

Article

An Application of Quantum Logic to Experimental Behavioral Science

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Abstract: In 1933, Kolmogorov synthesized the basic concepts of probability that were in general use at the time into concepts and deductions from a simple set of axioms that said probability was a σ -additive function from a boolean algebra of events into $[0, 1]$. In 1932, von Neumann realized that the use of probability in quantum mechanics required a different concept that he formulated as a σ -additive function from the closed subspaces of a Hilbert space onto $[0, 1]$. In 1935, Birkhoff & von Neumann replaced Hilbert space with an algebraic generalization. Today, a slight modification of the Birkhoff-von Neumann generalization is called “quantum logic”. A central problem in the philosophy of probability is the justification of the definition of probability used in a given application. This is usually done by arguing for the rationality of that approach to the situation under consideration. A version of the Dutch book argument given by de Finetti in 1972 is often used to justify the Kolmogorov theory, especially in scientific applications. As von Neumann in 1955 noted, and his criticisms still hold, there is no acceptable foundation for quantum logic. While it is not argued here that a rational approach has been carried out for quantum physics, it is argued that (1) for many important situations found in behavioral science that quantum probability theory is a reasonable choice, and (2) that it has an arguably rational foundation to certain areas of behavioral science, for example, the behavioral paradigm of Between Subjects experiments.



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1. Introduction

Quantum logic has been applied in the social sciences, especially in economics and psychology. It has provided insights and alternative explanations to many puzzling behavioral phenomenon. It is a generalization of the logic that is designed to handle quantum phenomena in physics, and it is also a generalization of the classical propositional calculus. It is this second generalization that is the focus of this article, where it is viewed as an event space for probability functions with an application to the behavioral science paradigm of Between Subjects experiments.

The following simple example nicely illustrates the properties of quantum logic that apply throughout this article.

Figure 1 is an example of the kind of phenomena and properties of quantum logic that will be used in this article. The Firefly Box is due to Foulis [1]. With a slight change in notation, he writes the following about it:

The box is to have two translucent (but not transparent) windows, one on the front and one on the side. The remaining four sides of the box are opaque. At any given moment, the firefly might or might not have its light on. If the light is on, it can be seen as a blip by looking at either of the two windows.

Looking directly at the front window of the box when the light is on, one can tell by the position of the blip whether the firefly is in the left (*a*) or right (*b*) half of the box. Likewise, looking directly at the side window when the light is on, one can tell whether the firefly is in the front (*e*) or back (*f*) half of the box. Because the

windows are not transparent, one cannot rely on depth perception to determine from the front window whether the firefly is in the front or back half of the box, nor from the side window, whether the firefly is in the left or right half of the box. Now consider two experimental procedures F and S . Procedure F is conducted by looking directly at the front window and recording a , b , or c according to whether the blip is on the left, on the right, or there is no blip, respectively. Procedure S is conducted by looking directly at the side window and recording e , f , or d according to whether the blip is in the front, the back, or there is no blip, respectively. One cannot conduct both procedure S and F at the same time because of the necessity of looking directly at one window or the other. Indeed, if one stands in a position to see both windows, parallax could spoil the accuracy of the observation. (Foulis, 1999)

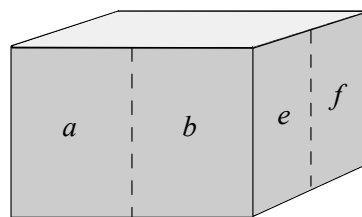


Figure 1. Firefly Box.

Let $X = \{a, b, c, d, e, f\}$ be the combined set of observations, and $\wp(X)$ stand for the power set of X . X generates a boolean algebra of events, $\mathfrak{X} = \langle \wp(X), X, \emptyset, \cup, \cap, -, \rangle$. Elements of $\wp(X)$, called “events”, and are partially ordered by the subset relation, $\subseteq$. Similarly, for $F = \{a, b, c\}$ let $\mathfrak{X}_1$ be the boolean algebra of events on $\wp(F)$, and for $S = \{d, e, f\}$, $\mathfrak{X}_2$ be the boolean algebra of events on $\wp(S)$.

Repeating procedure F provides complete data for the boolean algebra of events $\mathfrak{X}_1$; that is, the experimenter is able to count exactly the number of times each event in $\wp(X_1)$ has occurred *while viewing* F , and similarly for procedure S . However, this is not the case for the combined structure $\mathfrak{X}$: There are many events of $\wp(X)$ for which there is no data to account for their occurrence. An example is the intersection of the events $\{a\}$ and $\{f\}$: We can count the number of times a occurs and the number of times f occurs, but not the number of times they jointly occur; $\{a\} \cap \{f\}$ is not an observable event. Only observable events have number of occurrences *computed from data* associated with them. These are called *propositions*, and their number of (observed) occurrences is exact. Although non-observable events also have a number of occurrences associated with them—after all, they are events occurring in the real Firefly Box world—their numbers are not computable from data; such numbers are said to be “not exact”. A reasonable strategy for theory and data analysis is to only concern oneself with exact probabilities, that is, only concern oneself with propositions.

$\mathfrak{X}_1$ and $\mathfrak{X}_2$ have $2^3 = 8$ events; all are observable. $\mathfrak{X}$ has $2^6 = 64$ events. Because c if and only if d , that is when no light is observed in F , no light is observed S , and vice versa, and thus c and d can be identified. Call this identification k . This restriction reduces the 64 events of $\mathfrak{X}$ to 32. Of these, up to equivalence of membership, 12 are observable. We call them *definite events* because their members are known and therefore their size can be computed from data.

The event space for probability is generally assume to be a *boolean algebra of events*, which consists of a set X , called the *sure event*, and a nonempty subset $\mathcal{X}$ of subsets of X such that

- X and the empty set, $\emptyset$, are in $\mathcal{X}$,
- $X \neq \emptyset$, and
- if A and B are in $\mathcal{X}$, then $A \cup B$, $A \cap B$, and $-A$ are in $\mathcal{X}$, where $\cup$, $\cap$, and $-$ are, respectively, set-theoretic union, intersection, and complementation with respect to X .

