

Review

Relatively Undistorted Solutions to the Wave Equation: A Review

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Abstract

A review of simple solutions to the linear wave equation in three spatial variables, involving arbitrary functions, is given. Physicists' interest in these solutions is primarily motivated by the need for a convenient description of spatiotemporally localized wave beams and packets. Four families of closed-form solutions with arbitrary functions—relatively undistorted waves—and some of their applications, particularly in optics, are examined within a unified framework. These are classical plane waves and their simple generalizations, classical and complexified spherical waves known as “complex source fields”, the Bateman wave and its various generalizations, and quasi-spherical waves, which have recently been introduced into consideration. In connection with the latter, we discuss the relatively new concept of the unidirectionality of solutions. We also briefly review families of solutions generated from spherical, quasi-spherical, and Bateman-type waves by Lorentz transformations. The review updates literature from the past two decades to assist in modeling spatiotemporally localized wave phenomena, such as laser pulses, using arbitrary functions instead of complex simulations. The review focus on few-cycle pulses and localized beams matches crucial ongoing developments in laser physics. It also bridges the gap between recent exact solutions and modern paraxial and non-paraxial optics and fills a significant literature void.

Keywords: wave equation; exact solutions; relatively undistorted waves; spatiotemporal solutions; localized pulses; Bateman wave; complex source; quasi-spherical waves; unidirectional waves

1. Introduction

The present paper is concerned with exact solutions to the linear wave equation with three spatial variables

$$\square u \equiv u_{xx} + u_{yy} + u_{zz} - \frac{1}{c^2} u_{tt} = 0, \quad (1)$$

where $c = \text{const} > 0$ is the wave propagation speed and doubled subscripts denote the second partial derivatives with respect to the coordinates (x, y and z) and time t . Since the eighteenth century, simple solutions to Equation (1) involving arbitrary functions have remained a subject of continuing interest. Their significance has increased markedly in recent decades due to applications in the description of spatiotemporally localized wave phenomena. The arbitrariness of the functional parameters offers a powerful yet simple enough tool for constructing diverse wave beams and localized wave packets as closed-form expressions.



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This paper focuses on special solutions involving arbitrary functions, namely, relatively undistorted waves, and on their applications, primarily in the theory of localized waves. We review mainly the four basic families of such solutions containing a functional parameter. The first family comprises the classical plane waves and their straightforward generalizations. The second consists of complexified spherical waves, often referred to as complex-source fields. Further, we address various generalizations of the Bateman wave. Finally, we consider the recently introduced family of quasi-spherical waves. In this context, the fairly recent concept of wave unidirectionality is discussed; such waves are also referred to as forward-propagating or causal waves, although causality has long been understood in a different sense; see, for example, [1].

The reason for referring to these four solutions as basic is that spherical, quasi-spherical, and Bateman-type waves, together with their Lorentz-transformed counterparts and plane waves, encompass all classes of relatively undistorted waves that have thus far found applications in physics. By focusing on the four aforementioned families of solutions, we endeavor to bring some order to the overwhelming multitude of particular spatiotemporal solutions of the wave equation introduced in optics thus far.

Beyond the scope of the present paper are other intriguing solutions involving arbitrary functions, such as the functionally invariant solutions by Vladimir Smirnov and Sergey Sobolev [2] that are not relatively undistorted waves.

A comprehensive review of relatively undistorted solutions was published more than twenty years ago by Ioannis Besieres and Amr Shaarawi [3]. Beside the axisymmetric Bateman waves, that review considered families of solutions obtained by applying subluminal and superluminal Lorentz transformations to complexified spherical waves. A subsequent review by one of the authors of this paper [4] focused extensively on the correspondence between the complexified Bateman solutions and the paraxial solutions obtained within the framework of the parabolic equation method. Since these reviews were published, significant new results have become available, necessitating their further analysis and generalization. These include, in particular, astigmatic and tilted Bateman-type solutions, as well as quasi-spherical waves; the latter has only recently emerged as a subject of active research. Particular attention is paid to the singularities of the solutions under consideration and to the cases in which the solutions are spatially and temporally localized.

Regarding the most familiar case of plane waves, in addition to reviewing the commonly known results, we also discuss waves with polynomial amplitudes that arise in diffraction theory. Our interest in spherical waves is motivated primarily by their complexified versions, highly known since the 1970s as complex-source fields. Since their introduction, these fields have been extensively studied as simple exact solutions possessing Gaussian localization, which makes them particularly attractive for modeling laser radiation. We devote substantial attention to the singularities of complexified spherical waves in physical space and to the possibility of eliminating them by considering the difference of two such solutions. We point out that this approach naturally leads to the solution known as the splash pulse, which is frequently employed in the simulation of few-cycle wave packets.

The greatest attention is devoted to exact spatiotemporal solutions that can be viewed as modifications and generalizations of the Bateman wave. This family includes solutions exhibiting quite a wide variety of properties. We begin with Bateman's classical result, which, in its original form, is of primarily mathematical interest as quite a simple example of a solution to the homogeneous wave equation possessing a singularity at a single running point. We then proceed to complexified versions of Bateman-type solutions and to their various generalizations and modifications, which provide models for a broad range of spatiotemporal wave beams and wave packets. Among the solutions considered are

higher-order modes with axisymmetric phase, as well as solutions with astigmatic and tilted phases.

Interest in both Bateman-type solutions and complex-source fields is largely motivated by studies of Gaussian beams in the time-harmonic regime within the framework of the common paraxial theory based on the parabolic equation method; see, for example, [4,5]. Although the theory based on generalizations of Bateman's construction is primarily aimed at describing multi-cycle spatiotemporal wave fields that oscillate quite rapidly in the longitudinal direction and exhibit Gaussian localization in the transverse directions, the theory is not restricted to the paraxial regime and can describe solutions possessing a broad angular spectrum. A notable example is the splash pulse, which can equally well be interpreted within the framework of complex-source theory. We also discuss modifications of Bateman-type solutions that give rise to the so-called tilted waveobjects.

The final family of solutions discussed in this paper is that of quasi-spherical waves, a subject that has emerged only recently. These solutions are noteworthy both as an essential subfamily of unidirectional waves and as a useful tool for modeling few-cycle optical pulses. Quasi-spherical waves encompass spatiotemporal solutions with remarkably diverse properties. It is worth noting that they are closely related to the X waves that received considerable attention in the 1990s.

In this paper, we concentrate mainly on the case of three spatial dimensions, but also briefly address other dimensions.

2. Relatively Undistorted Waves

2.1. General Definitions

The classical monograph [6] introduces relatively undistorted progressing wave families as solutions of Equation (1) having the form

$$u = gf(\theta). \quad (2)$$

Here, the waveform f is an essentially arbitrary function of one variable, and the phase $\theta = \theta(x, y, z, t)$ and the amplitude $g = g(x, y, z, t)$ are specific functions. Other terms occur in the literature as well, in particular non-dispersive waves. We refer to the ansatz (2), containing the functional parameter f , as a relatively undistorted wave. Throughout the paper, we choose the z -axis as the preferred direction of wave propagation, calling the z -coordinate the longitudinal coordinate, and the x - and y -coordinates the transverse coordinates.

In fundamental monographs such as Refs. [6,7], the functional classes to which arbitrary functions f belong are rarely specified explicitly. For real-valued phases, one may work with classical solutions of the wave equation, assuming the waveforms to be twice continuously differentiable away from singularities. Alternatively, one may formulate the problem in the framework of distribution theory, or, in other terminology, the theory of generalized functions (see, e.g., [8,9]). For complex phases, however, we assume that the functions $f(\theta)$ are regular on the domain of the phase variable θ . By regularity, we mean analyticity at interior points of the domain and smoothness along its boundary. In what follows, we use the term “regularity” in that sense. The importance of the requirement of analyticity of waveforms of relatively undistorted waves with nonreal phases is justified in paper [10].

