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Review

The Uncertainty Principle and the Minimal Space–Time Length Element

David Escors * and Grazyna Kochan

Fundacion Miguel Servet, Public University of Navarre (UPNA), 31008 Pamplona, Spain
* Correspondence: descorsm@navarra.es

Abstract: Quantum gravity theories rely on a minimal measurable length for their formulations, which clashes with the classical formulation of the uncertainty principle and with Lorentz invariance from general relativity. These incompatibilities led to the development of the generalized uncertainty principle (GUP) from string theories and its various modifications. GUP and covariant formulations of the uncertainty principle are discussed, together with implications for space–time quantization.

Keywords: general relativity; uncertainty principle; geodesics; black hole singularity; quantum gravity; Planck star; Lorentz invariance violations

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1. Introduction, General Relativity, Quantum Mechanics and the Problem of a Minimal Length

General relativity (GR) and quantum mechanics (QM) constitute the two major pillars of modern physics. So far, these two theories in their various formulations have survived all experimental testing, which supports their role as fundamental theories of nature. While classical GR is a geometric theory for gravitation, classical QM describes phenomena other than gravitation at "Planck scales" by probability theory of states in Hilbert space. Owing to their fundamentality, one would expect that these two theories could be combined in a single, unified theory for quantum gravity. However, these two theories have major incompatibilities starting from their different frameworks, formulations and principles, which make their merging a daunting task. Nevertheless, attempts to unify these two fundamental theories have given rise to well-developed quantum gravity theories such as string theory and loop quantum gravity (LQG) [1,2].

GR is a Lorentz-covariant geometric theory for gravitation put forward by Albert Einstein in 1916 [3] (see [4] for English translation), in which a radical conceptual change was introduced to classical gravitation. In GR, the concept of classical gravitational force disappears and is substituted by a dynamical space–time geometry given by a pseudo-Riemannian manifold consisting of three spatial dimensions and a time dimension. The space–time manifold in GR presents a Lorentzian $(- + + +)$ signature and it is shaped by energy–momentum densities from an energy–momentum tensor in Einstein's field equations [3]. GR is also a background independent theory in which the space–time metric is the dynamical variable [5]. Space–time geometries are determined by mass, energy and momentum densities, and particles follow geodesic trajectories in the space–time manifolds, for which position and momentum are defined with absolute certainty. This is just not allowed in QM.

QM was developed through a process of tackling several inconsistencies, mainly in particle physics and thermodynamics, which could not be solved by classical principles of physics. Its foundation as a consistent theory rested on a collection of postulates not directly derived from first principles [6,7], and on three fundamental pillars: energy quantization; the concept and interpretation of the wave function; and the uncertainty principle. For the uncertainty principle, classical QM states that the position and momentum of a particle in a trajectory cannot be defined with absolute certainty, which is in direct contradiction with GR. This principle is further completed by a similar statement on energy–time uncertainty.

Classical QM evolved into quantum field theory during the 1930s, and with it the problem of ultraviolet divergences. These divergences were later taken care of by the development of renormalization mathematical techniques [8,9]. However, before that, in this context, the idea of a minimal measurable discrete length was put forward, with Heisenberg being one of the main advocates [10]. His main argument was the necessity for a discreet length to overcome the divergences in quantum field theories, and also for the description of the range of elementary known particles. The proposals for a minimal measurable length were met with skepticism because this concept was in direct contradiction with Lorenz invariance and general relativity. A minimal discrete length would imply the need of privileged reference frames. Snyder was the first to show that the two ideas—a minimal length and Lorentz invariance—could be combined by modifying the canonical commutators of position/momentum operators [11]. It was also realized relatively early that quantum uncertainties would affect the background space–time, leading to the necessity of its quantization in a quantum theory of gravity [12,13]. The proposal by Mead that Planck length constituted such a fundamental minimal length [14] initially was received with skepticism.