Elements of $\mathcal{X}$ are called *events*, and the empty set, $\emptyset$, is called the *impossible* or *null event*. Elements of X are interpreted as *possible states* of the world, or just *states* for short. Among the possible states of the world is the *true state*, which is generally unknown. Associated with $\mathcal{X}$ is a *boolean probability function* $\mathbb{P}$. For events A in $\mathcal{X}$, $\mathbb{P}(A)$ is interpreted as the degree of belief that the true state of the world is in A . $\mathbb{P}$ is assumed to have the following properties for all A and B in $\mathcal{X}$:

- $\mathbb{P}(X) = 1$, $\mathbb{P}(\emptyset) = 0$, and $0 \subseteq \mathbb{P}(A) \subseteq 1$.
- *boolean finite additivity*: If $A \cap B = \emptyset$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

For some applications, a stronger condition than boolean finite additivity is needed:

- *boolean σ -additivity*: For each countable set of pairwise disjoint events $A_1, A_2, \dots$ in $\mathcal{X}$ such that $\bigcup_{i=1}^{\infty} A_i$ is in $\mathcal{X}$,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

This article focuses on the more general case of boolean finite additivity.

1.1. Background: Lattices

The concept of a boolean algebra of events is first generalized to a “lattice algebra of events”, then later specialized to the kind of algebra of events used quantum mechanics, which is here called “quantum logic”. The key idea is to formulate the generalization of $\cup$ and $\cap$ in terms of the set-theoretic subset relation, $\subseteq$.

In a boolean algebra of events $\cup$ and $\cap$ has the following characterization: For all A and B in $\mathcal{X}$,

- $A \cup B$ = the $\subseteq$ -smallest set C in $\mathcal{X}$ such that $A \subseteq C$ and $B \subseteq C$, and
- $A \cap B$ = the $\subseteq$ -largest set D in $\mathcal{X}$ such that $D \subseteq A$ and $D \subseteq B$.

The above definition of $A \cup B$ says that C is the $\subseteq$ -least upper bound in $\mathcal{X}$ of A and B and that $A \cap B$ is the $\subseteq$ -greatest lower bound of A and B in $\mathcal{X}$. This suggests the following straightforward generalization:

Definition 1. $\mathfrak{X} = \langle \mathcal{X}, \sqcup, \sqcap, X, \emptyset \rangle$ is said to be a *lattice algebra of events* if and only if $X \neq \emptyset$, $\mathcal{X}$ is a set of subsets of X , and $\sqcup$ and $\sqcap$ are binary operations on $\mathcal{X}$ such that the following statements hold for all A and B in $\mathcal{X}$:

- X and $\emptyset$ are in $\mathcal{X}$ and $X \neq \emptyset$.
- $A \sqcup B$ is the $\subseteq$ -smallest element of $\mathcal{X}$ such that $A \subseteq A \sqcup B$ and $B \subseteq A \sqcup B$.
- $A \sqcap B$ is the $\subseteq$ -largest element of $\mathcal{X}$ such that $A \sqcap B \subseteq A$ and $A \sqcap B \subseteq B$.

The algebra inherent in boolean algebra of events is captured by the following definition and theorem.

Definition 2. Let $\mathfrak{X} = \langle \mathcal{X}, \sqcup, \sqcap, X, \emptyset \rangle$ be a lattice algebra of events.

$\dot{\cdot}$ is said to be a *complementation operation* on $\mathfrak{X}$ if and only if $\dot{\cdot}$ is a unary operation on $\mathcal{X}$ such that for all A in $\mathcal{X}$,

$$A \sqcup \dot{\cdot} A = X \text{ and } A \sqcap \dot{\cdot} A = \emptyset.$$

$\mathfrak{X}$ is said to be *complemented* if and only if there exists a complementation operation on $\mathfrak{X}$.

$\mathfrak{X}$ is said to be *distributive* if and only if for all A, B , and C in $\mathcal{X}$,

$$A \sqcap (B \sqcup C) = (A \sqcap B) \sqcup (A \sqcap C).$$

A lattice algebra of events is said to be *boolean* if and only if it is complemented and distributive. (Note the distinction between a boolean algebra of events and a boolean lattice algebra of events: The boolean algebra has the operations of set theoretic union, $\cup$, intersection, $\cap$, and com-

plementation, $\neg$, whereas a boolean lattice algebra has the more generally defined operations, $\sqcup$, $\sqcap$, and $\neg$.

Throughout this article, $\mathfrak{X} = \langle \mathcal{X}, \sqcup, \sqcap, \neg, X, \emptyset \rangle$, where $\mathfrak{X}$ is distributive and $\neg$ is a complementation operation will usually describe a boolean lattice algebra of events.

Stone [2] showed the following theorem:

Theorem 1. Each boolean lattice algebra of events is isomorphic to a boolean algebra of events.

Definition 3. Let $\mathfrak{X} = \langle \mathcal{X}, \sqcup, \sqcap, \neg, X, \emptyset \rangle$ be a complemented lattice algebra of events. Then

- $\mathfrak{X}$ is said to satisfy DeMorgan's Laws if and only if for all A and B in X ,

$$\neg(A \sqcap B) = \neg A \sqcup \neg B \text{ and } \neg(A \sqcup B) = \neg A \sqcap \neg B.$$

The following well-known theorem of lattice theory provides an often used alternative formulation of DeMorgan's Laws.

