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Article

On the Non-Uniqueness of Statistical Ensembles Defining a Density Operator and a Class of Mixed Quantum States with Integrable Wigner Distribution

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Abstract: It is standard to assume that the Wigner distribution of a mixed quantum state consisting of square-integrable functions is a quasi-probability distribution, i.e., that its integral is one and that the marginal properties are satisfied. However, this is generally not true. We introduced a class of quantum states for which this property is satisfied; these states are dubbed "Feichtinger states" because they are defined in terms of a class of functional spaces (modulation spaces) introduced in the 1980s by H. Feichtinger. The properties of these states were studied, giving us the opportunity to prove an extension to the general case of a result due to Jaynes on the non-uniqueness of the statistical ensemble, generating a density operator.

Keywords: mixed state; covariance matrix; Wigner distribution; modulation space

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1. Introduction

A useful device when dealing with density operators is the covariance matrix, whose existence is taken for granted in most elementary and advanced texts. However, a closer look shows that the latter only exists under rather stringent conditions on the involved quantum states. This question is related to another more general one. Assume that we are dealing with the mixed quantum state $\{(\psi_j, \alpha_j)\}$ with $\psi_j \in L^2(\mathbb{R}^n)$, $||\psi_j|| = 1$ and $\alpha_j \geq 0$, $\sum_j \alpha_j = 1$ the index j belonging to some countable set. The Wigner distribution of that state, which is by definition the convex sum of Wigner transforms:

$$\rho = \sum_j \alpha_j W\psi_j \tag{1}$$

is usually referred to as a "quasi-probability". However, such an interpretation only makes sense if:

$$\int_{\mathbb{R}^{2n}} \rho(z)dz = 1 \tag{2}$$

which is the case if $\psi_j \in L^2(\mathbb{R}^n)$, but it does not imply per se the absolute integrability of $W\psi_j$: there are examples of square-integrable functions ψ such that $W\psi \in L^2(\mathbb{R}^n)$ but $W\psi \notin L^1(\mathbb{R}^{2n})$ [1]. In [2], Cordero and Tabacco give a simple example in dimension $n = 1$: the function $\psi = \chi_{[-/2,1/2]}$ (the characteristic function of the interval $[-/2, 1/2]$ belongs to both $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ but we have $W\psi \notin L^1(\mathbb{R}^2)$. On the other hand, even when satisfied, condition (2) is not sufficient to ensure the existence of the covariances:

$$\sigma^2_{x_j p_k} = \int_{\mathbb{R}^{2n}} x_j p_k \rho(z)dz \tag{3}$$

since these involve the calculation of second moments, and the integrability of ρ does not guarantee the convergence of such integrals. These difficulties are usually ignored in the physical literature, or are dismissed by vague assumptions such as the "sufficiently fast" decrease in the Wigner distribution at infinity.

The aim of this paper was to remedy this vagueness by proposing an adequate functional framework for a rigorous analysis of mixed states and of the associated density operator. For this purpose, we will use the Feichtinger modulation spaces $M_s^1(\mathbb{R}^n)$,instead of $L^2(\mathbb{R}^n)$ as a "reservoir" for quantum states. In the simplest case, where $s = 0$, we have the so-called Feichtinger algebra $M_0^1(\mathbb{R}^n) = S_0(\mathbb{R}^n)$ which can be defined by

$$\psi \in S_0(\mathbb{R}^n) \Longleftrightarrow \left\{ \begin{array}{c} \psi \in L^2(\mathbb{R}^n) \\ W\psi \in L^1(\mathbb{R}^{2n}) \end{array} \right. . \tag{4}$$