The classical uncertainty principle, one of the pillars of QM, is not restricted to a minimal length or a minimal momentum if these are interpreted as uncertainties. Hence, the uncertainty in position or momentum can be arbitrarily small, leading to troublesome divergencies. Then, string theory came in the 1980s by deriving a generalized uncertainty principle, which stated the impossibility of measuring an arbitrarily small length [15–17]. In the 1990s, a modification of the position/momentum commutator relations of space–time to a Hopf algebra was introduced [18] and Kempf modified the commutator relations to accommodate a minimal length in quantum field theories [19–21]. The generalized uncertainty principle could be derived from these modifications [19]. This generalized uncertainty principle with Planck length as a minimal measureable length was proposed as a solution to ultraviolet divergencies in quantum gravity at Planck energies. However, another drawback appeared when GR was found to be apparently nonrenormalizable when formulated as a quantum field theory. The introduction of a Lorentz-covariant minimal length could be a way forward to tackle this issue [9].

Here, we review the uncertainty principle and its main modifications for adaptation to a minimal length element and to Lorentz covariance.

2. The Uncertainty Principle

The uncertainty principle originally proposed by Heisenberg is a general property of wave systems, and as such it is considered a fundamental law of nature. Heisenberg put forward this principle for the canonical conjugated variables of momentum and position in 1927 [22], which was later generalized as an inequality by Kennard for any arbitrary wave function [23]. In 1945, Mandelshtam and Tamm derived a similar nonrelativistic uncertainty principle between energy and time in the form of the Mandelshtam–Tamm inequality [24]. In this latter inequality, time still remains as an independent privileged variable. The current classical uncertainty principle thus consists of two inequalities:

$$|\Delta p||\Delta x| \geq \frac{\hbar}{2}, \quad |\Delta E||\Delta t| \geq \frac{\hbar}{2}, \tag{1}$$

where Δp denotes the change in magnitude of momentum parametrized by coordinate time, Δx is the change in magnitude of the position vector, ΔE and Δt denote the change in magnitude of energy and time, respectively, and $\hbar$ is the reduced Plank constant.

The uncertainty relations (1) are considered a fundamental principle in nature behind many quantum phenomena [25–27]. Although Heisenberg utilized the "observer effect" as an intuitive interpretation, this principle is fundamentally intrinsic to any wave system [27–29]. The momentum/position classical uncertainty principle is conveniently represented by the Heisenberg commutator algebra, which is a reflection of the noncommutability of momentum and position operators:

$$\left[\hat{p}^i, \hat{x}^j\right] = -i\hbar\delta^{ij}, \tag{2}$$

where the indices, denoted by Latin letters, take on the values 1, 2 and 3; $\hat{p}^i, \hat{x}^j$ represent momentum and position operators, and δ^{ij}, is the Kronecker delta.

The momentum/position commutator and the classical inequalities of the uncertainty principle were reinterpreted as standard deviations in momentum and position (σ_p, σ_x) by Kennard for any wave function [23,30]:

$$\sigma_p \sigma_x \geq \frac{\hbar}{2}. \tag{3}$$

One key consequence of the uncertainty relations in QM is that momentum–position phase space is quantized. However, this does not directly imply the existence of a minimal length since in inequalities (1) and (3), the actual uncertainty in position is unrestricted (Figure 1A). Uncertainty in position can be arbitrarily small, also leading to divergence in momentum, which is highly problematic. This was soon shown to be in conflict with quantum gravity theories such as string theories [31] and LQG [1,32]. Their formulations require a minimal length proportional to Planck length, ℓ_p [33–35]:

$$\ell_p = \sqrt{\frac{\hbar G}{c^3}}, \tag{4}$$

where G and c denote the universal gravitational constant and the speed of light, respectively.