To cover essential examples (see Section 3), the just given classical definition should be refined while allowing Equation (1) to be satisfied not on the entire spacetime $\mathbb{R}^3 \times \mathbb{R}$, but with the exception of lower-dimensional manifold [11].

Reference [6], while considering plane and spherical waves already known in eighteenth century to Jean-Baptiste d’Alembert and Leonard Euler, did not mention the Bateman waves which had been discovered by that time and whose modifications subsequently proved to be essential for modeling highly localized wave beams and packets. Another intriguing family of relatively undistorted waves, which we call quasi-spherical waves, arose in the current century.

2.2. Equations for the Phase and Amplitude—The Study by Friedlander

Let us discuss how much freedom there is in the choice of the phase in an expression of the form (2). As is generally known (see, for example, [3,10]), substituting the solutions (2) into Equation (1) and using the arbitrariness of f , one obtains that θ and g must simultaneously satisfy three equations:

$$(\nabla\theta)^2 - \frac{1}{c^2}\theta_t^2 = 0, \quad (3)$$

$$2\left(g_x\theta_x + g_y\theta_y + g_z\theta_z - \frac{1}{c^2}g_t\theta_t\right) + g\Box\theta = 0, \quad (4)$$

where the subscript denotes the partial derivative with respect to the variable shown, and

$$\Box g = 0. \quad (5)$$

The amplitude must satisfy Equations (4) and (5) simultaneously; this is possible only for a closely restricted class of phase functions.

Frederick Friedlander addressed a substantial special case (including plane and spherical waves):

$$\theta(x, y, z, t) = \tau(x, y, z) - ct, \quad (6)$$

where $\tau(x, y, z)$ is real and $g = g(x, y, z)$. In Ref. [12], it was established that, for the solvability of the system of Equations (4) and (5), the wave fronts—that is, the level surfaces $\tau = \text{const}$ of the function τ —must be algebraic surfaces of the fourth order known as cyclides of Dupin. Their special cases include planes and quadric surfaces, particularly spheres.

No general results are available on the existence of solutions of the form (2) for phases with more general time dependence than Equation (6). Two examples of such solutions that are of particular interest from the standpoint of Optics–Bateman-type waves and quasi-spherical waves are considered in Sections 5 and 6.

Friedlander also introduced a generalization of the solutions (2) in the form of finite sums (which reduces to the solutions (2) at $N = 1$)

$$u = \sum_{m=1}^N g_m(x, y, z, t) f_m(\theta), \quad f_{m-1}(\theta) = \frac{df_m(\theta)}{d\theta}, \quad (7)$$

where f_N is still an arbitrary function and θ and g_m are specialized functions. Examples of these are the fields of time-dependent multipoles and the solutions obtained from them by a Lorentz transformation [13].

Infinite series of the form (7), however, occur widely in mathematics in the study of the propagation of singularities; see, e.g., [6]. Such series construction imposes no strong enough restriction on the shape of wave fronts.

3. Plane Waves

The classical plane wave solution, having the form (2) with $g = 1$ and

$$\theta = z - ct, \quad (8)$$

where f is an arbitrary function of one real variable, describes waves running along the z -axis. One may equally well consider waves with $\theta = z + ct$ traveling in the opposite direction. Both such solutions attributed to d'Alembert certainly exist for a wave equation with any number of variables.

Essential generalization of Equation (8) is given through the complexification which requires that the number of spatial variables be at least two,

$$\theta = \sqrt{1 - p^2}x + pz - ct, \quad (9)$$

where we set $p > 1$ upon choosing a branch of the square root, for example as $\sqrt{1 - p^2} = i\sqrt{p^2 - 1}$. Regardless of the chosen branch of the square root, the phase (9) assumes all complex values, and we require the waveform f to be analytic throughout the entire complex plane. It should be noted that any such function other than a constant is unbounded. For the time-harmonic case characterized by the waveform

$$f(\theta) = e^{ik\theta}, \quad k = \text{const}, \quad (10)$$

such solutions are known as evanescent plane waves. Practical applications of solutions with complex phases are not restricted to case (10); such solutions also involve more general analytic waveforms, as, for example, in problems of pulse reflection and transmission at interfaces under conditions of total internal reflection; see, e.g., [14,15].

Following the terminology adopted in optics, the solutions (2), (8) with $g = \text{const}$ can be called fundamental modes, and solutions with $g \neq \text{const}$ can be called higher-order modes. As pointed out in Ref. [12], any solution of the 2D (two-dimensional) Laplace equation

$$g_{xx} + g_{yy} = 0 \quad (11)$$

can be taken for the amplitude. All non-constant solutions of Equation (11) unboundedly grow to infinity.

Solutions of the form (2), (8), (11) growing linearly in the transverse direction have been encountered in diffraction theory. For example, in Ref. [16], diffraction on a semi-infinite screen of an incident wave of the form

$$u = xH(z - ct) \quad (12)$$

was studied, where

$$H(s) = \begin{cases} 1, & s \geq 0, \\ 0, & s < 0. \end{cases} \quad (13)$$

is the Heaviside step function; see Figure 1. Waves with a plane wavefront and an arbitrary waveform, having linear growth in amplitude, were also met in the description of surface elastic waves in Ref. [17] and in anisotropic elasticity in Ref. [18].

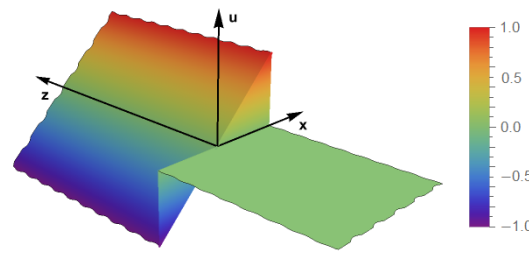


Figure 1. Plane wave with a linear transverse structure (12).

4. Spherical Waves

4.1. Classical Spherical Wave and Its Complexification

Classical divergent spherical wave

$$u = \frac{f(\theta)}{R} \quad (14)$$

has the form (2) with

$$\theta = R - ct, \quad (15)$$

where

$$R = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2} \quad (16)$$

is the distance from the origin, $r = \sqrt{x^2 + y^2}$, and

$$g = \frac{1}{R}. \quad (17)$$

This expression describes the field of a point source at the origin:

$$\square u = -4\pi\delta(x)\delta(y)\delta(z)f(-ct), \quad (18)$$

where $\delta(\cdot)$ is the Dirac delta function. There are no higher-order modes that are regular outside the origin, i.e., no such solutions with $g \neq \text{const}/R$ exist.

Solution (14) cannot be straightforwardly generalized to other spatial dimensions; see, e.g., [19]. In odd spatial dimensions greater than three, the analogues of the wave (14) take the form of the generalized relatively undistorted waves (7), whereas in even dimensions, no relatively undistorted waves with phase $\theta = R - ct$ are available.

The converging spherical wave is obtained from Equation (14) by replacing $R - ct$ with $R + ct$; for it, $f(-ct)$ is replaced with $f(ct)$ in Equation (18).

