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(2 + 1)-Maxwell Equations in Split Quaternions

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Abstract: The properties of spinors and vectors in (2 + 2) space of split quaternions are studied. Quaternionic representation of rotations naturally separates two $SO(2,1)$ subgroups of the full group of symmetry of the norms of split quaternions, $SO(2,2)$. One of them represents symmetries of three-dimensional Minkowski space-time. Then, the second $SO(2,1)$ subgroup, generated by the additional time-like coordinate from the basis of split quaternions, can be viewed as the internal symmetry of the model. It is shown that the analyticity condition, applying to the invariant construction of split quaternions, is equivalent to some system of differential equations for quaternionic spinors and vectors. Assuming that the derivatives by extra time-like coordinate generate triality (supersymmetric) rotations, the analyticity equation is reduced to the exact Dirac–Maxwell system in three-dimensional Minkowski space-time.

Keywords: split quaternions; triality; (2 + 1) electrodynamics

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1. Introduction

It is important to express physical variables and their relationships by maximally universal mathematical structures. In previous papers, I suggested that normed division algebras can play this role [1–8], since they provide a natural framework to describe space-time transformations, together with spinors and vectors and their equations.

It is known that besides of the usual real numbers, according to the Hurwitz theorem, there are three other unique normed division algebras A_n: complex numbers, quaternions and octonions [9–11]. These four division algebras have dimensions $n = 1$, 2, 4, and 8, respectively. The real numbers are ordered, commutative and associative, but for each next mentioned algebra one such property is lost.

An element X of any n-dimensional normed real algebra, A_n, can be written as the linear combination of the basis elements e_n with the real coefficients X_n,

$$X = X_n e_n \, . \tag{1}$$

The unit basis element, $e_0 = 1$, is real and the other $n-1$ elements are hyper-complex. The square of the unit element is always positive, while the squares of some hyper-complex basis units can be negative as well,

$$e_0^2 = 1 \, , \qquad e_{n-1}^2 = \pm 1 \, . \tag{2}$$

In normed algebras the conjugation operation is defined, which does not affect unit element but changes the sign of all hyper-complex basis units,

$$e_0^* = e_0 \, , \qquad e_{n-1}^* = -e_{n-1} \, . \tag{3}$$

Then, using the multiplication and conjugation rules of e_n, the quadratic form (norm) in A_n can be defined,

$$N = XX^* \, , \tag{4}$$

which for any two elements, X and Y, satisfy the condition

$$N(XY) = N(X)N(Y)\ . \tag{5}$$

In applications of normed algebras mainly the basis elements with the negative square $e_{n-1}^2 = -1$ (similar to the ordinary complex unit i), which correspond to Euclidean norms,

$$N = \sum_n X_n^2\ , \tag{6}$$

are used. Introducing the vector-like basis elements (with the positive squares $e^2 = 1$) leads to split algebras having an equal number of "+" and "–" sign terms in the definition of their norms. In particular, signatures of the quadratic forms of split algebras have the structures: (1 + 1) for hyper-numbers, (2 + 2) for split quaternions and (4 + 4) for split octonions.

Normed algebras A_n are closely related to Lie algebras in various guises. The norm of division and split algebras are preserved under the transformations of their elements X_n by the Lie algebras of $SO(n)$ and $SO(n/2, n/2)$, respectively; $n = 2, 4, 8$. Algebras A_n are characterized by the remarkable property of having vector, spinor and conjugate spinor representations of the same (real) dimensions, a property best known as triality. The vector, spinor and conjugate spinor, as well as the gamma matrices that transform them, are all represented by division algebra elements and their multiplication. One can show that there is a division algebra of dimension n if the double cover of the rotation group $SO(n)$ has a spinor representation whose dimension equals n.

The current paper concentrates on four-dimensional ($n = 4$) normed algebra of split quaternions. On physical application of quaternions, one can find thousands of publications [12]. The most commonly quaternions are used in the areas of computer graphics, navigation systems, quantum physics and kinematics (see [13–18] and references therein).

It is known that ordinary (Hamilton's) and split quaternions can be used to describe three-dimensional Euclidean and Minkowski spaces, respectively. Here, vector and spinor representations are considered in (2 + 2) space of split quaternions, which generates kinematics of three-dimensional Minkowski space-times [8]. There are several physical models employed in (2 + 1) spaces, such as the theory of graphene [19], black holes [20], quantum gravity [21], Anti-de Sitter/conformal field thery (AdS/CFT) correspondence [22], gauge theory of gravity [23,24], etc.

This paper is organized as follows. Section 2 briefly reviews properties of split quaternions and their applications to describe geometry of (2 + 2) space. It is shown that maximal rotations towards the second time-like coordinate in (2 + 2) space of split quaternions can be used to introduce Plank's constant and generate quantum uncertainty relation. In Section 3, the property of triality—rotations between quaternionic vectors and spinors—is discussed. In Section 4, the generalized Cauchy–Riemann analyticity condition for quaternionic functions is introduced. It is shown that the requirement of analyticity for triality invariant quaternionic construction leads to the system of equations that are equivalent to the Dirac–Maxwell system in three-dimensional Minkowski space-time. Finally, conclusions are presented in Section 5.

The paper is accompanied by several Appendices. In Appendix A, the $SL(2, R)$ and $SU(1, 1)$ matrix representations of the basis elements of split quaternions are presented. Appendix B is devoted to classification of split quaternions by the values of their norms. In Appendix C, a reminder on descriptions of vectors and spinors of (2 + 2) space is given. In Appendix D, quaternionic representations of rotations and boosts are considered. In Appendix E, decomposition of split quaternions by idempotent and nilpotent elements is described. In Appendix F, the quaternionic spinors are considered. It is shown also the validity of the so-called 3-ψ rule, the Fierz identity for quaternionic spinors.

2. Split Quaternions

Algebra of split quaternions over reals is an associative, non-commutative ring, the general element of which can be written as the linear combination of four basis elements, $e_0 = 1$ and e_k, $k = 1, 2, 3$, with the real coefficients q_0 and q_k,

$$q = q_0 + q^k e_k = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 \,. \tag{7}$$

The first basis element $e_0 = 1$ is real, while the other three, e_k, are anti-commuting hyper-complex units. Using algebra of e_k it can be shown that one of the hyper-complex basis elements, for example the last one e_3, is possible to define as the product of other two,

$$e_3 = e_1 e_2 = -e_2 e_1 \,. \tag{8}$$

Matrix representations of quaternionic basis elements is introduced in Appendix A.

In analogy with complex numbers, the quaternionic conjugation, which changes the sign of the hyper-complex basis elements, is defined:

$$e_0^* = e_0 \,, \quad e_1^* = -e_1 \,, \quad e_2^* = -e_2 \,, \quad e_3^* = (e_1 e_2)^* = e_2^* e_1^* = -e_1 e_2 = -e_3 \,. \tag{9}$$

Then, the inverted (conjugated) quaternion can be constructed:

$$q^* = q_0 - q_1 e_1 - q_2 e_2 - q_3 e_3 \,. \tag{10}$$

Note that the two main basis units of split quaternions have the feature of unit polar vectors,

$$e_1^2 = e_2^2 = 1 \,, \tag{11}$$

while the third hyper-complex basis element, $e_3 = e_1 e_2$, behaves like a complex unit,

$$e_3^2 = (e_1 e_2)(e_1 e_2) = -e_1^2 e_2^2 = -1 \,, \qquad e_3^* = -e_3 \,. \tag{12}$$

Because of these distinct properties of basis elements, the norm of a split quaternion,

$$N = \sqrt{|qq^*|} = \sqrt{|q_0^2 - q_1^2 - q_2^2 + q_3^2|} \,, \tag{13}$$

has (2 + 2)-signature and in general qq^* is not positively defined.

For geometrical applications, one can define the line element in the space of split quaternions in the form [8]:

$$s = \lambda + e_1 x + e_2 y + e_3 t \,, \tag{14}$$

where the four real parameters that multiply basis units, 1 and e_k, denotes: some time-like quantity λ, the spatial coordinates x and y and the time coordinate t (the speed of light is set as $c = 1$). Using the conjugation rules one can find that the norm of quaternion (14) (the interval),

$$ss^* = s^* s = \lambda^2 - x^2 - y^2 + t^2 \,, \tag{15}$$

has (2 + 2)-signature and in general is not positively defined. As in the standard relativity, it is required:

$$ss^* \geq 0 \,, \tag{16}$$

i.e., the consideration is restricted to time-like split quaternions, defined in Appendix B.

2.1. Vector-Type Rotations

Now we want to represent rotations of a split quaternion q by the products of other quaternions.

There are a lot of methods to represent three-dimensional rotations (the method of orthonormal matrices is considered in Appendix C), however, quaternions are known to be the most convenient ones [14–16]. The vector part (time-like or space-like) of unit split quaternions (see Appendix B) can be thought of as a vector about which rotation should be performed and scalar part specifies the amount of rotation that should be performed about the vector part. That is, only four numbers are enough to represent a rotation by quaternions and there is only one constraint—the unity of norm. This makes it possible to find solutions to some optimization problems involving rotations in three dimensions, which are hard to solve when using orthonormal matrices because of the six non-linear constraints to enforce orthonormality, and the additional constraint—the unity of determinant.

Rotations in the vector (2 + 1) space of split quaternions about the time-like axis, e_3, or about the space-like axes e_1 and e_2, can be expressed with two side products of the unit time-like quaternions with time-like vector part (A37), or unit time-like quaternions with space-like vector part (A35), respectively. Every unit half-angle split quaternion $\alpha(\theta/2)$ represents a vector-type rotation by two-side multiplication. The result of the product,

$$q' = \alpha\left(\frac{\theta}{2}\right) q\, \alpha^*\left(\frac{\theta}{2}\right), \tag{17}$$

is the quaternion q', which norm and scalar part are the same as for q and the vector part $V_{q'}$ is obtained by revolving V_q through the angle θ conically about the vector ϵ defined in Equation (A34).

The set of unit split quaternions α_t, α_x and α_y form the group of rotations in (2 + 1)-dimensional space-time, which algebra is isomorphic to $SU(1,1)$. Under this isomorphism the quaternion multiplication operation corresponds to the composition operation of rotations.

However, why vector-like quaternions have the 'double-cover' property (17), why there are two different quaternions (α and $\alpha^{-1} = \alpha^*$) that represent the same rotation instead of $\alpha(\theta)q$ in analogy with complex numbers? It turns out that multiplication by a single quaternion do represent double rotations of spinors in four-dimensional quaternionic space (see Appendix F) and not in a plane of the three-dimensional vector space V_q. In the case of complex numbers, one has just one dimension of rotation. In three dimensions of quaternionic vector parts, we talk regarding to rotating about an axis, but in reality it means rotating in a plane perpendicular to that axis. In four-dimensional space of quaternions there are enough dimensions that it's possible to rotate in two independent planes at once. These planes have no axes in common, they intersect only at a single point, which is the center of rotation. So, both rotations can take place without disturbing each other, which is not possible in three dimensions, where two planes always intersect in a line.

In Appendix D, it is shown that for quaternions one needs two distinct representations of vector-type rotations, $\alpha q \alpha^*$ and $\alpha q \alpha$, each corresponding to $SO(2,1)$-subgroups of $SO(2,2)$. Thus, the four components of a split quaternion q^a ($a = 0,1,2,3$) cannot be treated as a single 4-vector of $SO(2,2)$. The $\alpha q \alpha$-type $SO(2,1)$-transformations contain two independent hyperbolic rotations (boosts) of x and y axes towards the second time-like direction λ of split quaternions (2 + 2) space, and compact rotations in time-like $(\lambda - t)$-plain, while the products $\alpha q \alpha^*$, forming automorphism group of split quaternions $SO(2,1)$, give successfully representation of rotations in three dimensional Minkowski space-time with two space-like and one time-like coordinates (x, y, t) and can be used to define physical 3-vectors.

So, the standard representation of quaternion rotations $\alpha q \alpha^*$, which generate transformations of only vector part of split quaternions,

$$V_q = q_1 e_1 + q_2 e_2 + q_3 e_3 \qquad (e_1^2 = e_2^2 = -e_3^2 = 1), \tag{18}$$

naturally separates the Lorentz-like $SO(2,1)$ subgroup from the full group of symmetry of the split quaternion norms, $SO(2,2)$. Then, in field theory applications in three-dimensional Minkowski space-time, the second $SO(2,1)$ subgroup that generates rotations involved the second time-like coordinate, can be interpreted as corresponding to the internal symmetries of the model.

Note that to define coordinate transformations as quaternionic products, all four elements of a split quaternion (7) are needed, i.e., one should not remove the scalar part q_0 (see Appendix D). In general, interpretation of scalar parts of quaternions, which is not affected by $\alpha q \alpha^*$-type rotations, causes difficulties. Hamilton himself without notable success tried to interpret the scalar part of quaternions as an extra-spatial unit [25,26]. Later, when inspired by quaternions, vector algebra was introduced in physics, the scalar part of quaternions even was omitted. Introduction of vectors was successful, however because of removing of the scalar part of quaternions the division operation is not defined for vectors. There was lost also the property of quaternions, that they are rotation generators and expresses not only the final state achieved after a rotation, but the direction in which this rotation has been performed. It is this direction of rotation that the standard matrix representation of the rotation group fails to give.

2.2. Maximal Velocity and Uncertainty Principle

Analysis of the spinor-type one-side boosts of quaternionic intervals (14), can help in physical interpretation of the scalar parts of split quaternions λ. From definitions (A77) one notices that for the boosts along the positive x-direction, i.e., when x' increases (or the x-component of the momentum, p_x, increases), the scalar part λ' decreases and vice versa. In quantum mechanics the quantity with the dimension of length which is inversely proportional to the momentum is particle's wavelength. So, in geometrical application it is natural to interpret the scalar part of the split quaternion (14) as the wavelength describing the inertial properties of particle's reference frame.

Suppose the split quaternion (14) is used to describe motion of a particle with the wavelength λ along the x-axis with the velocity v_x in a laboratory coordinate system. Denoting the wavelength in particles own system by Λ, from the condition of invariance of the intervals,

$$ds = d\lambda\left(1 + \frac{dy}{d\lambda}e_2\right) + dt(1 - v_x e_2)e_3 = d\Lambda + d\tau e_3\,, \tag{19}$$

where τ is the proper time of the particle, one obtains the two conditions:

$$\left|\frac{d\tau}{dt}\right| = \sqrt{1 - v_x^2}\,, \qquad \left|\frac{d\Lambda}{d\lambda}\right| = \sqrt{1 - \frac{dy^2}{d\lambda^2}}\,. \tag{20}$$

From these conditions follows the relations:

$$|v_x| < 1\,, \qquad \left|\frac{dy}{d\lambda}\right| < 1\,, \tag{21}$$

which must be obeyed simultaneously. This means that, together with speed of light $c = 1$, there must exist the second fundamental constant (which can be extracted from λ) characterizing this critical property of the algebra.