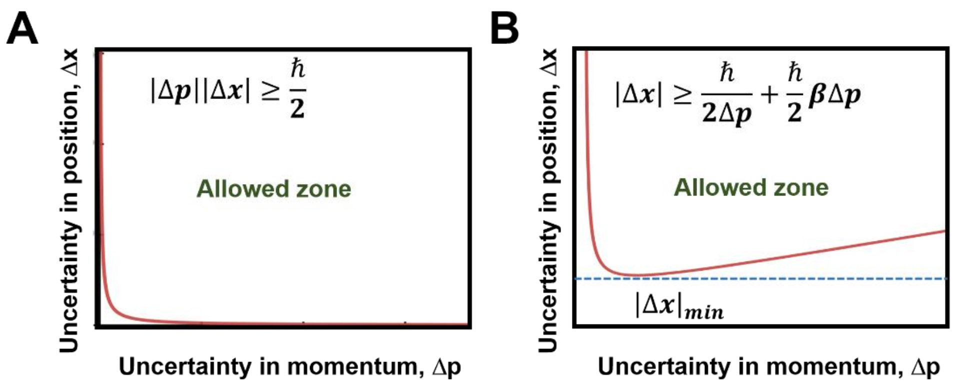

Figure 1. Classical uncertainty principle and generalized uncertainty principle (GUP). (**A**): Plot of the classical uncertainty momentum–position inequality (shown on top), indicating the allowed region. Uncertainties in position and momentum diverge to infinity. (**B**): Graph plot of a GUP representation of the uncertainty principle (shown on top). A minimum in the function is reached representing a minimal measurable length, $|\Delta x|_{\min}$. The allowed region by the inequality is shown. Plots are represented in relative units. See text for details.

For string theories, ℓ_p is a fundamental length element for string particles [2,15,36]. LQG is a quantum theory for gravitation that starts from classical GR in its Arnovitt–Deser–Misner (ADM) formalism, in which space–time is foliated and then space lattice quantization is introduced [37]. As a consequence, this lattice quantization leads to a minimum length, and for example, LQG area and volume operators are quantized and proportional to ${\ell_p}^2$ and ${\ell_p}^3$, respectively. However, this concept of a fixed, measurable minimal length not only clashes with the original formulation of Heisenberg's uncertainty principle, but also with Lorentz covariance. Nevertheless, the uncertainty principle provides a means to introduce a minimal length in relativity. As the gravitational field in classical GR depends on energy and momentum densities, the uncertainty principle would be expected to alter the background space–time geometry and introduce constraints to the classical space–time metric. Indeed, these constraints could be identified with a minimal length in quantum gravity. The starting point constitutes the extension of the position/momentum commutator relation from inequality (2) to the background Minkowski space–time. These modified commutator relations introduce a Lorentzian signature in the commutator, and are valid as a local projection of momentum and position operators on asymptotically noncurved tangent space [38]:

$$[\hat{p}^{\mu}, \hat{x}^{\nu}] = -i\hbar\eta^{\mu\nu}, \tag{5}$$

where the indices denoted by Greek letters take on the values 0 (time), 1, 2 and 3 (space) following standard tensor notation; $\eta^{\mu\nu}$ represents Minkowski space–time metric.

Hence, one of the first issues was to reconcile the classical uncertainty principle with the necessity for a measurable minimal length in quantum gravity theories. This gave rise to the generalized uncertainty principle and its variants.

3. Generalized Uncertainty Principle (GUP) and Its Modifications

The uncertainty principle inequalities as originally formulated (inequality (1)) imply a quantized momentum–position phase space, and subsequently, a quantized space, as discussed above. However, the momentum–position uncertainty relation as shown in Inequality (1) is not constrained to a minimal length (if considered as a nonzero uncertainty in position) and thus subject to ultraviolet divergences (Figure 1A). In this classical formulation, the space length represented as the uncertainty in position can asymptotically approach zero, making momentum diverge to infinity. This uncertainty relation is therefore unbound both in position and momentum uncertainties. This is in direct contrast with the need for a minimal length element, which is a common feature of gravity theories including string theory [1,15,31,36,39], LQG [2,32] and doubly special relativity in the Amelino–Camelia formulation [40].