We agree that it is not clear at all why the condition above should define a vector space—let alone an algebra!—since the Wigner transform is not additive; surprisingly enough, however, this is the case, and one sees that if we assume that the state $\{(\psi_j, \alpha_j)\}$ is such that $\psi_j \in S_0(\mathbb{R}^n)$ for every j, then the normalization condition (2) is satisfied. We will also see that the existence of covariances such as (3) (and hence the covariance matrix) is guaranteed if we make the sharper assumption that $\psi_j \in M_s^1(\mathbb{R}^n)$ for some $s \geq 2$. While the theory of modulation spaces has become a standard tool in time-frequency and harmonic analysis, it is somewhat less known in quantum physics. In [3], we applied it to deformation quantization; in [4], spectral and regularity results for operators in modulation spaces were studied. One of the reasons for which this theory is less popular in quantum mechanics might be that the usual treatments are given in terms of short-time Fourier transforms (also called Gabor transforms) instead of the Wigner transform as we do here, and this has the tendency to make the theory obscure for many physicists ignoring the simple relation between Wigner and Gabor transforms. We therefore speculate that this lack of communication between both communities is of a pedagogical nature. The use of Wigner transforms in the theory of modulation spaces actually has many advantages: for instance, it makes the symplectic invariance of these spaces become obvious and thus directly links them to the Weyl–Wigner–Moyal formalism.

The notation used here is standard: the phase space variable is $z = (x, p)$ with $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $p = (p_1, \ldots, p_n) \in (\mathbb{R}^n)^* \equiv \mathbb{R}^n$. The scalar product on $L^2(\mathbb{R}^n)$ is:

$$(\psi|\phi) = \int_{\mathbb{R}^n} \psi(x)\overline{\phi(x)}dx$$

(where $\overline{\phi(x)}$ is the complex conjugate of $\phi(x)$) and the associated norm is denoted by $||\psi||$.

2. The Modulation Spaces $M_s^1(\mathbb{R}^n)$

We give here a brief review of the main definitions and properties of the class of modulation spaces we will need. Modulation spaces were introduced by Feichtinger during the early 1980s [5–7]. The most complete treatment can be found in the book [1] by Gröchenig; also see the recent review paper [8]. In [9] (Chapters 16 and 17), modulation spaces are studied from the point of view of the Wigner transform which we use here.

We will denote by $W(\psi, \phi)$ the cross-Wigner function of:

$$(\psi, \phi) \in L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) ;$$

which is defined by

$$W(\psi, \phi)(z) = \left(\tfrac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}p\cdot y} \psi(x + \tfrac{1}{2}y)\overline{\phi(x - \tfrac{1}{2}y)}dy . \tag{5}$$

when $\psi = \phi$, one obtains the usual Wigner transform:

$$W\psi(z) = \left(\tfrac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}p\cdot y} \psi(x + \tfrac{1}{2}y)\overline{\psi(x - \tfrac{1}{2}y)}dy . \tag{6}$$

Recall [9,10] that $W(\psi,\phi)$ is a continuous function belonging to $L^2(\mathbb{R}^{2n})$ and that:

$$|W(\psi,\phi)(z)| \leq \left(\tfrac{2}{\pi\hbar}\right)^n ||\psi||\,||\phi|| \tag{7}$$

as well as

$$\int W(\psi,\phi)(z)dz = (\psi|\phi)\ . \tag{8}$$

Taking $\psi = \phi,$ it follows that, in particular:

$$\int W\psi(z)dz = ||\psi|| \tag{9}$$

hence, the integral condition (2) holds as soon as $\psi \in L^2(\mathbb{R}^n)$ is normalized to one.

In what follows, s is a non-negative real number: $s \geq 0$. We set $z = (x,p)$ and:

$$\langle z\rangle = (1+|z|^2)^{1/2}\ . \tag{10}$$

The function $z \longmapsto \langle z\rangle$ is the Weyl symbol of the elliptic pseudodifferential operator $(1-\Delta)^{1/2}$ where Δ is the Laplacian in the z variables. We denote by $L^1_s(\mathbb{R}^{2n})$ the weighted L^1-space, which is defined by

$$L^1_s(\mathbb{R}^{2n}) = \{\rho : \mathbb{R}^{2n} \longrightarrow \mathbb{C} : \langle z\rangle^s \rho \in L^1(\mathbb{R}^{2n})\}\ , \tag{11}$$

that is, $\rho \in L^1_s(\mathbb{R}^{2n})$ if and only if we have:

$$\|\rho\|_{L^1_s} = \int |\rho(z)|\langle z\rangle^s dz < \infty\ . \tag{12}$$

Due to the submultiplicativity of the weights, these spaces are in fact Banach algebras with respect to convolution (see [11]). The same is true for the spaces $M^1_s(\mathbb{R}^n)$.