In (2 + 2) space there are two different light-cones with two class of critical rotations and there must exist two fundamental constants characterizing this property of algebra.

It is known that the parameter with required properties is the reduced Planck constant, $\hbar$, which relates particles wavelength to its momentum,

$$\lambda = \frac{\hbar}{p}\,. \tag{22}$$

So, in the approach here considered, two fundamental physical constants, c and $\hbar$, have the algebraical origin and correspond to two kinds of critical signals in (2 + 2) space of split quaternions [8].

Furthermore, it was shown that for the boosts with the positive velocity, when p_x increases (p_y decreases), the quantity λ decreases, i.e., $d\lambda < d\Lambda$, and vice versa. So, for the change in the scalar part of the split quaternion, $\Delta\lambda$, when the particle momentum in x-direction increases, one can write

$$\Delta\lambda \sim -\frac{\hbar}{\Delta p_y}\,, \tag{23}$$

where p_y is the y-component of the momentum. When the variety of wave lengths becomes overall shorter, the overall magnitude of the variety of momenta must become greater, i.e., the shorter the wavelength, the higher will be its frequency and hence carry a greater amount of momentum. Similar relation can be obtained for the boost along the y-axis.

Inserting Equation (23) into Equation (21), one can conclude that the uncertainty principle,

$$\Delta x \Delta p_x \gtrsim \hbar\,, \tag{24}$$

probably has the same geometrical meaning as the existence of the maximal velocity c [8].

In the formalism here applied, one can reach the relation (24) using also stochastic approach [27]. The trajectory $X^n(t)$ $(n = x, y)$ of a classical test particle in (2 + 1) space, in general, depends on the second time-like coordinate λ. Then in the (2 + 2) space of split quaternions one gets a diffusion-like process:

$$dX^n = v^n dt + \frac{\partial x^n}{\partial \lambda} d\lambda \qquad (n = x, y), \tag{25}$$

where X^n is the random 2-position. The 2-velocity, $v^n = \partial x^n/\partial t$, and diffusion parameters toward the x and y directions, $\partial x^n/\partial\lambda$, in Equation (25) are restricted by condition (21).

The additional time-like coordinate of quaternionic (2 + 2) space, λ, can be imagined to represent a Wiener process (the most common example of a Wiener process is Brownian motion) if is characterized by the following properties:

- at a starting moment can be set to zero together with t;
- has independent increments for every $t > 0$;
- is normally distributed with mean 0 and have some finite variance;
- has continuous paths in t.

These requirements are indeed satisfied for physical processes in (2 + 1)-sub-subspace of the (2 + 2) space of split quaternions. In Appendix D it is shown that any rotation in the spatial $(x - y)$-plain (represented with one-side multiplications by a time-like unit split quaternions) is accompanied with the compact rotations (represented with the harmonic functions) in the temporal $(t - \lambda)$-plain. Thus, the variable λ in (2 + 1) space has continues path in t and its mean value is zero for sufficiently large time intervals. On the other hand, to a quaternionic boost, e.g., towards x, it is corresponded non-compact rotation in $(y - \lambda)$-plain with the restriction $dy/d\lambda < 1$, which follows from the positive norm condition (21). So, in (2 + 1) space a normal distribution of the continuous variable λ with the mean value 0 and the variance 1 can be assumed.

For a process described by the stochastic differential Equation (25) the probability density obeys the Fokker–Planck equation. Using a stationary solution to this equation one can define the standard deviations, σ_X and σ_P, of the random coordinate X and momentum P. Then, the Cauchy-Schwartz inequality,

$$\sigma_X \sigma_P \geq |Cov(X, P)|\,, \tag{26}$$

leads to the standard uncertainty relation (24) [28].

3. Quaternions and Triality

Lie groups corresponding to division algebras have the property of having vector, spinor and conjugate spinor representations of the same (real) dimensions, a property known as triality [10,29,30]. By split quaternions the basic operations involving vectors, spinors and scalars can be described. These include an operation that takes two spinors, ξ

and χ (defined in Appendix F using zero divisors of the algebra Appendix E), and forms a vector $A = \xi\bar{\chi}$, and an operation that takes a vector A and a spinor ξ and forms a spinor $\chi = A\xi$.

3.1. Vector and Spinor Transformations

In quaternionic (2 + 2) space the vector (for example the interval (14)) and the first and second kind spinors can be parameterize using the quaternionic basis (A1),

$$s = x^a e_a \, , \qquad \xi = \xi^a e_a \, , \qquad \chi = \chi^a e_a \, . \tag{27}$$

Thus, all three spinors are realized as split quaternion and their rotations will take a multiplicative form, different in each case. One can check that their indices can be raised/lowered by the matrix (A47) and that, in all three cases, the generators can be chosen to be real antisymmetric matrices. These are consequences of the triality property, which implies that one can just use a single index a for all three representations, which will be distinguished only by the symbols, s, ξ and χ.

Note that the (2×2) matrices

$$\gamma^a = e^a \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{28}$$

satisfy the Clifford algebra

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab} \, , \tag{29}$$

so rotations of spinors of the first and second kinds, ξ and χ, can be written by two sets of matrices e^a and e^{a*},

$$\begin{aligned} \xi' &= e^{\frac{1}{2}\epsilon^{ab}S_{ab}}\, \xi = \xi + \tfrac{1}{4}\epsilon^{ab} e_a e_b^* \xi \, , \\ \chi' &= \chi \, e^{\frac{1}{2}\epsilon^{ab}S_{ab}} = \chi + \tfrac{1}{4}\epsilon^{ab} e_a^* e_b \chi \, , \end{aligned} \tag{30}$$

where S_{ab} are defined in (A54).

Furthermore, one can write the $SL(2,R)$-matrix representation of the Lorentz-type transformations (A49), L_{nm} $(n, m = x, y, t)$, in the (2 + 1)-subspace (x, y, t) of the vector (14):

$$\begin{aligned} x'^n = L^n{}_m x^m \quad \Leftrightarrow \quad s' &= e^{\frac{1}{2}\epsilon^{mn}(L_{mn})_a{}^b} x^a e_b \\ &= s + \tfrac{1}{4}\epsilon^{mn}\left(\delta^b_m \delta_{na} - \delta^b_n \delta_{ma}\right) x^a e_b \\ &= s + \tfrac{1}{4}\epsilon^{mn}[e_m(e_n^* e_a) - e_a(e_m^* e_n)]x^a \\ &= s + \tfrac{1}{4}\epsilon^{mn}(e_m e_n^* s - s e_m^* e_n) \, . \end{aligned} \tag{31}$$

3.2. Triality Algebra

The triality algebra of split quaternions can be defined as:

$$\alpha_1(pq) = \alpha_2(p)q + p\alpha_3(q) \, , \tag{32}$$

where p and q denote some split quaternions and α_1, α_2 and α_3 are unit quaternions that generate $SO(2,1)$ rotations.

In the formulations of $SO(2,2)$ vector A, covariant spinor ξ and contravariant spinor χ representations in terms of quaternions (see Appendix F), the relationships between the three may be expressed without gamma matrices. For example, a covariant and a contravariant spinor can be used to form a vector

$$A = \xi\chi^* \, , \tag{33}$$

or a contravariant spinor can be made from a vector and a covariant spinor

$$\chi = A^*\xi\,, \tag{34}$$

or a covariant spinor can be made from a vector and a contravariant spinor

$$\xi = A\chi\,. \tag{35}$$

Supposing that all three objects A, ξ and χ are transformed by a priori unrelated unit quaternions α_1, α_2 and α_3 and still insisting that A is made from ξ and χ, these transformations would have to be related. This relation is exactly the condition that defines triality (32) and leads to the conclusion that the triality transformations is the largest group that preserves (33). However, these transformations are exactly of the form used in supersymmetry (see, for example [31]), so it is only natural that the overall symmetry of these theories is given by the triality algebras.

If A, ξ and χ are quaternionic vector and covariant and contravariant spinors, respectively, then, the triality construction

$$\mathrm{Tri} \equiv \chi A\xi \tag{36}$$

is $SO(2,1)$ invariant. This can be checked explicitly by applying to (36) spinor and vector transformation rules (see Appendices D and F),

$$\chi' = \chi\alpha^*\,, \qquad \xi' = \alpha\xi\,, \qquad A' = \alpha A\alpha^*\,. \tag{37}$$

where α is a unit quaternion, i.e., $\alpha\alpha^* = 1$.

The construction of division algebras from trialities has tantalizing links to physics. In the Standard Model of particle physics, all particles (other than the Higgs boson) transform either as vectors or spinors, and the interaction between matter and the forces is described by a trilinear map (36), involving two spinors and one vector. Moreover, split normed algebras naturally introduce pseudo-Euclidean spaces which are needed to describe physical spinors and vectors that are associated with Lorentz-type groups.

4. Quaternionic Analyticity and (2 + 1) Electrodynamics

In this section, the quaternionic analyticity condition, that is generalization of the Cauchy–Riemann equations from complex analysis, is derived. Then it is shown that this condition, written for the triality invariant element of split quaternions, leads to the system of equations for quaternionic vector and spinors, which is equivalent to the ordinary Dirac–Maxwell system in three-dimensional Minkowski space-time [1].

4.1. Quaternionic Gradient Operator

In physical applications it is important to define the quaternionic gradient operator. Although there have been some derivations of this operator in the literature with different level of details (see, [32–41] and references therein), it is still not fully clear how this operator can be written in the most general case and how it can be applied to various quaternion-valued functions.

Employ analogy with complex analysis, one can define the split quaternionic derivative operator in the form [35–37],

$$\frac{d}{ds} = \frac{1}{2}\left(\partial_\lambda + e_1\partial_x + e_2\partial_y + e_3\partial_t\right)\,, \tag{38}$$

with $\partial_a \equiv \partial/\partial_a$, such that its action upon interval quaternion (14) is equal to one,

$$\frac{ds}{ds} = \frac{1}{2}(1 + e_1^2 + e_2^2 + e_3^2) = 1\,, \tag{39}$$

while if applied to conjugated interval element,

$$s^* = \lambda - xe_1 - ye_2 - te_3 \,, \tag{40}$$

it gives zero,

$$\frac{ds^*}{ds} = \frac{1}{2}(1 - e_1^2 - e_2^2 - e_3^2) = 0 \,. \tag{41}$$

Similarly, the conjugated gradient can be defined by the operator

$$\frac{d}{ds^*} = \frac{1}{2}(\partial_\lambda - e_1\partial_x - e_2\partial_y - e_3\partial_t) \,, \tag{42}$$

which annihilates s. Thus, from the definitions of quaternionic gradients, (38) and (42), one finds:

$$\frac{ds^*}{ds} = \frac{ds}{ds^*} = 0 \,. \tag{43}$$

From these relations it is clear that the interval (15) is a constant function for the restricted left quaternionic gradient operators,

$$\begin{aligned} \frac{d}{ds}(s^*s) &= \left(\frac{ds^*}{ds}\right)s = 0 \,, \\ \frac{d}{ds^*}(ss^*) &= \left(\frac{ds}{ds^*}\right)s^* = 0 \,. \end{aligned} \tag{44}$$

4.2. Analyticity Condition

The main obstacle in physical applications of quaternions is that the real-valued functions of quaternion variables $\Phi(q)$ are not analytic according to quaternion analysis [32–41]. To bypass the issue of non-existent derivatives of real functions of quaternion variables, current applications typically rewrite $\Phi(q)$ in terms of the four real components $\phi^a(q^a)$ ($a = 0, 1, 2, 3$) of four quaternion variables:

$$\Phi(s, s^*) = \phi_\lambda + e_1\phi_x + e_2\phi_y + e_3\phi_t \,, \tag{45}$$

and take the real derivatives with respect to q^a [37]. Thus, analogously to the Cauchy–Riemann equations from complex analysis,

$$\partial_{z^*} f(z, z^*) = 0 \,, \qquad (z = x + iy) \tag{46}$$

the Cauchy–Riemann–Fueter condition of analyticity for quaternionic functions of quaternionic variables can be written as [35–37],

$$\frac{d\Phi(s, s^*)}{ds^*} = \frac{1}{2}(\partial_\lambda - e_1\partial_x - e_2\partial_y - e_3\partial_t)\Phi = 0 \,. \tag{47}$$

This statement, that quaternionic functions should be independent of the variable s^*, represents the condition that quaternionic derivative be independent of direction along which it is evaluated.

Note that the simple case of the quaternionic analyticity condition for interval vectors (43), due to the existence of the relations of the type (41), holds only for split quaternions. This justifies in split quaternionic calculus the use of the simple gradient operators (38) introduced in [35,36]. For ordinary quaternions, even the polynomial functions do not satisfy these Cauchy–Riemann–Fueter conditions and several generalizations are necessary [37–41].

The coordinate transformations, e.g., $q' = \alpha q$, in general do not preserve the property of analyticity, since unit split quaternions representing rotations and boosts are not analytic functions. Thus, to construct analytic and invariant structures (Lagrangians, Superpotentials, etc.), combinations of spinor-like and vector-like split quaternions are needed. For instance, if the transformation laws of χ, ξ and A have the forms,

$$\chi' = \chi\alpha^* , \qquad \xi' = \alpha\xi , \qquad A' = \alpha A\alpha^* , \tag{48}$$

the invariant constructions are products of the type $\chi\xi$ and $\chi A\xi$. Indeed,

$$\begin{aligned} (\chi\xi)' &= \chi'\xi' = \chi\, \alpha^*\alpha\, \xi = \chi\xi , \\ (\chi A\xi)' &= \chi\, \alpha^*\alpha\, A\, \alpha^*\alpha\, \xi = \chi A\xi . \end{aligned} \tag{49}$$

Then, using validity of the distribution law for quaternionic gradient operators (38) and (42) [35–37], one can write Cauchy–Riemann–Fueter analyticity condition (47) for the triality invariant construction (36) in the form:

$$\frac{d(\chi A\xi)}{ds^*} = \frac{d\chi}{ds^*}A\xi + \chi\frac{dA}{ds^*}\xi + \chi A\frac{d\xi}{ds^*} = 0 . \tag{50}$$

This condition is equivalent to the system of equations for quaternionic vector and spinors of the first and second kind (covariant and contravariant),

$$\frac{dA}{ds^*} = 0 , \qquad \frac{d\chi}{ds^*} = 0 , \qquad \frac{d\xi}{ds^*} = 0 . \tag{51}$$

Below, it is shown that this system can be reduced to the equations of (2 + 1) electrodynamics, i.e., to the system of standard Dirac and Maxwell equations in three-dimensional Minkowski space-time.