In order to derive exact solutions of Equation (1), exhibiting Gaussian beam asymptotics, critical for the description of fields in resonators, Alexey Izmet'ev [20] and George Deschamps [21] subjected the spherical wave to complexification via a complex shift of the variable z ,

$$z \mapsto z - ia,$$

where $a > 0$ is a free parameter. This resulted in

$$R \mapsto R_* = \sqrt{x^2 + y^2 + (z - ia)^2} = \sqrt{r^2 + (z - ia)^2}.$$

A naive expansion of the root near the z -axis readily gives

$$R_* \approx z - ia + \frac{r^2}{2(z - ia)}.$$

For the time-harmonic regime characterized by the waveform (10) with a real wavenumber k , it yields

$$u = \frac{e^{i(kR_* - \omega t)}}{R_*} \approx \frac{1}{z - ia} \exp i \left(k(z - ct) + \frac{z}{a} \frac{r^2}{\delta_\perp^2} \right) \exp \left(-\frac{r^2}{\delta_\perp^2} \right) e^{ka}, \quad (19)$$

where $\omega = kc$, and

$$\delta_\perp = \delta_\perp(z) = \sqrt{2 \frac{z^2 + a^2}{ak}}$$

is the transverse width. The last factor in Equation (19) is just the normalization constant. For $ka \gg 1$, the solution (19) describes a time-harmonic beam with a Gaussian transverse localization. Such a beam quite rapidly oscillates in the longitudinal direction and has close localization in the transverse directions, being the most narrow at the waist $z = 0$. The expression (19), known as the time-harmonic Gaussian beam, was previously arising in approximate paraxial consideration (see, for example, [4,5]). For $ka \gg 1$, the solution (19) is highly localized in the transverse directions and exhibits quite rapid oscillations in the longitudinal direction. It is typical that fast enough longitudinal oscillations give rise to quite a strong transverse confinement; see, e.g., [4].

The solution (19) immediately attracted the attention. This solution was called a “complex source field”, since it was implied that the source in Equation (18) had moved into the complex domain due to complexification. In Ref. [22], an interpretation of the corresponding wavefields in terms of complex sources and sinks was introduced. Such an approach remains relevant in the physics literature to this day; see, e.g., a very recent review [23]. Efforts have been made to formulate the terminology related to sources in complex space in mathematically rigorous terms; see, for instance, [24] and references therein.

Moving to the non-harmonic time dependence,

$$u = \frac{f(R_* - ct)}{R_*}, \quad (20)$$

one, as stressed in Section 2 earlier, must deal with waveforms f that are analytic in the range of values of θ . Detailed analysis shows that sources remain in the real space, and the complexified solution satisfies an equation of the form

$$\square u = F(x, y, z, t),$$

where F is a distribution depending on f and localized on a 2D manifold determined by choosing the square root in R_* . This manifold where R_* has jumps is called an antenna.

In general, an antenna that can be bounded or unbounded has, as its edge, the circle $\mathcal{C} = \{r = a, z = 0\}$, on which R_* vanishes, while the complexified spherical wave becomes infinite. Some antennas corresponding to different choices of the square-root branch are shown in Figure 2.

The antenna may be either a plane with a circular aperture, $\{z = 0, r \geq a\}$, or an infinitely thin disk $\{z = 0, r \leq a\}$. In the first case (which corresponds to the beam choice of the branch of square root such that $\text{Im } R_* \geq 0$; see, e.g., [25,26]), with an appropriate choice of the waveform f , the field takes the form of a Gaussian beam or wave packet propagating near the z -axis. In the second case (corresponding to the source choice with $\text{Re } R_* \leq 0$), the beam or packet is radiated by the antenna in the direction of positive z , while for $z < 0$, the wavefield decays rapidly enough. The time-harmonic wavefield for these two cases is shown in Figure 3.

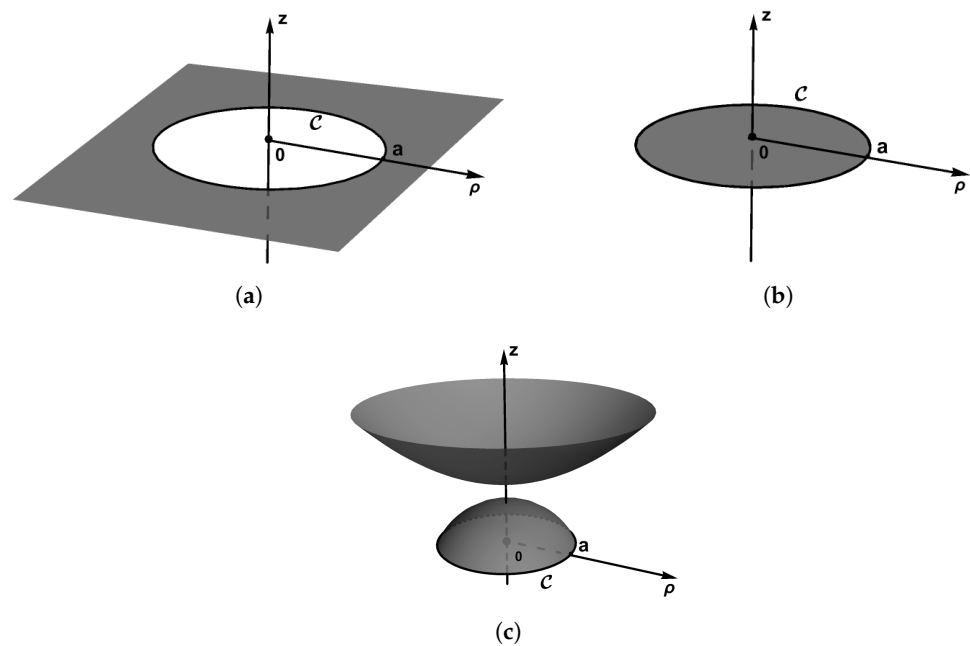


Figure 2. Different choices for antenna: (a) beam, (b) source, and (c) exotic choices.

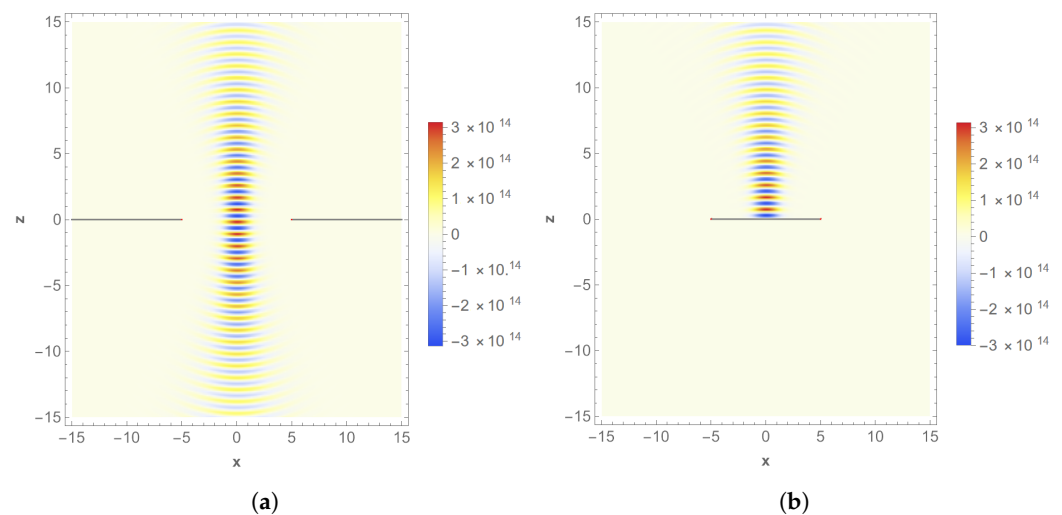


Figure 3. $\text{Re } u$ at the section $y = 0$ for $ka = 35$ for (a) beam and (b) source choices. See text for details.

A detailed analysis of the relevant sources in physical space $\mathbb{R}^3$ can be found in Refs. [26–28]. As established in Ref. [29], the energy density

$$\mathcal{E} = \frac{1}{2} \left(|\nabla u|^2 + \frac{1}{c^2} |u_t|^2 \right) \quad (21)$$

is non-integrable near $\mathcal{C}$. The time-harmonic wavefield in its vicinity is shown in Figure 4.

Highly localized spatiotemporal solutions based on solution (20) are considered in Ref. [30].

