

Article

Bessel Beams in Gyroanisotropic Crystals with Optical Activity

Yuriy Egorov ^{1,*}, Bogdan Sokolenko ¹, Aziz Aliev ^{1,2}, Ruslan Dzhemalyadinov ^{1,2}, Ervin Umerov ^{1,2} and Alexander Rubass ¹

- ¹ Physics and Technology Institute, Crimean Federal V.I. Vernadsky University, Vernadsky Prospect 4, 295007 Simferopol, Russia; simplexx.87@gmail.com (B.S.); alievaziz2704@gmail.com (A.A.); rus.dzhemalyadinov@mail.ru (R.D.); e.d.umerov@kipu-rc.ru (E.U.); alex.rubass@gmail.com (A.R.)
² Faculty of Engineering and Technology, Crimean Engineering and Pedagogical University, Uchebny Lane, 8, 295015 Simferopol, Russia
* Correspondence: yuriyegorov@cfuv.ru

Abstract: Using a Bessel beam as an example, it is shown that such a beam with the initial circular polarization, propagating through an optically active medium devoid of linear birefringence, generates an optical vortex in the orthogonally polarized component. It is shown that a medium with genuine optical activity allows the mutual conversion of beams with radial and azimuthal polarization distributions. It is also shown that a considerably weak perturbation of a medium with genuine optical activity by linear birefringence leads to the destruction of an isotropic point. In the vicinity of this point, the optical activity is suppressed by linear birefringence, so that a medium with a combined type of anisotropy responds as a medium with genuine linear birefringence. The structure of the fields of the eigenmodes of Bessel beams in a birefringent uniaxial crystal with optical activity is similar to the structure of the modes of a medium with genuine optical activity. The findings of the current study are believed to expand the understanding of physical processes in the developing field of polarization and the phase profilometry of materials.

Keywords: singular beams; gyrotropic crystals; topological charge



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1. Introduction

The propagation of electromagnetic beams in free space differs ally basically from their propagation in a birefringent medium. In general, a birefringent medium can be characterized using the birefringence tensor and the optical activity tensor [1]. Actually, such a medium has an optical axis along which the polarization state of a plane wave with an initial circular (right or left) polarization does not change. The properties of an electromagnetic beam can change fundamentally when the direction of its propagation axis relative to the crystal axis changes. Moreover, the properties of even a paraxial electromagnetic beam in a crystal differ significantly from the properties of a plane wave [2,3], since an optical beam can be represented as a set of plane waves propagating in different directions. The simplest case of a beam propagating in a birefringent medium with optical activity is a Bessel beam [4–7]. Such a beam is characterized by the axial symmetry of the field along the entire propagation axis, e.g., along the z axis.

Generally, by now, many solutions of the Helmholtz equation have been found for beams propagating in a homogeneous isotropic medium, the structure of which does not change during propagation (the so-called non-diffracting beams). The form of the solutions depends on the coordinates in which the original scalar Helmholtz equation is written. The simplest solution of this equation in Cartesian coordinates is a plane wave; in cylindrical

circular coordinates, those solutions representing Bessel beams. Solutions to the Helmholtz equation in elliptic coordinates represent Matthew beams [8,9], and solutions in parabolic coordinates represent parabolic beams [10].

Let us focus on Bessel beams in more detail. The Bessel beam obtains its name from the Bessel function, which describes the distribution of the scalar field of the beam in the cross-section in the form of a set of light and dark rings:

$$\Psi_l = J_l(U r) \exp(\pm i l \varphi) \exp(-i \beta z), \quad (1)$$

where $J_l(x)$ is the Bessel function of the first kind, U is the transverse wave number, r is the beam radius vector, φ is the radius vector inclination angle, β is the longitudinal wave number or propagation constant, related by the relation $k = \sqrt{U^2 + \beta^2}$ to k , the wave number corresponding to a plane wave propagating along the beam axis, and $l = 0, 1, 2, \dots$ is the azimuthal index or the topological charge of the beam.

Basically, a Bessel beam can contain an infinite number of rings (see Figure 1) occupying an infinitely large cross-sectional area. Consequently, such a beam must carry infinitely large energy, and creation of that beam also requires infinitely large energy.

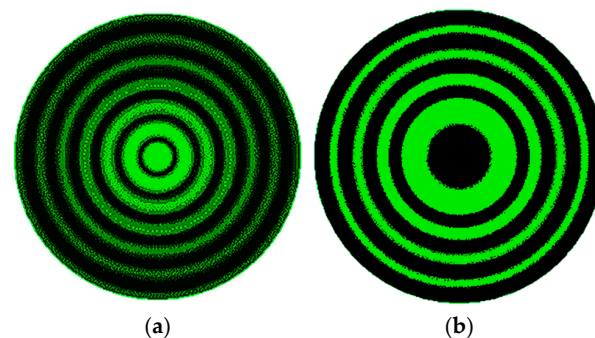


Figure 1. Intensity distribution in a Bessel beam for the azimuthal index $l = 0$ (a) and $l = 5$ (b). See text for details.

