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On Bell's Inequality in $\mathcal{PT}$-Symmetric Quantum Systems

Sarang S. Bhosale [1], Biswanath Rath [2] and Prasanta K. Panigrahi [1,*]

[1] Department of Physical Sciences, Indian Institute of Science Education and Research Kolkata, Mohanpur 741246, India; sarangbhosale11@iiserkol.ac.in
[2] Department of Physics, Maharaja Sriram Chandra Bhanja Deo University, Barupada 757003, India; biswanathrath10@gmail.com
* Correspondence: pprasanta@iiserkol.ac.in

Abstract: Bell's inequality is investigated in parity-time ($\mathcal{PT}$) symmetric quantum mechanics, using a recently developed form of the inequality by Maccone, with two $\mathcal{PT}$-qubits in the unbroken phase with real energy spectrum. It is shown that the inequality produces a bound that is consistent with the standard quantum mechanics even after using Hilbert space equipped with $\mathcal{CPT}$ inner product and therefore, the entanglement has identical structure with standard quantum mechanics. Consequently, the no-signaling principle for a two-qubit system in $\mathcal{PT}$-symmetric quantum theory is preserved.

Keywords: foundations of quantum mechanics; bell's inequality; $\mathcal{PT}$-symmetric quantum mechanics; entanglement invariance; no-signaling principle

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1. Introduction

Bell's inequality [1–4] has played a significant role in distinguishing the quantum theory and a classical theory with local hidden variables [2]. It has been studied extensively in many equivalent forms, involving the standard Dirac-von-Neumann inner product. The most familiar form of the Bell's inequality is Clauser-Horne-Shimony-Holt (CHSH) inequality [5], where expectation values of observables are calculated using bi-linear Pauli operators and if the local hidden variable assumption is considered, then the inequality has a bound of 2, while the maximum violation allowed by quantum mechanics is $2\sqrt{2}$, also known as Tsirelson's bound [6]. There has been a large number of experimental tests of Bell's inequality, where many subtle aspects of the underlying quantum correlations have been carefully probed [7–12]. It is important to note that, historically, the mathematical core of Bell's theorem goes back to the derivation of Boole's inequality in the probability theory [13,14]. Recently a simpler form of the inequality has been obtained in a succinct way by Maccone [15], following Preskill [16] and Mermin's suggestion [17].

The proof of the inequality in [15] considers two identical objects with the same values of all properties and takes Einstein's arguments [18] into account, implying that, the values of properties are initially known i.e., predetermined (or counterfactual-definite). Furthermore, it assumes that the values are independent of measurements, which suggests measuring a property of one object will not affect the measurement of the second object's property (i.e., locality). Assuming three, arbitrary two-valued properties A, B, C, satisfying both locality and counterfactual-definiteness and that each observer has two such objects, the Bell's inequality in Maccone's form [15] states that,

$$P_{same}(A,B) + P_{same}(A,C) + P_{same}(B,C) \geq 1. \tag{1}$$

Here, $P_{same}(A,B)$ is the probability that the property A of the first object and B of the second have the same values. If the probability sum is greater than or equal to one, it will indicate that, the theory will obey both locality and counterfactual-definiteness. However, quantum theory violates the Bell's inequality showing that, it is either non-local e.g., de Broglie-Bohm interpretation [19] or non-counterfactual-definite e.g., Copenhagen interpretation [15] and

it indicates that a deterministic quantum theory encompassing local hidden variables can not account for observations made from quantum physics [20].