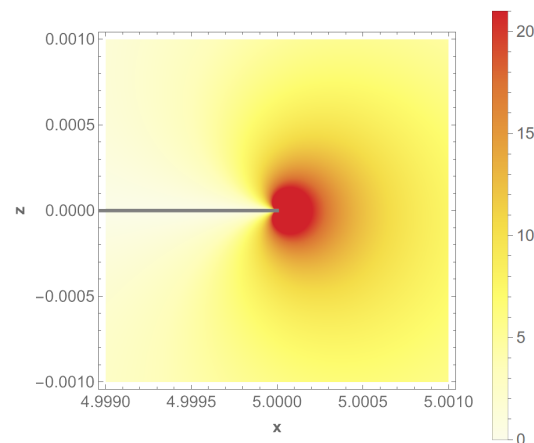


Figure 4. The same wavefield as in Figure 3b near the antenna edge.

4.2. Lorentz Transformation

It should be noted that both the complexified and classical spherical waves allow generalizations of the form (2) through transition to a moving frame of reference. Such generalizations were via a longitudinal Lorentz transformation

$$z \mapsto z' = \frac{z - \gamma ct}{\sqrt{1 - \gamma^2}}, \quad ct \mapsto ct' = \frac{ct - \gamma z}{\sqrt{1 - \gamma^2}}, \quad (22)$$

transforming one solution to Equation (1) into another. Here, γ is a real parameter of the transformation corresponding to the velocity ratio of the moving reference frame to c ; see, e.g., [31].

In Refs. [32,33], a subluminal transformation with $-1 < \gamma < 1$ was applied to Equation (20), and in Ref. [13], this was extended to multipoles of the form (7). Such solutions necessarily have singularities of the same structure as the solution (20), and their energy is also infinite. Superluminal, i.e., corresponding to $|\gamma| > 1$, Lorentz transformations of Equation (20) were considered in Refs. [3,32], giving solutions that can be free of singularities.

These studies were motivated by the highly discussed paper [34] that introduced the so-called *X* waves solutions (see also Refs. [35–39])—that is, solutions localized in the vicinity of a moving conical surface, whose discussion from the viewpoint of relatively undistorted waves we postpone until Section 6.4.

4.3. A Difference of Two “Complex-Source Solutions”

It was noticed by Colin Sheppard and Saiedeh Saghafi [22] for the time-harmonic case that the difference of diverging and converging complexified spherical waves

$$u = 2i \frac{\sin kR_*}{R_*} e^{-i\omega t}, \quad \omega = kc, \quad (23)$$

has no singularities in the physical space $\mathbb{R}^3$, which was interpreted as the result of the cancellation of effects of complex sources and sinks. In a detailed study of the solution, Sheppard and Saghafi made extensive use of these terms, whereas in Ref. [26], it was studied without using them.

In a recent paper [40], solutions of the form

$$u = \frac{f(R_* - ct) - f(-R_* - ct)}{2R_*}, \quad (24)$$

regularized at $R_* = 0$ by setting

$$u = f'(-ct) \text{ as } R_* = 0,$$

were introduced. These functions satisfy Equation (1) throughout the spacetime $\mathbb{R}^3 \times \mathbb{R}$ and are nonsingular, provided that the waveform f is regular within the strip $|\text{Im } \theta| \leq a$.

The family (24) contains both earlier known and new localized solutions of the wave Equation (1). The Sheppard–Saghafi solution (23) is described by Equation (24) for the waveform (10).

Let us also note another remarkable solution that can be represented in the form (24). Taking in Equation (24) the waveform profile as

$$f(\theta) = \frac{1}{2(\theta - ib)} \quad (25)$$

with a free real parameter b , $|b| > a$, one obtains the solution

$$u = \frac{1}{2R_*} \left(\frac{1}{R_* - ct + ib} + \frac{1}{R_* + ct - ib} \right) = \frac{1}{R_*^2 - (ct - ib)^2}, \quad (26)$$

known as the splash pulse [41].

Equation (26) defines a solution depending on two arbitrary parameters, which is convenient for optical pulse simulations. Not being quite strongly localized, however, the solution (26) has a finite energy, i.e.,

$$E = \iiint_{\mathbb{R}^3} \mathcal{E} \, dx \, dy \, dz < +\infty, \quad (27)$$

with $\mathcal{E}$ given by Equation (21). The solution (26) is used in the simulation of low-cycle scalar and electromagnetic wave pulses; see, e.g., [42]. We encounter this solution in the Section 5 in the role of a particular Bateman-type wave.

We also note that a non-complexified counterpart of Equation (24) with a non-time-harmonic waveform was used for the simulation of an optical wavefield near a focus in recent research [43]. Such solutions have also been considered in acoustics [44]. For the just above mentioned solutions obtained through the application of the subluminal Lorentz transformation, a similar procedure was employed in Ref. [32]. A combination of four such time-harmonic waves was applied to modeling acoustic beams in Ref. [45].

5. Bateman Waves

5.1. Axisymmetric Phase: Fundamental Mode

At the beginning of the twentieth century, Harry Bateman established the invariance of the wave equation under inversion transformations in the Minkowski metric. As an illustrative example, Bateman applied this transformation to a plane wave; see [7,46]. Bateman thus obtained a solution that is now known as the fundamental axisymmetric mode of the Bateman wave.

It is convenient to present it in the form (2) using characteristic variables

$$\alpha = z - ct, \beta = z + ct,$$

where θ is the axisymmetric (noncomplexified) Bateman phase

$$\theta = z - ct + \frac{x^2 + y^2}{z + ct} = \alpha + \frac{r^2}{\beta} \quad (28)$$

and

$$g = \frac{1}{\beta}. \quad (29)$$

It should be noted that in some studies, the opposite convention is used, with α and β interchanged.

As observed in Ref. [47], this solution admits a straightforward generalization to the case of any number of spatial variables starting from two, as well as to ultrahyperbolic equations.

Bateman was not concerned with the question of whether the function defined by Equations (2), (28) and (29), which has singularities, satisfies the wave Equation (1) everywhere. It has only recently been established in Ref. [48], using the methods of distribution theory [8], that this is true for smooth compactly supported waveforms.

Additionally, in Ref. [49], it was shown that a certain function generally known in mathematics [50], which exhibits a strong singularity at a single moving point but satisfies the homogeneous Equation (1) on $\mathbb{R}^3 \times \mathbb{R}$, coincides with the Bateman wave. However, the 2D analogue of the Bateman wave represents the field of a certain moving point source [11]. More references to studies investigating whether various modifications of the above Bateman wave, with both smooth and singular waveforms, satisfy the wave equation throughout the entire spacetime, can be found in Ref. [51].

The Bateman solution (2), (28) and (29), having singularity at $z = -ct$ and not being localized for any f , remained unnoticed by researchers for a long time. It was only in 1993 that Pierre Hillion [52] observed that certain localized solutions found by that time [41,53–55] were special cases of the axisymmetrically complexified version of the Bateman solution. Among these, the most significant were the focus wave mode and the splash pulse discussed below.

Hillion introduced a complex shift

$$\beta \mapsto \beta_* = \beta - ib,$$

where $b > 0$ is a free parameter in expressions (28) and (29), which yields solutions

$$u = \frac{1}{\beta_*} f(\theta), \quad (30)$$

$$\theta = \alpha + \frac{x^2 + y^2}{\beta_*}. \quad (31)$$

This satisfies Equation (1) on the entire spacetime under the assumption that f is regular in the upper half-plane where θ now takes its values. The solution (30) and (31) is said to represent a complexified axisymmetric fundamental Bateman mode.

5.2. Conservation of the L_2 -Norm

It was shown in Ref. [56] that the Bateman waves, as well as quasi-spherical waves (see Section 6), satisfy nontrivial conservation laws. Whenever the modulus of a Bateman-type wave is spatially square-integrable, the spatial integral of its squared modulus remains invariant in time. Let us demonstrate this following Ref. [56].