The angular spectrum of a Bessel beam is formed by a set of plane waves situating on the surface of a hollow conical surface with an angle 2θ at the cone apex given by an elementary expression: $\tan\theta = U / \beta$. The requirement for indefinitely large energy to create a Bessel beam is readily overcome in an experiment. For example, a Bessel beam can be formed using a conical prism (axicon [11]) with an angle at the apex $\pi - 2\Gamma$, where $\Gamma = \theta / (n - 1)$. Generally, a beam formed by an axicon changes its shape during propagation. However, as shown in Ref. [5], the dimensions of the central spot in the beam after the axicon do not undergo significant changes at a distance $z_{\max} = w_0 / [(n - 1)\Gamma]$, where w_0 denotes the beam waist. In Ref. [11], a method for the creation of Bessel beams carrying optical vortices based on a helical axicon, as well as a conventional axicon and a helical phase plate located in series, is analyzed experimentally and theoretically. Improvements to this technique for practical purposes are indicated in Refs. [7,12,13].

Bessel beams, similar to Gaussian beams, can be divided into two relatively large groups, namely paraxial and non-paraxial beams. Paraxial beams include beams for which $U \ll \beta$. In the case in which the transverse number becomes comparable to the wave number of a plane wave, the beam becomes non-paraxial. If $U \geq k$, the beam is transformed into an evanescent wave, rapidly attenuating in the direction of beam propagation [14]. Non-paraxial Bessel beams can no longer be described within the framework of the scalar approximation. A vector representation of the Bessel beam field based on an analysis of the vector potential is given in Refs. [15,16]. The electric and magnetic fields can no longer be described within the framework of only one Bessel

function with a given number l . In general, those fluctuations represent the superposition of Bessel functions with different azimuthal indices. Moreover, the structure of the electric and magnetic fields of such beams is different. As soon as U decreases, the structure of the fields becomes similar, and, in the limit $U \ll k$, Bessel beams in a homogeneous isotropic medium can be described within the framework of the scalar approximation.

One of the key problems in modern singular optics is the control of the properties of non-diffracting beams—in particular, the control of the spatial positions of the optical vortices carried by these beams. A suitable medium for this purpose is birefringent crystals. Light beams propagating in such crystals are described exclusively by vector fields. References [17,18] were the first drawing attention to the feature that a circularly polarized Bessel beam propagating along the optical axis of a uniaxial birefringent crystal without optical activity generates a beam carrying an optical vortex in the orthogonally polarized component. The analysis was based on the spectral integral method. The method allows representing the beam as the superposition of plane waves and find particular solutions of Maxwell's equations for specific field distributions in the original plane $z = 0$. The method cannot be used to find all possible groups of singular beams in anisotropic media. In Ref. [19], the method has been strengthened using the stereoscopic projection method and analyzed the fine structures of polarization singularities for several special cases of birefringent dichroic crystals with optical activity. Note also that in Ref. [20] the propagation of Bessel beams was analyzed in the media with natural and magnetic gyrotropy without linear birefringence and showed that the fields of such beams are represented as the superposition of transverse electrical (TE) and transverse magnetic (TM) modes (beams with azimuthal and radial distributions of the electric field, respectively). Examples of optically active crystals for the visible range include paratellurite or lithium iodate. In this paper, we use a crystal SiO_2 that is active in the visible frequency range. When propagating through optically active media, a Bessel beam with a radial distribution can transform into a beam with an azimuthal distribution and vice versa.

2. Bessel Beams in Uniaxial Birefringent Crystals with Optical Activity

Let us consider the propagation of non-diffracting electromagnetic beams in crystals whose properties are described by diagonal tensors of permittivity and optical activity. Generally, the characteristic properties of non-diffracting Bessel beams carrying optical vortices in free space and a homogeneous isotropic medium were considered in Refs. [21,22]. In general, the structure of such beams during propagation does not undergo any cardinal changes during propagation or a change in the refractive index of the medium. However, the presence of even quite weak linear birefringence leads to structural changes in the Bessel beam field [17,18]. At the same time, the presence of even weaker optical activity in a birefringent medium can lead to significant changes in the structure of the field and the wave parameters of the beams. Let us focus on the case in which the active medium is perturbed by weak uniaxial birefringence. Let the optical activity tensor be among the diagonal elements:

$$\widehat{\varepsilon} = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}, \quad \widehat{\gamma} = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix} = k\widehat{g}, \quad (2)$$

where $\widehat{\varepsilon}$ is the permittivity tensor, $\widehat{\gamma}$ is the crystal gyrotropy tensor, k is the wave number in free space, $\widehat{g}$ is the optical activity tensor [1], and the relations

$$g_3 g < 0, \quad |g_3| > |g| \quad (3)$$

are satisfied.

For all other cases, including the case of a medium with isotropic optical activity: the difference between the elements of the optical activity tensor, $\Delta g = |g - g_3| = 0$, i.e., there are no isotropic points over the entire range of traveling waves. In addition, the difference $\Delta \varepsilon = |\varepsilon - \varepsilon_3|$ between the elements of the permittivity tensor is much smaller than that of the gyrotropy tensor $\Delta \gamma = |\gamma - \gamma_3|$: $\Delta \varepsilon \ll \Delta \gamma$. If $\Delta \varepsilon = 0$, then, in the uniaxial crystal–non-diffracting beam system, there is an isotropic point U_{is} at which the propagation constants $\beta_+(U) = \beta_-(U)$ for the two (left-handed (“−”) and right-handed (“+”)) beam circulation components. If a weak perturbation occurs in the crystal: $\Delta \varepsilon \neq 0$, then $\beta_{\pm}$ are described by the general relations

$$\beta_{\pm}^2 = k^2(\varepsilon + \gamma^2) - (\varepsilon_3 + \varepsilon - 2\gamma\gamma_3)/(2\varepsilon_{\gamma}) U^2 \pm \sqrt{D}, \quad (4)$$

where D denotes the dispersion.