Theorem 2. Let $\mathfrak{X} = \langle \mathcal{X}, \subseteq, \sqcup, \sqcap, \neg, X, \emptyset \rangle$ be a complemented lattice. Then the following two statements are equivalent:

1. $\mathfrak{X}$ satisfies DeMorgan's Laws.
2. For all A and B in $\mathcal{X}$,
 - (i) $A \subseteq B$ if and only if $\neg B \subseteq \neg A$, and
 - (ii) $\neg \neg A = A$.

The following is also a well-known theorem:

Theorem 3. Suppose $\mathfrak{X} = \langle \mathcal{X}, \sqcup, \sqcap, \neg, X, \emptyset \rangle$ is a boolean lattice algebra of events. Then $\mathfrak{X}$ satisfies DeMorgan's Laws.

The lattice literature calls a complementation operation satisfying De Morgan's Laws an "orthocomplementation operation", and the special symbol $^\perp$ is used to denote it. A lattice algebra of events with an orthocomplementation operation $^\perp$ is called an "ortholattice". Formally, we have the following definitions.

Definition 4. $\mathfrak{X} = \langle \mathcal{X}, \sqcup, \sqcap, ^\perp, X, \emptyset \rangle$ is said to be an ortholattice with orthocomplementation operation $^\perp$ if and only if it is a complemented lattice algebra of events and $^\perp$ satisfies De Morgan's Laws, that is, for all A and B in $\mathcal{X}$,

$$(A \sqcup B)^\perp = A^\perp \sqcap B^\perp \text{ and } (A \sqcap B)^\perp = A^\perp \sqcup B^\perp.$$

1.2. Background: Quantum Logic

Theorem 3 below suggests that a complemented lattice satisfying DeMorgan's laws with a generalization of the distributive law is one natural starting place to find candidates for generalizing boolean event space. For such candidates to be proper generalizations of boolean event spaces, they must be nondistributive. Birkhoff & von Neumann [3] took this approach for their logic for quantum mechanics. Their candidate logic had, in addition to De Morgan's Laws, is the following generalization implied by distributivity: For all events A , B , and C in the algebra,

$$\textbf{Modular Law:} \text{ If } A \subseteq B \text{ then } B \sqcap (A \sqcup C) = (B \sqcap A) \sqcup (B \sqcap C). \quad (1)$$

Quantum physics is often formulated in terms of Hilbert space. Finite dimensional Hilbert space is an ortholattice of the form, $\langle \mathcal{X}, \sqcup, \sqcap, ^\perp, X, \emptyset \rangle$, where $\mathcal{X}$ is the set of subspaces of the Hilbert space, and for A and B in $\mathcal{X}$, $A \sqcup B$ = the subspace spanned by A and B , and $^\perp$ is the operation of orthogonal complementation. (Note the use here of the

set-theoretic intersection operation $\cap$ in the definition of Hilbert Space ortholattice.) For quantum physics, infinite dimensional Hilbert space is sometimes required, and for that case $\mathcal{X}$ is the set of all closed subspaces. (In the finite dimensional case, all subspaces are closed.) Finite dimensional Hilbert space satisfies the Modular Law. Infinite dimensional Hilbert space requires a weaker law called “orthomodularity”. An ortholattice satisfying orthomodularity is called a “quantum logic”:

Definition 5. An ortholattice $\mathfrak{X} = \langle \mathcal{X}, \sqcup, \sqcap, ^\perp, X, \emptyset \rangle$, is said to be orthomodular if and only if for all A and B in $\mathcal{X}$,

$$\text{orthomodular law: if } B \subseteq A \text{ then } A = B \sqcup (B^\perp \sqcap A).$$

A lattice is said to be a quantum logic if and only if it is an ortholattice that is orthomodular.

It is well known that distributivity implies modularity and modularity implies orthomodularity.

Quantum logics have a rich algebraic structure that makes some of them interesting candidates for event spaces for probability functions. Many concepts in orthomodular lattice theory are algebraic abstractions of concepts of Hilbert space. One of the most important is “orthogonality”.

Definition 6. Let $\mathfrak{X} = \langle \mathcal{X}, \sqcup, \sqcap, ^\perp, X, \emptyset \rangle$ be a quantum logic and A and B be in $\mathcal{X}$. Then A is said to be orthogonal to B , in symbols, $A \perp B$, if and only if $B \subseteq A^\perp$.

Definition 7. Let $\mathfrak{X} = \langle \mathcal{X}, \sqcup, \sqcap, \div, X, \emptyset \rangle$ be lattice algebra of events.

- A in $\mathcal{X}$ is said to be an atom (of $\mathcal{X}$) if and only if $\emptyset \subset A$ and for all B in $\mathcal{X}$, if $\emptyset \subseteq B \subseteq A$ then $B = \emptyset$ or $B = A$.
- $\mathfrak{X}$ is said to be atomic if and only if for each B in X , if $B \neq \emptyset$ then there exists an atom A of $\mathfrak{X}$ such that $A \subseteq B$.

It is well-known that finite lattices are atomic and that a finite quantum logic satisfies the property that for each of its elements A ,

$$A = \{a_1\} \sqcup \{a_2\} \sqcup \dots \sqcup \{a_{n-1}\} \sqcup \{a_n\}, \text{ where } \{a_1, \dots, a_n\} \quad (2)$$

is the set of atoms a such that $a \subseteq A$.

The next section constructs a quantum logic where $^\perp = -$. The following theorem of Calude, Hertling, and Svozil [4] shows that in general this is permissible:

Theorem 4. Let $\mathfrak{X} = \langle \mathcal{X}, \sqcup, \sqcap, ^\perp, X, \emptyset \rangle$ be a quantum logic. Then there is an isomorphic quantum logic,

$$\mathfrak{X}' = \langle \mathcal{X}', \sqcup', \sqcap', -, X', \emptyset \rangle,$$

with $-$ as its orthocomplementation operator.

