{(\gamma+\gamma')}{2}}\,. \tag{74}$$

This yields the multiplication operator

$$(A_h^{\varpi}\psi)(\gamma) = \left(\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\hat{h}(n)\,\varpi(n,0)e^{\mathrm{i}\,n\gamma}\right)\psi(\gamma) := h^{\varpi}(\gamma)\psi(\gamma)\,. \tag{75}$$

We observe that the coefficients of Fourier series of the factor $h^{\varpi}(\gamma)$ are the those of the original h multiplied by $\varpi(n,0)$.

6. Semi-Classical Portraits

Given a function $\varpi(m,\theta)$ on the phase space Γ, normalized at $\varpi(0,0)=1$, and yielding a positive unit trace operator, i.e., a density operator, M^{ϖ}, the quantum phase space portrait of an operator A on $L^2(\Gamma,\mathrm{d}\gamma)$ reads as:

$$\check{A}(m,\theta) := \mathrm{Tr}\Big(A\,U(m,\theta)\,\mathsf{M}^{\varpi}U^{\dagger}(m,\theta)\Big) = \mathrm{Tr}\big(A\,\mathsf{M}^{\varpi}(m,\theta)\big)\,. \tag{76}$$

The most interesting aspect of this notion in terms of probabilistic interpretation holds when the operator A is precisely the quantized version A^{ϖ}_f of a classical $f(m,\theta)$ with the same function ϖ (actually, we could define the transform with two different ones, one for the "analysis" and the other for the "reconstruction"). Then, with the use of the composition rule (33), we compute the transform:

$$\begin{aligned}
f(m,\theta) \mapsto \check{f}(m,\theta) &\equiv \check{A}^{\varpi}_f(m,\theta) = \mathrm{Tr}\Big(A^{\varpi}_f\,\mathsf{M}^{\varpi}(m,\theta)\Big)\\
&= \frac{1}{2\pi}\sum_{m'\in\mathbb{Z}}\int_{\mathbb{S}^1}\mathrm{d}\theta'\,f(m',\theta')\mathrm{Tr}\Big(U(m',\theta')\mathsf{M}^{\varpi}U^{\dagger}(m',\theta')\,U(m,\theta)\mathsf{M}^{\varpi}U^{\dagger}(m,\theta)\Big)\\
&= \frac{1}{2\pi}\sum_{m'\in\mathbb{Z}}\int_{\mathbb{S}^1}\mathrm{d}\theta'\,f(m',\theta')\mathrm{Tr}\big(\mathsf{M}^{\varpi}\,U(m-m',\theta-\theta')\mathsf{M}^{\varpi}U(-(m-m'),-(\theta-\theta'))\big)\\
&= \frac{1}{2\pi}\sum_{m'\in\mathbb{Z}}\int_{\mathbb{S}^1}\mathrm{d}\theta'\,f(m-m',\theta-\theta')\mathrm{Tr}\Big(\mathsf{M}^{\varpi}\,U(m',\theta')\mathsf{M}^{\varpi}U^{\dagger}(m',\theta')\Big)\\
&= \frac{1}{2\pi}\sum_{m'\in\mathbb{Z}}\int_{\mathbb{S}^1}\mathrm{d}\theta'\,f(m-m',\theta-\theta')\mathrm{Tr}\big(\mathsf{M}^{\varpi}\,\mathsf{M}^{\varpi}(m',\theta')\big)\,.
\end{aligned}$$

Finally, we get:

Proposition 7. *The semi-classical portrait of the operator A^{ϖ}_f with respect to the weight ϖ is given by:*

$$\check{f}(m,\theta) = \frac{1}{2\pi}\sum_{m'\in\mathbb{Z}}\int_{\mathbb{S}^1}\mathrm{d}\theta'\,f(m-m',\theta-\theta')\mathrm{Tr}\big(\mathsf{M}^{\varpi}(m',\theta')\,\mathsf{M}^{\varpi}\big)\,. \tag{77}$$