Equation (4) shows that each solution corresponds to the two eigenmodes with propagation constants β_+ and β_- , respectively. Each propagation constant $\beta_{\pm}$ corresponds to a pair of beam amplitude parameters $(A_{\pm}, B_{\pm})$ [3]. Following the method for the solution of a system of homogeneous equations, let us assume that $A_{\pm} = 1$. In this case,

$$B_{\pm} = (\beta_{\pm}/k) \frac{k^2(\varepsilon - \gamma^2) - \beta_{\pm}^2 - U^2(\varepsilon_3 - \gamma^2)/\varepsilon_{\gamma}}{2\beta_{\pm}^2\gamma + U^2(\varepsilon_3 - \gamma^2)(\gamma + \gamma_3)/\varepsilon_{\gamma}}, \quad (5)$$

where ε_{γ} is the permittivity depending on gyrotropy tensor.

It is understandable that the behavior of the curves in the vicinity of the point $U = U_{is}$ changes basically, and the point itself U_{is} no longer describes the isotropic case, although its value does not change:

$$U_{is}^2 \rightarrow U_g^2 = -2k^2(\gamma/\Delta\gamma) \varepsilon_{\gamma}. \quad (6)$$

where U_{is} is beam point with isotropic properties, U_g is beam point under the influence of crystal gyrotropy. Moreover, the difference in the propagation constants is equal to

$$\delta\beta_{\pm}^2 = \beta_+^2 - \beta_-^2 = k^2(2\gamma/\Delta\gamma) \Delta\varepsilon. \quad (7)$$

Figure 2 shows the spectral curves for the coefficients $B_{\pm}(U)$ and $\delta\beta_{\pm}^2(U) = \pm\sqrt{D}$. A small enough perturbation $\Delta\varepsilon$ results in the coefficients undergoing significant changes $B_{\pm}(U)$, namely

$$B_-(U_g) = 0, \quad \lim_{U \rightarrow U_g - 0} B_+(U_g) \rightarrow -\infty, \quad \lim_{U \rightarrow U_g + 0} B_+(U_g) \rightarrow \infty. \quad (8)$$

Note that, in the system of a uniaxial crystal and a non-diffracting beam ($\Delta\varepsilon = 0$), the coefficients B_+ and B_- experience a jump (Figure 2), i.e., at a special point $U = U_{is}$, the beam does not experience the optical activity of the crystal and responds only to linear birefringence. As a result, eigenmode beams with a spiral polarization distribution are transformed into mode beams with an azimuthal (TE) and radial (TM) polarization distribution [23].

Relatively small changes in the difference in the diagonal elements of the optical activity matrix lead to significant changes in the amplitude parameters $B_{\pm}$ of the beam, as shown in Figure 3. The coefficient B_+ undergoes a discontinuity of the second kind, changing sign when passing through a special point U_g , but the position of this specific point also changes with decreasing $|\Delta g|$, shifting to the region of the critical parameter (to the region of evanescent waves). At a certain (critical) value $U_g = U_{cr}$, the critical point disappears, and the curve B_+ has no discontinuity, having a minimum in the region of negative values.

As Δg grows, the curve B_+ is smoothed out and gradually tends towards its critical value $B_+ = B_- = 0$ at the cutoff frequency.

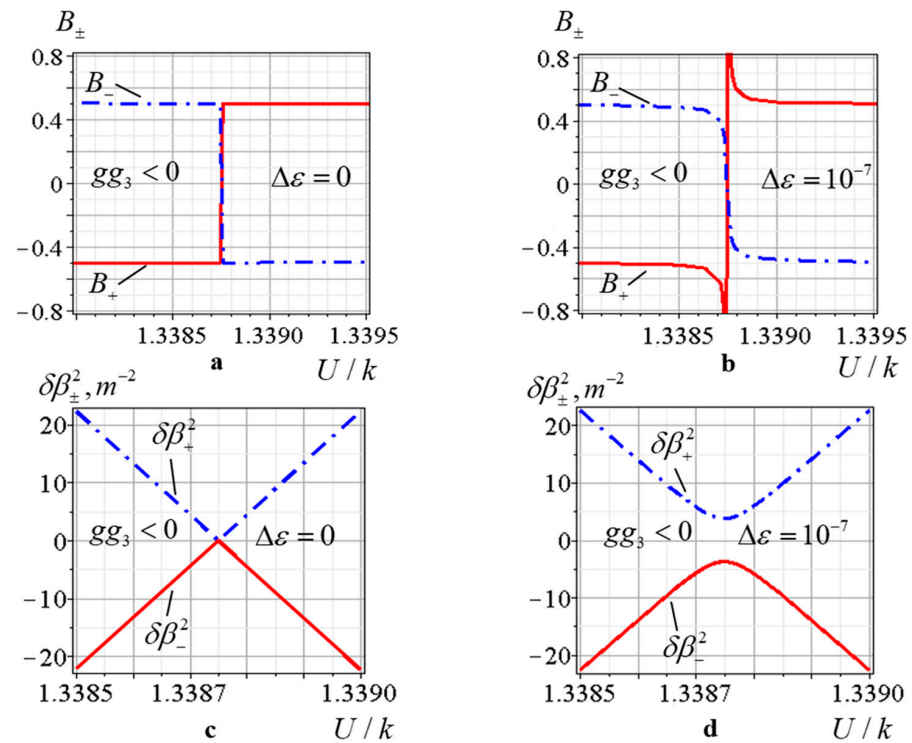


Figure 2. $B_{\pm}$ (a,b) and $\delta\beta_{\pm}^2$ (c,d) as a function of U/k for non-diffracting beams in an optically active medium perturbed by weak anisotropy $\Delta\epsilon$ ($\Delta\epsilon = 0$ (a,b) and 10^{-7} (c,d)), $g = 3 \cdot 10^{-10}$ m, $g_3 = -8 \cdot 10^{-10}$ m. See text for details.