4.3. Quaternionic Dirac Equation

Let us demonstrate that the algebraic Cauchy–Riemann–Fueter condition (51) for the covariant spinor ξ can be understood as the Dirac equation. Analogous logic can be applied for the case of the second kind (contravariant) spinor χ to obtain Dirac equation for conjugated spinors.

Consider the first kind of spinor, ξ^+ (A128),

$$\xi^+ = (q_0 - iq_3)D^+ + (q_2 + iq_1)G^- , \tag{52}$$

expressed in terms of the idempotent and nilpotent elements in $SU(1,1)$ representation (A92), where q^n represents real functions of λ, x, y and t. One can replace

$$\xi^+(\lambda, x, y, t) \quad \rightarrow \quad e^{m\lambda}\xi^+(x, y, t) , \tag{53}$$

where m is a real parameter. Then, the condition (51) for the covariant spinor ξ takes the form:

$$(\partial_\lambda - e_1\partial_x - e_2\partial_y - e_3\partial_t + m)\xi^+(x, y, t) = 0 . \tag{54}$$

Next let us assume that the derivative of the covariant split quaternion $\xi^+(x, y, t)$ by the extra time-like coordinate λ generates the supersymmetric (triality) transformation:

$$\partial_\lambda\xi^+ = B\chi^{+*} , \tag{55}$$

where B is some vector-type split quaternion, while χ^{+*} is the conjugated contravariant quaternionic spinor (A129) in $SU(1,1)$ representation:

$$\chi^{+*} = (q_0 - iq_3)D^+ - (q_2 + iq_1)G^- . \tag{56}$$

Then the quaternionic Cauchy–Riemann condition (54) obtains the form:

$$(e_1\partial_x + e_2\partial_y + e_3\partial_t - m)\xi^+ - B\chi^{+*} = 0 , \tag{57}$$

or in matrix notation,

$$\left[\begin{pmatrix} -i\partial_t + m & \partial_y - i\partial_x \\ \partial_y + i\partial_x & i\partial_t + m \end{pmatrix} + \begin{pmatrix} B_0 - iB_3 & B_2 - iB_1 \\ B_2 + iB_1 & B_0 + iB_3 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} q_0 - iq_3 \\ q_2 + iq_1 \end{pmatrix} = 0\,. \tag{58}$$

On the other hand, the (2 + 1)-dimensional Dirac equation for a massive particle has the form,

$$\left[i\gamma^k(\partial_k + iA_k) - m\right]\Psi(t,x,y) = 0 \qquad (k = 1,2,3), \tag{59}$$

where i denotes the standard complex unit, A_k is the vector-potential and γ^3 plays the role of the γ^0 matrix. As the (2 + 1) gamma matrices the split quaternionic basis elements can be used in $SU(1,1)$-matrix representation,

$$i\gamma^1 = e_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad i\gamma^2 = e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad i\gamma^3 = e_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \tag{60}$$

and identify the complex Dirac spinors $\Psi(t,x,y)$ with the complex $SU(1,1)$ representation of the quaternionic spinor $\xi^+(t,x,y)$. Then, the matrix form of (2 + 1) Dirac Equation (59) obtains the form:

$$\left[\begin{pmatrix} -i\partial_t + m & \partial_y - i\partial_x \\ \partial_y + i\partial_x & i\partial_t + m \end{pmatrix} + \begin{pmatrix} A_t & A_x + iA_y \\ -A_x + iA_y & -A_t \end{pmatrix}\right]\begin{pmatrix} q_0 - iq_3 \\ q_2 + iq_1 \end{pmatrix} = 0\,, \tag{61}$$

where $q^n(t,x,y)$ denote the four real components of the complex Dirac spinor $\xi^+(t,x,y)$.

Than assuming

$$B_0 = A_t\,, \qquad B_1 = A_y\,, \qquad B_2 = -A_x\,, \qquad B_3 = 0\,, \tag{62}$$

the quaternionic analyticity conditions (58) becomes equivalent to the complex (2 + 1) Dirac Equation (61) [1].

Note that Dirac's theory in (2 + 1) space has some novel features. For example, there exists two inequivalent representations of Dirac gamma matrices, without which parity operation and its conservation would not have been possible. Further, when an external gauge field is introduced, it induces an 'anomalous' current which can be related to the anomalous divergence of an axial current and to the topological Chern-Simons charge.

4.4. (2 + 1) Maxwell Fields

Firs let us remind ordinary Maxwell's equations in three-dimensional Minkowski space-time. To write the full set of Maxwell's equations in (2 + 1) spaces with the signature

$$\eta_{km} = \mathrm{diag}(+1,+1,-1) \qquad (k,m = x,y,t), \tag{63}$$

let us define the three potential $A^k = (A^x, A^y, A^t)$ and the Faraday tensor,

$$F^{km} = \partial^k A^m - \partial^m A^k\,, \tag{64}$$

which has only three independent components. The relation between the vector potential and the magnetic and electric fields can be done through

$$H = \partial_x A_y - \partial_y A_x\,, \qquad E_x = \partial_x A_t - \partial_t A_x\,, \qquad E_y = \partial_y A_t - \partial_t A_y\,. \tag{65}$$

Note that in (2 + 1) space, Faraday's tensor has only three independent components and the magnetic field is no longer a vector—it becomes a (pseudo) scalar field. Indeed, in a world where all electric effects are confined to two planes, the magnetic field would be along the perpendicular direction. One must notice also that the dual tensor,

$$\tilde{F}^k = \epsilon^{klm} F_{lm}\,, \tag{66}$$

will become a vector.

In covariant form, Maxwell's equations in (2 + 1) spaces are then

$$\partial_k F^{mk} = j^m \,, \qquad \partial_k \tilde{F}^k = 0 \,, \tag{67}$$

where j_t is the surface charge and j_x and j_y are surface currents. One can then work out the differential Maxwell's equations in (2 + 1) spaces in terms of the fields,

$$\begin{aligned} \partial_x E_x + \partial_y E_y = j_t \,, \qquad \partial_t E_x - \partial_y H = -j_x \,, \\ \partial_x E_y - \partial_y E_x + \partial_t H = 0 \,, \qquad \partial_t E_y + \partial_x H = -j_y \,. \end{aligned} \tag{68}$$

One can see that there is no equivalent to the three-dimensional magnetic Gauss' law. Electrodynamics become complete with the Lorentz force law $f^k = F^{km} v_m$,

$$f_x = e(E_x + v_y H) \,, \qquad f_y = e(E_y - v_x H) \,, \tag{69}$$

where e is the charge and $v_m = (v_x, v_y, 0)$ represents components of the velocity in the $(x-y)$-plain.

An immediate consequence of (2 + 1) Maxwell's equations in is the existence of electromagnetic waves in two spatial dimensions,

$$\begin{aligned} (\partial_x^2 + \partial_y^2 - \partial_t^2) E_x = \partial_x j_t + \partial_t j_x \,, \\ (\partial_x^2 + \partial_y^2 - \partial_t^2) E_y = \partial_y j_t + \partial_t j_y \,, \\ (\partial_x^2 + \partial_y^2 - \partial_t^2) H = \partial_y j_x - \partial_x j_y . \end{aligned} \tag{70}$$

In the Lorenz gauge,

$$\partial_k A^k = 0 \,, \tag{71}$$

Equation (70) are the wave equations with sources for the components of A^k ,

$$(\partial_x^2 + \partial_y^2 - \partial_t^2) A_k = -j_k \,. \tag{72}$$

Short discussions about the electrodynamics in (2 + 1) space can be found in [42–45], where it is noted that the model faces a few issues:

- The Coulomb force in (2 + 1) space must change, the electric field of a point charge now falls off as the inverse of the distance which entails a logarithmic electrostatic potential. This dramatically alters the phenomenology, since the attractive potential between opposite charges becomes confining, i.e., an infinite amount of energy would be required to extract the electron from the hydrogen atom, for example.
- Part of the vector calculus must change. The absence of a right-hand rule is obvious and the magnetic field must be qualitatively different; it turns out that it cannot be a vector any more—it becomes a scalar field.
- One of the main new features is connected to the retarded potentials. The reason for this is directly linked to the Huygens principle, which states that every point on a wave front is itself the source of (spherical) waves and relies on the fact that all waves propagate with a single speed. In (2 + 1) space, however, a solution to the wave equation can be understood as a superposition of waves travelling with speeds ranging from zero to the maximum value c, with which the first wave front travels.

4.5. First-Order Maxwell System

Let us write down the split quaternion that contains the electromagnetic potentials as

$$A = A_0 + e_1 A_x + e_2 A_y + e_3 A_t \,, \tag{73}$$

where A_x, A_y and A_t are components of the (2 + 1) vector and A_0 corresponds to extra degree of freedom in the quaternionic algebra. Using quaternionic gradient operator (42) the Cauchy–Riemann–Fueter analyticity condition (51) for the vector potential (73) can be written in the form:

$$(\partial_\lambda - e_1\partial_x - e_2\partial_y - e_3\partial_t)A = \left[\partial_\lambda A - (e_1\partial_x + e_2\partial_y + e_3\partial_t)A_0 - F\right] = 0\,. \tag{74}$$

Here F denotes the quaternionic electro-magnetic field,

$$F = (e_1\partial_x + e_2\partial_y + e_3\partial_t)(e_1A_x + e_2A_y + e_3A_t) = -\partial_k A^k - e_1E_y + e_2E_x + e_3H\,, \tag{75}$$

where H, E_x and E_y are components of the magnetic and electric fields (65).

As in Equation (53), the variable λ is separated in the scalar part of the vector-potential (73) (with the unit 'charge', $m = 1$),

$$\partial_\lambda A_0 = A_0\,. \tag{76}$$

Similar to the case of spinors (55), it is also assumed here that the derivative of the vector part of (73) by the extra time-like coordinate λ generates the supersymmetric (triality) transformations ($A \to \xi\chi$),

$$\partial_\lambda(e_1A_x + e_2A_y + e_3A_t) = e_1j_x + e_2j_y - e_3j_t\,, \tag{77}$$

where j_k are the components of the current vector, which is constructed by the covariant and contravarint spinors $j \sim \xi\chi$.

Using Equations (76) and (77) the Cauchy–Riemann–Fueter analyticity condition (74) can be written as the first-order Maxwell system [46,47]:

$$\begin{aligned} F &= A_0\,,\\ (e_1\partial_x + e_2\partial_y + e_3\partial_t)A_0 &= -e_1j_x - e_2j_y + e_3j_t\,. \end{aligned} \tag{78}$$

Utilize the auxiliary function A_0 this system can be transferred to the single second-order Maxwell's equation in (2 + 1) space,

$$(e_1\partial_x + e_2\partial_y + e_3\partial_t)F = -e_1j_x - e_2j_y + e_3j_t\,. \tag{79}$$

Indeed, the left side of this equation under the Lorenz gauge,

$$\partial_k A^k = \partial_x A_x + \partial_y A_y - \partial_t A_t = 0\,, \tag{80}$$

in $SU(1,1)$-matrix representation has the form:

$$\begin{aligned} &\begin{pmatrix} -i\partial_t & (\partial_y - i\partial_x)\\ (\partial_y + i\partial_x) & i\partial_t \end{pmatrix}\begin{pmatrix} -iH & (E_x + iE_y)\\ (E_x - iE_y) & iH \end{pmatrix}\\ &= (\partial_yE_x - \partial_xE_y - \partial_tH)e_0 + (\partial_tE_x - \partial_yH)e_1 + (\partial_tE_y + \partial_xH)e_2 + (\partial_xE_x + \partial_yE_y)e_3\,. \end{aligned} \tag{81}$$

Equating to zero the coefficients in front of the four quaternionic basis units, one obtains the system of four real equations which are identical to the system of the Maxwell Equation (68) [1].

Note that the algebra of quaternions was applied to the study of Maxwell's equations starting from the work of Maxwell himself [48]. Maxwell's equations also were obtained long ago using Cauchy–Riemann-type analyticity conditions for functions of real quaternions [49]. However, in previous studies, first-order quaternionic equations were considered for the fields (65) and not second-order equations for vector potentials (78). A review of different applications of quaternionic analysis to the Maxwell equations can be found in [50]. Furthermore, usually it is considered sourceless quaternionic Maxwell's equation, while in this paper, coupled quaternionic Dirac–Maxwell system is obtained receiving Dirac's current from triality transformations.

5. Conclusions

Spinors and vectors in (2 + 2) space of split quaternions were studied and connections of their algebraic and physical properties were noted.

We think that some properties of physical models, such as invariance of space-time intervals, non-commutativity and half-angle spinor representations, are encoded in the structure of split quaternions. In the approach considered, two fundamental physical constants (light speed and Planck's constant) have similar geometrical meanings and appear from the positive definiteness of quaternionic norms.

Quaternionic representation of rotations naturally separates two $SO(2,1)$ subgroups of the full group of symmetry of the norms of split quaternions, $SO(2,2)$. One of them represents symmetries of the three-dimensional Minkowski space-time, while the extra $SO(2,1)$ rotations by the second time-like coordinate can be viewed as internal symmetries of the model.

It is shown that the quaternion analyticity condition, analog of the Cauchy–Riemann equations from complex analysis, applied to the triality invariant construction of split quaternions, is equivalent to some system of differential equations for quaternionic spinors and vectors. Assuming that derivatives by the extra time-like coordinate of quaternionic (2 + 2) space generate triality (supersymmetric) rotations of vectors and spinors, the analyticity equations is reduced to the exact Dirac–Maxwell system in three-dimensional Minkowski space-time.

We should state that we do not know of any physical system which can be described by the (2 + 1) electrodynamics considered in this paper. Rather, the main purpose was to consider split quaternionic analysis as a toy-model on the way of developing the realistic field theoretical models using split octonions, in the spirit of [2–7].