This expression, which can be viewed as a convolution on the phase space, has the meaning of an averaging of the classical f. The function

$$(m,\theta) \mapsto \mathrm{Tr}\big(\mathsf{M}^{\varpi}(m,\theta)\mathsf{M}^{\varpi}\big)\,, \tag{78}$$

is a true probability distribution on Γ, i.e., it is positive and with integral on Γ equal to 1, as demonstrated by the resolution of the identity and the fact that M^{ϖ} is chosen as a density operator. Expression (78) is actually a kind of Husimi function. The expression of $\mathrm{Tr}(\mathsf{M}^{\varpi}(m,\theta)\mathsf{M}^{\varpi})$ is easily derived, and reads as

$$\begin{aligned}
\mathrm{Tr}\big(\mathsf{M}^{\varpi}(m,\theta)\mathsf{M}^{\varpi}\big) &= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\langle\, e_n|\big(\,\mathsf{M}^{\varpi}(m,\theta)\mathsf{M}^{\varpi}\big)\,e_n\rangle\\
&= \frac{1}{2\pi}\sum_{m'\in\mathbb{Z}}\int_{\mathbb{S}^1}d\theta'|\varpi(m',\theta')|^2 e^{\mathrm{i}\,(m\theta'-m'\theta)}\,.
\end{aligned} \tag{79}$$

Integrating this expression on Γ, we get 1, which means that $\check{1}=1$, as expected.

7. Quantization with Various Weights

7.1. Weight Related to Coherent States for $\mathrm{H}_1^{\mathrm{dc}}$

The weight ϖ_ψ corresponding to the projection operator $|\psi\rangle\langle\psi|$ of a square integrable function ψ, with unit norm, on $\mathbb{S}^1$ is found through the trace formula given by Proposition 3:

$$\varpi_\psi(m,\theta) = \mathrm{Tr}[U^\dagger(m,\theta)|\psi\rangle\langle\psi|] = \langle U(m,\theta)\psi|\psi\rangle \tag{80}$$

In this case, from $\varpi_\psi = \langle U(m,\theta)\psi|\psi\rangle$, and from the Fourier expansion of $\psi \in L^2(\mathbb{S}^1, \mathrm{d}\gamma)$,

$$\psi = \sum_{m\in\mathbb{Z}} \hat{\psi}(m) e_m\,, \quad \hat{\psi}(m) = \langle e_m|\psi\rangle = \int_{\mathbb{S}^1} \mathrm{d}\gamma\, \frac{e^{-\mathrm{i}m\gamma}}{\sqrt{2\pi}}\,\psi(\gamma)\,, \tag{81}$$

for $\Omega_\psi(\gamma) = \varpi_\psi(0,\gamma)$, we have:

$$\Omega_\psi(\gamma) = \langle U(0,\gamma)\psi|\psi\rangle = \sum_{m\in\mathbb{Z}} e^{\mathrm{i}\,m\gamma}|\hat{\psi}(m)|^2 = \left\langle e^{\mathrm{i}\,m\gamma}\right\rangle_{|\hat{\psi}|^2}, \tag{82}$$

$$\Omega'_\psi(0) = \mathrm{i}\sum_{m\in\mathbb{Z}} m\,|\hat{\psi}(m)|^2 = \mathrm{i}\langle m\rangle_{|\hat{\psi}|^2}\,, \tag{83}$$

$$\Omega''_\psi(0)) = -\sum_{m\in\mathbb{Z}} m^2|\hat{\psi}(m)|^2 = -\langle m^2\rangle_{|\hat{\psi}|^2}\,, \tag{84}$$

where $\langle\cdot\rangle_{|\psi|^2}$ means the average of the random variable "$\cdot$" with respect to the discrete probability distribution $\mathbb{Z} \ni m \mapsto |\hat{\psi}(m)|^2$ (ψ has unit norm). We then derive the following quantizations of the angular momentum, of its square, and a function of the angle only.