Indeed, it follows from Equations (30) and (31) that, under analytic continuation with respect to the variable t into the lower half-plane, the phase and amplitude of the solution (30) remain regular. At the same time, the phase θ takes values in the upper half-plane, where the waveform f is regular. Thus, the function (30) admits an analytic

continuation into the lower half-plane. Therefore, if it is square-integrable over $\mathbb{R}^3$, its Fourier representation is given by

$$u(x, y, z, t) = \iiint_{\mathbb{R}^3} U(\mathbf{k}) e^{-i[\omega t - k_z z - k_x x - k_y y]} d^3 \mathbf{k},$$

with $\mathbf{k} = (k_x, k_y, k_z)$, $d^3 \mathbf{k} = dk_x dk_y dk_z$, and $\omega = kc = |\mathbf{k}|c$. Here, by the Parseval identity (see, e.g., [9]), $U(\mathbf{k})$ is a square-integrable function of $\mathbf{k}$, and

$$\iiint_{\mathbb{R}^3} |u(x, y, z, t)|^2 d^3 \mathbf{R} = (2\pi)^3 \iiint_{\mathbb{R}^3} |U(\mathbf{k}) e^{-i\omega t}|^2 d^3 \mathbf{k} = (2\pi)^3 \iiint_{\mathbb{R}^3} |U(\mathbf{k})|^2 d^3 \mathbf{k}. \quad (32)$$

The expression on the most right of the equality (32) is independent of time, and thus we have proved the conservation of the L_2 norm of the solution (30). A differential analogue of the conservation law for the L_2 norm (the continuity equation) was also obtained in Ref. [56].

5.3. Examples of Localized Solutions

Let us list three waveform choices of particular interest in expression (30). The first two lead to comparably strongly localized solutions, convenient for simulating multicycle wave beams and packets. The third has found application as a model of a few-cycle pulse.

In the case of the choice given by Equation (10), with k an arbitrary positive parameter, an exact solution arises in the entire spacetime,

$$u = \frac{1}{\beta_*} \exp i \left(k\alpha + \frac{\beta}{b} \frac{r^2}{\Delta_{\perp}^2} \right) \exp \left(-\frac{r^2}{\Delta_{\perp}^2} \right), \quad (33)$$

describing a beam propagating along the z -axis with Gaussian transverse localization, cf. Equation (19). Here,

$$\Delta_{\perp} = \Delta_{\perp}(\beta) = \sqrt{\frac{\beta^2 + b^2}{kb}} \quad (34)$$

is the transverse width which is minimal at the waist, described by $z = -ct$. The solution (33) has infinite energy because it decays too slowly as soon as $z \rightarrow \pm\infty$. The solution (33) is known as the focus wave mode [53]; its 2D analog was presented in Ref. [54].

Hillion's observation provided an impetus for constructing simple solutions describing wave packets that are Gaussian-localized both in the transverse and longitudinal directions. A most fit choice [57] is a Gaussian wave packet with a finite energy, defined as

$$f(\theta) = \exp \left[2kb \left(1 - \sqrt{1 - \frac{i\theta}{b}} \right) \right] \quad (35)$$

and the branch of the square root is chosen such that $\text{Re} \sqrt{\dots} \geq 0$. When $kb \gg 1$, such a solution quite rapidly oscillates and is localized in the Gaussian manner around a peak point $\{r = 0, \alpha = 0\}$, where $\theta = 0$. Its asymptotic behavior near the peak is obtained by expanding the square root up to quadratic terms and has the form

$$u \approx \frac{1}{\beta_*} \exp i \left(k\alpha + \frac{\beta}{b} \frac{r^2}{\Delta_{\perp}^2} \right) \exp \left(-\frac{r^2}{\Delta_{\perp}^2} - \frac{\alpha^2}{\Delta_{\parallel}^2} \right), \quad (36)$$

where the longitudinal width $\Delta_{\parallel}$ equals

$$\Delta_{\parallel} = 2\sqrt{\frac{b}{k}},$$

which is constant. Note that the right-hand side of Equation (36) differs from that of Equation (33) only by the presence of the term with $\alpha^2/\Delta_{\parallel}^2$ describing the packet localization around the running point $\{r = 0, z = ct\}$. The packet is shown in Figure 5. One can also find a collection of functions with similar properties in Ref. [57].

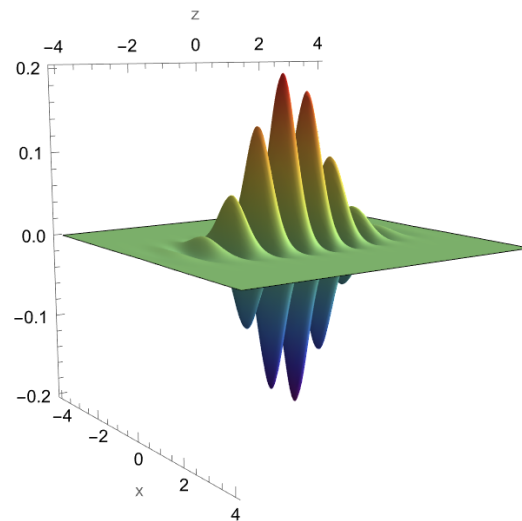


Figure 5. The Gaussian wavepacket (36): $\text{Re } u$ for $kb = 45$ at $t = 0$.

Taking in Equation (30)

$$f(\theta) = \frac{1}{\theta + ia}$$

with $a > 0$, one obtains, after a straightforward calculation, the expression

$$u = \frac{1}{\beta_*} \frac{1}{\theta + ia} = \frac{1}{r^2 + \left[z - i\frac{b-a}{2}\right]^2 - \left[ct - i\frac{b+a}{2}\right]^2}, \quad (37)$$

which is identical to Equation (26) upon setting

$$b - a = 2a, \quad b + a = 2b.$$

One can see that the splash pulse arises both in the theory of complexified spherical waves and in Bateman's wave theory.

More waveforms yielding localized solutions can be found in studies [3,41,58].

5.4. Axisymmetric Phase: Higher-Order Modes

To construct higher modes of interest in optics, i.e., solutions of the form (2), (28) with

$$g = \frac{W}{\beta_*},$$

where the modulating factor W is non-constant, we introduce the variables

$$X = \frac{x}{\beta_*}, \quad Y = \frac{y}{\beta_*}.$$

If one assumes that the modulating factor has the form

$$W = W(X, Y),$$

then W can be taken as any solution of the 2D Laplace equation

$$W_{XX} + W_{YY} = 0,$$

while the waveform f can be chosen arbitrarily. Such solutions u are called Laplace modes [4,59]. As noted in Ref. [60], among such solutions are some that describe the propagation of Gaussian beams along the edge of a wedge with an arbitrary opening angle.

For a special waveform given by Equation (10), a modulation factor depending on β_* ,

$$W = W(X, Y, \beta_*),$$

can be found, satisfying the equation

$$W_{XX} + W_{YY} + 4ik\beta_*W_{\beta_*} = 0.$$

By substituting $W = \frac{1}{\beta_*} \exp\left(\frac{iK^2}{4k\beta_*}\right) \tilde{W}(X, Y)$ with an arbitrary constant K , this reduces to the 2D Helmholtz equation

$$\tilde{W}_{XX} + \tilde{W}_{YY} + K^2\tilde{W} = 0, \quad (38)$$

and one arrives at Helmholtz modes [4,59]

$$u = \frac{1}{\beta_*} \exp\left(ik\theta + \frac{iK^2}{4k\beta_*}\right) \tilde{W}(X, Y). \quad (39)$$

The best-known solutions of this type are Bessel–Gauss pulses with

$$\tilde{W} = J_m\left(\frac{Kr}{\beta_*}\right) e^{im\phi},$$

where ϕ is a polar coordinate defined by

$$x = r \cos \phi, \quad y = r \sin \phi,$$

and J_m is the Bessel function. The solution for $m = 0$ was found by a cumbersome method in Ref. [61]; a straightforward derivation for the general case just described is given in Ref. [59]. Other essential examples of Laplace and Helmholtz modes can be found, for example, in Ref. [4].