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Appendix A. Matrix Representation of Split Quaternions

The quaternion algebra is associative and therefore can be represented by matrices. For example, one gets the simplest non-trivial representation of the basis element of split quaternions by the unit matrix and the three traceless (2×2) matrices of the $SL(2,R)$ algebra,

$$e_0 = (1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e_3 = e_1 e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A1}$$

In this representations split quaternion (7) uniquely corresponds to the 2×2 real matrix,

$$q_{SL(2,R)} = \begin{pmatrix} (q_0 + q_1) & (q_2 + q_3) \\ (q_2 - q_3) & (q_0 - q_1) \end{pmatrix}. \tag{A2}$$

Conversely, an arbitrary real matrix:

$$p = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, \tag{A3}$$

where a_n and b_n ($n = 1, 2$) are some real parameters, uniquely corresponds to a quaternion:

$$p = \frac{1}{2}\left[(a_1 + b_2) + (a_1 - b_2)e_1 + (b_1 + a_2)e_2 + (b_1 - a_2)e_3\right] . \tag{A4}$$

To represent the basis units of split quaternions in (A1) one can use, instead of the three real $SL(2, R)$ matrices, the complex $SU(1, 1)$ matrices,

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad e_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \tag{A5}$$

where i is the ordinary complex unit. The basis units e_0 and e_2 in matrices (A5), coincides with the corresponding ones in matrices (A1), while for the remaining two basis units, one has $e_1 \to ie_3$ and $e_3 \to ie_1$. So, any split quaternion (7) has the following $SU(1, 1)$-matrix representation:

$$q_{SU(1,1)} = \begin{pmatrix} (q_0 - iq_3) & (q_2 - iq_1) \\ (q_2 + iq_1) & (q_0 + iq_3) \end{pmatrix}, \tag{A6}$$

which can be obtained from the $SL(2, R)$-matrix representation (A2) by the replacements $q_1 \to -iq_3$ and $q_3 \to -iq_1$.

The representations (A2) and (A6) have the following properties:

1. the norm of a quaternion is expressed by the determinant of associated matrix,

$$\det(q) = N^2 = q_0^2 - q_1^2 - q_2^2 + q_3^2 ; \tag{A7}$$

2. the spur (trace) of the associated matrices is equal to $2q_0$;
3. the conjugated quaternion is associated with the quaternionic matrices,

$$q^*_{\mathrm{SL(2,R)}} = \begin{pmatrix} (q_0 - q_1) & -(q_2 + q_3) \\ -(q_2 - q_3) & (q_0 + q_1) \end{pmatrix}, \qquad q^*_{\mathrm{SU(1,1)}} = \begin{pmatrix} (q_0 + iq_3) & -(q_2 - iq_1) \\ -(q_2 + iq_1) & (q_0 - iq_3) \end{pmatrix}. \tag{A8}$$

Appendix A.1. $SL(2, R)$ and $SU(1, 1)$ Groups

A split quaternion q (with non-zero norm $N_q \neq 0$) can be written in the form:

$$q = N_q \alpha , \tag{A9}$$

where α is the split quaternion with the unit norm $N_\alpha = 1$. Any split quaternion q has (2×2)-matrix representation, thus to α corresponds a matrix with the norm equal to one. On the other hand, it is known that the set of all (2×2) matrices with the unit determinant,

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \alpha^* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad (ad - bc = 1), \tag{A10}$$

forms a group. The parameters a, b, c and d in (A10), in general, can be complex.

Matrix groups consisting of matrices with unit determinant are called special linear, but which particular group obtained depends on the nature of the elements in (A10). When a, b, c, and d – all are real, one gets the $SL(2, R)$ group, while for the complex elements and when $d = a^*$, and $c = b^*$, the $SU(1, 1)$ group is obtained.

The matrices (A2) and (A6) with unit determinant represent the elements of the groups $SL(2, R)$ and $SU(1, 1)$, respectively. However, it is known that algebras of $SU(1, 1)$ and $SL(2, R)$ are isomorphic. Indeed the complex $SU(1, 1)$ matrices (A5) can be obtained from the real $SL(2, R)$ matrices (A1) by the two-side products with

$$\frac{1}{\sqrt{2}}(1 + ie_2) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}(1 - ie_2) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}. \tag{A11}$$

i.e., any quaternion in real $SL(2,R)$-matrix representation can be transformed to the complex $SU(1,1)$ representation,

$$q_{SU(1,1)} = \frac{1}{2}(1+ie_2)q_{SL(2,R)}(1-ie_2)\ . \tag{A12}$$

Note that, in contrast with the ordinary Pauli matrices that are used to represent basis of Hamilton's quaternions, the squares of the $SL(2,R)$ and $SU(1,1)$ matrices (A1) and (A5), give the unit matrix with different signs,

$$e_1^2 = e_2^2 = (1)\ , \qquad e_3^2 = -(1)\ . \tag{A13}$$

Conjugations of the hyper-complex basis elements of split quaternions means Hermitian conjugation of corresponding matrices, i.e.,

$$e_1e_1^* = e_1(-e_1) = -(1) \qquad e_3e_3^* = (1)\ . \tag{A14}$$

Appendix A.2. Complex-like Representation

In the representation used, one of the hyper-complex basis units of split quaternions, e_3, has properties of the complex unit i, in the sense that $e_3^2 = -1$ and $e_3^* = -e_3$. Then one can rewrite elements of the split-quaternion algebra (7) in the forms:

$$q = z_1 + z_2e_2 = z_1 - z_2e_1 \qquad (e_1^2 = e_2^2 = 1), \tag{A15}$$

where

$$z_1 = q_0 + q_3e_3\ , \qquad z_2 = q_2 + q_1e_3 \tag{A16}$$

are 'quaternionic complex numbers'. Using the relations

$$ze_2 = e_2z^*\ , \qquad ze_1 = e_1z^*\ , \tag{A17}$$

for any complex number z, the definition of conjugate split quaternions can be written as follows:

$$q^* = z_1^* + z_2e_2 = z_1^* - z_2e_1\ . \tag{A18}$$

One can correspond to a split quaternion in the representation (A15) the complex matrix

$$q = \begin{pmatrix} z_1 & z_2 \\ z_2^* & z_1^* \end{pmatrix}, \tag{A19}$$

determinant of which is the norm of the split quaternion q,

$$N_q^2 = N_{z_1}^2 - N_{z_2}^2\ . \tag{A20}$$

Then it follows that the set of all split-quaternions,

$$\alpha_n^k = \begin{pmatrix} z_1 & z_2 \\ z_2^* & z_1^* \end{pmatrix} \qquad (n,k=1,2)\ , \tag{A21}$$

with the unit norm, i.e., when the complex numbers z_1 and z_2 in quaternions (A21) are restricted by the unimodularity condition

$$z_1z_1^* - z_2z_2^* = 1\ , \tag{A22}$$

constitute the algebra of non-compact complex group $SU(1,1)$ that is isomorphic to the algebra of the special linear group $SL(2,R)$.

Appendix B. Classification of Split Quaternions

Split quaternions are characterized by pseudo-Euclidean norms (13) and thus can be grouped in various classes.

The two basis units of split quaternions, e_0 and e_3 (and the corresponding real parameters q_0 and q_3) to be called time-like, with the positive norms, $e_0e_0^* = e_3e_3^* = 1$. The remaining two basis units, e_1 and e_2 (and the real parameters q_1 and q_2), are called space-like, since their norms $e_1e_1^* = e_2e_2^* = -1$. Then, one can distinguish the space-like, time-like and light-like split quaternions [51,52]:

$$\begin{aligned} &N^2 < 0 \quad \text{(space-like)},\\ &N^2 > 0 \quad \text{(time-like)},\\ &N^2 = 0 \quad \text{(light-like)} \qquad \left(N^2 = q_0^2 - q_1^2 - q_2^2 + q_3^2\right). \end{aligned} \tag{A23}$$

Space-like and time-like quaternions have multiplicative inverses,

$$q^{-1} = \frac{q^*}{N^2}\,, \tag{A24}$$

with the property:

$$qq^{-1} = q^{-1}q = 1\,, \tag{A25}$$

while light-like quaternions have no inverses.

Appendix B.1. Scalar and Vector Parts

Another useful representation of the split quaternion (7) is:

$$q = S_q + V_q\,, \qquad q^* = S_q - V_q\,, \tag{A26}$$

where the symbols

$$S_q = q_0\,, \qquad V_q = q_1e_1 + q_2e_2 + q_3e_3\,, \tag{A27}$$

are called the scalar and vector parts, respectively. The real part of the quaternion S_q is then the one that is invariant by the action of the quaternionic conjugation, $S_q^* = S_q$, while the vector part V_q is the one that flips the sign under this operation, $V_q^* = -V_q$.

In the representation (A26) the product of q and another split quaternion $p = S_p + V_p$ is given by

$$qp = S_qS_p + (V_q \cdot V_p) + S_qV_p + S_pV_q + V_q \wedge V_p\,, \tag{A28}$$

where

$$\begin{aligned} S_p &= p_0\,,\\ V_p &= p_1e_1 + p_2e_2 + p_3e_3\,,\\ (V_q \cdot V_p) &= q_1p_1 + q_2p_2 - q_3p_3\,,\\ V_q \wedge V_p &= (q_3p_2 - q_2p_3)e_1 + (q_3p_1 - q_1p_3)e_2 + (q_1p_2 - q_2p_1)e_3\,. \end{aligned} \tag{A29}$$

The vector part V_q of any space-like split quaternions is space-like, since in this case

$$V_qV_q^* = -q_1^2 - q_2^2 + q_3^2 < -q_0^2\,. \tag{A30}$$

But the vector part of a time-like quaternion can be space-like, time-like, or light-like, when the quantity

$$V_qV_q^* = -q_1^2 - q_2^2 + q_3^2 > -q_0^2\,. \tag{A31}$$

is negative, positive, or null, respectively.

The set of space-like split quaternions is not a group since it is not closed under multiplication. That is, the product of two space-like quaternions is time-like, whereas the set of time-like quaternions forms a group under the split quaternion product and are useful to represent $SO(2,1)$-rotations of Minkowski 3-space.

Appendix B.2. The Polar Form

The unit split quaternion α can be obtained from any non-lightlike quaternion q by the division on its norm,

$$\alpha = \frac{q}{N} \,. \tag{A32}$$

Using this definition, one can construct polar forms of space-like and time-like split quaternions as the products of their norms and the corresponding unit quaternions [51,52]:

- Every space-like quaternion can be written in the form:

$$q = N(\sinh\theta + \epsilon\cosh\theta) \,, \tag{A33}$$

where

$$\sinh\theta = \frac{q_0}{N} \,, \qquad \cosh\theta = \frac{\sqrt{q_1^2 + q_2^2 - q_3^2}}{N} \,, \qquad \epsilon = \frac{q_1e_1 + q_2e_2 + q_3e_3}{\sqrt{q_1^2 + q_2^2 - q_3^2}} \,, \tag{A34}$$

and ϵ is a unit ($\epsilon^2 = 1$) space-like three vector.
- Every time-like quaternion with the space-like vector part can expressed as

$$q = N\alpha_x = N(\cosh\theta + \epsilon\sinh\theta) \,, \tag{A35}$$

where the functions cosh and sinh have switched places compared to the previous case, and

$$\cosh\theta = \frac{q_0}{N} \,, \qquad \sinh\theta = \frac{\sqrt{q_1^2 + q_2^2 - q_3^2}}{N} \,, \tag{A36}$$

ϵ again is a unit space-like three vector as in Equations (A34).
- Every time-like quaternion with the time-like vector part can be written in the form:

$$q = N\alpha_t = N(\cos\theta + \varepsilon\sin\theta) \,, \tag{A37}$$

where

$$\cos\theta = \frac{q_0}{N} \,, \qquad \sin\theta = \frac{\sqrt{-q_1^2 - q_2^2 + q_3^2}}{N} \,, \qquad \varepsilon = \frac{q_1e_1 + q_2e_2 + q_3e_3}{\sqrt{-q_1^2 - q_2^2 + q_3^2}} \,, \tag{A38}$$

now ε is a unit time-like three vector.

Appendix B.3. Exponential Maps

Using the classification scheme of split quaternions, one can see that any unit split quaternion,

$$\alpha = \alpha_0 + \alpha^k e_k \qquad (N_\alpha = \alpha_0^2 - \alpha_1^2 - \alpha_2^2 + \alpha_3^2 = 1) \,, \tag{A39}$$

has the equivalent exponential representations, which for the case of time-like vector part has the form:

$$\alpha_t = \cos\theta + \varepsilon\sin\theta = e^{\varepsilon\theta} \qquad (\alpha_3^2 > \alpha_1^2 + \alpha_2^2) \,, \tag{A40}$$

where θ is some real 'angle'. Since each component of $e^{\varepsilon\theta}$ is a differentiable function of θ, then

$$\frac{d}{d\theta}e^{\varepsilon\theta} = -\sin\theta + \varepsilon\cos\theta = \varepsilon e^{\varepsilon\theta} = e^{\varepsilon\theta}\varepsilon \,. \tag{A41}$$

For time-like unit split quaternion with the space-like vector part, one gets the other representation:

$$\alpha_x = \cosh\phi + \epsilon \sinh\phi = e^{\epsilon\phi} \qquad (\alpha_3^2 < \alpha_1^2 + \alpha_2^2)\,, \tag{A42}$$

where ϕ corresponds to some 'velocities'. The differential of $e^{\epsilon\phi}$ is

$$\frac{d}{d\phi} e^{\epsilon\phi} = \sinh\phi + \epsilon\cosh\phi = \epsilon e^{\epsilon\phi} = e^{\epsilon\phi}\epsilon\,. \tag{A43}$$

Appendix C. Matrix Representation of (2 + 2)-Rotations

Every point of (2 + 2) space can be represented by coordinates x^a, $a = 0,1,2,3$, with respect of a frame defined by a set of linearly independent unit time-like and space-like basis vectors l_a. Keeping the notation of Equation (14), contravariant and covariant coordinate vectors can be written as row and column matrices,

$$x^a = \begin{pmatrix} \lambda \\ x \\ y \\ t \end{pmatrix}, \qquad l_a = (l_\lambda, l_x, l_y, l_t)\,. \tag{A44}$$

The choice of x^a and l_a is entirely arbitrary and one has to worry about what aspects of physical fields $\phi(x)$ are artifacts of the representation of the coordinate system considered. There is important class of symmetries, forming the group $SO(2,2)$, and coordinate transformations that leave the norm (15) invariant. It is crucial that components of fields $\phi^a(x)$ and the basis vectors l_a in general are not invariant under these transformations. In fact, based on the different behavior of $\phi^a(x)$ under $SO(2,2)$, The physical field can be divided into several types—scalar, spinor and vector (tensor) fields. Specifically, any field (as an element of a linear space with the basis l_a) can be written as

$$\phi(x) = \phi_0 l_\lambda + \phi_1 l_x + \phi_2 l_y + \phi_3 l_t\,. \tag{A45}$$

If a $SO(2,2)$-transformation U of ϕ^a is performed, then the basis l_a should transform inversely by U^{-1}, in order to make invariant the combination

$$\phi = U\phi^a U^{-1} l_a\,. \tag{A46}$$

The metric tensor of the (2 + 2) space (15) is given by

$$\eta_{ab} = \mathrm{diag}(1,-1,-1,1)\,, \tag{A47}$$

and the physics convention for Lie algebras and the exponential mapping can be used. One possible choice of basis for the Lie algebra of $SO(2,2) \cong SO(1,2)\times SO(1,2)$ group,

$$L_{ab} = -L_{ba} \qquad (a,b = 0,1,2,3), \tag{A48}$$

is the vector representation by (4×4) matrices:

$$\begin{aligned}
L_{01} &= i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad
L_{02} = i\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad
L_{03} = i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\
L_{12} &= i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad
L_{13} = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad
L_{23} = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},
\end{aligned} \tag{A49}$$

Consequently, all (2 + 2)-transformations can be written under the form

$$U = \exp\{-\tfrac{i}{2}\epsilon^{ab}L_{ab}\}\ , \tag{A50}$$

where ϵ^{ab} are the elements of a real and antisymmetric matrix.