Collisions of strings at Planckian energies also required a minimal length, leading to a modification of the classical uncertainty inequalities into what is known as GUP [15–17,41–44]. GUP formulations included boundaries to both momentum and position [15,45]. However, the first forms of GUP led to corrections in inequality (1) that bounded only uncertainties in position by adding quadratic forms of momentum [20,45]:

$$|\Delta p||\Delta x| \geq \frac{\hbar}{2} + \frac{\hbar}{2}\beta\Delta p^2 + \frac{\hbar}{2}\gamma, \tag{6}$$

where β and γ represent the functions dependent on the expectation value of momentum and position [19]. This reformulation of the uncertainty principle presents a minimum of uncertainty in position, below which the uncertainty relation is not allowed (Figure 1B). By modeling string collisions at Planck energies, an explicit quadratic-momentum GUP formulation arises with expressions dependent on a fundamental quadratic length on Planck scale, $\delta{\ell_p}^2$ [14,41–43,46]:

$$|\Delta x| \geq \frac{\hbar}{2\Delta p} + \frac{\alpha G}{c^3}\Delta p, \quad |\Delta x| \geq \frac{\hbar}{2\Delta p} + \delta{\ell_p}^2\Delta p, \tag{7}$$

where α and δ are constants.

An uncertainty relation in the framework of quantum geometry theory can be derived for any accelerating particle in the absence of a gravitational field. The uncertainty relation perturbs the background Minkowski space–time through acceleration, and the particle experiences gravitation via a perturbation over the background Minkowski metric [38]. This perturbation can be reflected by local quantum deviations from the background flat space at high-energy collisions, for example:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \left(1 + c^4\frac{\ddot{x}^{\alpha}\ddot{x}_{\alpha}}{A^2}\right), \tag{8}$$

where $g_{\mu\nu}$ and $h_{\mu\nu}$ represent the covariant pseudo-Riemannian metric tensor and a metric perturbation, respectively, $\ddot{x}^{\alpha}$ and $\ddot{x}_{\alpha}$ represent, respectively, contravariant and covariant components of acceleration, and A denotes maximal acceleration. By incorporating the perturbed metric from Equation (8) into the canonical position–momentum commutator in Minkowski space, GUP in the momentum-quadratic form is recovered as a function of the particle mass, m, the maximal proper acceleration, A, and the quadratic form of a space–time length element, δs [38]:

$$|\Delta x| \geq \frac{\hbar}{2\Delta p} + \frac{\hbar c^2}{m^2 A^2 \delta s^2}\Delta p, \tag{9}$$

This reformulation of the uncertainty principle is equivalent to GUP, as shown by inequality (7), by equating δs to the particle's Compton length [38].

The inequality formulations for GUP can be expressed as commutator relationships between momentum and position operators by introducing functions of quadratic momentum, $f\left(\vec{p}\right)^2$, as follows:

$$\left[\hat{p}^i, \hat{x}^j\right] = -i\hbar\delta^{ij}\left(1 + f\left(\vec{p}\right)^2\right), \tag{10}$$

If one considers the commutator (10) as an example, where the quadratic momentum is multiplied by a function β, then the smallest uncertainty in position that could be related to a minimum length, $\Delta x_{\min}$, is given by [18,19]

$$\left[\hat{p}^i, \hat{x}^j\right] = -i\hbar\delta^{ij}\left(1 + \beta\left(\vec{p}\right)^2\right), \quad \Delta x_{\min} = \hbar\sqrt{\beta}. \tag{11}$$

This minimal length can then be related to quadratic length elements on the order of Planck length, as shown in inequality (7).

4. Relativistic Formulations of GUP

The second main issue to be solved was the apparent incompatibility between a minimal measurable length and Lorentz covariance. However, it had already been shown by Snyder that quantizing space–time does not necessarily imply the breaking of Lorentz covariance [11].