5.5. Astigmatic Phases

The Bateman wave can be generalized by incorporating dependence on the transverse coordinates into the phase through a quadratic form (dependent on β). In optics, such a type of phase is referred to as astigmatic [62]. Such solutions can be obtained by setting

$$\theta = \alpha + \mathbf{r}^T \Gamma(\beta) \mathbf{r} \quad (40)$$

and

$$g = \sqrt{\det \Gamma(\beta)}, \quad (41)$$

where $\mathbf{r} = (x, y)^T$, with ^{T} denoting transposition, and

$$\Gamma(\beta) = (\Gamma_0^{-1} + \beta \mathbf{I})^{-1}. \quad (42)$$

Here, $\Gamma_0 = \Gamma(0)$ is a symmetric 2×2 matrix with positive definite imaginary part $\text{Im } \Gamma_0$, and $\mathbf{I}$ is the unit matrix. The square root in the definition understood in an arbitrary manner. Positive definiteness of $\text{Im } \Gamma_0$ implies positive definiteness of $\text{Im } \Gamma(\beta)$ for all values of β . Proofs of this significant result for the theory of astigmatic waveobjects, presented with varying degrees of generality and detail, can be found, for example, in Refs. [62,63]. Therefore, the functions (2), (40) and (41) are free of singularities throughout the entire spacetime if $f(\theta)$ is regular in the half-plane $\text{Im } \theta \geq 0$.

An expression of the form (2), (40) and (41) describes a relatively undistorted wave generalizing the Bateman wave [62]. This object is referred to as the complexified astigmatic fundamental mode of the Bateman wave.

In the case

$$\Gamma(\beta) = \frac{1}{\beta_*} \mathbf{I}$$

one has the axisymmetric solution introduced in Section 5.1 above. If

$$\Gamma(\beta) = \begin{pmatrix} 1/q_1 & 0 \\ 0 & 1/q_2 \end{pmatrix} \quad (43)$$

with $q_{1,2} = \beta - z_{1,2} - ib_{1,2}$, where $z_{1,2}$ and $b_{1,2} > 0$ are free real constants, then the corresponding solution exhibits simple astigmatism. Solutions with matrices of the form (43) were first introduced in Ref. [57]. A solution (2), (40) and (41) is called general astigmatic if the matrix $\Gamma(\beta)$ is not diagonalizable by an orthogonal transformation. Such matrices can be obtained from Equation (43) by a rotation through an imaginary angle $i\Psi$, where Ψ is a real parameter [62,64], yielding

$$\Gamma(\beta) = \begin{pmatrix} q_1^{-1} \cosh^2 \Psi - q_2^{-1} \sinh^2 \Psi & i(q_1^{-1} - q_2^{-1}) \sinh \Psi \cosh \Psi \\ i(q_1^{-1} - q_2^{-1}) \sinh \Psi \cosh \Psi & -q_1^{-1} \sinh^2 \Psi + q_2^{-1} \cosh^2 \Psi \end{pmatrix}. \quad (44)$$

General solutions of the form (2), (40) and (41) were addressed in Ref. [62], where their specifications with proper waveforms modeling Gaussian-localized wave beams and wavepackets were considered.

In Figure 6, red denotes the level surface of the modulus of the real part of the astigmatic Gaussian wavepacket (2), (35), (40) and (41), and pink denotes the level surface of the modulus of the corresponding Gaussian beam (2), (10), (40) and (41).

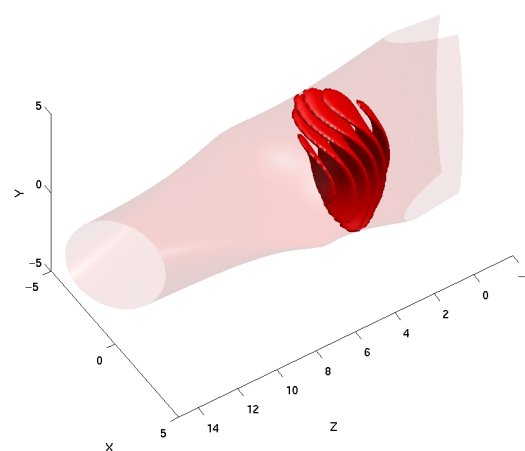


Figure 6. Astigmatic wavepacket versus astigmatic wave beam with $z_1 = 3$, $b_1 = 4$, $z_2 = 10$, $b_2 = 1$, $k = 5$, $\Psi = 0.19$, $b = 2.5$, and $ct = 30$. See text for details.

The astigmatic solutions (2), (40) and (41) allow a straightforward generalization to an arbitrary spatial dimension, starting from 4D [65]. The related 4D solutions found an application in the construction of highly localized solutions of the 3D Klein–Gordon–Fock equation [66].

5.6. Tilted Solutions

Within the framework of the approximate paraxial theory, the so-called tilted Gaussian beams were introduced for the time-harmonic case in Ref. [67]. However, only quite small tilt angles could be considered there. The extension to Bateman-type solutions, in which the smallness of the tilt is not required, was first presented in Ref. [68].

Under a complex shift of the transverse variable

$$x \mapsto x - ih,$$

with $h > 0$, in the solution (30), the phase (31) transforms to

$$\theta = \alpha + \frac{(x - ih)^2 + y^2}{\beta - ib} = \hat{\alpha} + \frac{[x - (z + ct) \tan \psi]^2 + y^2}{z + ct - ib} - i \frac{h^2}{b}, \quad (45)$$

where

$$\hat{\alpha} = \frac{z \cos 2\psi + x \sin 2\psi - ct}{\cos^2 \psi} \quad (46)$$

with $\psi = \arctan(h/b)$. The last item in Equation (45) is just a constant phase shift. In contrast to the paraxial theory, ψ is not assumed quite small.

We restrict ourselves to the examples of a tilted Gaussian beam and a tilted Gaussian wave packet.

In the case of the waveform (10), one has

$$u = \frac{1}{\beta_*} \exp i \left(k \hat{\alpha} + \frac{\beta}{b} \frac{\hat{r}^2}{\Delta_{\perp}^2} \right) \exp \left(- \frac{\hat{r}^2}{\Delta_{\perp}^2} \right) e^{kh^2/b}. \quad (47)$$

Here,

$$\hat{r}^2 = [x - (z + ct) \tan \psi]^2 + y^2, \quad (48)$$

and $\Delta_{\perp}$ is defined by Equation (34). This solution describes a beam that is Gaussian-localized, at each instant of time, around a straight line inclined at an angle ψ to the z -axis. The beam waist, as in the case of Equation (33), is located at the point $\{x = y = 0, z = -ct\}$, which runs along the z -axis with velocity c . The term $\hat{\alpha}$ in the phase corresponds to a plane wave propagating at the speed c in a direction that makes an angle 2ψ with the z -axis.

For localized pulses, it is convenient to introduce a phase of the form

$$\hat{\theta} = \theta + i \frac{h^2}{b}.$$

If the waveform is localized near $\hat{\theta} = 0$, the solution describes a pulse centered at the point $\{x = ct \sin 2\psi, y = 0, z = ct \cos 2\psi\}$ running with speed c in a direction inclined at an angle 2ψ to the z -axis. In particular, for the waveform (35), one has

$$u \approx \frac{1}{\beta_*} \exp i \left(k \hat{\alpha} + \frac{\beta}{b} \frac{\hat{r}^2}{\Delta_{\perp}^2} \right) \exp \left(- \frac{\hat{r}^2}{\Delta_{\perp}^2} - \frac{\hat{\alpha}^2}{\Delta_{\parallel}^2} \right). \quad (49)$$

A detailed discussion of the properties of tilted beams and packets can be found in Ref. [69], where those beams and packets were constructed for the general astigmatic case. For localized pulses, their energies and angular momenta were also evaluated there.