The vector field A is the field whose field representation is vector representation. Thus, the components transform as

$$U(A^c) = \exp\Big\{-\tfrac{i}{2}\epsilon^{ab}L_{ab}\Big\}_d^{\ c} A^d\ , \tag{A51}$$

The transformation of the basis l_c follows the trivial representation which is defined as

$$U(l_c) = \exp\{-\tfrac{i}{2}\epsilon^{ab}L_{ab}\}_c^{\ d}\ l_d\ . \tag{A52}$$

Therefore,

$$A = U(A^c)\cdot U^{-1}(l_c) = A^c \exp\Big\{-\tfrac{i}{2}\epsilon^{ab}L_{ab}\Big\}_c^{\ d}\cdot\exp\Big\{\tfrac{i}{2}\epsilon^{ab}L_{ab}\Big\}_d^{\ k} l_k = A^c\cdot l_c\ , \tag{A53}$$

where the dot can be "viewed" as the matrix product, while it is known that this interpretation fails for spinors.

Now note that matrices (A49) is not the only "basis" of $SO(2,2)$, generators are also the matrices

$$S_{ab} = \frac{i}{4}[\gamma_a, \gamma_b]\ . \tag{A54}$$

Here, the four γ_a matrices satisfy

$$\{\gamma_a, \gamma_b\} = \gamma_a\gamma_b + \gamma_b\gamma_a = -2\eta_{ab}\ , \tag{A55}$$

where η_{ab} is the metric of the (2 + 2) space (A47). Therefore, along with transformation (A50), the $SO(2,2)$-transformation can be expressed as

$$U = \exp\{-\tfrac{i}{2}\epsilon^{ab}S_{ab}\}\ , \tag{A56}$$

which is called spinor representation of the $SO(2,2)$ group. The spinor field $\psi = \psi^a l_a$ is the field whose components representation is spinor representation,

$$U(\psi^c) = \exp\Big\{-\tfrac{i}{2}\epsilon^{ab}L_{ab}\Big\}\psi^c = \exp\Big\{-\tfrac{i}{2}\epsilon^{ab}S_{ab}\Big\}_d^{\ c}\psi^d\ . \tag{A57}$$

If the (2 + 2)-transformation of the spinor ψ is written explicitly, namely,

$$\psi^c \exp\Big\{\tfrac{i}{2}\epsilon^{ab}S_{ab}\Big\}_c^{\ d}\cdot\exp\Big\{-\tfrac{i}{2}\epsilon^{ab}L_{ab}\Big\}_d^{\ k} l_k = \psi^a\cdot l_a\ , \tag{A58}$$

one can see that the parameters ϵ^{ab} are the same in the two sides of dot, but the "basis" is different. Thus, if the dot is still considered as the matrix product, the equality will no longer hold. Hence, a new definition will be needed in this case, which means that under the new product, there is

$$\exp\Big\{\tfrac{i}{2}\epsilon^{ab}S_{ab}\Big\}\cdot\exp\Big\{-\tfrac{i}{2}\epsilon^{ab}L_{ab}\Big\} \quad\to\quad \text{identity matrix.} \tag{A59}$$

In Fibre Bundle theory this is achieved by introducing an equivalence class.

Quaternionic Basis in Matrix Representation

The interval vector of quaternionic (2 + 2) space (14) is partitioned into the identity matrix and the split quaternionic basis matrices e_k,

$$s = x^a e_a . \qquad (a = \lambda, x, y, t) \tag{A60}$$

The basis vectors of (2 + 2) space have the components:

$$\begin{aligned} e^a &= e^*_a = (1, e_1, e_2, e_3) , \\ e_a &= e^{a*} = (1, -e_1, -e_2, -e_3) , \end{aligned} \tag{A61}$$

and corresponding matrices satisfy the algebra:

$$\begin{aligned} e^a e^{b*} + e^b e^{a*} &= 2\eta^{ab} , \\ e^{a*} e^b + e^{b*} e^a &= 2\eta^{ab} , \end{aligned} \tag{A62}$$

where the metric tensor η^{ab} has the signature (A47).

Then, the (2 + 2)-vectors (A44) can equally well be described by the matrices, for example in $SL(2,R)$-matrix representation (A1),

$$x^a \quad \Leftrightarrow \quad \begin{cases} x = x^a e^*_a = \begin{pmatrix} \lambda + x & y + t \\ y - t & \lambda - x \end{pmatrix} , \\ \bar{x} = x^a e_a = -\begin{pmatrix} -\lambda + x & y + t \\ y - t & -\lambda - x \end{pmatrix} , \end{cases} \tag{A63}$$

thus

$$x\bar{x} = \bar{x}x = x^a x_a (1) . \tag{A64}$$

Here,

$$x^a x_a = (\lambda + x)(\lambda - x) - (y + t)(y - t) = \det x \tag{A65}$$

is the Minkowski square of the 4-vector (A63), with the signature $(+, -, -, +)$.

The Minkowski inner product of two 4-vectors x^a and y^a, corresponding to real 2×2 matrices x and y, can be written as

$$x^a y_a = \frac{1}{2} \mathrm{Tr}(x\bar{y}) , \tag{A66}$$

where the index-lowering operation y_a is accomplished in matrix form by

$$\bar{y} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{A67}$$

where y^T denotes transpose matrix.

Appendix D. Quaternionic Rotations

Let us consider various types of quaternionic rotations in the (2 + 2) space of a split quaternions.

Appendix D.1. Compact Rotations

Suppose a split quaternion q is rotated around the time-like e_3-axis in space-like plain defined by e_1 and e_2. Rotations are described by a time-like unit quaternion with time-like vector part (which is analogous to ordinary complex numbers),

$$\alpha_t = \cos\frac{\theta}{2} + \sin\frac{\theta}{2} e_3 \qquad (N_\alpha = 1), \tag{A68}$$

and the spherical compact rotation angle θ. The result of the left multiplication,

$$q^L = \alpha_t q = \left(\cos\frac{\theta}{2}q_0 - \sin\frac{\theta}{2}q_3\right) + \left(\cos\frac{\theta}{2}q_1 + \sin\frac{\theta}{2}q_2\right)e_1 + \\ + \left(\cos\frac{\theta}{2}q_2 - \sin\frac{\theta}{2}q_1\right)e_2 + \left(\cos\frac{\theta}{2}q_3 + \sin\frac{\theta}{2}q_0\right)e_3\,, \tag{A69}$$

gives simultaneous rotations in $(q_0 - q_3)$ and $(q_1 - q_2)$ planes by the same angle $\theta/2$. Note that the left products of q by a conjugate unit quaternions α^* change the direction of rotations,

$$\alpha_t^*(\theta)q = \alpha_t(-\theta)q\,. \tag{A70}$$

So, the single product on a unit quaternion α always rotates q by the same angle in two independent planes at once. Moreover, one of the two planes always include the axis of unit element q_0. This is not what we want for three-dimensional vector rotation; we need to be able to rotate in just one plane. It turns out that swapping the order of multiplication of two quaternions, i.e., the right product,

$$q^R = q\alpha_t = \left(\cos\frac{\theta}{2}q_0 - \sin\frac{\theta}{2}q_3\right) + \left(\cos\frac{\theta}{2}q_1 - \sin\frac{\theta}{2}q_2\right)e_1 + \\ + \left(\cos\frac{\theta}{2}q_2 + \sin\frac{\theta}{2}q_1\right)e_2 + \left(\cos\frac{\theta}{2}q_3 + \sin\frac{\theta}{2}q_0\right)e_3\,, \tag{A71}$$

will reverse the direction of rotation in one of the two planes—namely in $(q_1 - q_2)$ plane that does not contain the axis of unit element q_0. At the same time,

$$q\alpha_t^*(\theta) = q\alpha_t(-\theta)\,. \tag{A72}$$

Thus, to get a rotation of vector part of split quaternion that only affects a single plane, one has to apply the quaternion twice, multiplying on both the left and on the right (with its inverse), as in transformation (17). The rotation we want gets done twice (and the one we do not want gets canceled out), so one has to halve the angle θ going in to make up for it. Finally, for rotation of vector split quaternion:

$$\begin{aligned}\alpha_t\, q\, \alpha_t^* &= \cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}q_0 - \sin\frac{\theta}{2}q_3\right) + \sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}q_3 + \sin\frac{\theta}{2}q_0\right) \\ &+ \left[\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}q_1 + \sin\frac{\theta}{2}q_2\right) + \sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}q_2 - \sin\frac{\theta}{2}q_1\right)\right]e_1 \\ &+ \left[\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}q_2 - \sin\frac{\theta}{2}q_1\right) - \sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}q_1 + \sin\frac{\theta}{2}q_2\right)\right]e_2 \\ &+ \left[\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}q_3 + \sin\frac{\theta}{2}q_0\right) - \sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}q_0 - \sin\frac{\theta}{2}q_3\right)\right]e_3 \\ &= q_0 + (\cos\theta q_1 + \sin\theta q_2)e_1 + (\cos\theta q_1 - \sin\theta q_2)e_2 + q_3 e_3\,,\end{aligned} \tag{A73}$$

Appendix D.2. Boosts

Now let us consider the boosts along the space-like axes e_1 and e_2 represented by the unit quaternions α_y and α_x, respectively, which can be obtained from the definition (A68) by replacing the basis element e_3 with e_1 or e_2 and θ with a hyperbolic angle ϕ. For example, rotations of q around the space-like axis e_2 in the $(e_1 - e_3)$-plain, or boosts along e_1 by the 'velocity' v, can be done using the time-like unit split quaternion with the space-like vector part,

$$\alpha_y = \cosh\frac{\phi}{2} - \sinh\frac{\phi}{2}e_2\,. \tag{A74}$$

Here, the hyperbolic angle ϕ relates to the velocity by the standard relativistic expressions,

$$\cosh\phi = \frac{1}{\sqrt{1-v^2}}\,, \qquad \sinh\phi = vs.\cosh\phi\,, \tag{A75}$$

where the units $c = 1$ is used. Left transformation of q by α_y has the form:

$$q' = \alpha_y q = q_0' + q_1' e_1 + q_2' e_2 + q_3' e_3\,, \tag{A76}$$

where

$$\begin{aligned}
q_0' &= \cosh\frac{\phi}{2} q_0 - \sinh\frac{\phi}{2} q_2\,,\\
q_1' &= \cosh\frac{\phi}{2} q_1 + \sinh\frac{\phi}{2} q_3\,,\\
q_2' &= \cosh\frac{\phi}{2} q_2 - \sinh\frac{\phi}{2} q_0\,,\\
q_3' &= \cosh\frac{\phi}{2} q_3 + \sinh\frac{\phi}{2} q_1\,.
\end{aligned} \tag{A77}$$

For the right boosts, one has:

$$q'' = q\alpha_y = q_0'' + q_1'' e_1 + q_2'' e_2 + q_3'' e_3\,, \tag{A78}$$

where

$$\begin{aligned}
q_0'' &= \cosh\frac{\phi}{2} q_0 - \sinh\frac{\phi}{2} q_2\,,\\
q_1'' &= \cosh\frac{\phi}{2} q_1 - \sinh\frac{\phi}{2} q_3\,,\\
q_2'' &= \cosh\frac{\phi}{2} q_2 - \sinh\frac{\phi}{2} q_0\,,\\
q_3'' &= \cosh\frac{\phi}{2} q_3 - \sinh\frac{\phi}{2} q_1\,.
\end{aligned} \tag{A79}$$

Acting from both side on q by α_y and α_y^*, the hyperbolic rotations in $(q_0 - q_2)$-plane cancels out and we left with the one parameter (the hyperbolic angle ϕ) group of boosts in $(q_3 - q_1)$-plane.

For completeness note that, analogous to the case of rotations (A70) and (A72), for boosts there exist the relations

$$\alpha_y^*(v) q = \alpha_y(-v) q\,, \qquad q\alpha_y(v) = q\alpha_y^*(-v)\,. \tag{A80}$$

Appendix D.3. Boosts by Extra 'Time'

Above, it was shown that the $\alpha q \alpha^*$-type two-side products by unit quaternions α and their conjugates α^* represent rotations of the vector parts of quaternions (which not affect the unit element q_0). These transformations form automorphism group of quaternions, which for the case of split quaternions is $SO(2,1)$, the sub-group of $SO(2,2) \cong SO(2,1) \times SO(2,1)$.

The second $SO(2,1)$ subgroup of $SO(2,2)$, which mix the real time-like axis q_0 with the other three axes e_1, e_2 and e_3, can be represented by two-side products with the same unit quaternion, i.e., $\alpha q \alpha$. Indeed, the two-side product of q by the unit quaternion (A68) rotates into each other the two time-like axes q_0 and q_3:

$$\begin{aligned}
\alpha_t q \alpha_t = {} & \cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}q_0 - \sin\frac{\theta}{2}q_3\right) - \sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}q_3 + \sin\frac{\theta}{2}q_0\right) \\
& + \left[\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}q_1 + \sin\frac{\theta}{2}q_2\right) - \sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}q_2 - \sin\frac{\theta}{2}q_1\right)\right]e_1 \\
& + \left[\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}q_2 - \sin\frac{\theta}{2}q_1\right) + \sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}q_1 + \sin\frac{\theta}{2}q_2\right)\right]e_2 \\
& + \left[\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}q_3 + \sin\frac{\theta}{2}q_0\right) + \sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}q_0 - \sin\frac{\theta}{2}q_3\right)\right]e_3 \\
= {} & (\cos\theta q_0 - \sin\theta q_3) + q_1 e_1 + q_2 e_2 + (\cos\theta q_3 + \sin\theta q_0)e_3\,,
\end{aligned} \tag{A81}$$

Analogous relations are valid for boosts along the axes e_1 and e_2.

Appendix E. Decomposition of Split Quaternions

In order to define quaternionic spinors, one needs to decompose a split quaternions (7), which in general contain four distinct real parameters q_0 and q_k. To do this, at first let us introduce special elements of split quaternions, called zero divisors.