One way to obtain relativistic, Lorentz-covariant formulations implies modifications of the commutator relations in Minkowski space–time (Equation (5)). One first step is its generalization to curved space through a differential local perturbation over the Minkowski metric [38],

$$[\hat{p}^\mu, \hat{x}^\nu] = -i\hbar g^{\mu\nu}, \;\; g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}. \tag{12}$$

Such perturbation approaches have been used in semiclassical quantum gravity. For example, by defining a metric tensor operator decomposed into a pseudo-Riemannian metric tensor plus a fluctuating tensor operator of quantum origin, $\delta\hat{g}_{\mu\nu}$, that can be identified with a classical energy–momentum tensor, $T_{\mu\nu}$ [47]:

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + \delta\hat{g}_{\mu\nu}, \qquad \langle\delta\hat{g}_{\mu\nu}\rangle \equiv T_{\mu\nu}\,. \tag{13}$$

The necessity for a fixed, measurable minimal space–time length in quantum gravity theories clashes with Lorentz covariance, an inherent property of relativity [48,49] (see English translation [50] of Ref. [49]). Quantum gravity theories thus operate under a privileged frame of reference, which have restricted the application of GUP mainly to nonrelativistic problems. Whereas in some instances, the minimal length in LQG can be considered a free parameter subject to Lorentz covariance [51,52], the need for a covariant formulation for GUP has led to correcting its canonical commutator for Minkowski space [52]. For example, Quesne and Tkachuk generalized Kempf's deformed commutator algebra in D-dimensions [19,53] to make it Lorentz-covariant [54]. In this procedure, the quadratic forms of momentum and products of momentum and position were replaced by their contracted tensor formulations. The resulting commutator algebra is invariant under classical Lorentz transformations, and used to solve the relativistic Dirac oscillator [54,55]:

$$\begin{aligned}
&[\hat{p}^{\mu}, \hat{x}^{\nu}] = i\hbar[(1-\beta p^{\alpha}p_{\alpha})g^{\mu\nu} - \beta' p^{\mu}p^{\nu}], \\
&[\hat{x}^{\mu}, \hat{x}^{\nu}] = i\hbar(p^{\mu}p^{\nu} - p^{\nu}p^{\mu})\frac{2\beta-\beta'-(2\beta+\beta')\beta p^{\alpha}p_{\alpha}}{1-\beta p^{\alpha}p_{\alpha}}, \\
&[\hat{p}^{\mu}, \hat{p}^{\nu}] = 0.
\end{aligned} \tag{14}$$

where in the context of these equations, β, β' correspond to non-negative deforming parameters. In this modified relativistic GUP, the smallest uncertainty in position is given by

$$\left(\Delta x^{i}\right)_{\min} = \hbar\sqrt{(D\beta+\beta')\left[1-\beta\left\langle (P^{0})^{2}\right\rangle\right]}. \tag{15}$$

A similar strategy was undertaken by Todorinov et al. to comply with Lorentz covariance in Minkowski space–time [25,52]:

$$[\hat{p}^{\mu}, \hat{x}^{\nu}] = -i\hbar\Big(1+(\varepsilon-\alpha)\lambda^{2}p^{\rho}p_{\rho}\Big)\eta^{\mu\nu} - i\hbar(\beta+2\xi)\lambda^{2}p^{\mu}p^{\nu}, \tag{16}$$

where in the context of this equation, α, β, ε and ξ are dimensionless parameters to be adjusted to the specific problem, and λ a parameter with dimensions of inverse momentum. This formulation was applied to three relativistic systems: the Klein–Gordon equation for the hydrogen atom; the Schrödinger equation for a particle in a box and a linear harmonic oscillator; and the Dirac equation [52]. For these examples, GUP corrections were obtained only for the Schrödinger equation.

Recently, an approximation towards a GUP formulation in pseudo-Riemannian curved spaces has been proposed using normal coordinates defined in tangent space [56]:

$$[x^{a}, \hat{p}_{b}] = i\hbar(\alpha K^{a}{}_{b} - u^{a}u_{b}), \tag{17}$$

where x^{a} denotes normal coordinates, α is a constant, K^{a}_{b} represents components of the extrinsic curvature tensor associated with the equigeodesics, and u^{a} and u_{b} denote, respectively, the contravariant and covariant components of the proper velocity 4-vector.