- $g(m) = m$, and from (71), we get:

$$A_m^{\varpi_\psi} = -\mathrm{i}\frac{\partial}{\partial\gamma} + \langle m\rangle_{|\hat{\psi}|^2} = L + \langle m\rangle_{|\hat{\psi}|^2}\,. \tag{85}$$

- $g(m) = m^2$ and from (72), we get:

$$A_{m^2}^{\varpi_\psi} = L^2 + 2\langle m\rangle_{|\hat{\psi}|^2} L + \langle m^2\rangle_{|\hat{\psi}|^2}\,. \tag{86}$$

- $h(\theta)$ and from (80) we get the multiplication operator:

$$\begin{aligned}\left(A_h^{\varpi_\psi}\phi\right)(\gamma) &= \left[\sum_{m\in\mathbb{Z}} \hat{h}(m)\,\langle\, U(m,0)\psi|\psi\rangle\, e^{\mathrm{i}\,m\gamma}\right]\phi(\gamma)\\ &= \left[\sum_{m\in\mathbb{Z}} \hat{h}(m)\,\langle e^{-\mathrm{i}m\gamma}\rangle_{|\psi|^2}\, e^{\mathrm{i}\,m\gamma}\right]\phi(\gamma)\,.\end{aligned} \tag{87}$$

Note that for the CS weight (80), the expression (78) $\mathrm{Tr}(\mathsf{M}^{\varpi}(m,\theta)\mathsf{M}^{\varpi})$ is the Fourier transform of the Husimi function associated with ψ.

7.2. Weight Related to the Angular Parity Operator

The weight ϖ_{P} corresponding to the parity operator $\mathsf{P}\psi(\gamma) = \psi(2\pi-\gamma)$ on a square integrable function ψ on $\mathbb{S}^1$ is simply:

$$\varpi_{\mathsf{P}}(m,\theta) = 1\,. \tag{88}$$

Proof.

$$
\begin{aligned}
&2\,\mathrm{Tr}[U^\dagger(m,\theta)\mathsf{P}] = 2\sum_{n\in\mathbb{Z}}\langle\, e_n|U(m,\theta)^\dagger\mathsf{P}\, e_n\rangle\\
&= 2\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_{\mathbb{S}^1} d\gamma e^{-\mathrm{i}\,n\gamma}e^{-\mathrm{i}\frac{m\theta}{2}}e^{-\mathrm{i}\,m\gamma}e^{-\mathrm{i}\,n(\theta+\gamma)} = 2\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_{\mathbb{S}^1} d\gamma e^{-\mathrm{i}\,m(\gamma+\frac{\theta}{2})}e^{-\mathrm{i}\,n(\theta+2\gamma)}\\
&= 2\frac{1}{2\pi}\int_{\mathbb{S}^1} d\gamma\, e^{-\mathrm{i}\,m(\gamma+\frac{\theta}{2})}2\pi\delta(2\gamma+\theta) = 1\,.
\end{aligned}
$$

□

Defining a Wigner-like distribution of ψ on the phase $\mathbb{Z}\times\mathbb{S}^1$ as

$$
W_\psi(m,\theta) = \frac{1}{2\pi}\int_{\mathbb{S}^1} \mathrm{d}\gamma\, e^{-\mathrm{i}\,m\,\gamma}\overline{\psi\Big(\theta-\frac{\gamma}{2}\Big)}\psi\Big(\theta+\frac{\gamma}{2}\Big)\,, \tag{89}
$$

let us see how it is linked to the parity operator P and its corresponding uniform weight $\varpi_{\mathsf{P}}(m,\theta) = 1$. This was derived by Mukunda in 1979 [3]. It is interesting to compare the following result derived from the $\mathrm{H}_1^{\mathrm{dc}}$ symmetry with the one issued from the Weyl–Heisenberg H_1 symmetry [26].

Proposition 8. *The Wigner-like distribution of ψ on the phase $\mathbb{Z}\times\mathbb{S}^1$ is the mean value $\times 1/2\pi$ of the Weyl–Heisenberg transport of the parity operator in the state ψ :*

$$
W_\psi(m,\theta) = \frac{1}{2\pi}\langle\psi|\mathsf{P}(m,\theta)\psi\rangle\,,\quad \textit{with}\quad \mathsf{P}(m,\theta) = U(m,\theta)\mathsf{P}U(m,\theta)^\dagger\,. \tag{90}
$$

Proof.