In standard quantum mechanics, to obtain real energy eigenvalues and to maintain the unitarity of the evolution, the condition of the Hermiticity of the Hamiltonian (i.e., $\mathcal{H} = \mathcal{H}^\dagger$) is indispensable. Although in the past few years, it has been predicted that this requirement of Hermiticity which is generally stated as an axiom in quantum theory, can be replaced by the less mathematical and more physical conditions on Hamiltonians without compromising on the physical core of the quantum theory. Recently, the complex extension of quantum mechanics has been put forward by Bender, Brody, and Jones [21], which includes replacement of mathematical condition of Hermiticity of Hamiltonians by the condition of $\mathcal{PT}$-symmetry, to obtain the corresponding real energy eigenvalue spectrum [22]. Physically, the $\mathcal{PT}$-symmetric Hamiltonians are not isolated like Hermitian Hamiltonians; rather, they are in contact with the environment leading to the non-Hermiticity character. If this contact is constrained such that the gain from the environment is exactly balanced by the loss, then the $\mathcal{PT}$-symmetric Hamiltonians will have real energy eigenvalues, leading to the unbroken $\mathcal{PT}$-symmetric phase. Consequently, the $\mathcal{PT}$-symmetric quantum theory behaves like Hermitian quantum theory in equilibrium. For the general case, it can have complex eigenvalues, in the broken $\mathcal{PT}$-symmetric phase, and in this case, the $\mathcal{PT}$-symmetric systems behave like out of equilibrium systems.

The unique feature of the non-Hermitian system is merging different eigenvalues and eigenvectors. This singularity in parametric space, where this merging happens, is known as an Exceptional point. Additionally, non-Hermitian systems do not obey conservation laws as they exchange energy with the surroundings. To understand such systems, $\mathcal{PT}$-symmetric Hamiltonians play a crucial role. Physically, the $\mathcal{PT}$-symmetric Hamiltonians are being used to understand optical gain and loss in photonics by treating them as non-conservative ingredients [23]. The first experimental realization of $\mathcal{PT}$-symmetric systems was observed in an electrical circuit system [24]. Consequently, it was further explored into conservative coupled systems, which consists of balanced gain and loss, e.g., optical microcavities [25], optical systems with atomic media [26], optical waveguides [27] and mechanical systems [28].

It has been previously shown that, the basic properties of entanglement can be violated under local $\mathcal{PT}$-symmetric operations [29–33] i.e., if one qubit is subjected to $\mathcal{PT}$-symmetric Hamiltonian evolution and the other is in the conventional world, then entanglement increases under local operation. We use the version of Bell's inequality from [15,16] and consider both the qubits in $\mathcal{PT}$-symmetric framework along with $\mathcal{CPT}$ inner product, which results in obtaining a consistent Bell's bound as in Hermitian quantum mechanics. It needs to be emphasised that, the Bell's inequality is a fundamental measure of non-locality and it becomes evident from our result that, the non-locality is consistent in $\mathcal{PT}$-symmetric quantum theoretic framework. It is worth pointing out that, in the unbroken $\mathcal{PT}$-symmetric phase, Bell-CHSH inequality has been found to be consistent with Hermitian quantum mechanics [34].

The paper is organized as follows: In Section 2 we discuss the general $\mathcal{PT}$-symmetric Hamiltonian, the corresponding eigenvectors and introduce $\mathcal{CPT}$ inner product with the individual definitions of $\mathcal{C}, \mathcal{P}, \mathcal{T}$ operators. We deduce the proof of Bell's inequality for $\mathcal{PT}$-symmetric quantum systems and show that the bound is independent of non-hermiticity parameter (which further leads to the conservation of no-signaling theorem) in Section 3. In Section 4 we summarize our findings, compare with the previous works and conclude with the future directions.

2. $\mathcal{PT}$-Symmetric Qubits

For the purpose of illustration, we consider the $\mathcal{PT}$-symmetric Hamiltonian [21],

$$H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix}, \tag{2}$$

with the eigenvalues,

$$E_{\pm} = rcos\theta \pm \sqrt{s^2 - r^2 sin^2\theta}. \tag{3}$$

Based on the above eigenvalues, there are two parametric regions, the broken $\mathcal{PT}$-symmetric region in which energy eigenvalues form a complex conjugate pair since, $s^2 < r^2 sin^2\theta$ and the region of unbroken $\mathcal{PT}$-symmetry, where energy eigenvalues are real because, $s^2 \geqslant r^2 sin^2\theta$ [21]. The eigenvectors for the unbroken $\mathcal{PT}$-symmetric case are,

$$|a_0\rangle = |\psi_+\rangle = \frac{1}{\sqrt{2cos\alpha}} \begin{pmatrix} e^{i\alpha/2} \\ e^{-i\alpha/2} \end{pmatrix}, \tag{4}$$

$$|a_1\rangle = |\psi_-\rangle = \frac{i}{\sqrt{2cos\alpha}} \begin{pmatrix} e^{-i\alpha/2} \\ -e^{i\alpha/2} \end{pmatrix}. \tag{5}$$