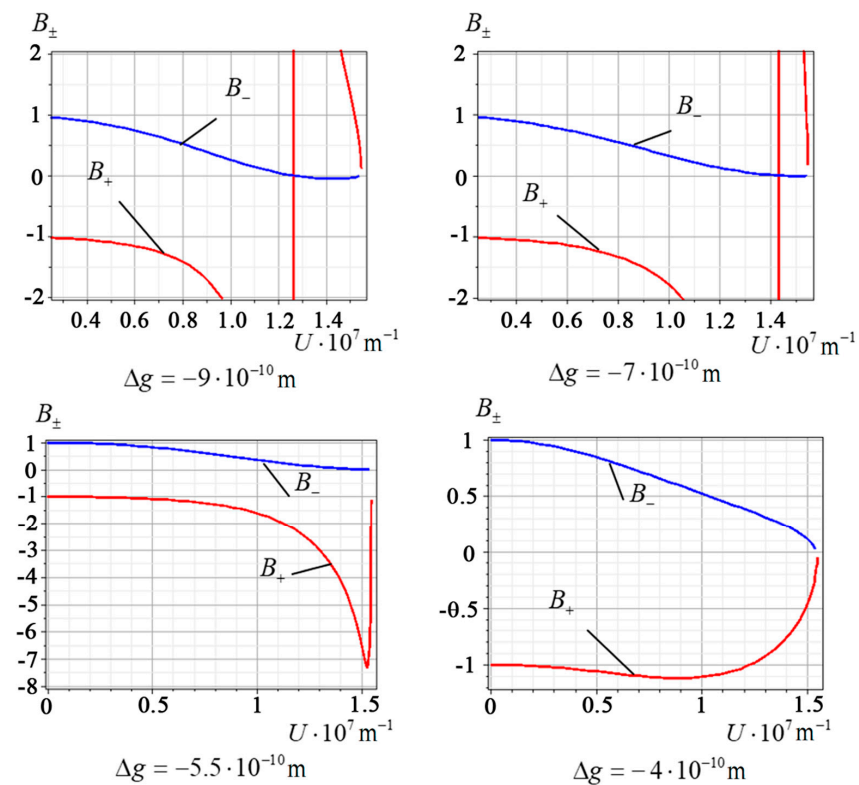


Figure 3. Parameters $B_{\pm}(U)$ for crystals with $n_0 = 1.544$, $n_3 = 1.551$, $g = 3 \cdot 10^{-10}$ m for different Δg values, as indicated. See text for details.

It is of interest to trace the change in the beam parameters depending on the change in the linear birefringence $\Delta n = n_3 - n_o$, where $n_3 = \sqrt{\varepsilon_3}$ and n_o is refractive index for an ordinary beam. As one can find from Figure 4, for $\Delta n = 0$ and the amplitude parameters $B_{\pm}$, the relations (8) are satisfied. A small decrease $\Delta n < 0$ leads to an increase in the parameter B_+ to a certain sufficiently large value, while $B_- = 0$. With an increase in the difference in the refractive indices, the amplitude parameters change the behavior. The propagation constants in this region behave as follows. For negative values $\Delta n < 0$, the propagation constant β_+ is almost independent of the refractive index, and β_- increases linearly as it approaches Δn zero. At the point $\Delta n = 0$, the propagation constants coincide, and, for positive values Δn , β_- practically does not depend on the change in the difference in the refractive indices, but β_+ starts increasing linearly, while the propagation constants continue as straight lines, mutually changing.

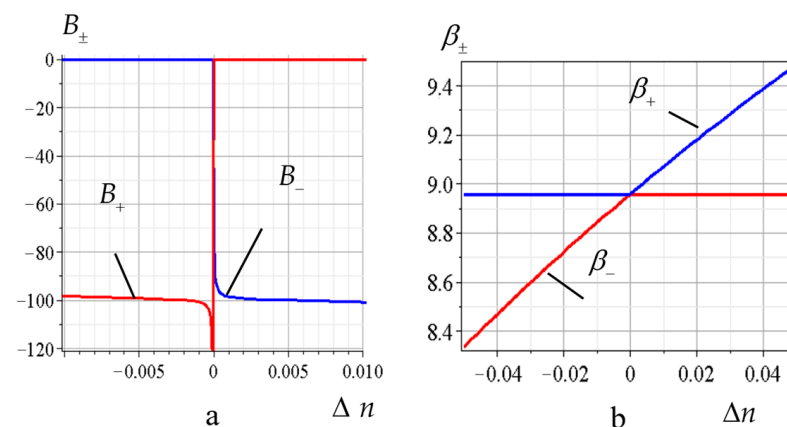


Figure 4. Parameters $B_{\pm}(\Delta n)$ (a) and $\beta_{\pm}(\Delta n)$ (b) in the crystal with $n_o = 1.55$, $g = 3 \cdot 10^{-10}$ m, $\Delta g = -9 \cdot 10^{-10}$ m and $U = 1.26 \cdot 10^7$ m $^{-1}$. See text for details.