$$
\begin{aligned}
W_\psi(m,\theta) &= \frac{1}{2\pi}\langle\psi|\,\mathsf{P}(m,\theta)\psi\rangle = \frac{1}{2\pi}\langle\psi|\,U(m,\theta)\,\mathsf{P}\,U^\dagger(m,\theta)\psi\rangle\\
&= \frac{1}{2\pi}\langle\, U^\dagger(m,\theta))\psi|\,\mathsf{P}\,U^\dagger(m,\theta)\psi\rangle\\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{S}^1}\mathrm{d}\gamma\, e^{\mathrm{i}\frac{m\theta}{2}}e^{\mathrm{i}\,m\gamma}\overline{\psi(\gamma+\theta)}\times\\
&\times\sum_{n\in\mathbb{Z}}\int_{\mathbb{S}^1}\mathrm{d}\varphi\, e^{\mathrm{i}\,n\,\gamma}e^{-\mathrm{i}\frac{n\varphi}{2}}e^{-\mathrm{i}\frac{m\theta}{2}}\,e^{-\mathrm{i}\,m\,(\gamma-\varphi)}\psi(\gamma-\varphi+\theta)\\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{S}^1}\mathrm{d}\gamma\int_{\mathbb{S}^1} d\varphi\, e^{\mathrm{i}\,m\,\varphi}\overline{\psi(\gamma+\theta)}\psi(\gamma-\varphi+\theta)\sum_{n\in\mathbb{Z}} e^{\mathrm{i}\,n(\gamma-\frac{\varphi}{2})}\\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{S}^1}\mathrm{d}\gamma\int_{\mathbb{S}^1} d\varphi\, e^{\mathrm{i}\,m\,\varphi}\overline{\psi(\gamma+\theta)}\psi(\gamma-\varphi+\theta)\,(2\pi)\,\delta\Big(\gamma-\frac{\varphi}{2}\Big)\\
&= \frac{1}{2\pi}\int_{\mathbb{S}^1}\mathrm{d}\varphi\, e^{\mathrm{i}\,m\,\varphi}\overline{\psi\Big(\theta+\frac{\varphi}{2}\Big)}\psi\Big(\theta-\frac{\varphi}{2}\Big)\,.
\end{aligned}
$$

□

Given a state ψ, one can also show that its Wigner distribution can be retrieved through the symplectic Fourier transform of its reproducing kernel; that is:

$$
\begin{aligned}
&\frac{1}{(2\pi)^2}\sum_{m\in\mathbb{Z}}\int_{\mathbb{S}^1}\mathrm{d}\gamma\, e^{\mathrm{i}\,(m'\theta-m\theta')}\langle\phi_{(m',\theta')}|\psi\rangle\\
&= \frac{1}{(2\pi)^2}\sum_{m'\in\mathbb{Z}}\int_{\mathbb{S}^1}\mathrm{d}\theta'\, e^{\mathrm{i}\,(m'\theta-m\theta')}\int_{\mathbb{S}^1}\mathrm{d}\gamma\, e^{\mathrm{i}\frac{m'\theta'}{2}}e^{-\mathrm{i}\,m'\gamma}\overline{\psi(\gamma-\theta')}\psi(\gamma)\\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{S}^1}\mathrm{d}\theta'\, e^{-\mathrm{i}\,m\theta'}\int_{\mathbb{S}^1}\mathrm{d}\gamma\,\overline{\psi(\gamma-\theta')}\psi(\gamma)2\pi\delta\Big(\theta+\frac{\theta'}{2}-\gamma\Big)\\
&= \frac{1}{2\pi}\int_{\mathbb{S}^1}\mathrm{d}\theta'\, e^{-\mathrm{i}\,m\theta'}\,\overline{\psi\Big(\theta-\frac{\theta'}{2}\Big)}\psi\Big(\theta+\frac{\theta'}{2}\Big)\,.
\end{aligned}
$$

Remark 2. *We observe that the Wigner function for ψ, which is given in Equation (89), is the m-th Fourier coefficient of the 2π-periodic complex valued function $\overline{\psi(\theta - \frac{\cdot}{2})}\psi(\theta + \frac{\cdot}{2})$. Due to Equation (63) and the normalization $\|\psi\|^2 = 1$, we have*

$$\frac{1}{2\pi}\sum_{\mathbb{Z}}\int_{\mathbb{S}^1} \mathrm{d}\theta\, W_\psi(m,\theta) = 1\,. \tag{91}$$

Hence, as in the case of the standard phase space $\mathbb{R}\times\mathbb{R}$, it is a normalized, not necessarily non-negative distribution on the discrete cylinder [3]. However, while the Wigner function for the former case is non-negative for Gaussian states, and more generally for standard coherent states. This is not true in the case of the discrete cylinder, as is well asserted by the recent study [17] on the circular CS, the Gaussian CS and the Gaussian–Fourier on the circle. As a matter of fact, it was proved in [27] that the Wigner function of a pure state $|\psi\rangle$ is non-negative if—and only if—$|\psi\rangle$ is an eigenstate of the angular momentum operator.