5. Covariant Reformulation of the Classical Uncertainty Principle

The uncertainty principle arises as a fundamental property of wave functions, and in truth, there is no objective reason why it should be constrained to a minimal length and/or minimal momentum. GUP was developed from most quantum gravity theories, which do have a minimal noncovariant length in their mathematical framework. Hence, if one decides that a minimal length is a "sine qua non" condition for a quantum gravity theory, then there is some basis to justify the modification of the uncertainty principle. However, this modification may not be a requirement for quantum gravity theories, as it is the case of doubly special relativity with de Sitter invariance [57–60]. It could be possible that there is no a need to restrict the uncertainty principle to minimal lengths and momentums.

Hence, we [61] recently reformulated the uncertainty principle in a covariant form, but without introducing restrictions to minimal lengths and/or momenta such as those in GUP. This approach would make the uncertainty principle compatible with GR as well. To achieve this, we tried a generalization of the classical uncertainty principle inequalities strictly from covariant tensor formulations. We assumed that the following (or modified) statement as a starting point:

$$\left|\Delta P^{\mu}\Delta x_{\mu}\right| \geq f(\hbar), \tag{18}$$

where $f(\hbar)$ represents a function of the reduced Planck constant. Such a formulation would introduce a Lorentz-covariant constraint through a contraction of the change in relativistic momentum and position 4-vectors. However, it turned out that such formulation did not recover the two classical inequalities. Hence, we decided to re-express the classical inequalities in a covariant form, allowing for its application as a mathematical constraint over GR geodesics [61,62]. This formulation extended the uncertainty inequality to a differential length of relativistic proper space–time line element, $d\tau^2$, as a function of Planck length, ℓ_p, and a geodesic-related scalar, G_{geo} :

$$\left|G_{\text{geo}}\, d\tau^2\right| \geq (1+\gamma)\ell_p^2, \tag{19}$$

where the gamma factor, γ, and G_{geo} are defined in terms of the total energy, E, of the particle, its mass, m, and Christoffel connectors, $\Gamma^{\ \mu}_{\alpha\beta}$, in units of c set to 1:

$$\begin{gathered} \gamma = \tfrac{dt}{d\tau} \equiv \tfrac{E}{m}, \\ G_{\text{geo}} \equiv 2Gm\left|u_0 \Gamma^{\ 0}_{\alpha\beta}\ u^\alpha u^\beta\ \right| + 2Gm\left|u_j \Gamma^{\ j}_{\alpha\beta}\ u^\alpha u^\beta\ \right|. \end{gathered} \tag{20}$$

This covariant reformulation of the classical uncertainty principle sets a length limit for the quadratic proper space–time line element. Its application as a constraint to Minkowski space requires the introduction of a time-dependent differential perturbation, ε, to the g_{00} component of the metric [61,62]:

$$g_{00} = \eta_{00} + h_{00} = -1 - \varepsilon(t). \tag{21}$$

This correction to the metric establishes a limit to the space–time quadratic distance in terms of energy fluctuations, $\dot{E} = dE/dt$, arising from the uncertainty principle as follows:

$$\left|d\tau^2\right| \geq \frac{2c^5 \ell_p^2}{G\left|\dot{E}\right|}, \tag{22}$$

When applied to the metric of an expanding universe, as represented by the Friedmann–Robertrson–Walker (FRW) metric [63–65], the quadratic space–time line element was calculated in terms of two functions [62]. The first one derived from energy fluctuations from the uncertainty principle, E_{un}, and the second from the expansion rate, H_{ex}, of the universe:

$$\begin{gathered} E_{\text{un}} \equiv u^0 u_o\, \dot{\varepsilon}, \quad H_{\text{ex}} \equiv 6u^1 u_1 H, \\ \left|d\tau^2\right| \geq (1+\gamma)\frac{\ell_p^2}{Gp^0|H_{\text{ex}} - E_{\text{un}}|}, \end{gathered} \tag{23}$$

where $\dot{\varepsilon}$ represents energy fluctuations of quantum origin, and H is Hubble's function.