Proof. Section 2.4 of Calude, Hertling, and Svozil (1999). $\square$

Calude, Hertling, and Svozil (1999) also show similar results for $\cup$ and $\cap$:

Theorem 5. Let $\mathfrak{X} = \langle \mathcal{X}, \sqcup, \sqcap, ^\perp, X, \emptyset \rangle$ be a quantum logic. Then there are isomorphic quantum logics of the form,

$$\mathfrak{X}'' = \langle \mathcal{X}'', \cup, \cap, ^{\perp''}, X'', \emptyset \rangle$$

and

$$\mathfrak{X}''' = \langle \mathcal{X}''', \cap, \cup, ^{\perp'''}, X''', \emptyset \rangle.$$

Thus, one can always choose one of the operations $\cup, \cap, -$ in place of, respectively, $\mathbb{U}, \mathbb{M}, \perp$. For example, in modeling quantum physics, $\cap$ is chosen for $\mathbb{M}$. One cannot model non-distributive quantum logic with $\cup$ and $-$, because, by DeMorgan's Laws, the result would be a distributive logic; similarly for $\cap$ and $-$. The quantum logic discussed in the following section chooses $-$ for $\perp$.

Harding & Pták [5] show the following theorem.

Theorem 6. Let $\mathfrak{X} = \langle \mathcal{X}, \mathbb{U}, \mathbb{M}, \perp, X, \emptyset \rangle$ be a quantum logic and $\mathcal{B}$ be a boolean lattice subalgebra of $\mathfrak{X}$. Then there exist a set Y and a mapping φ from X into the powerset of Y , $\wp(Y)$, such that for all A and B in $\mathcal{B}$,

- $\varphi(X) = Y$;
- $\varphi(A^\perp) = -A$;
- $A \perp B$ if and only if $\varphi(A) \cap \varphi(B) = \emptyset$;
- $\varphi(A \mathbb{U} B) = \varphi(A) \cup \varphi(B)$.

The following section concerns a social science application of quantum logic. It uses a representation for a quantum logic where $\perp = -$. Its events have probabilities associated with them through relative frequencies, and it is argued in that section that the system of resulting probabilities is coherent, that is, displays rationality. Section 3 also provides concluding remarks.

2. Between Subjects Experiments

Modeling methods inspired by quantum mechanics have recently entered into the social sciences in order to explain many of the context and order effects routinely found in social science experiments. These effects often produce empirically observed failures of boolean finite additivity. This section explores an empirically based theory designed to avoid order and context effects—the popular “Between Subjects paradigm”. The ideas and method generalize to more complicated situations—even ones involving infinitely many experiments, each with infinitely many outcomes. Going from the simple finite case described in this section to more general ones just requires additional technical methods, e.g., Abraham Robinson’s “nonstandard analysis”.

The experimental situation considered throughout this article is similar to those in [6,7], except that this article’s definition of a quantum probability function applies to more situations than the orthoprobability functions of Narens (2016 a.b), and changes to notation and proofs have been made to accommodate for this.

The experimental situation considered in this section assumes a large population of subjects and an overall experiment E involving six distinct outcomes, a, b, c, d, e, f . E is divided into two sub-experiments, E_1 and E_2 . A large number of subjects for E is assumed, half randomly assigned to E_1 and half to E_2 . E_1 has outcomes, a, b, c , and E_2 outcomes d, e, f . Each subject selects one outcome from her assigned experiment. Attached to subject i , are binary pairs $(x, y)_i$, where x is in $\{a, b, c\}$ and y is in $\{d, e, f\}$. The interpretation of $(x, y)_i$ is that before experiment E is performed, subject i will choose x if put in E_1 and will choose y if put in E_2 .

The collected data about E comes from the two sub-experiments. This results that some events A in E do not have data that informs the number of times A has occurred. This is because that that information is non-accessible to the observer. For example, data has been collected in E_1 that tells how many times the outcome a has occurred, and data in E_2 about how many times the outcome f has occurred, but no data collected about E tells many times the joint event $\{a, f\}$ has occurred. The joint event $\{a, f\}$ is not observable in experiment E . In this article, we take the position that even though the exact number of times $\{a, f\}$ has occurred is *non-accessible*, it *did* occur some exact number of times. Increasing the number of subjects, does not help in estimating that number.

With slight abuse of notation, we will identify throughout this article the outcome x of an experiment with the event $\{x\}$ of that experiment.

Convention: Throughout the rest of this section, let S stand for the set of subjects participating in E , S_1 stand for those participating in E_1 , and S_2 stand for those participating in E_2 . Let $X = \{a, b, c, d, e, f\}$, $X_1 = \{a, b, c\}$, $X_2 = \{d, e, f\}$, and $X_1 \cap X_2 = \emptyset$,

$\mathfrak{X} = \langle \wp(X), \cup, \cap, -, X, \emptyset \rangle$ be the boolean algebra of events for E ,

$\mathfrak{X}_1 = \langle \wp(X_1), \cup, \cap, -, X_1, \emptyset \rangle$ be the boolean algebra of events for E_1 ,

$\mathfrak{X}_2 = \langle \wp(X_2), \cup, \cap, -, X_2, \emptyset \rangle$, be the boolean algebra of events for E_2 .

For each finite set Z , let $|Z|$ stand for the number of elements in Z . Let S be the set of subjects participating in experiment $\mathfrak{X}$. Then, by the way S_1 and S_2 were chosen,

$$|S_1| = |S_2| = \frac{1}{2}|S|. \quad (3)$$

For each i in S and x and y in X , we say that $(x, y)_i$ is the *choice set of i* if and only if before running experiment $\mathfrak{X}$, i would choose x if placed in S_1 and would choose y if placed in S_2 .

For each A in $\wp(X)$, let

- $\ll A \gg = \{i \mid \text{for } i\text{'s choice set, } (x, y)_i, \text{ either } x \in A \text{ or } y \in A\}$.