In this case of unit weight $\varpi_{\mathsf{P}} = 1$, we trivially have for $\Omega(\gamma) = \varpi_{\mathsf{P}}(0,\gamma) = 1$: $\Omega'(\gamma) = \Omega''(\gamma) = 0$. Hence, for the quantizations of elementary functions, we get

- $g(m) = m$, $A^{\varpi_{\mathsf{P}}}_m = L$,
- $g(m) = m^2$, $A^{\varpi_{\mathsf{P}}}_{m^2} = L^2$,
- $h(\theta)$,

$$(A^{\varpi_{\mathsf{P}}}_h \psi)(\gamma) = \left(\sum_{m\in\mathbb{Z}} \hat{h}(m)\,\frac{e^{\mathrm{i}\,m\gamma}}{\sqrt{2\pi}}\right)\psi(\gamma) = h(\gamma)\psi(\gamma)\,. \tag{92}$$

which means that:

$$(A^{\varpi_{\mathsf{P}}}_{\sin\theta}\psi)(\gamma) = \sin\gamma\,\psi(\gamma)\,, \quad (A^{\mathsf{P}}_{\cos\theta}\psi)(\gamma) = \cos\gamma\,\psi(\gamma)\,. \tag{93}$$

In this case, we can consider the position operator through the multiplication by the smooth Fourier exponential $e^{\mathrm{i}\gamma}$.

Alternatively, we can quantize the periodized angle function $(\mathrm{Per}\,\mathsf{A})(\theta)$. One finds the multiplication operator defined by the same discontinuous $(\mathrm{Per}\,\mathsf{A})(\gamma)$. There is no regularization. On the contrary, the choice of a smooth weight function allows us to obtain a multiplication via smooth regularization of this angle function.

Finally, for the angular parity weight, the expression in Equation (78) reduces to

$$\mathrm{Tr}\big(\mathsf{M}^{\varpi_{\mathsf{P}}}(m,\theta)\mathsf{M}^{\varpi_{\mathsf{P}}}\big) = \delta_{m0}\delta(\theta)\,.$$

In this case, the semi-classical portrait of $A^{\varpi_{\mathsf{P}}}_f$ is f, $\check{f}(m,\theta) = f(m,\theta)$.

8. Conclusions

In this paper, we have established a covariant integral quantization for systems moving on the circle with integral momentum, i.e, for systems whose phase space is the discrete cylinder $\mathbb{Z}\times\mathbb{S}^1$. The symmetry group of this phase space is the discrete and compact version of the Weyl–Heisenberg group, namely the central extension of the abelian group $\mathbb{Z}\times\mathrm{SO}(2)$, and the phase space can be viewed as the right coset of the group with its center. The existence of a non-trivial unitary irreducible representation of this group on the phase space, as acting on $L^2(\mathbb{S}^1)$, allowed us to derive interesting results. First, we have established the concomitant resolution of the identity and subsequent properties, such as the Gabor transform on the circle and its inversion, the reproducing kernel, and the fact that any square integrable function on the circle is a fiducial vector. Moreover, picking a weight function on the discrete cylinder with suitable properties allows us to build a bounded self-adjoint operator on $L^2(\mathbb{S}^1)$. The Weyl transported versions of this operator yield a resolution of the identity and the subsequent covariant quantization of functions or distributions defined on the discrete cylinder.

There are noticeable results related to the quantization of a point (m, θ) in the phase space according to two standard choices of the weight.