In split algebras the special singular objects, zero divisors, can be constructed [9]. These critical elements of the algebra, which are similar to light-cone variables in Minkowski space-time, could serve as the unit signals characterizing physical events. The norms of split quaternions (13) have (2 + 2)-signature and, thus, one gets two different types of 'light-cones'. Correspondingly, two types of zero divisors, idempotent elements (projection operators) and nilpotent elements (Grassmann numbers) [9].

Appendix E.1. Idempotents

At first let us consider the idempotent quaternions, which have the property that they coincide with their squares. A non-zero split quaternion D fulfills this condition if and only if $N_D = 0$ and $q_0 = 1/2$. In the algebra there exist two classes (totally four) primitive idempotents,

$$D_1^{\pm} = \frac{1}{2}(1 \pm e_1)\,, \qquad D_2^{\pm} = \frac{1}{2}(1 \pm e_2)\,. \tag{A82}$$

These two classes do not commute with each other. The commuting ones with the standard properties of projection operators:

$$D^{\pm}D^{\mp} = 0\,, \qquad D^{\pm}D^{\pm} = D^{\pm}\,, \tag{A83}$$

are only the pairs (D_1^+, D_1^-), or (D_2^+, D_2^-).

The operators $D_{1,2}^+$ and $D_{1,2}^-$ differ from each other by the reflection of hyper-complex basis element and thus correspond to the direct and reverse critical signals along one of the two real directions, e_1 or e_2, and turn into each other by quaternionic conjugations,

$$(D_1^{\pm})^* = \frac{1}{2}(1 \mp e_1) = D_1^{\mp}\,, \qquad (D_2^{\pm})^* = \frac{1}{2}(1 \mp e_2) = D_2^{\mp}\,. \tag{A84}$$

The conjugations satisfy the following conditions:

$$D_1^+ + D_1^- = D_2^+ + D_2^- = 1\,. \tag{A85}$$

One can characterize D_1 and D_2 as primitive idempotent quaternions, since, in contrast to 1, may no longer be decomposed into the sum of idempotents and non-zero quaternions. One can show that D_1^- and D_2^- are the only primitive quaternion that is independent of D_1^+ and D_2^+, respectively.

Appendix E.2. Nilpotents

In the algebra of split quaternions, there are also two classes (totally four) of primitive nilpotents,

$$G_1^{\pm} = \frac{1}{2}(e_2 \pm e_3) = \frac{1}{2}(1 \pm e_1)e_2\,, \qquad G_2^{\pm} = \frac{1}{2}(e_1 \mp e_3) = \frac{1}{2}(1 \pm e_2)e_1\,, \tag{A86}$$

with the properties

$$\begin{aligned} G^{\pm}G^{\mp} = D^{\pm}\,, \qquad D^{\pm}G^{\pm} = G^{\pm}D^{\mp} = G^{\pm}\,, \\ G^{\pm}G^{\pm} = D^{\pm}G^{\mp} = G^{\pm}D^{\pm} = 0\,, \end{aligned} \tag{A87}$$

for each index, 1 or 2. Quaternionic conjugations of the nilpotents give the same element with the opposite signs,

$$(G_1^{\pm})^* = -\frac{1}{2}(e_2 \pm e_3) = -G_1^{\pm}\,, \qquad (G_2^{\pm})^* = -\frac{1}{2}(e_1 \mp e_2) = -G_2^{\pm}\,. \tag{A88}$$

Appendix E.3. Matrix Representation of Zero Divisors

By means of commuting zero divisors any split quaternion q can be written in the form:

$$\begin{aligned} q &= \big[(q_0 + q_1) + G_1^{+}(q_2 + q_3)\big]D_1^{+} + \big[(q_0 - q_1) + G_1^{-}(q_2 - q_3)\big]D_1^{-} \\ &= \big[(q_0 + q_2) + G_2^{+}(q_1 + q_3)\big]D_2^{+} + \big[(q_0 - q_2) + G_2^{-}(q_1 - q_3)\big]D_2^{-}\,. \end{aligned} \tag{A89}$$

Using the matrix representation of quaternionic units, (A1) and (A5), one can find the matrix form of idempotents and nilpotents. For the case of real $SL(2,R)$-matrix representation (A1), idempotents and nilpotents (A82) and (A86) labeled by 1 are:

$$\begin{aligned} D_1^{+} &= \frac{1}{2}(1 + e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad D_1^{-} = \frac{1}{2}(1 - e_1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\ G_1^{+} &= \frac{1}{2}(e_2 + e_3) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad G_1^{-} = \frac{1}{2}(e_2 - e_3) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{A90}$$

Similar $SL(2,R)$-matrix representation of zero divisors labeled by the second index 2 has the form:

$$\begin{aligned} D_2^{+} &= \frac{1}{2}(1 + e_2) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad D_2^{-} = \frac{1}{2}(1 - e_2) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \\ G_2^{+} &= \frac{1}{2}(e_1 + e_3) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \qquad G_2^{-} = \frac{1}{2}(e_1 - e_3) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}. \end{aligned} \tag{A91}$$

In the $SU(1,1)$-matrix representation (A5) the zero divisors (A82) and (A86) labeled by 1 are:

$$\begin{aligned} D_1^{+} &= \frac{1}{2}(1 + e_1) = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \qquad D_1^{-} = \frac{1}{2}(1 - e_1) = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \\ G_1^{+} &= \frac{1}{2}(e_2 + e_3) = \frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}, \qquad G_1^{-} = \frac{1}{2}(e_2 - e_3) = \frac{1}{2}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}. \end{aligned} \tag{A92}$$

Similar $SU(1,1)$-matrix representation of zero divisors labeled by the index 2 has the form:

$$
\begin{aligned}
D_2^+ &= \frac{1}{2}(1+e_2) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad D_2^- = \frac{1}{2}(1-e_2) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \\
G_2^+ &= \frac{1}{2}(e_1+e_3) = \frac{i}{2}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad G_2^- = \frac{1}{2}(e_1-e_3) = \frac{i}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.
\end{aligned} \tag{A93}
$$

Note that a matrix D represents a primitive idempotent quaternion if and only if its determinant is zero and the trace equals 1,

$$
\det D = 0\,, \qquad \mathrm{Tr} D = 1\,, \tag{A94}
$$

while for the primitive nilpotent quaternion matrix G, both the determinant and the trace should be zero,

$$
\det G = \mathrm{Tr} G = 0\,. \tag{A95}
$$

Appendix E.4. Left Decomposition

Using primitive idempotent quaternions $D^{\pm}$, where, for simplicity, the index is omitted, one can define two left-invariant and two right-invariant subalgebras of split quaternions. Most obvious is to do this in matrix representation. Then spinors can be represented as ideal elements of quaternions, what is almost identical to their representation as column matrices. The correspondence of a quaternion with a column spinor is most easily accomplished by means of the following algorithm: express quaternion in matrix form, then column spinor is given by its product with idempotent element.

Let us as an example consider decomposition of a split quaternion q by the idempotent $D_1^{\pm}$ (A90) in $SL(2,R)$-matrix representation (A2). Corresponding complex $SU(1,1)$-matrix representation can be obtained using the transformations (A12). Equation (A85) allows one to write:

$$
\begin{aligned}
q &= q(D^+ + D^-) = qD^+ + qD^- = \mathbf{q}^+ + \mathbf{q}^- \\
&= \begin{pmatrix} (q_0+q_1) & (q_2+q_3) \\ (q_2-q_3) & (q_0-q_1) \end{pmatrix}\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] \\
&= \begin{pmatrix} (q_0+q_1) & 0 \\ (q_2-q_3) & 0 \end{pmatrix} + \begin{pmatrix} 0 & (q_2+q_3) \\ 0 & (q_0-q_1) \end{pmatrix} \\
&= \frac{1}{2}[(q_0+q_1)+(q_2-q_3)e_2](1+e_1) + \frac{1}{2}[(q_0-q_1)+(q_2+q_3)e_2](1-e_1)\,.
\end{aligned} \tag{A96}
$$

The split quaternions $\mathbf{q}^+$ and $\mathbf{q}^-$ fulfill the equations:

$$
\begin{aligned}
\mathbf{q}^+D^+ &= \begin{pmatrix} (q_0+q_1) & 0 \\ (q_2-q_3) & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (q_0+q_1) & 0 \\ (q_2-q_3) & 0 \end{pmatrix} = \mathbf{q}^+\,, \qquad \mathbf{q}^+D^- = 0\,, \\
\mathbf{q}^-D^- &= \begin{pmatrix} 0 & (q_2+q_3) \\ 0 & (q_0-q_1) \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & (q_2+q_3) \\ 0 & (q_0-q_1) \end{pmatrix} = \mathbf{q}^-\,, \qquad \mathbf{q}^-D^+ = 0\,,
\end{aligned} \tag{A97}
$$

which define two left-invariant subalgebras, left ideals inside of the algebra of split quaternions. Since, any split quaternion q can be uniquely decomposed into quaternions that belongs to left ideals the algebra of split quaternions is the sum of the two left-invariant subalgebras.

Appendix E.5. Right Decomposition

By means of the idempotent quantity (A90) one can also perform the right decomposition of an arbitrary split quaternion q, namely:

$$\begin{aligned} q &= (D^+ + D^-)q = D^+q + D^-q = {}^+\mathbf{q} + {}^-\mathbf{q} \\ &= \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} (q_0+q_1) & (q_2+q_3) \\ (q_2-q_3) & (q_0-q_1) \end{pmatrix} \\ &= \begin{pmatrix} (q_0+q_1) & (q_2+q_3) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ (q_2-q_3) & (q_0-q_1) \end{pmatrix} \\ &= \frac{1}{2}(1+e_1)[(q_0+q_1)+(q_2+q_3)e_2] + \frac{1}{2}(1-e_1)[(q_0-q_1)+(q_2-q_3)e_2]\,. \end{aligned} \tag{A98}$$

Here, the split quaternions ${}^+\mathbf{q}$ and ${}^-\mathbf{q}$ fulfill the equations:

$$\begin{aligned} D^+\cdot{}^+\mathbf{q} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} (q_0+q_1) & (q_2+q_3) \\ 0 & 0 \end{pmatrix} = {}^+\mathbf{q}\,, \qquad D^-\cdot{}^+\mathbf{q} = 0\,, \\ D^-\cdot{}^-\mathbf{q} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ (q_2-q_3) & (q_0-q_1) \end{pmatrix} = {}^-\mathbf{q}\,, \qquad D^+\cdot{}^-\mathbf{q} = 0\,, \end{aligned} \tag{A99}$$

which define two right-invariant subalgebras, right ideals inside of the algebra of split quaternions.

Appendix F. Quaternionic Spinors

The study of spinors is of great physical relevance, since fermionic fields are represented by spinorial fields or by tensor products of spinor fields. Therefore, spinors play a central role in the theory of particles and fields. Particularly, in supersymmetric theories the parameters that label the supersymmetry transformations are always given by spinors. In addition, spinors are of great geometrical relevance, since they carry the fundamental representation of the space-time group and can be used to build all other representations of this group. In this sense, spinors are the most fundamental objects of a space endowed with a metric.

Spinors are numbers which represents a rotation and cannot be defined without reference to vectors, since commonly definition of spinor involves the idea of a 2π rotation resulting in some sort of an inversion, so that a 4π rotation is needed to recover identity. This property of spinors explains the half-angle form of the two side vector-type transformations, $\alpha q\alpha^*$, which is generated by the spinorial one-side products with unit half-angle split quaternions $\alpha(\theta/2)$. Indeed, the definition of quaternionic vector transformations (17) suggests the introduction of two types of split quaternions, which transform according to the following schema:

$$\xi' = \alpha\xi\,, \qquad \xi'^* = \xi^*\alpha^*\,, \tag{A100}$$

and

$$\chi' = \chi\alpha^*\,, \qquad \chi'^* = \alpha\chi^*\,. \tag{A101}$$

Split quaternions that transforms according to schema (A100) and (A101) will be called spinors of the first and second kind, respectively. From transformations (A100) and (A101), one can see that for the case of the real quaternions, transformation laws of the conjugated spinor split quaternions are identical with the laws for the dual split quaternions,

$$\xi'^* \quad \Leftrightarrow \quad \chi'\,, \qquad \chi'^* \quad \Leftrightarrow \quad \xi'\,. \tag{A102}$$

Then, the 4-component quantity,

$$\Psi = \begin{pmatrix} \xi \\ \chi \end{pmatrix}, \tag{A103}$$

will be the Majorana spinor in (2 + 1) space. Thus, quaternions are able to give spinor representation of the $SO(2,2)$ group (and its subgroup $SO(2,1)$) by one-side products, which transform all four components of split quaternions.

The transformations (A100) and (A101) are orthogonal, since α are unit split quaternions. The norms of the split quaternion of the first and second kinds, $\xi^*\xi$ and $\chi^*\chi$, are then invariants under $SO(2,2)$. Furthermore, for the spinor quaternions of the first and second kinds, the product $\chi\xi$ is an invariant,

$$(\chi\xi)' = \chi\,\alpha^*\alpha\,\xi = \chi\xi\,, \tag{A104}$$

while $\xi\chi$ (observe the order of the terms) transforms like a vector quaternion,

$$(\xi\chi)' = \alpha(\xi\chi)\alpha^*\,. \tag{A105}$$

Appendix F.1. Matrix Representation

To obtain matrix representation of the covariant quaternionic spinors ξ let us perform the left decomposition of a split quaternion q in the $SL(2,R)$-matrix representation (A3) by idempotents (A96):

$$q = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}(1+e_1) + \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}(1-e_1) = \xi^+ + \xi^-\,, \tag{A106}$$

where

$$\begin{aligned} a_1 &= \frac{1}{2}(q_0+q_1)\,, \qquad & a_2 &= \frac{1}{2}(q_2-q_3)\,, \\ b_1 &= \frac{1}{2}(q_2+q_3)\,, \qquad & b_2 &= \frac{1}{2}(q_0-q_1)\,, \end{aligned} \tag{A107}$$

are some real parameters and

$$\xi^+ = \begin{pmatrix} a_1 & 0 \\ a_2 & 0 \end{pmatrix}, \qquad \xi^- = \begin{pmatrix} 0 & b_1 \\ 0 & b_2 \end{pmatrix} \tag{A108}$$

are the elements of the two left ideals of the algebra of split quaternions.