Because S is a large set and X is a small set (i.e., has six elements) and equal numbers of subjects in S were randomly assigned to experiments E_1 and E_2 , there exists a *near* one-to-one correspondence α between S_1 and S_2 such that for nearly all individuals i in S_1

$$(x, y)_i \text{ iff } (x, y)_{\alpha(i)}.$$

That is, for all x in X_1 ,

if $i \in S_1$ and i chooses x , then $\alpha(i)$ is in S_2 and $\alpha(i)$ would also have chosen x if placed in S_1 instead of S_2 ,

and similarly, for all y in X_2 ,

if $\alpha(i) \in S_2$ and $\alpha(i)$ chooses y , then i is in S_1 and i would also have chosen y if placed in S_2 instead of S_1 ,

In experiment E_1 , if x is a choice of i , then x is also a choice of $\alpha(i)$ in experiment E_2 and vice versa, and similarly for experiment E_2 . “Near” is used here to indicate there could be a small error in the one-to-one correspondence that goes to 0 as the relative frequencies go to ∞ . For purposes of convenience and exposition, we ignore this error. The correspondence $\alpha(i)$ captures the idea that the set of choice behaviors of subjects in S_1 and S_2 are going to be the same due to the equal and random split of subjects.

For each A in $\wp(X)$, let $\#A$ be the number of subjects that chose A when experiment $\mathfrak{X}$ was run, and let,

$$\mathbb{P}(A) = \frac{\#A}{\#X}.$$

For each B in $\wp(X_1)$, let $\#B$ be the number of subjects that chose B when experiment E_1 was run, and let,

$$\mathbb{P}_1(B) = \frac{\#B}{\#X_1}.$$

Furthermore, for each C in $\wp(X_2)$, let $\#C$ be the number of subjects that chose C when experiment E_2 was run, and let,

$$\mathbb{P}_2(C) = \frac{\#C}{\#X_2}.$$

Then $\mathbb{P}$, $\mathbb{P}_1$, and $\mathbb{P}_2$, are, respectively, boolean probability functions on $\mathfrak{X}$, $\mathfrak{X}_1$, and $\mathfrak{X}_2$. Furthermore, because $|S_1| = |S_2| = \frac{1}{2}|S|$ (Equation (3)), $\mathbb{P}$ restricted to X_1 is $\mathbb{P}_1$ and $\mathbb{P}$ restricted to X_2 is $\mathbb{P}_2$.

The probability $\mathbb{P}$ can be computed from data for some events in $\wp(X)$, e.g., the events $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{e\}$, $\{f\}$, but not for others, e.g., event the $\{a, f\}$. Those that it can be computed for are called “propositions”. We will show that set of propositions form a quantum logic. To do this, we introduce the following notation and definitions:

- For each G in $\wp(X)$, let

$$\ll G \gg = \{i \in S \mid \text{for some } (x, y)_i, \text{ either } x \in G \text{ or } y \in G\}.$$

- Events A and B are said to be *equivalent*, in symbols, $A \equiv B$, if and only if $\ll A \gg = \ll B \gg$. It easily follows that $\equiv$ is an equivalence relation.
- It is convenient and useful to identify each $\equiv$ -equivalent class $\mathcal{K}$ with its $\subseteq$ -largest element. (Such an element exists, because $\bigcup\{F \mid F \in \mathcal{K}\}$ is in $\mathcal{K}$ and is $\mathcal{K}$'s largest member.)
- For each $G \in \wp(X)$, let $\lambda(G)$ be the $\subseteq$ -largest element of the $\equiv$ -equivalent class to which G belongs.
- For each $G \in \wp(X)$, let $\sigma(\ll G \gg) = \lambda(G)$.
- H is said to be a *proposition* if and only if
 - $\ll H \gg$ is (exactly) computable from data, and
 - $H = \lambda(H)$.

For each F in $\wp(X)$ such that $\ll F \gg$ is (exactly) computable from data, there exists a proposition that is $\equiv$ to F . In some cases different F can have the same proposition equivalent to it. For example, $X_1 \neq X_2$, but

$$\sigma(\ll X_1 \gg) = \sigma(\ll X_2 \gg) = \sigma(\ll X \gg).$$

- Let P stand for the set of propositions.

This article takes the perspective that $\mathbb{P}$ is the correct function for describing probabilities on $\mathfrak{X}$. Although correct, it is somewhat problematic: We know some of its probabilities exactly while others only vaguely with their probabilities somewhere between two exact ones. Furthermore, increasing the number of subjects does not change this, including taking the relative frequency approach to the limit. This suggests restricting the probability function to events with exact probabilities; that is, restricting the probability function to propositions. This article shows that for Between Subjects experiments, a surprising rich theory results: a quantum logic with a probability function that assign to events their relative frequencies. The following describes the steps in achieving this.

First the operations $\mathbb{U}$, $\mathbb{M}$, and $\perp$ are defined on P as follows:

Definition 8. For all propositions F and G in P ,

- $F^\perp = \sigma(- \ll F \gg)$.
- $F \mathbb{U} G = \sigma(\ll F \gg \cup \ll G \gg)$.
- $F \mathbb{M} G = \sigma(\ll F \gg \cap \ll G \gg)$.