Following [21], we have set $sin\alpha = (r/s)sin\theta$, where, α is the non-Hermiticy parameter. It is noted that, the condition $s^2 = r^2 sin^2\theta$ yields the $\mathcal{PT}$-symmetric degenerate states.

The $\mathcal{CPT}$ inner product is defined as [21],

$$\langle\psi_+| = [\mathcal{CPT}\,|\psi_+\rangle]^t, \tag{6}$$

where, $\mathcal{C}$ is the charge conjugation operator and t is the matrix transposition operation. The operator $\mathcal{C}$ has eigenvalues ± 1 and it yields the sign of the $\mathcal{PT}$-norm of the state.

This follows,

$$\mathcal{CPT}\,|\psi_\pm\rangle = \pm\,|\psi_\pm\rangle \tag{7}$$

It is seen that $\mathcal{C}^2 = 1$, provided;

$$\mathcal{C} = \frac{1}{cos\alpha} \begin{pmatrix} isin\alpha & 1 \\ 1 & -isin\alpha \end{pmatrix}. \tag{8}$$

The parity operator is defined as [21],

$$\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{9}$$

with, $\mathcal{P}^2 = 1$. Operators $\mathcal{P}$ and $\mathcal{C}$ do not commute with each other and if the non-Hermiticity parameter $\alpha \to 0$ then $C \to P$.

The time reversal operator $\mathcal{T}$ is an anti-linear operator, which changes i to $-i$ i.e., $\mathcal{T}$ implements the action of complex conjugation. It is important to note that, $\mathcal{T}$ belongs to the class of operators known "involutional" operators [35];

$$\mathcal{T}^2 = \eta I, \quad \eta = \pm 1, \tag{10}$$

provided, $\mathcal{T}$ explicitly satisfies, $\mathcal{T}i\mathcal{T}^{-1} = -i$.

Using above definition of inner product and $\mathcal{C}, \mathcal{P}, \mathcal{T}$ operators, one observes that the orthonormality conditions are satisfied for eigenvectors of $\mathcal{PT}$-symmetric Hamiltonian;

$$\begin{aligned} \langle a_0|a_0\rangle = \langle a_1|a_1\rangle = \langle\psi_+|\psi_+\rangle = \langle\psi_-|\psi_-\rangle = 1. \\ \langle a_0|a_1\rangle = \langle\psi_+|\psi_-\rangle = 0. \end{aligned} \tag{11}$$

This immediately follows,

$$\langle b_0|b_1\rangle = 0 = \langle c_0|c_1\rangle \tag{12}$$

3. Proof of Bell's Inequality in $\mathcal{PT}$-Symmetric Quantum Theory

To obtain Bell's inequality in $\mathcal{PT}$-symmetric quantum theory, we consider a quantum system that violates inequality (1). Consider three objects A, B and C having two valued properties defined by the following set of eigenstates [15].

For A, we define the two valued properties as states $|a_0\rangle$ and $|a_1\rangle$, while for B and C we define their corresponding two valued properties in terms of states given as,

$$\begin{aligned} |b_0\rangle &= \frac{1}{2}|\psi^+\rangle + \frac{\sqrt{3}}{2}|\psi^-\rangle, \\ |b_1\rangle &= \frac{\sqrt{3}}{2}|\psi^+\rangle - \frac{1}{2}|\psi^-\rangle, \\ |c_0\rangle &= \frac{1}{2}|\psi^+\rangle - \frac{\sqrt{3}}{2}|\psi^-\rangle, \\ |c_1\rangle &= \frac{\sqrt{3}}{2}|\psi^+\rangle + \frac{1}{2}|\psi^-\rangle. \end{aligned} \tag{13}$$