Definition 1. *The modulation space $M^1_s(\mathbb{R}^n)$ consists of all $\psi \in L^2(\mathbb{R}^n)$ such that:*

$$W(\psi,\phi) \in L^1_s(\mathbb{R}^{2n}) \tag{13}$$

for every $\phi \in \mathcal{S}(\mathbb{R}^n)$ (the Schwartz space of test functions decreasing rapidly at infinity).

It turns out that it suffices to check that condition (13) holds for *one* function $\phi \neq 0$ for it then holds for *all*; moreover, the mappings $\psi \longmapsto ||\psi||_{\phi,M^1_s}$ defined by

$$||\psi||_{\phi,M^1_s} = ||W(\psi,\phi)||_{L^1_s(\mathbb{R}^{2n})} = \int_{\mathbb{R}^{2n}} |W(\psi,\phi)(z)|\langle z\rangle^s dz$$

form a family of equivalent norms, and the topology on $M^1_s(\mathbb{R}^n)$ thus defined makes it into a Banach space. We have the chain of inclusions:

$$\mathcal{S}(\mathbb{R}^n) \subset M^1_s(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) \cap \mathcal{F}(L^1(\mathbb{R}^n)) \subset L^1(\mathbb{R}^n) \cap C^0(\mathbb{R}^n) \tag{14}$$

where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of test functions and $\mathcal{F}(L^1(\mathbb{R}^n))$ is the space of Fourier transforms $\mathcal{F}\psi$ of the elements ψ of $L^1(\mathbb{R}^n)$. Observe that:

$$M^1_s(\mathbb{R}^n) \subset M^1_{s'}(\mathbb{R}^n) \text{ if and only if } s \geq s'\ ;$$

and one proves [1] that:

$$\textstyle\bigcap_{s\geq 0} M^1_s(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n)\ .$$

Recall that the metaplectic group $\mathrm{Mp}(n)$ is the unitary representation on $L^2(\mathbb{R}^n)$ of the symplectic group. $\mathrm{Sp}(n)$. The modulation spaces $M^1_s(\mathbb{R}^n)$ are invariant under the action of the metaplectic group: if $\widehat{S} \in \mathrm{Mp}(n)$ and $\psi \in M_s^1(\mathbb{R}^n)$, then $\widehat{S}\psi \in M_s^1(\mathbb{R}^n)$. This property actually follows from the symplectic covariance property:

$$W(\widehat{S}\psi)(z) = W\psi(S^{-1}z) \tag{15}$$

of the Wigner transform, where $S \in \mathrm{Sp}(n)$ is the projection of $\widehat{S} \in \mathrm{Mp}(n)$ (see [9] for a detailed study of the metaplectic representation and symplectic covariance).

When $s = 0$, we write $M_0^1(\mathbb{R}^n) = S_0(\mathbb{R}^n)$ (the Feichtinger algebra); clearly $M_s^1(\mathbb{R}^n) \subset S_0(\mathbb{R}^n)$ for all $s \geq 0$. In addition to being a vector space, $S_0(\mathbb{R}^n)$ is a Banach algebra for both pointwise multiplication and convolution. It is actually the smallest Banach algebra invariant under the action of the metaplectic group $\mathrm{Mp}(n)$ and phase space translations. As mentioned in the introduction, the Feichtinger algebra can be characterized by the condition (4): $S_0(\mathbb{R}^n)$ is the vector space of all $\psi \in L^2(\mathbb{R}^n)$ such that $W\psi \in L^1(\mathbb{R}^{2n})$.