Using these definitions, the following theorem results:

Theorem 7. $\mathfrak{P} = \langle P, \mathbb{U}, \mathbb{M}, \perp, X, \emptyset \rangle$ is a quantum logic, where $\perp$ is set-theoretic complementation, $-$, and $\mathbb{P}$ assigns to propositions in P their relative frequencies, and $\mathbb{P}$ is additive on P in the sense that

$$\text{if } C \text{ and } D \text{ in } P \text{ are such that } D \subseteq -C, \text{ then } \mathbb{P}(C \mathbb{U} D) = \mathbb{P}(C) + \mathbb{P}(D).$$

Proof. Lemma A6 of the Appendix A shows $\mathfrak{P}$ is an ortholattice. Thus, to show it is a quantum logic, it needs only to be shown that the orthomodular law holds. Suppose A and B are in P and $B \subseteq A$. We need to show that

$$B \mathbb{U} (A \mathbb{M} B^\perp) = A.$$

Thus, will be done by noting that both $B \uplus (A \mathbin{\lrcorner} B^\perp)$ and A are propositions and showing,

$$\ll B \uplus (A \mathbin{\lrcorner} B^\perp) \gg = \ll A \gg.$$

By the definitions of “ $\uplus$ ” and “ $\mathbin{\lrcorner}$ ”,

$$\ll B \uplus (A \mathbin{\lrcorner} B^\perp) \gg = \ll B \cup (A \mathbin{\lrcorner} B^\perp) \gg \text{ and } \ll (A \mathbin{\lrcorner} B^\perp) \gg = \ll A \cap B^\perp \gg.$$

Thus, because $B \subseteq A$ and $^\perp = -$,

$$\ll B \uplus (A \mathbin{\lrcorner} B^\perp) \gg = \ll B \cup (A \cap -B) \gg = \ll A \gg.$$

$\mathbb{P}$ is additive because it is a boolean probability function on $\mathfrak{X}$ and, in $\mathfrak{P}$, for all C and D in $\mathfrak{P}$ such that $D \subseteq -C$,

$$\ll C \uplus D \gg \text{ iff } \ll C \cup D \gg \text{ iff } \ll C \gg \cup \ll D \gg,$$

and thus that

$$C \uplus D \text{ iff } C \cup D,$$

because C and D are propositions. $\square$

Note that Theorem 7 results from empirical considerations about Between Subjects experiments: All the definitions of the relations making up $\mathfrak{P}$ are empirically based and are computable from the collected data from a Between Subjects experiment.

However, there is more. $\mathbb{P}$ is also a boolean probability function on

$$\mathfrak{X} = \langle \wp(X), \cup, \cap, -, X, \emptyset \rangle.$$

Section 3 argues that this makes $\mathbb{P}$ a *rational* assignment of probabilities to $\mathfrak{P}$.

Figure 2 diagrams the lattice of propositions for experiment E with outcomes c and d satisfying $c \text{ iff } d$. It is also a diagram of the lattice of propositions resulting from the firefly example of the Introduction. In the diagram, k is used to replace c and d , because the choices k provide the same information across experiments as c combined with d . (This idea can be used to extend the Between Subjects paradigm to include cases with the experiments having common outcomes.)

Figure 2 is a Hasse diagram of the lattice $\mathfrak{P} = \langle \mathfrak{P}, \uplus, \mathbin{\lrcorner}, ^\perp, X, \emptyset \rangle$. The set-theoretic boolean algebra generated by X has $2^6 = 64$ elements. The elements at the bottom of Figure 2 are called *atoms*. They are lattice elements F such that there does not exist a lattice element G such that $\emptyset \subset G \subset F$. (Note that these atoms of $\mathfrak{P}$ are in boldface because they are not just events; they are propositions, e.g., $\mathbf{a} = \{a, e, f\}$). Figure 2 has 5 atoms, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{k}$, $\mathbf{e}$, $\mathbf{f}$. These atoms generate a set-theoretic boolean algebra consisting of $2^5 = 32$ elements. $\mathfrak{P}$ has 12 elements—a substantial reduction from 32.

In Figure 2, the *lattice-theoretic intersection* $\mathbin{\lrcorner}$ of atoms, e.g., $\mathbf{a} \mathbin{\lrcorner} \mathbf{f}$, is the proposition $\emptyset$. However, the *set-theoretic intersection* of atoms need not be $\emptyset$, e.g., as propositions $\mathbf{a} = \lambda(a) = \{a, e, f\}$ and $\mathbf{f} = \lambda(f) = \{a, b, f\}$, and thus $\mathbf{a} \cap \mathbf{f} = \{a, f\}$, but $\{a, f\}$ is not in the displayed lattice; the largest $\subseteq$ -element in the lattice that is displayed as a proper subset of $\{a, f\}$ is $\emptyset$.

Busemeyer & Bruza [8] and others have noted that there are many experimental designs in the social sciences that are structurally similar to those in quantum mechanics. Various researchers have taken advantage of this, and have applied methods of quantum mechanics to model social science phenomena. Quantum-like models like these often provide different insights into the mechanisms that generate the experimental data, while often providing equal or better fit to the data.

A criticism of this kind of “quantum mechanical” modeling in the social sciences is that its foundational principles are not adequately justified. For the social sciences, much of fundamental symmetry inherent in quantum physics phenomena is obscure at best and

likely does not exist. This is not surprising, because the foundation of quantum mechanics was designed to model the kinds of symmetry found in physical quantum phenomena. An exception appears to be the use of closed subspaces of a Hilbert space as an event space. It applies to many other phenomena. It also provides a simple and useful way to model some dynamic aspects of cognitive phenomena.