Let us summarize the obtained results. In crystals for which the conditions (8) are satisfied, the beam field structure depends significantly on the value of the transverse wave number U . For a value $U < U_g$, the crystal responds to the beam in a normal way. That is, if $g > 0$, the crystal has positive optical activity. If $U > U_g$, the crystal's response to the beam becomes anomalous: a crystal with positive optical activity acquires the properties of a crystal with negative optical activity. This dual behavior of the crystal is illustrated by Figure 5, which shows a diagram of the crystal response in coordinates Δg and U . As an example of what is shown in Figure 5, one can use a SiO₂ crystal, which has positive or negative optical activity depending on the external parameters. In Figure 5a, the curve $\Delta g(U)$ divides the $(\Delta g-U)$ -plane into two regions: the upper region, the crystal response corresponds to positive optical activity, and, the lower part, it corresponds to negative optical activity. The points on the boundary of the regions correspond to the condition $B_+(\Delta g, U) = 0$. In this case, B_- experiences a discontinuity of the second kind and changes signs upon crossing the boundary (for different values g in Figure 5b), that is, the eigenmodes are TE and TM mode beams. On this boundary curve, the medium loses its optical activity properties and the medium response is equivalent to the response of a medium with genuine uniaxial birefringence.

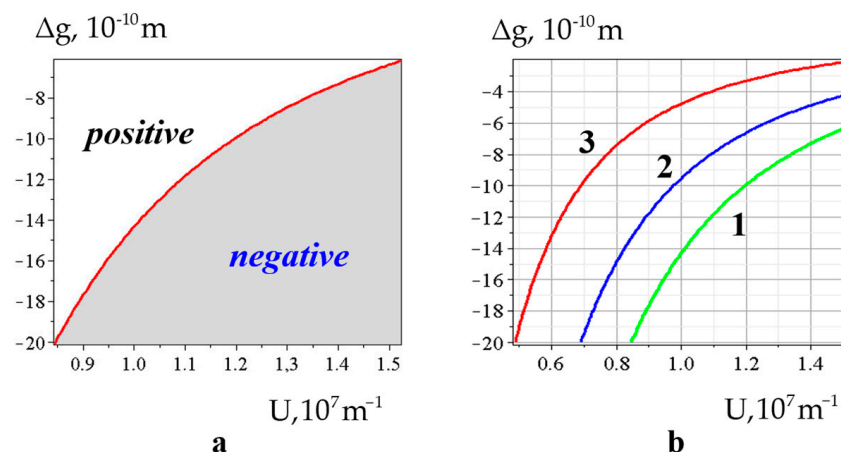


Figure 5. (a) The response diagram of the crystal-beam system to optical activity (positive and negative, as indicated) for a crystal SiO_2 with $n_o = 1.551$, $n_3 = 1.544$, $g = 3 \cdot 10^{-10}$ m. (b) The dependence of the difference between the diagonal elements Δg on the transverse wave number U of the optical activity tensor for $n_o = 1.551$, $n_3 = 1.544$, and $g = 3 \cdot 10^{-10}$ m (curve 1), $g = 2 \cdot 10^{-10}$ m (curve 2), $g = 1 \cdot 10^{-10}$ m (curve 3).

The problem of Bessel beams propagating in a genuine birefringent medium has been considered quite often in the literature—see, e.g., Refs. [17,18]—and we do not dwell on this issue here. In the vicinity of a singular point $U = U_g$ in a crystal, processes with genuine linear birefringence dominate over processes with circular birefringence. Meanwhile, for $U \ll U_g$, processes with circular birefringence suppress processes with linear birefringence, and a Bessel beam in such a crystal propagates as in a medium with genuine optical activity. For anisotropic media for which the conditions (8) are not satisfied, there is no singular point $U = U_g$, and the crystal response does not change the sign of the optical activity at any value of the transverse wave number $U < U_{\text{cr}}$. Nevertheless, in the paraxial region, processes associated with circular birefringence dominate in the crystal for such beams. Meanwhile, in the non-paraxial region, linear birefringence processes dominate in the case of real crystals ($\Delta\epsilon \gg \gamma^2$).

3. Propagation of a Circularly Polarized Bessel Beam in an Anisotropic Medium with Optical Activity

Since linear birefringence changes only the values of the propagation constants $\beta_{\pm}$ and amplitude parameters $B_{\pm}$, the recording of the eigenmodes do not change. Close to the point $U = U_g$, for azimuth index $m = 0$, the eigenmodes have the form shown in Figure 6. In Figure 6a, the field has a form close to the TM mode and in Figure 6b, the field has a form of the TE mode [23].

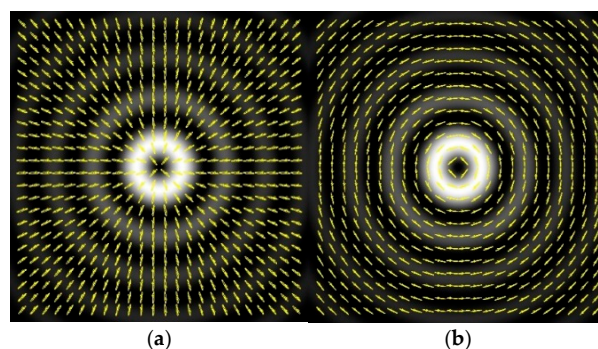


Figure 6. View of own (a) TM and (b) TE modes for crystal with $n_o = 1.551$, $n_3 = 1.544$, $g = 3 \cdot 10^{-10}$ m, $\Delta g = -9 \cdot 10^{-10}$ m, $U = 1.2 \cdot 10^7 \text{ m}^{-1}$ ($U_g = 1.257 \cdot 10^7 \text{ m}^{-1}$). See text for details.