3. Feichtinger States

Consider, as in the introduction, a mixed quantum state $\{(\psi_j, \alpha_j)\}$ and denote by

$$\widehat{\rho} = \sum_j \alpha_j \widehat{\rho}_j \ , \ \rho = \sum_j \alpha_j W\psi_j$$

the corresponding density operator and its Wigner distribution: $\widehat{\rho}_j$ is the orthogonal projection on the ray $\mathbb{C}\psi_j$, which is:

$$\widehat{\rho}_j \psi = (\psi|\psi_j)\psi_j, \quad \psi \in L^2(\mathbb{R}^n).$$

Observe that $(2\pi\hbar)^n \rho$ is the Weyl symbol of the operator $\widehat{\rho}$ [9].

Definition 2. *A mixed state $\{(\psi_j, \alpha_j)\}$ is a Feichtinger state if and only we have $\psi_j \in M_s^1(\mathbb{R}^n)$ for some $s \geq 0$.*

Feichtinger states are preserved by the action of the metaplectic group:

Proposition 1. *Let $\{(\psi_j, \alpha_j)\}$ be a Feichtinger state and $\widehat{S} \in \mathrm{Mp}(n)$. Then, $\{(\widehat{S}\psi_j, \alpha_j)\}$ is also a Feichtinger state.*

Proof. It is an immediate consequence of the metaplectic invariance of the modulation spaces $M_s^1(\mathbb{R}^n)$. □

The Wigner distribution ρ of a Feichtinger state is a *bona-fide* quasi-distribution:

Proposition 2. *Let $\{(\psi_j, \alpha_j)\}$ be a Feichtinger state. Then, $\rho \in L^1(\mathbb{R}^{2n})$ and the marginal properties:*

$$\int_{\mathbb{R}^n} \rho(z)dp = \sum_j \alpha_j |\psi_j(x)|^2 \tag{16}$$

$$\int_{\mathbb{R}^n} \rho(z)dx = \sum_j \alpha_j |F\psi_j(x)|^2 \tag{17}$$

hold, and we have:

$$\int_{\mathbb{R}^{2n}} \rho(z)dz = 1\ . \tag{18}$$

Proof. Since $M_s^1(\mathbb{R}^n) \subset M_0^1(\mathbb{R}^n) = S_0(\mathbb{R}^n)$, we automatically have $W\psi_j \in L^1(\mathbb{R}^{2n})$ for every j, hence $\rho \in L^1(\mathbb{R}^{2n})$. On the other hand, we know that the marginal conditions:

$$\int_{\mathbb{R}^n} W\psi_j(z)dp = |\psi_j(x)|^2 \tag{19}$$

$$\int_{\mathbb{R}^n} W\psi_j(z)dx = |F\psi_j(p)|^2 \tag{20}$$

hold if both ψ_j and $F\psi_j$ are integrable [10], and this is precisely the case here in view of the inclusion (14). It follows that:

$$\int_{\mathbb{R}^{2n}} \rho(z)dz = \sum_j \alpha_j \int_{\mathbb{R}^{2n}} W\psi_j(x,p)dpdx = 1$$

since the α_j sum up to one and the ψ_j are unit vectors in $L^2(\mathbb{R}^n)$. $\square$

Remark 1. *The assumption that* $\{(\psi_j,\alpha_j)\}$ *is a Feichtinger state is crucial since it ensures that the marginal properties (19) and (20) hold.*

Modulation spaces also allow to rigorously define the covariance matrix Σ of a state. It is the symmetric $2n \times 2 \times n$ matrix defined by

$$\Sigma = \int_{\mathbb{R}^{2n}} (z-\overline{z})(z-\overline{z})^T \rho(z)dz \tag{21}$$

where $\overline{z}$, the expectation vector, is given by

$$\overline{z} = \int_{\mathbb{R}^{2n}} z\rho(z)dz \tag{22}$$

(all vectors z are viewed as column matrices in these definitions). Assuming $\overline{z} = 0$, the covariance matrix is explicitly given by

$$\Sigma = \begin{pmatrix} \Sigma_{XX} & \Sigma_{XP} \\ \Sigma_{PX} & \Sigma_{PP} \end{pmatrix} \ , \ \Sigma_{PX} = \Sigma_{XP}^T$$

where $\Sigma_{XP} = (\sigma^2_{x_j p_k})_{1\leq j,k,\leq n}$ with $\sigma^2_{x_j p_k}$ given by (3), etc.