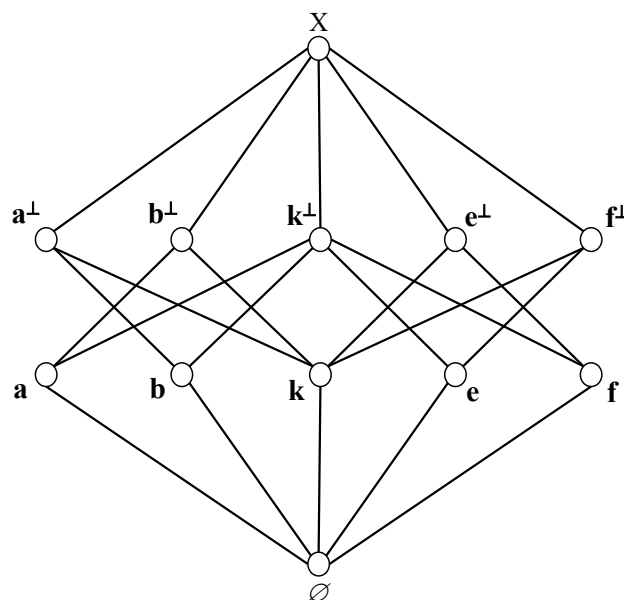


Figure 2. The lattice of propositions $\mathfrak{P} = \langle P, \sqcup, \sqcap, \perp, X, \emptyset \rangle$ for experiment E (with $\perp = -, k$ replacing c and d , $X_1 = \{a, b, k\}$, and $X_2 = \{k, e, f\}$, because of the theoretical assumption c iff d). In this figure, the $\subseteq$ relation is illustrated by moving along a line upward from an open circle (representing a proposition) to another open circle. $X = \{a, b, k, e, f\}$ is the $\subseteq$ -largest element of the lattice, $\emptyset$ the $\subseteq$ -smallest element, and $\perp$ is an ortho-complementation operation. Boldface letters represent propositions, e.g., **a** represents the proposition $\lambda(a) = \{a, e, f\}$.

3. Concluding Remarks

Heyting algebras, Hilbert space, and quantum logics are three kinds of event spaces that have recently appeared in the literature for modeling probabilistic social science phenomena. They all appear to be useful in providing alternative explanations for unintuitive experimental findings in the cognitive science literature as well providing insights into underlying cognitive mechanisms. All three provide richer mathematical event spaces and probabilistic concepts than boolean probability theory. In psychology, they have been mainly used to explain phenomena where boolean explanations appear inadequate. Although different from one another, the three logics have been used to explain the same troublesome behavioral phenomena.

This article shows that one kind of behavioral science experimental paradigm—Between Subjects experiments—is naturally modeled as a probabilistic quantum logic using concepts derived from collected data. Its probabilities are empirical relative frequencies.

There has been several results, and some controversy, in the literature concerning the relationship of probability on quantum logics to probability on boolean algebra of events. For the special case of Between Subjects experiments, as treated in this article, the following shows it satisfies the Dutch Book argument for rationality—a standard argument in the literature for the rationality of boolean probability theory.

The Dutch Book Argument [9] revolves around the betting between the Bookie and the Bettor. The Bookie assigns tickets to events from a set of tickets, $\mathcal{T}$. For each ticket A in

$\mathcal{T}$, A has the form, win \$1 if A occurs and lose \$0 if A does not occur, where A is from the boolean algebra of events,

$$\mathfrak{W} = \langle \mathcal{W}, \cup, \cap, -, W, \emptyset \rangle, \text{ where } W \text{ is finite and } \mathcal{W} \subseteq \wp(W).$$

The bookie sets a price p_A for each $A \in \mathcal{T}$. The Bettor can buy from the Bookie or sell to the Bookie any finite number of tickets at the Bookie's set prices. A *Dutch Book* is said to be made against the Bookie, if it is possible for the Bettor to buy and sell tickets to the Bookie at the Bookie's set prices, so that for each state of the world that arises, the Bettor cannot lose money, but for some states of the world she can make a profit. Interpreting prices as probabilities, a probability function on $\mathcal{W}$ results. The Bookie's assignment of probabilities is said to be *coherent* if and only if no Dutch Book can be made against the Bookie. The following well-known theorem due to de Finetti (1972) results: The Bookie's assignment of probabilities is coherent if and only if they form a boolean probability function for $\mathfrak{W}$. (Many researchers use the term "rational" in place of "coherent", because they consider the Theorem to be an argument that shows rationality.)

The proof makes use of the fact in an essential way that $\mathfrak{W}$ is boolean algebra of events. This leaves open the issue of, "What if the underlying event space is not a boolean algebra of events, for example, a quantum logic?" I propose the following generalization of the Dutch Book Argument to handle these cases: For an event space having Y as its states of the world, extend the Bettor's probability assignments to the boolean algebra of events of all events in $\wp(Y)$ in such a way that the extended assignments produce a boolean probability function $\mathbb{B}$. Then the Dutch Book Argument applied to $\wp(Y)$ and $\mathbb{B}$ says no Dutch Book can be made. However, if it cannot be made for $\wp(Y)$ and $\mathbb{B}$, it cannot be made for any subset of $\wp(Y)$ and $\mathbb{B}$ restricted to that subset. That is, the Dutch Book Argument would work for the Bookie's probability assignments if such extensions could be made. For the Between Subjects experimental paradigm discussed earlier in this article, such extensions exist for the quantum logic and its additive probability function: $\mathbb{P}$ is a boolean probability function on $\wp(X)$ that is an extension of $\mathbb{P}$ restricted to the set of propositions. The crux of the argument for rationality is the following: If a rational person assigns $\mathbb{B}$'s probabilities to events in $\wp(Y)$, then the restriction of $\mathbb{B}$ to any subset of $\wp(Y)$ is a rational assignment of probabilities, no matter whether a rational person or the Bookie made them.

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Appendix A. Additional Proofs

Throughout this Appendix, we use definitions and concepts developed earlier in the article. In particular, let $\mathfrak{P} = \langle \mathcal{P}, \mathbb{U}, \mathbb{M}, ^\perp, X, \emptyset \rangle$, where $\mathcal{P}$ is the set of propositions arising from a Between Subjects experiment, $^\perp$ is set-theoretic complementation, $-$, and $S, \mathbb{U}, \mathbb{M}$, etc., have their definitions as given earlier.

Lemma A1. *Let D be a proposition. Then*

- (1) $D = \sigma(\ll D \gg)$, and
- (2) $\ll -D \gg = - \ll D \gg$ and $-D$ is a proposition.

Proof. (1) $D = \sigma(\ll D \gg)$ follows from the definitions of "proposition" and σ .

(2) Let i be an arbitrary subject of S such that there exists y in X such that either $(x, y)_i$ or $(y, x)_i$.