Let us now consider how a Bessel beam carrying a single optical vortex, circularly polarized at the input to the crystal, will behave. For definiteness, we choose a right-circular direction on the input face of the anisotropic medium, with a single negative topological charge on the beam axis.

For this case, the evolution of the polarization distribution in the beam cross-section for the case of Figure 6 is shown in Figure 7. For example, it is seen from Figure 7 that the beat length in the case of a crystal with linear birefringence is about 1000 times greater than in the case of only optical activity. For a transverse wave number U (near U_g) of one eighth of the beat length, a spiral beam with virtually linear polarization in the cross-section is obtained. At a quarter of the beat length, a field with azimuthal integral lines arises, while the beam ellipticity has the opposite sign to the ellipticity in the initial section $z = 0$ and is close to -1 .

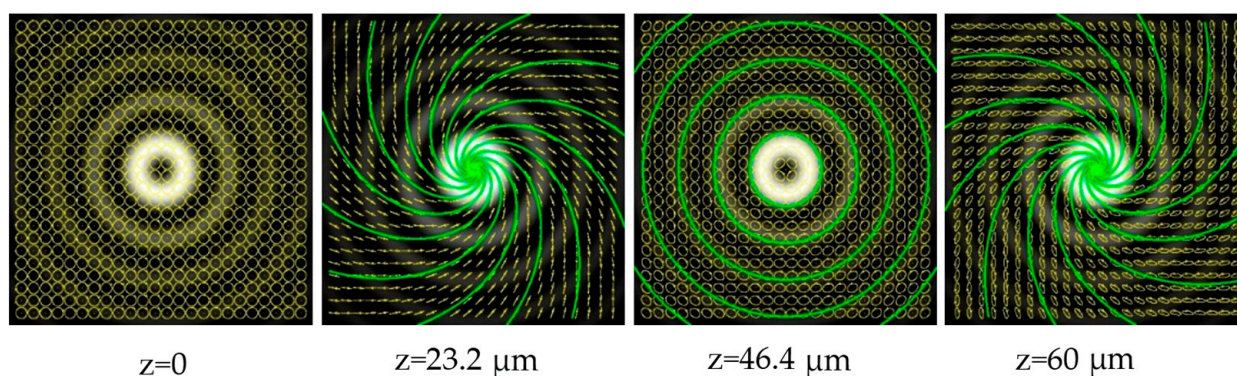


Figure 7. Polarization distribution against the background intensity in a Bessel beam carrying a single optical vortex for different initial section z , as indicated. See text for details.

4. Propagation of a Linearly Polarized Bessel Beam in an Anisotropic Medium with Optical Activity

Since, for an initially circularly polarized Bessel beam, the polarization transformation occurs similarly to the case of a medium with genuine optical activity, and the field remains axially symmetric during propagation, we study the propagation of a linearly polarized Bessel beam in a medium with axial anisotropy and optical activity. For definiteness, let us consider the beam be polarized along the x -axis in the plane $z = 0$.

Let us consider two cases. In the first case, there is a linearly polarized Bessel beam without singularities on the input face of the crystal, and, in the second case, there is a transporting unit optical vortex with topological charge $l = 1$. During propagation, the field of such a beam can be represented as an expansion in two initially circularly-polarized fields consisting of a pair of eigenmodes. In the case of a smooth field for a right-handed circular field and a field with left-handed circulation [15]:

$$E_{+,0}^{(2)} = -2i(1 + B_-)e^{-i2\varphi}e^{-i\tilde{\beta} \cdot z} \sin(\delta\beta \cdot z)J_2(Ur) , \quad (9)$$

$$E_{-,0}^{(2)} = \left[(1 - B_+) e^{-i\beta_+ \cdot z} - \frac{(1 + B_+)(1 - B_-)}{(1 + B_-)} e^{-i\beta_- \cdot z} \right] J_0(Ur), \quad (10)$$

where the subscript “(0)” indicates that the input field does not contain a topological charge, the superscript “(2)” indicates that when the beam passes through an anisotropic medium, an optical vortex with a topological charge $l = 2$ is formed on the beam axis, J_2 and J_0 are the Bessel functions of the second kind and of zero order, respectively, $\tilde{\beta} = (\beta_+ + \beta_-)/2$, and $\beta \cdot z$ represents the change in the value of the propagation constant along the z -axis. Note that during its propagation, a beam with a right-handed circular smooth field

(9) $E_{+,0}^{(1)}(z = 0) \sim J_0(Ur)$ at the input induces an optical vortex with a positive (double topological charge in the left-handed circular component $E_{-,0}^{(1)}(z) \sim e^{i2\varphi} J_2(Ur)$, where the upper index “(1)” indicates that when a beam passes through an anisotropic medium, an optical vortex with a topological charge is formed on the beam axis. Meanwhile, a field (10) $E_{-,0}^{(2)}(z = 0) \sim J_0(Ur)$ with left-handed circular polarization at the input generates a double negative vortex $E_{+,0}^{(2)}(z) \sim e^{-i2\varphi} J_2(Ur)$ in the right-handed circular component during propagation. For a beam with a single optical vortex, these are fields with right-handed circulation polarization,