Proposition 3. *Assume that* $\{(\psi_j,\alpha_j)\}$ *is a Feichtinger state with* $s \geq 2$*: (i) Then, the covariance matrix* Σ *is well defined; (ii) the Fourier transform* $F\rho$ *of the Wigner distribution of* $\widehat{\rho}$ *is twice continuously differentiable:* $F\rho \in C^2(\mathbb{R}^{2n})$*.*

Proof. It suffices to assume that $s = 2$; we then have:

$$\int_{\mathbb{R}^{2n}} |\rho(z)|(1+|z|^2)dz < \infty\,. \tag{23}$$

Setting $z_\alpha = x_\alpha$ if $1 \leq \alpha \leq n$ and $z_\alpha = p_\alpha$ if $n+1 \leq \alpha \leq 2n$, we have $\langle z \rangle = (\langle z_1 \rangle, \ldots, \langle z_n \rangle)$ where $\langle z_\alpha \rangle$ is given by the absolutely convergent integral:

$$z_\alpha = \int_{\mathbb{R}^{2n}} z_\alpha \rho(z)dz$$

similarly, the integral

$$\overline{z_\alpha z_\beta} = \int_{\mathbb{R}^{2n}} z_\alpha z_\beta \rho(z)dz$$

is also absolutely convergent in view of the trivial inequalities $|z_\alpha z_\beta| \leq 1 + |z|^2$. We have:

$$F\rho(z) = \left(\tfrac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}z\cdot z'}\rho(z')dz'\ ;$$

differentiating twice under the integration sign, we obtain:

$$\partial_{z_\alpha} F\rho = -\tfrac{i}{\hbar} F(z_\alpha \rho) \ , \ \partial_{z_\alpha}\partial_{z_\beta} F\rho = \left(-\tfrac{i}{\hbar}\right)^2 F(z_\alpha z_\beta \rho)$$

hence, the estimates:

$$|\partial_{z_\alpha} F\rho(z)| \leq \tfrac{1}{\hbar}\left|\int_{\mathbb{R}^{2n}} z_\alpha \rho(z)dz\right| < \infty$$
$$|\partial_{z_\alpha}\partial_{z_\beta} F\rho(z)| \leq \left(\tfrac{1}{\hbar}\right)^2\left|\int_{\mathbb{R}^{2n}} z_\alpha z_\beta \rho(z)dz\right| < \infty \,.$$

□

4. Independence of the Statistical Ensemble

Several distinct statistical ensembles can give rise to the same density matrix. For instance, given an arbitrary mixed state $\{(\psi_j, \alpha_j)\}$ as above, the density matrix $\widehat{\rho} = \sum_j \alpha_j \widehat{\rho}_j$ where $\widehat{\rho}_j$ is the orthogonal projection on the rays $\mathbb{C}\psi_j$ which can be written, using the spectral decomposition theorem, as $\widehat{\rho} = \sum_j \lambda_j \widehat{\rho'_j}$ where the λ_j are the eigenvalues of $\widehat{\rho}$ and the $\widehat{\rho'_j}$ is the orthogonal projections on the rays $\mathbb{C}\phi_j$ the ϕ_j being the eigenvectors corresponding to the λ_j.

The main result of this section is a generalization to the infinite-dimensional case of a result due to Jaynes [12]. Recall that a partial isometry is an operator whose restriction to the orthogonal complement of its null-space is an isometry [13].

Proposition 4. *Let $\{(\psi_j, \lambda_j)\}$ be a mixed state and (ϕ_j) be the orthonormal basis of $L^2(\mathbb{R}^n)$ and write $\psi_j = \sum_k a_{jk}\phi_k$. (i) The operator $\widehat{A}$ defined by*

$$\widehat{A}\phi_j = \sum_k \lambda_j^{1/2} a_{jk}\phi_k$$

is a Hilbert–Schmidt operator and $\widehat{\rho} = \widehat{A}\widehat{A}^*$ is the density matrix of the state $\{(\psi_j, \lambda_j)\}$. (ii) Two mixed states $\{(\psi_j, \lambda_j)\}$ and $\{(\psi'_j, \lambda'_j)\}$ generate the same density matrix $\widehat{\rho} = \widehat{A}\widehat{A}^*$ if and only if there exists a partial isometry $\widehat{U}$ of $L^2(\mathbb{R}^n)$ such that $\widehat{A} = \widehat{A'}\widehat{U}$ where $\widehat{A'}$ is defined in terms of $\{(\psi'_j, \lambda'_j)\}$.