Suppose $i \in \ll -D \gg$. Then i is not in $\ll D \gg$, i.e., $i \in - \ll D \gg$. Because D is a proposition, it has an exact empirical probability. Thus, $-D$ has an exact empirical probability, because $S = \ll D \gg \cup \ll -D \gg$ and thus $\mathbb{P}(-D) = 1 - \mathbb{P}(D)$. Thus, $-D$ is a proposition.

Suppose $i \in - \ll D \gg$. Then $i \notin \ll D \gg$. Then, because $(x, y)_i$, $x \notin D$ and $y \notin D$. Thus, $x \in -D$ and $y \in -D$, and therefore $i \in \ll -D \gg$. $\square$

Lemma A2. Let D be a proposition. Then $D^\perp$ is a proposition and $^\perp$ is a complementation operation on $\mathfrak{P}$.

Proof. By Lemma A1 and $^\perp = -$, $D^\perp$ is a proposition. By Lemma A1 and the definitions of $\mathbb{U}$ and $\mathbb{M}$,

$$D^\perp \mathbb{U} D = \sigma[\ll -D \gg \cup \ll D \gg] = \sigma[(\ll -D \gg) \cup \ll D \gg] = \sigma(S) = X,$$

and

$$D^\perp \mathbb{M} D = \sigma[\ll -D \gg \cap \ll D \gg] = \sigma[(\ll -D \gg) \cap \ll D \gg] = \sigma(\emptyset) = \emptyset.$$

$\square$

Lemma A3. $\mathbb{U}$ is the $\subseteq$ -least upper bound operation on P .

Proof. Let F and G be arbitrary elements of P . Then, because

$$\ll F \gg \cup \ll G \gg = \ll F \cup G \gg,$$

it follows that

$$F \mathbb{U} G = \sigma(\ll F \cup G \gg)$$

and therefore $F \mathbb{U} G$ is a proposition. Thus,

$$F = \sigma(\ll F \gg) \subseteq \sigma(\ll F \gg \cup \ll G \gg)$$

and

$$G = \sigma(\ll G \gg) \subseteq \sigma(\ll F \gg \cup \ll G \gg),$$

making $F \mathbb{U} G$ an upper bound of F and G . Suppose H in P is such that

$$\ll F \gg \subseteq \ll H \gg \quad \text{and} \quad \ll G \gg \subseteq \ll H \gg.$$

Then

$$\ll F \gg \cup \ll G \gg = \ll F \cup G \gg \subseteq \ll H \gg,$$

and therefore,

$$F \mathbb{U} G = \sigma(\ll F \cup G \gg) \subseteq \sigma(\ll H \gg) = H,$$

showing that $F \mathbb{U} G$ is the least upper bound of F and G . $\square$

Lemma A4. $\mathbb{M}$ is the $\subseteq$ -greatest lower bound operation on P .

Proof. Let F and G be arbitrary elements of P . Then, because $\ll F \gg \cap \ll G \gg = \ll F \cap G \gg$, it follows that

$$F \mathbb{M} G = \sigma(\ll F \cap G \gg),$$

and therefore $F \mathbb{M} G$ is a proposition. Thus,

$$F = \sigma(\ll F \gg) \supseteq \sigma(\ll F \gg \cap \ll G \gg)$$

and

$$G = \sigma(\llbracket G \rrbracket) \supseteq \sigma(\llbracket F \rrbracket \cap \llbracket G \rrbracket),$$

making $F \sqcap G$ a lower bound of F and G . Suppose H in P is such that

$$\llbracket F \rrbracket \supseteq \llbracket H \rrbracket \quad \text{and} \quad \llbracket G \rrbracket \supseteq \llbracket H \rrbracket.$$

Then

$$\llbracket F \rrbracket \cap \llbracket G \rrbracket = \llbracket F \cap G \rrbracket \supseteq \llbracket H \rrbracket,$$

and therefore,

$$F \sqcap G = \sigma(\llbracket F \cap G \rrbracket) \supseteq \sigma(\llbracket H \rrbracket) = H,$$

showing that $F \sqcap G$ is the greatest lower bound of F and G . $\square$

Lemma A5. $\mathfrak{P} = \langle P, \sqcup, \sqcap, ^\perp, X, \emptyset \rangle$ is a complemented lattice event algebra.

Proof. X is clearly the $\subseteq$ -largest element of P and $\emptyset$ is clearly the $\subseteq$ -smallest element of P .

Because by Lemmas A3 and A4 $\sqcup$ and $\sqcap$ are, respectively, the $\subseteq$ -least upper bound and $\subseteq$ -greatest lower bound operators on P , $\mathfrak{P}$ is a lattice event algebra, and Lemma A2 shows that $^\perp$ is a complementation operation on $\mathfrak{P}$. $\square$

Lemma A6. The complemented lattice event algebra $\mathfrak{P} = \langle P, \sqcup, \sqcap, ^\perp, X, \emptyset \rangle$ satisfies DeMorgan's Laws.

Proof. Let E and F be arbitrary elements of P . By Lemma A2, $E^\perp$ is a proposition. Thus, by the same lemma, $E^{\perp\perp}$ is a proposition. By the definition of $E^\perp$ and $^\perp = -$,

$$E^{\perp\perp} = [\sigma(\llbracket -E \rrbracket)]^\perp = \sigma[-\llbracket -E \rrbracket] = \sigma[-\llbracket E \rrbracket] = \sigma(\llbracket E \rrbracket) = E$$

i.e.,

$$E^{\perp\perp} = E. \quad (\text{A1})$$

By set theory,

$$E \subseteq F \text{ iff } -F \subseteq -E \text{ iff } \sigma(-F) \subseteq \sigma(-E) \text{ iff } F^\perp \subseteq E^\perp. \quad (\text{A2})$$

Theorem 2 says Equations (A1) and (A2) are equivalent to DeMorgan's Laws. $\square$

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