$$E_{+,1}^{(1)} = \left[(1 - B_+) e^{-i\beta_+ \cdot z} - \frac{(1 + B_+)(1 - B_-)}{(1 + B_-)} e^{-i\beta_- \cdot z} \right] e^{i\varphi} J_1(Ur), \quad (11)$$

$$E_{-,1}^{(1)} = -2i(1 + B_-) e^{i3\varphi} e^{-i\tilde{\beta} \cdot z} \sin(\delta\beta \cdot z) J_3(Ur), \quad (12)$$

as well as fields with left-handed polarization circulation:

$$E_{+,1}^{(2)} = 2i(1 - B_+) e^{-i\varphi} e^{-i\tilde{\beta} \cdot z} \sin(\delta\beta \cdot z) J_1(Ur), \quad (13)$$

$$E_{-,1}^{(2)} = \left[(1 + B_+) e^{-i\beta_+ \cdot z} - \frac{(1 - B_+)(1 + B_-)}{(1 - B_-)} e^{-i\beta_- \cdot z} \right] e^{i\varphi} J_1(Ur). \quad (14)$$

In the case of a beam carrying an optical vortex, right-handed circularly-polarized in plane $z = 0$ (11), the field containing a single optical vortex with $l = +1$: $E_{+,1}^{(1)}(z = 0) \sim e^{i\varphi} J_1(Ur)$ induces in the left-handed circular component an optical vortex with triple topological charge $E_{-,1}^{(1)}(z) \sim e^{i3\varphi} J_3(Ur)$ during propagation. On the other hand, a single optical vortex in a left-circularly polarized field $E_{-,1}^{(2)}(z = 0) \sim e^{i\varphi} J_1(Ur)$ in plane $z = 0$ creates a negatively-charged vortex $E_{+,1}^{(2)}(z) \sim e^{-i\varphi} J_1(Ur)$ during propagation in the right-handed circular component. The linearly polarized components $\mathbf{E}^{(L)}$ of the beam field in a linearly-polarized basis are represented as $E_x = (E_+^{(L)} + E_-^{(L)})/2$, $E_y = i(E_+^{(L)} - E_-^{(L)})/2$. Adding the fields with fields (9) and (10) for a smooth Bessel beam at the input (as well as fields (11)–(14) for a beam with a single optical vortex at $z = 0$) with different parameters and equating $E_y(z = 0) = 0$, one finds the form of the fields as follows.

In the case of a smooth Bessel beam ($E_x(z = 0) = J_0(Ur)$, $E_y(z = 0) = 0$), the field reads

$$E_{+,0}^{(L)} = \frac{e^{-i\tilde{\beta} \cdot z}}{2(1 - B_-)} \{ 2i(1 + B_-)(1 + B_+) \sin(\delta\beta \cdot z) e^{-i2\varphi} J_2(Ur) + [(1 + B_-)(1 - B_+) e^{i\delta\beta \cdot z} - (1 - B_-)(1 + B_+) e^{-i\delta\beta \cdot z}] J_0(Ur) \}, \quad (15)$$

$$E_{-,0}^{(L)} = \frac{e^{-i\tilde{\beta} \cdot z}}{2(1 - B_-)} \{ -2i(1 - B_-)(1 - B_+) \sin(\delta\beta \cdot z) e^{i2\varphi} J_2(Ur) + [(1 - B_-)(1 + B_+) e^{i\delta\beta \cdot z} - (1 + B_-)(1 - B_+) e^{-i\delta\beta \cdot z}] J_0(Ur) \}, \quad (16)$$

and, for the case of a beam with an optical vortex,

$$E_{+,1}^{(L)} = \frac{J_1(Ur) e^{-i\tilde{\beta} \cdot z}}{2(1 - B_-)} \{ 2i(1 + B_-)(1 + B_+) \sin(\delta\beta \cdot z) e^{-i\varphi} + [(1 + B_-)(1 - B_+) e^{i\delta\beta \cdot z} - (1 - B_-)(1 + B_+) e^{-i\delta\beta \cdot z}] e^{i\varphi} \}, \quad (17)$$

$$E_{-,1}^{(L)} = \frac{e^{-i\tilde{\beta} \cdot z}}{2(1 - B_-)} \{ 2i(1 - B_-)(1 - B_+) \sin(\delta\beta \cdot z) e^{i3\varphi} J_3(Ur) + [(1 + B_-)(1 - B_+) e^{-i\delta\beta \cdot z} - (1 - B_-)(1 + B_+) e^{i\delta\beta \cdot z}] e^{i\varphi} J_1(Ur) \}. \quad (18)$$

Since the eigenmodes of a linearly-polarized beam propagate with different phase velocities, the field amplitude oscillates along the crystal with a period $\Lambda = 2\pi/\delta\beta$.

Figure 8 illustrates the changes in the field component E_x of a field component during propagation along a crystal with $g_3g < 0$. Figure 8 (upper) shows the evolution of a Bessel beam with a single optical vortex with a topological charge $l = 1$ in the plane $z = 0$, and Figure 8 (lower) shows the evolution of a Bessel beam without a vortex. It should be noted that, since fields (15)–(18) contain a combination of many optical vortices, a small shift from the $z = 0$ plane into the crystal results in a uniformly linearly-polarized field with or without a single optical vortex being transformed into a complex array of optical vortices in each field component. In the plane $z = 0$, one observes a single optical vortex (Figure 8 (upper)) or a smooth field (Figure 8 (lower)) surrounded by concentric circles. The displacement into the crystal results in the emergence of multiple dipoles of 0° , $\pm 45^\circ$ and 90° (along the x -axis), which form the blurred outline of a dark cross.