The results obtained in this Section can also be obtained by means of a Lorentz transformation in the transverse direction [69,70]. A transformation along the z -axis alters the parameters of the solution but does not change its qualitative properties [69].

6. Quasi-Spherical Waves

6.1. Representation in the Form (2)

Relatively undistorted waves, which we call quasi-spherical waves, have only begun to be studied in the current millennium. Our interest in these solutions was motivated by their potential application to the modeling of few-cycle pulses for which the traditional parabolic-equation approach is inadequate. They can be presented in the form (2) with

$$\theta = S - z - ib, \quad (50)$$

where

$$S = \sqrt{(ct + ib)^2 - r^2} \quad (51)$$

and

$$g = \frac{1}{S} \quad (52)$$

with $b > 0$ being a free parameter. We choose the branch of the square root with a positive imaginary part. The phase (50) takes values in the upper half-plane, $\text{Im } \theta \geq 0$, and the waveform f is required to be regular in this domain. To note is that the amplitude (52) is known as the symmetric 2D splash pulse [71]. Up to a numerical factor, it is obtained by integrating the function (26) with respect to z over infinite limits.

Solutions defined by Equations (2), (50)–(52) seem to have first appeared in Ref. [72] and were later considered in Refs. [73–75]. It is observed in Ref. [74] that these solutions can be derived from spherical waves through a transformation

$$z \mapsto i(ct + ib), \quad ct \mapsto i(z + ib).$$

Higher-order modes for quasi-spherical solutions were constructed in Ref. [76]. In particular, they can be of the form (2), (50) and (51) with

$$g = \frac{\vartheta^n}{S}, \quad (53)$$

where

$$\vartheta = \frac{re^{\pm i\phi}}{S + ct + ib} \quad (54)$$

and $n = 0, 1, \dots$.

6.2. Unidirectionality and Integral Representations

The interest in solution (2), (50)–(52) is explained, on the one hand, by their unidirectionality along the z axis. Unidirectional waves are defined, for example, in Ref. [74] as those represented by their expansions in homogeneous plane-waves

$$e^{i(\omega t - k_x x - k_y y - k_z z)}, \quad \omega = c\sqrt{k_x^2 + k_y^2 + k_z^2} \geq 0,$$

in the form

$$u(x, y, z, t) = \int_{-\infty}^{+\infty} dk_x \int_{-\infty}^{+\infty} dk_y \int_0^{+\infty} \tilde{u}(\mathbf{k}) e^{i[kct - k_z z - k_x x - k_y y]} dk_z \\ = \iiint_{\mathbb{R}^3} H(k_z) \tilde{u}(\mathbf{k}) e^{i[\omega t - k_z z - k_x x - k_y y]} d^3 \mathbf{k}, \quad (55)$$

with a suitable weight $\tilde{u}$ ensuring the convergence of the integral. Here, all plane-wave constituents run in directions forming an angle with the z axis that does not exceed $\pi/2$. The spatial spectrum of u equals, up to a constant factor, $\tilde{u}(\mathbf{k})H(k_z)$. A more general definition of unidirectionality, which includes plane waves with negative frequencies, was proposed in Ref. [56].

This property expresses the natural requirement, from a physical point of view, that the mathematical model of the pulse describes its propagation strictly from the source. Such solutions are also referred to as causal or forward-propagating; see, e.g., Refs. [73,77,78]. Attention to unidirectional solutions was attracted by Ref. [79].

It is noteworthy that the unidirectionality understood in the above sense does not exclude the possibility that in some spatiotemporal regions, the projection of the energy flow vector on the chosen direction may turn out to be negative; see, e.g., [76,80] and references therein.

In the case of axial symmetry, the representation given by Equation (55) reduces to

$$u(t, r, z) = \int_0^{\infty} d\omega \int_0^{\omega/c} A(k_z, \omega) e^{i(\omega t - k_z z)} J_0\left(r\sqrt{\omega^2/c^2 - k_z^2}\right) dk_z$$

with some A that ensures the convergence, $\omega = kc$, and J_0 as the zero-order Bessel function. The relationship between the functions $\tilde{u}$, A , and f was described in Ref. [74].

For square integrable unidirectional solutions, the conservation of their L_2 norm in time is proved in Ref. [56], as well as a differential analogue of the L_2 -norm conservation law for quasi-spherical waves.

6.3. Simulation of Low-Cycle Pulses

Another incentive to study quasi-spherical waves was the progress made in generating low-cycle laser pulses.

The simplest low-cycle quasi-spherical pulse with finite energy is found in Refs. [81,82] by an elementary splitting of the splash pulse into two unidirectional parts. It has the form

$$u = \frac{1}{S(S - z - i\zeta)}, \quad (56)$$

where ζ is a free parameter assumed real, and S is defined by Equation (51). The solution has no singularities if

$$\zeta < b.$$

Plane-wave expansion of the function (56), as shown in Ref. [83], reads

$$u = -\frac{1}{2\pi} \iiint_{\mathbb{R}^3} H(k_z) \frac{1}{k} e^{i[k(ct+ib) - k_z(z+i\zeta) - k_x x - k_y y]} d^3 \mathbf{k},$$

i.e., it is represented in the form (55) where

$$\tilde{u}(\mathbf{k}) = -\frac{1}{2\pi k} e^{-kb + k_z \zeta}.$$

With different ratios between the free parameters, the function (56) describes pulses of different types. If b and ζ are close, then the pulse looks like a pancake, and at $|\zeta| \ll b$, it looks like a ball. If $\zeta < 0$ and $|\zeta| \gg b$, then the shape of the pulse in the focal area resembles a needle; see [81,82]. This solution, for the special case $\zeta = 0$, was independently obtained in Ref. [80]. The expression (56) was used for simulating electromagnetic pulses in Ref. [83].

Figure 7 shows the imaginary part of the function (56) for various values of the parameter ζ . Figure 8 presents the modulus of its spatial spectrum for the same values of the parameters.

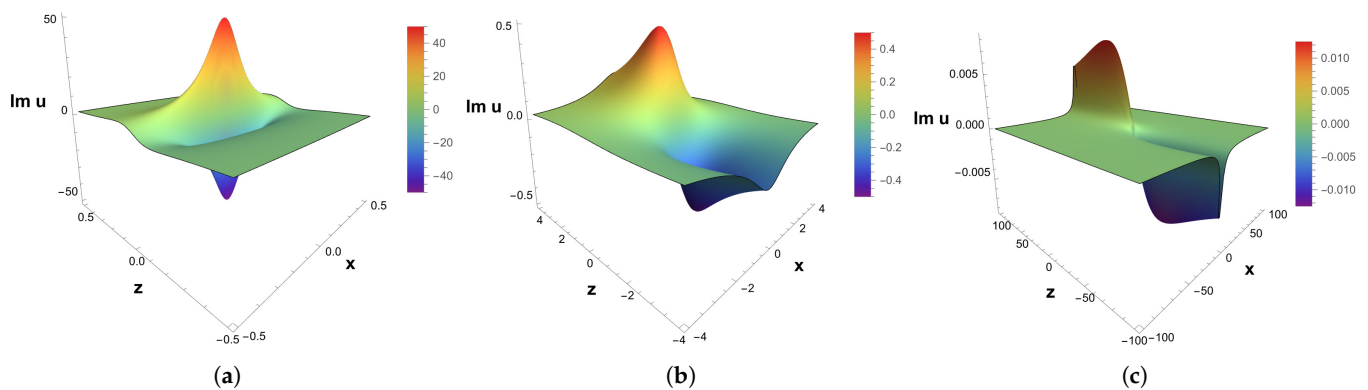


Figure 7. $\text{Im } u$ for the function (56) with $b = 1$ at $t = 0$ for (a) pancake with $\zeta = 0.99$, (b) ball with $\zeta = 0$ and (c) needle with $\zeta = -39$.