- With the parity weight, the quantization of the momentum is the expected angular momentum operator L.

$$m \mapsto \hat{m} = L\,, \quad L\psi(\gamma) = -\mathrm{i}\frac{\partial}{\partial\gamma}\psi(\gamma)\,. \tag{94}$$

  while the quantization of the angle yields the multiplication operator via the angle.

$$\theta \mapsto \hat{\theta}\,, \quad \hat{\theta}\psi(\gamma) = \gamma\psi(\gamma)\,. \tag{95}$$

  This is, of course, unacceptable. Alternatively, one can quantize the periodized angle function $(\mathrm{Per}\,\mathsf{A})(\theta)$. One finds the multiplication operator defined by the same discontinuous $(\mathrm{Per}\,\mathsf{A})(\gamma)$. There is no regularization.
- With the coherent state weight, one obtains the quantization of the momentum as the usual L plus an additional term, i.e., a kind of covariant derivative on the circle whose topology is now taken into account,

$$m \mapsto \hat{m} = L + \langle\, m\rangle_{|\psi|^2}\,, \tag{96}$$

  whereas the quantization of the periodized function theta leads to its smooth regularization,

$$(\mathrm{Per}\,\mathsf{A})(\theta) \mapsto \left[\sum_{m\in\mathbb{Z}} \hat{h}(m)\,\langle e^{-\mathrm{i}m\gamma}\rangle_{|\psi|^2}\, e^{\mathrm{i}\,m\gamma}\right]\phi(\gamma)\,. \tag{97}$$

In forthcoming work, we will extend the results of the present paper in three directions.

- Analyzing circular data (see, for instance, [28–30]). We expect that the formalism we have developed above will be useful for circular data or circular statistics. Some of the fiducial vectors we have considered here are probability densities in these areas, namely the uniform distribution, the shifted gaussian, and the Von Mises, Fejér and Poisson.
- Quantum systems for which the configuration space is $\mathrm{SO}(3)$: the group of rotations in three dimensions [31]. In this case [32], although the corresponding phase space is not a coset arising from a group, the Weyl formalism still applies.
- Stellar representation. In forthcoming work [33], we intend to link this formalism for both SO(2) and SO(3) to the so-called stellar representations [34,35] .

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Abbreviations

The following abbreviations are used in this manuscript:

POVM	Positive operator-valued measure
UIR	Unitary irreducible representation
CS	Coherent state

Appendix A. Some Fiducial Vectors & Reproducing Kernels

For simplicity in the table we denote $A(m, m', \theta, \theta')$ by A:

	Fiducial Vector	Reproducing Kernel
General	$\phi(\gamma)$	$\langle \phi_{(m,\theta)} \vert \phi_{(m',\theta')} \rangle$
Constant	$\frac{1}{\sqrt{2\pi}}$	$\delta_{m\,m'} e^{\mathrm{i}\,m\,(\theta-\theta')}$
Basis	$\frac{1}{\sqrt{2\pi}} \times e^{\mathrm{i}\,n\gamma}$	$A\,\delta_{m\,m'}$
Shifted gaussian	$(2\pi)^{-\frac{1}{2}} \theta\left(\frac{\gamma}{2\pi}, \frac{\sigma}{\sqrt{2\pi}}\right)$	$4\pi^2 A\, e^{-\frac{(m-m')^2}{2}} \sum_{n\in\mathbb{Z}} e^{-n^2} e^{\mathrm{i}\,n\,(\mathrm{i}(m-m')+(\theta-\theta'))}$
Dirichlet	$\frac{\sin(n+\frac{1}{2})\varphi}{\sin\frac{1}{2}\varphi}$	$\frac{A}{2\pi(n+1)} \frac{\sin((n+\frac{1}{2})(\theta-\theta'))}{\sin(\frac{\theta-\theta'}{2})}$
Fejer	$\frac{1}{n+1}\left[\frac{\sin((\frac{n+1}{2})\gamma)}{\sin\frac{1}{2}\gamma)}\right]^2$	$A\sum_{k=-n}^{k=n}\left(1-\frac{\vert k\vert}{n+1}\right)\left(1-\frac{\vert k-(m-m')\vert}{n+1}\right) e^{\mathrm{i}\,k(\theta-\theta')}$
Von Mises	$\frac{e^{\lambda\cos(\gamma)}}{\sqrt{2\pi\, I_0(2\lambda)}}$	$\frac{A}{2\pi\, I_0(2\lambda)} I_{m-m'}\left(2\lambda\, \cos\left(\frac{\theta-\theta'}{2}\right)\right)$

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