To define the transformation properties of (A108) note that the unit split quaternions, which are used to represent rotations, themselves are $SL(2,R)$ matrices with the determinant equal to one,

$$\alpha = \begin{pmatrix} (\alpha_0+\alpha_1) & (\alpha_2+\alpha_3) \\ (\alpha_2-\alpha_3) & (\alpha_0-\alpha_1) \end{pmatrix} \qquad (\alpha_0^2-\alpha_1^2-\alpha_2^2+\alpha_3^2=1). \tag{A109}$$

Then, any left product of ξ^+ by a unit split quaternion α,

$$\alpha\xi^+ = \xi'^+ = \begin{pmatrix} a'_1 & 0 \\ a'_2 & 0 \end{pmatrix} = \begin{pmatrix} (\alpha_0+\alpha_1)a_1 + (\alpha_2+\alpha_3)a_2 & 0 \\ (\alpha_2-\alpha_3)a_1 + (\alpha_0-\alpha_1)a_2 & 0 \end{pmatrix}, \tag{A110}$$

generates the covariant spinor-type transformations

$$\begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} = \alpha\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} (\alpha_0+\alpha_1)a_1 + (\alpha_2+\alpha_3)a_2 \\ (\alpha_2-\alpha_3)a_1 + (\alpha_0-\alpha_1)a_2 \end{pmatrix}, \tag{A111}$$

or

$$\xi'^+_A = \alpha_A^{\ B}\xi^+_B\,, \qquad (A,B=1,2) \tag{A112}$$

where the spinorial indices, $A, B = 1, 2$, are introduced.

Similarly, for the second covariant spinor-like object $\mathbf{q}^-$:

$$\begin{pmatrix} b'_1 \\ b'_2 \end{pmatrix} = \alpha \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} (\alpha_0+\alpha_1)b_1 + (\alpha_2+\alpha_3)b_2 \\ (\alpha_2-\alpha_3)b_1 + (\alpha_0-\alpha_1)b_2 \end{pmatrix}, \tag{A113}$$

and

$$\xi'^{-}_A = \alpha_A^{\ B} \xi^-_B \,. \tag{A114}$$

Thus, the general spinor split quaternion of the first kind ξ, which transforms according to the rule (A100), depends on four real parameters and has the matrix representation:

$$\xi = \xi^+ + \xi^- = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}. \tag{A115}$$

By means of idempotent quantities $D^\pm$, one can also perform the right decomposition of an arbitrary split quaternion q:

$$q = (1+e_1)\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} + (1-e_1)\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \chi^+ + \chi^- \,, \tag{A116}$$

where

$$\chi^+ = \begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \qquad \chi^- = \begin{pmatrix} 0 & 0 \\ a_2 & b_2 \end{pmatrix}. \tag{A117}$$

Then, the transformation of χ^+ by the inverse unit matrix,

$$\alpha^{-1} = \begin{pmatrix} (\alpha_0+\alpha_1) & (\alpha_2+\alpha_3) \\ (\alpha_2-\alpha_3) & (\alpha_0-\alpha_1) \end{pmatrix}^{-1} = \alpha^* = \begin{pmatrix} (\alpha_0-\alpha_1) & -(\alpha_2+\alpha_3) \\ -(\alpha_2-\alpha_3) & (\alpha_0+\alpha_1) \end{pmatrix}, \tag{A118}$$

takes the form:

$$\chi'^+ = \begin{pmatrix} a'_1 & b'_1 \\ 0 & 0 \end{pmatrix} = \chi^+ \alpha^* = \begin{pmatrix} (\alpha_0-\alpha_1)a_1 - (\alpha_2-\alpha_3)b_1 & -(\alpha_2+\alpha_3)a_1 + (\alpha_0+\alpha_1)b_1 \\ 0 & 0 \end{pmatrix}, \tag{A119}$$

which is equivalent to the contravariant spinor-type transformations,

$$\chi'^{+A} = \begin{pmatrix} a'_1 \\ b'_1 \end{pmatrix} = \alpha^* \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} (\alpha_0-\alpha_1)a_1 - (\alpha_2-\alpha_3)b_1 \\ -(\alpha_2+\alpha_3)a_1 + (\alpha_0+\alpha_1)b_1 \end{pmatrix} = (\alpha^*)_B^{\ A} \chi^{+B} \,. \tag{A120}$$

Analogously for the second contravariant spinor χ^-:

$$\chi'^{-A} = \begin{pmatrix} a'_2 \\ b'_2 \end{pmatrix} = \begin{pmatrix} (\alpha_0-\alpha_1)a_2 - (\alpha_2-\alpha_3)b_2 \\ -(\alpha_2+\alpha_3)a_2 + (\alpha_0+\alpha_1)b_2 \end{pmatrix} = (\alpha^*)_B^{\ A} \chi^{-B} \,. \tag{A121}$$

Thus, the general spinor split quaternion of the second kind χ, which transforms according the rule (A101), also depends on four real parameters and has the matrix representation:

$$\chi = \chi^+ + \chi^- = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix}. \tag{A122}$$

Since the unit split quaternion α^* is the inverse of α:

$$\alpha_A^{\ B} (\alpha^*)_B^{\ C} = \delta_A^{\ C} \,, \tag{A123}$$

the contraction $\chi^A \xi_A$ is an invariant,

$$\chi^A \xi_A = \chi^C (\alpha^*)_C^{\ A} \alpha_A^{\ D} \xi_D = \chi^C \xi_C \,. \qquad (A,B,C,D = 1,2) \tag{A124}$$

The rules for raising and lowering spinor indices are:

$$\xi^A = \varepsilon^{AB}\xi_B\,, \qquad \xi_A = \varepsilon_{AB}\xi^B\,, \tag{A125}$$

where the 'metric spinor'

$$\varepsilon^{AB} = \varepsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \varepsilon^{AC}\varepsilon_{BC} = \delta_B^A\,, \tag{A126}$$

is left invariant by spin transformations.

Appendix F.2. Quaternions in Cone Basis

Using idempotent and nilpotent elements, the general vector-type or spinor-type split quaternion (7) can be written in the form:

$$q = q_0 + q^n e_n = a_1 D^+ + a_2 G^- + b_1 G^+ + b_2 D^-\,, \tag{A127}$$

where the real parameters $a_{1,2}$ and $b_{1,2}$ were introduced in Equations (A107). Then, the covariant quaternionic spinors have the general structure,

$$\begin{aligned} \xi^+ &= a_1 D^+ + a_2 G^- \qquad (q_0 = q_1,\ q_2 = -q_3)\,, \\ \xi^- &= b_1 D^- + b_2 G^+ \qquad (q_0 = -q_1,\ q_2 = q_3)\,, \end{aligned} \tag{A128}$$

while the contravariant quaternionic spinors are:

$$\begin{aligned} \chi^+ &= c_1 D^- + c_2 G^- \qquad (q_0 = -q_1,\ q_2 = -q_3)\,, \\ \chi^- &= d_1 D^+ + d_2 G^+ \qquad (q_0 = q_1,\ q_2 = q_3)\,. \end{aligned} \tag{A129}$$

One can check that products of only covariant or only contravariant quaternionic spinors, (A128) and (A129), do not give a vector-type quaternion (A127), for example,

$$\begin{aligned} \xi^+\xi^- &= (a_1 D^+ + a_2 G^-)(b_1 D^- + b_2 G^+) = a_2 b_2 D^- + a_1 b_2 G^+ = \xi'^-\,, \\ \chi^+\chi^- &= (c_1 D^- + c_2 G^-)(d_1 D^+ + d_2 G^+) = c_2 d_2 D^- + c_2 d_1 G^- = \chi'^+\,. \end{aligned} \tag{A130}$$

At the same time the left products of covariant quaternionic spinors on contravariant spinors with different labels,

$$\begin{aligned} \xi^+\chi^- &= (a_1 D^+ + a_2 G^-)(d_1 D^+ + d_2 G^+) = a_1 d_1 D^+ + a_2 d_1 G^- + a_1 d_2 G^+ + a_2 d_2 D^-\,, \\ \xi^-\chi^+ &= (b_1 D^- + b_2 G^+)(c_1 D^- + c_2 G^-) = b_2 c_2 D^+ + b_1 c_2 G^- + b_2 c_1 G^+ + b_1 c_1 D^-\,, \end{aligned} \tag{A131}$$

lead to the structures of the vector split quaternion (A127), while the left products of covariant spinors on contravariant quaternionic spinors with same labels are zero,

$$\xi^+\chi^+ = \xi^-\chi^- = 0\,. \tag{A132}$$

The first and second kind quaternionic spinors, ξ and χ, contain the same number of parameters (four) than that of a quaternionic vector q. Then, depended on the transformation properties, the four real parameters, q_0 and q_k, may form a vector or spinors. Transformations of these quantities may be effected by using of the unit quaternion multiplications α, which for the case of the vector transformations (by means of two side half-angle products) naturally separates $SO(2+1)$ subgroup of $SO(2,2)$, rotations of only vector part of the split quaternion (7).

Appendix F.3. Spinor Basis

Now, let us find spinorial representations of the vector-like basis units of split quaternions. The $SL(2,R)$-matrix representations of the idempotent and nilpotent elements of split quaternions (A90) labeled by the index 1 can be written as a direct (tensor) product of the covariant $\xi^{\pm}$ (columns) and contravariant $\chi^{\pm}$ (rows) unit spinors,

$$\begin{aligned} \xi^{+} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi^{+} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \\ \xi^{-} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \chi^{-} = \begin{pmatrix} 0 & 1 \end{pmatrix}, \end{aligned} \tag{A133}$$

which constituting a normalized bi-orthogonal basis satisfying the conditions,

$$\chi^{\pm}\xi^{\pm} = 1\,, \qquad \chi^{\pm}\xi^{\mp} = 0\,, \tag{A134}$$

in the form:

$$\begin{aligned} D^{+} &= \frac{1}{2}(1+e_1) = \xi^{+}\chi^{+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ D^{-} &= \frac{1}{2}(1-e_1) = \xi^{-}\chi^{-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\ G^{+} &= \frac{1}{2}(e_2+e_3) = \xi^{+}\chi^{-} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ G^{-} &= \frac{1}{2}(e_2-e_3) = \xi^{-}\chi^{+} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{A135}$$

Using representations (A135), it is straight to verify that one couple of spinors is a sufficient basis for construction of the complete set of quaternionic hyper-complex units, which in $SL(2,R)$-matrix representation have the forms:

$$\begin{aligned} 1 &= \xi^{+}\chi^{+} + \xi^{-}\chi^{-}\,, \qquad e_1 = \xi^{+}\chi^{+} - \xi^{-}\chi^{-}\,, \\ e_2 &= \xi^{+}\chi^{-} + \xi^{-}\chi^{+}\,, \qquad e_3 = \xi^{+}\chi^{-} - \xi^{-}\chi^{+}\,. \end{aligned} \tag{A136}$$

To compare with the standard complex spinors, one finds, using the transformation law (A12), the $SU(1,1)$-matrix representation of basis elements by the spinors (A133),

$$\begin{aligned} 1 &= \xi^{+}\chi^{+} + \xi^{-}\chi^{-}\,, \qquad e_1 = i\xi^{+}\chi^{-} - i\xi^{-}\chi^{+}\,, \\ e_2 &= \xi^{+}\chi^{-} + \xi^{-}\chi^{+}\,, \qquad e_3 = i\xi^{+}\chi^{+} - i\xi^{-}\chi^{-}\,. \end{aligned} \tag{A137}$$

However, arbitrary complex $SU(1,1)$-transformation matrices are thyself quaternions,

$$\begin{aligned} \alpha &= \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2^* & \alpha_1^* \end{pmatrix} \qquad (\alpha_1\alpha_1^* - \alpha_2\alpha_2^* = 1)\,, \\ \alpha^{-1} &= \begin{pmatrix} \alpha_1^* & -\alpha_2 \\ -\alpha_2^* & \alpha_1 \end{pmatrix} \qquad (\alpha_1 = q_0 - iq_3,\ \alpha_2 = q_2 - iq_1)\,, \end{aligned} \tag{A138}$$

and can be written in terms of the spinors $\xi^{\pm}$ and $\chi^{\pm}$. The transformations of covariant spinors $\xi^{\pm}$ acquire the form:

$$\begin{aligned} \xi'^{+} &= \alpha\xi^{+} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \alpha_1\xi^{+} + \alpha_2\xi^{-}\,, \\ \xi'^{-} &= \alpha\xi^{-} = \begin{pmatrix} \alpha_2^* \\ \alpha_1^* \end{pmatrix} = \alpha_2^*\xi^{+} + \alpha_1^*\xi^{-}\,. \end{aligned} \tag{A139}$$

Meantime, for the transformations of contravariant spinors $\chi^{\pm}$, one has:

$$\begin{aligned} \chi'^{+} &= \chi^{+}\alpha^{-1} = \begin{pmatrix} \alpha_1^* & -\alpha_2 \end{pmatrix} = \alpha_1^*\chi^{+} - \alpha_2\chi^{-} \,, \\ \chi'^{-} &= \chi^{-}\alpha^{-1} = \begin{pmatrix} -\alpha_2^* & \alpha_1 \end{pmatrix} = -\alpha_2^*\chi^{+} + \alpha_1\chi^{-} \,. \end{aligned} \tag{A140}$$

For example, consider the vector-type rotations around the time-like axis e_3 by the unit split quaternion,

$$\alpha_t = \cos\frac{\theta}{2} + e_3\sin\frac{\theta}{2}\,, \tag{A141}$$

for which

$$q_0 = \cos\frac{\theta}{2}\,, \qquad q_1 = q_2 = 0\,, \qquad q_3 = \sin\frac{\theta}{2}\,, \tag{A142}$$

or

$$\alpha_1 = \cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\,, \qquad \alpha_2 = 0\,. \tag{A143}$$

One can see that the vector transformations by quaternion (A141) leave the vector e_3 unchanged,

$$\alpha_t e_3 \alpha_t^* = \left(\cos\frac{\theta}{2} + e_3\sin\frac{\theta}{2}\right)e_3\left(\cos\frac{\theta}{2} - e_3\sin\frac{\theta}{2}\right) = e_3\,. \tag{A144}$$

But the transformations (A139) and (A140) of the spinors $\xi^{\pm}$ and $\chi^{\pm}$ belonging (as eigenvectors) to the unchanged vector e_3 are not identical:

$$\begin{aligned} \xi'^{\pm} &= \alpha_t\xi^{\pm} = \left(\cos\frac{\theta}{2} \mp i\sin\frac{\theta}{2}\right)\xi^{\pm} = e^{\mp i\theta/2}\xi^{\pm}\,, \\ \chi'^{\pm} &= \chi^{\pm}\alpha_t^{-1} = \left(\cos\frac{\theta}{2} \pm i\sin\frac{\theta}{2}\right)\chi^{\pm} = e^{\pm i\theta/2}\chi^{\pm}\,, \end{aligned} \tag{A145}$$

they are subject to a phase rotation by a half-angle. Equations (A145) inserted into basis elements (A137) yield an ordinary rotation of the other two basis units by the angle θ,

$$\begin{aligned} 1' &= \xi'^{+}\chi'^{+} + \xi'^{-}\chi'^{-} = \xi^{+}\chi^{+} + \xi^{-}\chi^{-} = 1\,, \\ e_1' &= i\xi'^{+}\chi'^{-} - i\xi'^{-}\chi'^{+} = ie^{i\theta}\xi^{+}\chi^{-} - ie^{-i\theta}\xi^{-}\chi^{+} \\ &= i\cos\theta\left(\xi^{+}\chi^{-} - \xi^{-}\chi^{+}\right) - \sin\theta\left(\xi^{+}\chi^{-} + \xi^{-}\chi^{+}\right) = \cos\theta e_1 - \sin\theta e_2\,, \\ e_2' &= \xi'^{+}\chi'^{-} + \xi'^{-}\chi'^{+} = \cos\theta e_2 + \sin\theta e_1\,, \\ e_3' &= i\xi'^{+}\chi'^{+} - i\xi'^{-}\chi'^{-} = i\xi^{+}\chi^{+} - i\xi^{-}\chi^{-} = e_3\,. \end{aligned} \tag{A146}$$

The phase transformation of spinors, eigenvectors of a quaternionic unit directed along the axis of instant rotation (e_3 in the case considered), comprises in itself full information on any arbitrarily complicated rotation of a frame. Analogous vector and spinor Lorentz-type transformations can be found around the space-like axes e_1 and e_2.