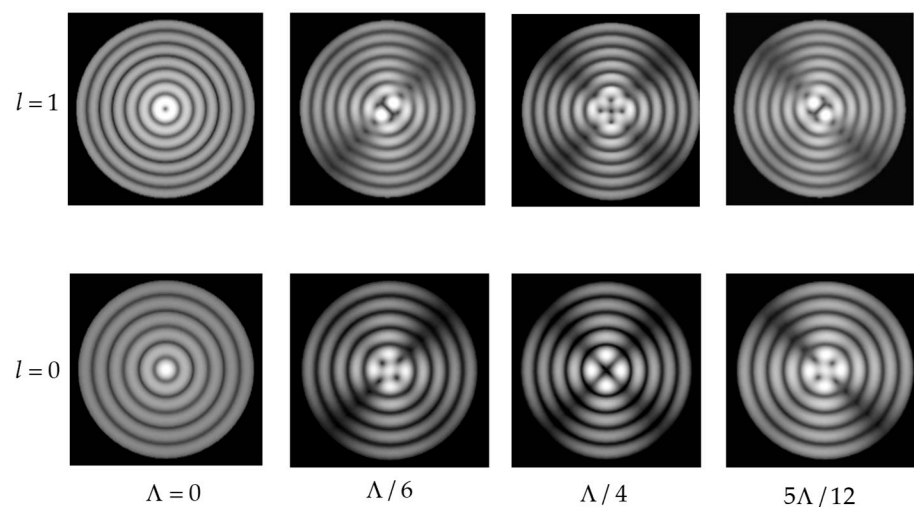


Figure 8. Intensity distribution in the linearly polarized E_x component of a Bessel beam with an optical vortex ($l = 1$) and without it ($l = 0$), for field amplitude period $\Lambda = 0$ and its quarters, as indicated, with the parameters $g = 3 \cdot 10^{-10}$ m, $g_3 = -9 \cdot 10^{-10}$ m, $U = 1.3 \cdot 10^7$ m $^{-1}$. See text for more details.

At lengths that are multiples of a quarter of the beat length, $L_p = (2p + 1) \Lambda/4$, ($p = 0, 1, 2, \dots$), the outlines of the cross become the most contrasting. The highest contrast can be achieved at the point $U = U_g$ with $z = L_p$, where linear birefringence predominates. The main difference in a Bessel beam without a vortex is that, at the length $z = L_p$ in the beam cross-section, there arises not a set of optical vortices but two crossed edge dislocations. Strong displacement from the point $U \neq U_g$ leads to the strong blurring of the contours of the cross.

5. Conclusions

Using the example of a Bessel beam, it is shown for the first time that such a beam with initial circular polarization, propagating through an optically active medium devoid of linear birefringence, generates an optical vortex in the orthogonally polarized component, the charge of which differs by two units from the charge of the initial beam, and its sign is determined by the sign of the circulation of the initial polarization. For example, an initially topologically neutral Bessel beam propagating in a saturated sugar solution generates a beam with a double topological charge with energy efficiency $\xi = 0.3$ over a length $z = 3 \cdot 10^{-2}$ m at a transverse wave number $U = 1.2 \cdot 10^7$ m $^{-1}$. This ability of Bessel beams to form optical vortices in a medium with genuine optical activity has much in common with the processes of the formation of optical vortices in Bessel beams propagating through media with genuine linear birefringence [17,18].

It is also shown for the first time that a medium with genuine optical activity allows the mutual conversion of beams with radial and azimuthal polarization distributions.

It is shown that, in anisotropic crystals with optical activity represented as a diagonal tensor whose elements obey the condition $g_3g < 0$, $|g_3| > |g|$, the beam field structure depends significantly on the transverse wave number U . For the value $U < U_g$, the crystal responds to the beam in a normal way. That is, if $g > 0$, then the crystal has positive optical activity. However, when $U > U_g$, the crystal's response to the beam becomes anomalous: a crystal with positive optical activity along the propagation axis acquires the properties of a crystal with negative optical activity. For a beam with $U = U_{is}$ (the so-called isotropic point in the structure of dispersion curves), the crystal medium turns out to be isotropic, i.e., the beam propagation velocity does not depend on the polarization of the beam.

It is also shown for the first time that a considerably weak perturbation of a medium with genuine optical activity by linear birefringence leads to the destruction of the isotropic point. In the vicinity of this point, the optical activity is suppressed by linear birefringence, so that a medium with a combined type of anisotropy responds as a medium with genuine linear birefringence.

It is shown, for the first time as well, that, in a Bessel beam with initial linear polarization, a cross-shaped conoscopic pattern is observed in the orthogonal component, formed by an array of optical vortices. In the component with the initial polarization, as the beam propagates, the conoscopic pattern periodically changes, restoring at lengths equal to a quarter of the beat length.

The structure of the fields of the eigenmodes of Bessel beams in a birefringent uniaxial crystal with optical activity is similar to the structure of the modes of a medium with genuine optical activity. However, the beam parameters differ significantly.

The methods of studying and analyzing singular beams that have passed through a gyroanisotropic medium presented in this paper can be extended to other areas of physics and instrumentation. Examples of the use of this research are the analysis of weakly turbulent atmospheric media [24] and the polarization and phase profilometry [25] of materials and surface roughness, as well as in the areas of singular optics to study the phase and polarization properties of topological charges [26–28].

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