Consider two level systems ($\mathcal{PT}$-qubits) in the joint entangled state,

$$|\psi_{AB}\rangle = \frac{1}{2\sqrt{2}}[|a_0\rangle(|b_0\rangle + \sqrt{3}|b_1\rangle) + |a_1\rangle(\sqrt{3}|b_0\rangle - |b_1\rangle)]. \tag{14}$$

With the corresponding bra vector for the above $\mathcal{PT}$-symmetric joint entangled state; [30,31,36],

$$\langle\psi_{AB}| = [(\mathcal{CPT}\otimes\mathcal{CPT})|\psi_{AB}\rangle]^t. \tag{15}$$

where, t is the matrix transposition operation.

Similarly, $|\psi_{AC}\rangle$ and $|\psi_{BC}\rangle$ are defined as,

$$\begin{aligned} |\psi_{AC}\rangle &= \frac{1}{2\sqrt{2}}[|a_0\rangle(|c_0\rangle + \sqrt{3}|c_1\rangle) - |a_1\rangle(\sqrt{3}|c_0\rangle - |c_1\rangle)], \\ |\psi_{BC}\rangle &= \frac{1}{4\sqrt{2}}[(|b_0\rangle + \sqrt{3}|b_1\rangle)(|c_0\rangle + \sqrt{3}|c_1\rangle) - (\sqrt{3}|b_0\rangle - |b_1\rangle)(\sqrt{3}|c_0\rangle - |c_1\rangle)]. \end{aligned} \tag{16}$$

The corresponding bra vectors are,

$$\langle\psi_{AC}| = [(\mathcal{CPT}\otimes\mathcal{CPT})|\psi_{AC}\rangle]^t \quad , \langle\psi_{BC}| = [(\mathcal{CPT}\otimes\mathcal{CPT})|\psi_{BC}\rangle]^t. \tag{17}$$

Using the $\mathcal{CPT}$ inner product defined in (15), one obtains,

$$\langle\psi_{AB}|\psi_{AB}\rangle = \langle\psi_{AC}|\psi_{AC}\rangle = \langle\psi_{BC}|\psi_{BC}\rangle = 1. \tag{18}$$

We now calculate the probability amplitude of obtaining 0 or 1, for both the properties A and B or A and C or B and C (for example the property can be the results of any dichotomic systems), by using $\mathcal{CPT}$ inner product. Given all the above prerequisites (The explicit calculation for one case is provided in the Appendix A), one obtains,

$$
\begin{aligned}
\langle a_0 b_0|\psi_{AB}\rangle &= [(\mathcal{CPT}\otimes\mathcal{CPT})\,|a_0 b_0\rangle]^t\,|\psi_{AB}\rangle = \frac{1}{2\sqrt{2}},\\
\langle a_1 b_1|\psi_{AB}\rangle &= [(\mathcal{CPT}\otimes\mathcal{CPT})\,|a_1 b_1\rangle]^t\,|\psi_{AB}\rangle = -\frac{1}{2\sqrt{2}},\\
\langle a_0 c_0|\psi_{AC}\rangle &= [(\mathcal{CPT}\otimes\mathcal{CPT})\,|a_0 c_0\rangle]^t\,|\psi_{AC}\rangle = \frac{1}{2\sqrt{2}},\\
\langle a_1 c_1|\psi_{AC}\rangle &= [(\mathcal{CPT}\otimes\mathcal{CPT})\,|a_1 c_1\rangle]^t\,|\psi_{AC}\rangle = \frac{1}{2\sqrt{2}},\\
\langle b_0 c_0|\psi_{BC}\rangle &= [(\mathcal{CPT}\otimes\mathcal{CPT})\,|b_0 c_0\rangle]^t\,|\psi_{BC}\rangle = -\frac{1}{2\sqrt{2}},\\
\langle b_1 c_1|\psi_{BC}\rangle &= [(\mathcal{CPT}\otimes\mathcal{CPT})\,|b_1 c_1\rangle]^t\,|\psi_{BC}\rangle = \frac{1}{2\sqrt{2}}.
\end{aligned}
\tag{19}
$$