One can test whether a covariant formulation without restrictions to minimal lengths/momenta is sufficient to calculate a minimal covariant length at the point-like singularity from the Schwarzschild solution. Therefore, this covariant expression was applied to Schwarzschild's metric [66] (for English translation, see [67]) for a point mass as it contains a singularity at radial position 0 [64,68,69]. The imposition of the covariant formulation of the classical uncertainty principle defined an exclusion zone around the singularity at $R = 0$ below, at which no GR geodesic is allowed [61]. This condition ensured a minimal nonzero uncertainty in radial distance right at the singularity, which corresponded to

$$dR^2 = \frac{2Mc}{mu^1}\ell_p^2, \tag{24}$$

where M is the mass of the black hole. Therefore, it turns out that unrestricted covariant formulations of the uncertainty principle prevent a point-like singularity in the Schwarzschild solution. Hence, there would be no need to restrict the uncertainty principle to obtain solutions with minimal, covariant length elements. If interpreted as a standard deviation, the average R coordinate position of a particle at the singularity will still be 0, but allowing for a radius of uncertainty that would counteract the information paradox at the singularity [31,70]. A calculation of this minimal uncertainty of dR for a stellar mass black hole provides a value within the range of 10^{-15} to 10^{-16} cm. The uncertainty principle had been previously proposed to be the source of a repulsion force that prevents particles from reaching the singularity within the framework of LGQ and string theory. Both these theories rely on a minimal measurable length in their formulations. The matter contained within a black hole would form a "fuzzball" [71] or a "Planck star" [72]. The repulsion force by the uncertainty principle as described by LQG would occur when reaching Planck density [73], and this leads to the radius of a Planck star:

$$R \sim \left(\frac{M}{m_p}\right)^n \ell_p, \tag{25}$$

where in the context of this equation, M corresponds to the mass of the Planck star and m_p to Planck mass. Considering scenarios where $n = 1/3$ or 1, the radius of a Planck star would comprise between 10^{-10} to 10^{-14} cm [72], close to our calculations [61] for a minimal radial uncertainty at the singularity.

6. Conclusions

As discussed above, most (but not all) quantum gravity theories rely on a minimal measurable length element that clashes with both the uncertainty principle, and Lorentz covariance. To solve this issue, the classical uncertainty relations have been modified to include length and momentum restrictions leading to GUP. Most GUP formulations consider a restriction only in minimal length but not in momentum. Considering that the uncertainty principle derives from fundamental properties of wave functions, it seems a rather arbitrary decision to modify it to comply with current quantum gravity theories. An alternative consideration would be to derive uncertainty relations with covariant formulations that will nevertheless lead to minimal covariant length elements.

Thanks to their formulations, gravity theories such as string theory and LQG predict Lorentz invariance violations (LIVs) due to the discreet nature of space–time and a minimal "noncovariant" measurable length. This is the rationale behind the experiments attempting to detect LIVs such as in vacuo dispersion of photons and neutrinos, or deviations of polarization over astronomical distances [1,15,74–79]. The experimental detection of LIVs and the energy scales in which LIVs might be detected (if indeed they are detected) could help to either confirm the existence of minimal noncovariant length elements in the space–time structure, or at least discard scenarios incompatible with the experimental data [74]. Detection (or not) of LIVs could help with lattice quantization in LQG, the time problem and the choice for privileged reference frames [8,32,80,81]. If proven, LIVs could demonstrate space–time quantization and set up the proper length scales and energies for quantum gravity with a minimal measureable length [79]. However, the experimental detection of LIV is controversial. Several studies have attempted to quantify upper limits to LIV constraints [75–78]. Measurement of energy and helicity-dependent photon propagation velocities over astronomical distances could uncover quantum gravity effects such as space quantization [72]. By measuring deviations of GRB 041219A gamma ray burst photons, an upper limit on the vacuum birefringence of 1.1 10^{-14} was estimated, which would correspond to spatial volume units of less than 10^{-42} m^3 [77]. While a number of recent studies are reporting LIV violations at different energy orders, other studies estimate stringent constraints, or even fail to detect LIVs [74,76,77,82,83]. These experimental results may reinforce the idea that space–time quantization can be compatible with Lorentz-covariant length elements without restrictions to minimal lengths or momenta, as shown by other covariant formulations such as de Sitter symmetries in doubly special relativity [58] and the recent formulations by us [61,62].

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