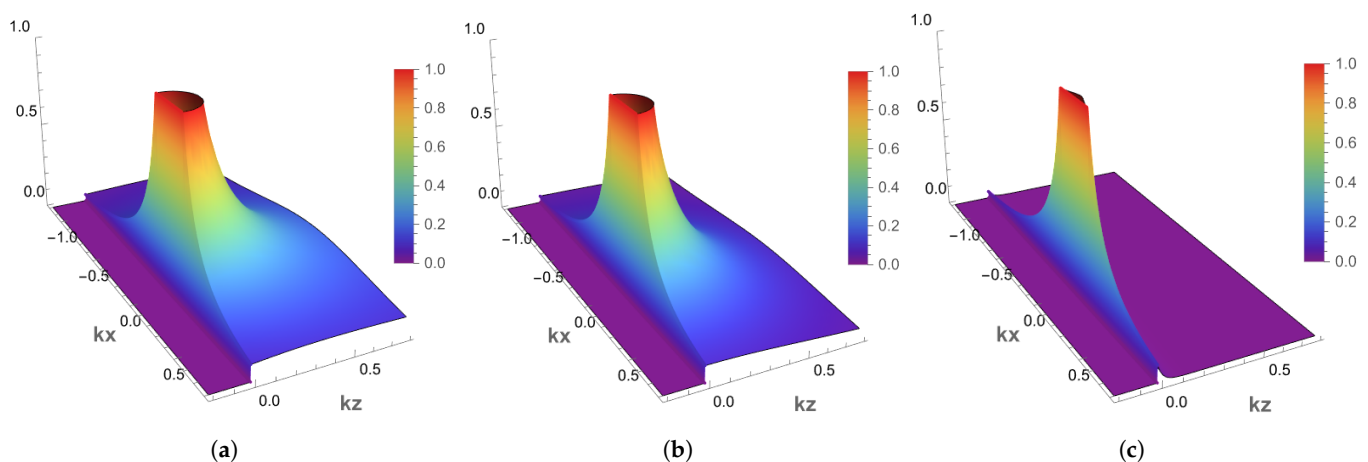


Figure 8. Spatial spectrum of the function (56) with $b = 1$ at $t = 0$ for (a) pancake with $\zeta = 0.99$, (b) ball with $\zeta = 0$ and (c) needle with $\zeta = -39$.

A number of other quasi-spherical waves with more complicated analytical representations, and a few of the form (7), have been studied keeping in mind applications in optics in Refs. [75,76].

6.4. Lorentz Transformation, Strong Unidirectionality, and X Waves

The application of the Lorentz transformation (22) to the unidirectional solutions of Equation (1) in the case

$$0 < \gamma < 1$$

yields solutions termed in Ref. [84] strongly unidirectional: the propagation directions of their plane-wave components make acute angles with the z -axis whose values do not exceed

$$\psi_0 = \arccos \gamma < \frac{\pi}{2}. \quad (57)$$

In particular, the quasi-spherical wave solution $u = 1/S$, corresponding to the constant waveform $f \equiv 1$, is transformed into the solution

$$u = \frac{1}{S'} = \frac{1}{\sqrt{(ct' + ib)^2 - r^2}}, \quad (58)$$

where t' is defined by Equation (22). The expression (58) is, up to notation and normalization, identical to the “nondiffracting superluminal zero-order X wave” found in Ref. [34] (see also Ref. [32]) through integral transforms. At each instant of time, the function (58) is localized near a conical surface whose generatrices make an angle $\pi/2 - \psi_0$ with the z -axis. The cross-section of this surface by the xOz -plane has the shape of the letter X, which motivates the term X wave. This term has also been extended to other solutions exhibiting a similar localization [32,34].

Applying the transformation (22) to quasi-spherical waves with fundamental-mode amplitude of the form (52) yields relatively undistorted strongly unidirectional waves for which the X wave serves as the amplitude:

$$u = \frac{1}{S'} f(\theta'), \quad (59)$$

where θ' and S' are obtained by replacing z and t with z' and t' in Equations (50) and (51), respectively. In Ref. [84], such solutions were employed for the simulation of few-cycle optical pulses. Similar expressions can be obtained by applying, as was done in Refs. [3,32], a superluminal (i.e., with $\gamma > 1$) Lorentz transformation to a spherical wave, including its complexified version.

Finally, by applying the transformation (22) to higher-order modes with amplitudes of the form (53), one arrives at the solutions

$$u = \frac{(\theta')^n}{S'} f(\theta'). \quad (60)$$

The obtained class of solutions can be easily seen to coincide with that derived in Ref. [79] by applying integral transform methods. Solutions given by Equations (59) and (60) are free of singularities.

7. Discussion

In this paper, we have reviewed the four principal families of relatively undistorted wave solutions and briefly mentioned several additional families that arise naturally from them via Lorentz transformations. An essential feature of these solutions, apart from plane waves, is their suitability for modeling spatiotemporally localized wave fields, also referred to as nondiffracting waves (see, e.g., [34,39,85,86]) in the form of wave beams and pulses. Such wave objects can also be modeled using other mathematical approaches, including, for instance, integral representations (see, e.g., [32,55]). These approaches sometimes yield solutions in closed form. The same applies to X waves [34–39,87,88]: some of them admit closed-form representations, whereas others do not. Not all of these closed-form solutions appear to be relatively undistorted waves.

It seems to us that, strictly speaking, among the solutions considered, only non-complexified spherical waves with a sufficiently rapidly decaying waveform can be regarded as physically realizable. The remaining solutions either possess infinite energy, as do plane waves and “complex-source” solutions, or have infinite duration, as is the case for all complexified solutions with an analytic waveform, or exhibit both features. The latter include the complexified Bateman solutions and quasi-spherical waves whose waveform does not decay sufficiently rapidly.

Nevertheless, simple solutions describing wave fields that cannot be generated throughout the entire spacetime by any physically realizable source are highly employed in the modeling of wave phenomena. For instance, the classical plane wave provides an excellent approximation to real wave fields over finite spatial and temporal intervals, which largely accounts for the centuries-long interest in this solution. Moreover, the transversely unbounded function (13) has been used in Ref. [16] to solve a diffraction problem motivated by the theory of crack propagation in an elastic medium.

The same applies to the other solutions considered here, which can provide suitable models of wave fields, although only within a finite spacetime region rather than throughout the entire spacetime. In particular, the “complex-source fields”, which have a nonintegrable energy in the vicinity of the antenna edge, are used to model a Gaussian beam in the vicinity of the optical axis (see, e.g., [23,24] and references therein). Similarly, unidirectional waves—including quasi-spherical waves and their modifications, such as X-waves—can be used to model, in a half-space, pulses generated by sources located on the plane bounding that half-space.

The solutions of the wave equation discussed above provide a convenient basis for constructing electromagnetic pulses when used as components of the Hertz vector. In particular, the function (56) was employed in Ref. [83] to construct few-cycle electromagnetic wave packets with spherical, pancake-like, needle-like, and toroidal geometries. The same methodology can be extended to the construction of solutions to the equations of elasticity theory and other systems of partial differential equations.

It is worth emphasizing that, among the relatively undistorted wave solutions, quasi-spherical waves, introduced only recently, remain the least investigated. Their asymptotic behavior at large times and distances has not been adequately studied; their tilted counterparts have not yet been analyzed; and it remains unknown which waveforms are most suitable for simulating few-cycle optical pulses. While the interaction of spherical and classical plane waves with quite a variety of obstacles has been extensively studied, and considerable attention has also been devoted to the propagation of Gaussian beams and X-waves through finite apertures and interfaces (see, e.g., [23,24,34,36–38,87,88]), corresponding problems for quasi-spherical and Bateman waves remain completely unexplored.

During the last three centuries, relatively undistorted waves have found quite a wide range of applications in mathematics and in wave physics. Their study continues to stay an active field.

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