Using,

$$P = |\,\langle\psi|\phi\rangle\,|^2,$$

the probabilities of obtaining 0 or 1 for both the properties are,

$$P(a_0 b_0) = P(a_1 b_1) = P(a_0 c_0) = P(a_1 c_1) = P(b_0 c_0) = P(b_1 c_1) = \frac{1}{8}. \tag{20}$$

Therefore,

$$P_{same}(A,B) + P_{same}(A,C) + P_{same}(B,C) = \frac{3}{4}. \tag{21}$$

This is the Bell's bound for $\mathcal{PT}$-symmetric quantum mechanics using Maccone's form and it identically matches with the Hermitian quantum mechanics. Remarkably, the above equation has no dependence on non-Hermiticity parameter.

4. Conclusions

To summarize, we show that, in the unbroken phase of $\mathcal{PT}$-symmetry, using the $\mathcal{PT}$-symmetric qubits and physically accepted $\mathcal{CPT}$ inner product, the Bell's inequality will be violated exactly in the same manner as in the conventional quantum mechanics. This verifies that $\mathcal{PT}$-symmetric quantum theory is a genuine complex extension of conventional quantum mechanics. Further, it signifies that, the no-signaling theorem is not violated under $\mathcal{PT}$-symmetric operations if both qubits are taken as $\mathcal{PT}$-qubits along with the consistent definition of $\mathcal{CPT}$ inner product. Our result is consistent with [34] using a much more simpler setting of Bell's inequality. Hence, $\mathcal{PT}$-symmetric Hamiltonians can be used as a powerful tool for quantum information, quantum computing and quantum communications. In short, if Alice and Bob both live in a $\mathcal{PT}$-symmetric quantum world, and use $\mathcal{PT}$-symmetric qubits and properly defined $\mathcal{CPT}$ inner product, then they will find that, the extent of violation for Bell's inequality is same as that in standard Hermitian quantum mechanical world. Analysing Bell's inequality for the broken $\mathcal{PT}$-symmetric case [37] and the dependence of the Bell's bound on the dimension of the $\mathcal{PT}$-symmetric systems are works in progress. The experimental realization of our result on IBM quantum experience will be the matter of forthcoming research.

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Appendix A

We begin with the $\mathcal{PT}$-symmetric eigenvectors of the Hamiltonian in (2).
The two orthonormal states of the properties A and B are respectively given as,

$$|a_0\rangle = \frac{1}{\sqrt{2cos\alpha}}\begin{pmatrix} e^{i\alpha/2} \\ e^{-i\alpha/2} \end{pmatrix} \qquad |a_1\rangle = \frac{i}{\sqrt{2cos\alpha}}\begin{pmatrix} e^{-i\alpha/2} \\ -e^{i\alpha/2}. \end{pmatrix} \tag{A1}$$

$$|b_0\rangle = \frac{1}{2\sqrt{cos\alpha}}\begin{pmatrix} \sqrt{\frac{3}{2}}e^{-i\alpha/2} + \frac{1}{\sqrt{2}}e^{i\alpha/2} \\ \frac{1}{\sqrt{2}}e^{-i\alpha/2} - \sqrt{\frac{3}{2}}ie^{i\alpha/2} \end{pmatrix} \qquad |b_1\rangle = \frac{1}{2\sqrt{cos\alpha}}\begin{pmatrix} -\frac{1}{\sqrt{2}}e^{-i\alpha/2} + \sqrt{\frac{3}{2}}e^{i\alpha/2} \\ \sqrt{\frac{3}{2}}e^{-i\alpha/2} + \frac{1}{\sqrt{2}}ie^{i\alpha/2}. \end{pmatrix} \tag{A2}$$