Proof. *(i)* We have:

$$(\widehat{A}\phi_j|\widehat{A}\phi_j) = \lambda_j \sum_{k,\ell} a_{jk}\overline{a_{j\ell}}(\phi_k|\phi_\ell) = \lambda_j \sum_k |a_{jk}|^2 = \lambda_j$$

since $||\psi_j||^2 = \sum_k |a_{jk}|^2 = 1$. It follows that:

$$\sum_j (\widehat{A}\phi_j|\widehat{A}\phi_j) = \sum_j \lambda_j = 1 \tag{24}$$

hence, $\widehat{A}$ is a Hilbert–Schmidt operator. It follows that $\widehat{A}^*$ is also Hilbert–Schmidt, hence $\widehat{A}\widehat{A}^*$ is a trace class operator with unit trace:

$$\mathrm{Tr}(\widehat{A}\widehat{A}^*) = \mathrm{Tr}(\widehat{A}^*\widehat{A}) = 1$$

the second equality in view of (24). Let us show that, in fact, $\widehat{\rho} = \widehat{A}\widehat{A}^*$, that is

$$\widehat{\rho}\psi = \sum_k \lambda_k (\psi|\psi_k)\psi_k$$

for every $\psi \in L^2(\mathbb{R}^n)$. It is sufficient to show that this identity holds for the basis vectors ϕ_j, that is:

$$\widehat{\rho}\phi_j = \sum_k \lambda_k(\phi_j|\psi_k)\psi_k \tag{25}$$

for every j. Using the expansions:

$$\psi_k = \sum_m a_{km}\phi_m = \sum_\ell a_{k\ell}\phi_\ell$$

we can rewrite this identity as

$$\widehat{\rho}\phi_j = \sum_{k,\ell} \lambda_k \overline{a_{kj}} a_{k\ell}\phi_\ell \ .$$

On the other hand, by definition of $\widehat{A}$, we have:

$$\widehat{A}^*\phi_j = \sum_k \lambda_k^{1/2}\overline{a_{kj}}\phi_j$$

hence, by linearity:

$$\widehat{A}\widehat{A}^*\phi_j = \sum_k \lambda_k^{1/2}\overline{a_{kj}}\widehat{A}\phi_j = \sum_{k,\ell} \lambda_k \overline{a_{kj}} a_{k\ell}\phi_\ell \tag{26}$$

which is the same thing as $\widehat{\rho}\phi_j$. *(ii)* Let $\widehat{U}$ be a partial isometry of $L^2(\mathbb{R}^n)$; then $\widehat{A'}\widehat{A'}^* = \widehat{A}\widehat{A}^* = \widehat{\rho}$. Conversely, suppose that $\widehat{A'}\widehat{A'}^* = \widehat{A}\widehat{A}^*$. A classical result from the theory of Hilbert spaces (Douglas' lemma [14]) tells us that there exists a partial isometry $\widehat{U}$ such that $\widehat{A'} = \widehat{A}\widehat{U}$—thus, we are done. □

Remark 2. *The result above has been considered and proven by [15–17] in the case of quantum states having a finite number of elements. Their proofs do not immediately extend to the infinite dimensional case.*

An immediate consequence is that the property of being a Feichtinger state is invariant under transformations preserving the density matrix.