The joint entangled state for A and B is then,

$$\begin{aligned} |\psi_{AB}\rangle &= \frac{1}{\sqrt{2}}\left[\frac{1}{2}|a_0b_0\rangle + \frac{\sqrt{3}}{2}|a_0b_1\rangle + \frac{\sqrt{3}}{2}|a_1b_0\rangle - \frac{1}{2}|a_1b_1\rangle\right] \\ &= \frac{1}{32cos\alpha}\left[\begin{pmatrix} (\sqrt{3}i + e^{i\alpha}sec\alpha) \\ sec\alpha + \sqrt{3}(-i + tan\alpha) \\ sec\alpha + \sqrt{3}(i + tan\alpha) \\ -(\sqrt{3}i + e^{-i\alpha})sec\alpha e^{-i\alpha/2} \end{pmatrix} + \begin{pmatrix} (\sqrt{3}i + e^{i\alpha}sec\alpha) \\ sec\alpha + \sqrt{3}(-i + tan\alpha) \\ sec\alpha + \sqrt{3}(i + tan\alpha) \\ -(\sqrt{3}i + e^{-i\alpha})sec\alpha e^{-i\alpha/2} \end{pmatrix}\right] \\ &\quad + \frac{1}{32cos\alpha}\left[\begin{pmatrix} (i - \sqrt{3}e^{-i\alpha})sec\alpha \\ sec\alpha + \sqrt{3}(-i + tan\alpha) \\ (i + \sqrt{3} + sec\alpha + tan\alpha) \\ -(i + \sqrt{3}e^{i\alpha})sec\alpha e^{-i\alpha/2} \end{pmatrix} + \begin{pmatrix} (\sqrt{3}i + e^{-i\alpha})sec\alpha \\ -sec\alpha + \sqrt{3}(i + tan\alpha) \\ (-i + \sqrt{3}e^{i\alpha})sec\alpha \\ (-1 + \sqrt{3}e^{i\alpha})sec\alpha e^{-i\alpha/2} \end{pmatrix}\right] \\ &= \begin{pmatrix} \frac{itan\alpha}{\sqrt{2}} & \frac{sec\alpha}{\sqrt{2}} & \frac{sec\alpha}{\sqrt{2}} & \frac{-itan\alpha}{\sqrt{2}} \end{pmatrix}^t \end{aligned} \tag{A3}$$

Then inner product can be obtained as,

$$\langle a_0b_0| = [(\mathcal{CPT} \otimes \mathcal{CPT})|a_0b_0\rangle]^t \qquad , \langle a_1b_1| = [(\mathcal{CPT} \otimes \mathcal{CPT})|a_1b_1\rangle]^t. \tag{A4}$$

The quantity $\langle a_0b_0|\psi_{AB}\rangle$ yields the probability amplitude for obtaining 0 for both A and B;

$$\begin{aligned} \langle a_0b_0|\psi_{AB}\rangle &= [(\mathcal{CPT} \otimes \mathcal{CPT})|a_0b_0\rangle]^t.|\psi_{AB}\rangle \\ &= \frac{1}{\sqrt{2}(2 + 2e^{2i\alpha})}\begin{pmatrix} \frac{1}{4}(1 - \sqrt{3}isec\alpha + itan\alpha)(2 + 2e^{2i\alpha})^{-1} \\ e^{i\alpha} + \sqrt{3}ie^{2i\alpha} \\ -\sqrt{3}i + e^{i\alpha} \\ 1 + \sqrt{3}ie^{i\alpha} \end{pmatrix}^t \cdot \begin{pmatrix} itan\alpha \\ sec\alpha \\ sec\alpha \\ -itan\alpha \end{pmatrix} \\ &= \frac{1}{2\sqrt{2}}. \end{aligned} \tag{A5}$$

Therefore, the probability is,

$$P(a_0b_0) = \frac{1}{8}. \tag{A6}$$

Similarly, the probability amplitude for getting 1 for both A and B as,

$$\langle a_1b_1|\psi_{AB}\rangle = \frac{1}{2\sqrt{2}}. \tag{A7}$$

The total probability of obtaining the same property for both A and B is found as,

$$P(a_0b_0) + P(a_1b_1) = \frac{1}{4} \tag{A8}$$

Likewise, the total probability of getting same property for both A and C or B and C, can be calculated;

$$P_{same}(A,B) + P_{same}(A,C) + P_{same}(B,C) = \frac{3}{4}. \tag{A9}$$

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