Corollary 1. *Let* $\{(\psi_j, \alpha_j) : j \in J\}$*be a Feichtinger state where* J *is a finite set of indices. Then, every state* $\{(\phi_j, \beta_j)$ *generating the same density matrix* $\widehat{\rho}$ *is also a Feichtinger state. In particular, the spectral decomposition* $\widehat{\rho} = \sum_{j\in J}\lambda_j\widehat{\rho}_j$ *consists of orthogonal projections* $\widehat{\rho}_j$ *on rays* $\mathbb{C}\psi_j$ *where* $\psi_j \in M_s^1(\mathbb{R}^n)$*.*

Proof. Assume that $\psi_j \in M_s^1(\mathbb{R}^n)$ for every j. In view of Proposition 4, there exist finite linear relations $\phi_k = \sum_j a_{jk}\psi_j$ for each index k, hence, $\phi_k \in M_s^1(\mathbb{R}^n)$ for every k since $M_s^1(\mathbb{R}^n)$ is a vector space. □

Remark 3. *The proof does not trivially extend to the general case where the index set* J *is infinite because of convergence problems. The question of whether the Corollary extends to the general case remains open.*

As a bonus, we obtain the following new result about the convex sums of Wigner distributions.

Corollary 2. *Let $(\psi_j)_j$ be a sequence in $M_s^1(\mathbb{R}^n)$. Let $(\phi_j)_j$ be a sequence of functions in $L^2(\mathbb{R}^n)$ and sequences (α_j) and (β_j) of positive numbers such that $\sum_j \alpha_j = \sum_j \beta_j = 1$. We assume that $||\psi_j|| = ||\phi_j|| = 1$ for all j. If we have:*

$$\sum_j \alpha_j W\psi_j = \sum_j \beta_j W\phi_j$$

then $\phi_k \in M_s^1(\mathbb{R}^n)$ for every k.

Proof. Both series are absolutely convergent in view of (7); for instance:

$$\sum_j \alpha_j |W\psi_j| \leq \left(\tfrac{2}{\pi\hbar}\right)^n \sum_j \alpha_j = \left(\tfrac{2}{\pi\hbar}\right)^n .$$

The function:

$$\rho = (2\pi\hbar)^n \sum_j a_j W\psi_j = (2\pi\hbar)^n \sum_j \beta_j W\phi_j$$

is the Wigner distribution of a density matrix generated by the Feichtinger state $\{(\psi_j, \alpha_j)\}$. In view of Corollary 1, we must then have $\phi_j \in M_s^1(\mathbb{R}^n)$ for every j. □

5. Discussion

We saw that the class of modulation spaces $M_s^1(\mathbb{R}^n)$ provides us with a very convenient framework for the study of the Wigner distribution of density matrices; it is far less restrictive than the conventional use of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ which requires that the functions and all their derivatives be zero at infinity. In addition, the topology of modulation spaces is simpler since they are defined by a norm making them Banach spaces, while that of $\mathcal{S}(\mathbb{R}^n)$ is defined by a family of semi-norms making it to a Fréchet space. Another particularly attractive feature of modulation spaces is that they enable introducing a useful class of Banach Gelfand triples (see [18]). For instance:

$$(S_0(\mathbb{R}^n), L^2(\mathbb{R}^n), S_0'(\mathbb{R}^n))$$

where $S_0'(\mathbb{R}^n)$ is the dual space of the Feichtinger algebra, $S_0(\mathbb{R}^n)$ is a such triple. $S_0'(\mathbb{R}^n)$ consists of all $\psi \in S'(\mathbb{R}^n)$ such that $W(\psi, \phi) \in L^\infty(\mathbb{R}^{2n})$ for one (and hence all) $\phi \in S_0(\mathbb{R}^n)$; the duality bracket is simply given by the pairing:

$$(\psi, \psi') = \int_{\mathbb{R}^{2n}} W(\psi, \phi)(z)\overline{W(\psi', \phi)(z)} dz . \tag{27}$$

Since $S_0(\mathbb{R}^n)$ is the smallest Banach space, isometrically invariant under the action of the metaplectic group, its dual is essentially the largest space of distributions with this property. The use of such triples makes the use of the Dirac bra-ket notation much more natural and rigorous. For instance, objects such as $\langle\psi|\phi\rangle$ automatically have a meaning for all $\phi \in S_0(\mathbb{R}^n)$ and all $\psi \in S_0'(\mathbb{R}^n)$.

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