Appendix F.4. 3-ψ Rule

Now we want to show that the fundamental Fierz identity for spinors in some distinguished dimensions,

$$(\bar{\psi}\gamma_\mu\psi)\gamma^\mu\psi = 0\,, \tag{A147}$$

appears as a simple consequence of the properties of the quaternionic spinors. This '3-ψ's rule [53], which restricts appearances of multi-spinor products and makes supersymmetry in split quaternions (2 + 2) space, can be written as

$$(\xi\xi^*)\xi = \chi(\chi^*\chi) = 0\,, \tag{A148}$$

where brackets select the vector-type objects obtained from the quaternionic spinors. Let us show validity of this identity for the real 2-component spinors $\xi^{\pm}$, the general proof you can find in [29].

According to definitions (A108), the quaternions for which $\xi D^{\pm} = \xi$ and $D^{\pm}\chi = \chi$ are true, lie in planes defined by a_A and b_A, $A = 1, 2$. These planes remain invariant under all maps $\xi' = \alpha\xi$ and $\chi' = \chi\alpha^*$, since the quaternions ξ' and χ' also lie in that planes. Furthermore, any spinor quaternion of the first (second) kind can be decomposed into two special spinor quaternions of the first (second) kind, each of which lies in an invariant plane. This decomposition may be accomplished with the help of the primitive idempotent quaternions $D^{\pm}$ by the formula

$$\xi = \xi D^+ + \xi D^- \tag{A149}$$

for a spinor quaternion of the first kind and by the formula

$$\chi = D^+\chi + D^-\chi \tag{A150}$$

for the spinor quaternion of the second kind.

The spinor quaternion of the first (second) kind that lie in an invariant plane thus have only two independent components. In the decomposition (A149) and (A150) the planes of the spinors of the first kind ($\xi^{\pm}$) and of the second kind ($\chi^{\pm}$) are orthogonal to each other, i.e., vector type quaternions constructed by the products of the type $\xi^+\xi^{-*}$ and $\chi^+\chi^{-*}$ are light-like, i.e., have zero norms. For instance, the covariant spinor quaternion ξ (with the real components a_A and b_A) can be represented by a singular $SL(2, R)$-matrix,

$$\xi = \xi^+ + \xi^- = \begin{pmatrix} a_1 & 0 \\ a_2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b_1 \\ 0 & b_2 \end{pmatrix}. \qquad \left(\mathbf{det}\,\xi = a_1 b_2 - b_1 a_2 = 0\right) \tag{A151}$$

The ideals $\xi^{\pm}$ have the following representations:

$$\begin{aligned} \xi^+ &= a_1 D^+ + a_2 G^- = \begin{pmatrix} a_1 & 0 \\ a_2 & 0 \end{pmatrix}, \qquad & \xi^{+*} &= a_1 D^- - a_2 G^- = \begin{pmatrix} 0 & 0 \\ -a_2 & a_1 \end{pmatrix}, \\ \xi^- &= b_2 D^- + b_1 G^+ = \begin{pmatrix} 0 & b_1 \\ 0 & b_2 \end{pmatrix}, \qquad & \xi^{-*} &= b_2 D^+ - b_1 G^+ = \begin{pmatrix} b_2 & -b_1 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{A152}$$

One can see that

$$\xi^+\xi^{+*} = \xi^{+*}\xi^+ = 0\,, \qquad \xi^-\xi^{-*} = \xi^{-*}\xi^- = 0\,, \tag{A153}$$

and

$$\xi\xi^* = \xi^+\xi^{-*} + \xi^-\xi^{+*}\,, \tag{A154}$$

where the products $\xi^+\xi^{-*}$ and $\xi^-\xi^{+*}$ correspond to vector-type split quaternions. Then, using the relations (A153), the validity of the quaternionic 3-ψ rule (A148) can be checked:

$$(\xi\xi^*)\xi = \xi^+\xi^{-*}\xi^+ + \xi^-\xi^{+*}\xi^- = \xi^+\begin{pmatrix} a_1 b_2 - b_1 a_2 & 0 \\ 0 & 0 \end{pmatrix} + \xi^-\begin{pmatrix} 0 & 0 \\ 0 & a_1 b_2 - b_1 a_2 \end{pmatrix} = 0\,. \tag{A155}$$

Similarly, the validity of the second 3-ψ identity (A148) can be shown for the spinor of the second kind χ.

References

1. Gogberashvili, M. Split-quaternion analyticity and (2 + 1)-electrodynamics. *PoS* **2021**, *Regio2020*, 007. [CrossRef]
2. Gogberashvili, M. Standard Model Particles from Split Octonions. *Prog. Phys.* **2016**, *12*, 30–33.
3. Gogberashvili, M.; Sakhelashvili, O. Geometrical applications of split octonions. *Adv. Math. Phys.* **2015**, *2015*, 196708. [CrossRef]
4. Gogberashvili, M. Octonionic electrodynamics. *J. Phys. A Math. Gen.* **2006**, *39*, 7099. [CrossRef]
5. Gogberashvili, M. Octonionic version of Dirac equations. *Int. J. Mod. Phys. A* **2006**, *21*, 3513–3523. [CrossRef]
6. Gogberashvili, M. Octonionic geometry. *Adv. Appl. Clifford Algebras* **2005**, *15*, 55–66. [CrossRef]

7. Gogberashvili, M. Observable algebra. *arXiv* **2002**, arXiv:hep-th/0212251.
8. Gogberashvili, M. Split quaternions and particles in (2 + 1)-space. *Eur. Phys. J. C* **2014**, *74*, 3200. [CrossRef]
9. Schafer, R.D. *An Introduction to Nonassociative Algebras*; Dover: New York, NY, USA, 2017. Available online: https://www.gutenberg.org/ebooks/25156 (accessed on 20 February 2022).
10. Dixon, G.M. *Division Algebras: Octonions Quaternions Complex Numbers and the Algebraic Design of Physics*; Springer Science & Business Media: Boston, MA, USA, 1994. [CrossRef]
11. Conway, J.H.; Smith, D.A. *On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry*; AK Peters/CRC Press: Boca Raton, FL, USA, 2003. [CrossRef]
12. Gsponer, A.; Hurni, J.P. Quaternions in mathematical physics. 1. Alphabetical bibliography. *arXiv* **2005**, arXiv:math-ph/0510059.
13. Hanson, A.J. *Visualizing Quaternions*; Morgan Kaufmann/Elsevier: San Francisco, CA, USA, 2006.
14. Altmann, S. *Rotations, Quaternions, and Double Groups*; Claredon Press: Oxford, UK, 1986.
15. Adler, S.L. *Quaternionic Quantum Mechanics and Quantum Fields*; Oxford Univercity Press: Oxford, UK, 1995.
16. Kuipers, J.B. *Quaternions and Rotation Sequences*; Princeton University Press: Princeton, NJ, USA, 1999.
17. Chanyal, B.C. Quaternionic approach on the Dirac–Maxwell, Bernoulli and Navier–Stokes equations for dyonic fluid plasma. *Int. J. Mod. Phys. A* **2019**, *34*, 1950202. [CrossRef]
18. Chanyal, B.C.; Karnatak, S. A comparative study of quaternionic rotational Dirac equation and its interpretation. *Int. J. Geom. Meth. Mod. Phys.* **2020**, *17*, 2050018. [CrossRef]
19. Castro Neto, A.H.; Guinea, G.; Peres, N.M.R.; Novoselov, K.S.; Geim, A.K. The electronic properties of graphene. *Rev. Mod. Phys.* **2009**, *81*, 109–162. [CrossRef]
20. Banados, M.; Henneaux, M.; Teitelboim, C.; Zanelli, J. Geometry of the (2 + 1) black hole. *Phys. Rev. D* **1993**, *48*, 1506–1525. [CrossRef]
21. Carlip, S. *Quantum Gravity in 2 + 1 Dimensions*; Cambridge University Press: Cambridge, UK, 2003. [CrossRef]
22. Witten, E. Three-dimensional gravity revisited. *arXiv* **2007**, arXiv:0706.3359.
23. Witten, E. 2 + 1 dimensional gravity as an exactly soluble system. *Nucl. Phys. B* **1988**, *311*, 46–78. [CrossRef]
24. Achucarro, A.; Townsend, P.K. A Chern-Simons action for three-dimensional anti-de Sitter supergravity theories. *Phys. Lett. B* **1986**, *180*, 89–92. [CrossRef]
25. Stillwell, J. *Mathematics and Its History*; Springer: New York, NY, USA, 1989. [CrossRef]
26. Needham, T. *Visual Complex Analysis*; Oxford University Press: Oxford, UK, 2000.
27. Khrennikov, A. *Beyond Quantum*; Jenny Stanford Publishing: New York, NY, USA, 2014. [CrossRef]
28. Lindgren, J.; Liukkonen, J. The Heisenberg uncertainty principle as an endogenous equilibrium property of stochastic optimal control systems in quantum mechanics. *Symmetry* **2020**, *12*, 1533. [CrossRef]
29. Baez, J.C.; Huerta, J. Division algebras and supersymmetry I. *Proc. Symp. Pure Math.* **2010**, *81*, 65–80.
30. De Andrade, M.A.; Rojas, M.; Toppan, F. The Signature triality of Majorana-Weyl space-times. *Int. J. Mod. Phys. A* **2001**, *16*, 4453–4479. [CrossRef]
31. Anastasiou, A.; Borsten, L.; Duff, M.J.; Hughes, L.J.; Nagy, S. Super Yang–Mills, division algebras and triality. *JHEP* **2014**, *8*, 80. [CrossRef]
32. Deavours, C.A. The quaternion calculus. *Am. Math. Mon.* **1973**, *80*, 995–1008. [CrossRef]
33. Sudbery, A. Quaternionic analysis. *Math. Proc. Camb. Philos. Soc.* **1979**, *85*, 199–225. [CrossRef]
34. Weisz, J.F. Comments on mathematical analysis over quaternions. *Int. J. Math. Edu. Sci. Tech.* **1991**, *22*, 499–506. [CrossRef]
35. Fueter, R. Die Funktionentheorie der Differentialgleichungen $\triangle$u=0 und $\triangle\triangle$u=0 mit vier reellen Variablen. *Comment. Math. Helv.* **1934**, *7*, 307–330. [CrossRef]
36. Fueter, R. Über die analytische Darstellung der regulären Funktionen einer Quaternionenvariablen. *Comment. Math. Helv.* **1935**, *8*, 371–378. [CrossRef]
37. Mandic, D.P.; Jahanchahi, C.; Took, C.C. A quaternion gradient operator and its applications. *IEEE Signal Process. Lett.* **2011**, *18*, 47–50. [CrossRef]
38. Gentili, G.; Struppa, D.C. A new approach to Cullen-regular functions of a quaternionic variable. *C. R. Math. Acad. Sci. Paris* **2006**, *342*, 741–744. [CrossRef]
39. Gentili, G.; Struppa, D.C. A new theory of regular functions of a quaternionic variable. *Adv. Math.* **2007**, *216*, 279–301. [CrossRef]
40. De Leo, S.; Rotelli, P.P. Quaternionic analyticity. *App. Math. Lett.* **2003**, *16*, 1077–1081. [CrossRef]
41. Frenkel, I.; Libine, M. Split quaternionic analysis and separation of the series for SL(2,R) and SL(2,C)/SL(2,R). *Adv. Math.* **2011**, *228*, 678–763. [CrossRef]
42. Boito, D.; de Andrade, L.N.S.; de Sousa, G.; Gama, R.; London, C.Y.M. On Maxwell's electrodynamics in two spatial dimensions. *Rev. Bras. Ensino Fís.* **2020**, *42*, e20190323. [CrossRef]
43. Lapidus, I.R. One- and two-dimensional hydrogen atoms. *Am. J. Phys.* **1981**, *49*, 807. [CrossRef]
44. Lapidus, I.R. Classical electrodynamics in a universe with two space dimensions. *Am. J. Phys.* **1982**, *50*, 155–157. [CrossRef]
45. Asturias, F.J.; Aragón, S.R. The hydrogenic atom and the period table of the elements in two spatial dimensions. *Am. J. Phys.* **1985**, *53*, 893–899. [CrossRef]
46. Moses, H.E. A spinor representation of Maxwell's equations. *Nuovo Cim. Suppl.* **1958**, *7*, 1–18. [CrossRef]

47. Moses, H.E. Solution of Maxwell's equations in terms of a spinor notation: The direct and inverse problem. *Phys. Rev.* **1959**, *113*, 1670–1678. [CrossRef]
48. Maxwell, J.C. *Treatise on Electricity and Magnetism*; Dover: New York, NY, USA, 1954.
49. Imaeda, K. A new formulation of classical electrodynamics. *Nuovo Cim.* **1976**, *32*, 138–162. [CrossRef]
50. Gürlebeck, K.; Sprössig, W. *Quaternionic and Clifford Calculus for Physicists and Engineers*; Wiley & Sons: Chichester, UK, 1997.
51. Özdemir, M.; Ergin, A.A. Rotations with unit timelike quaternions in Minkowski 3-space. *J. Geom. Phys.* **2006**, *56*, 322–336. [CrossRef]
52. Kula, L.; Yayli, Y. Split quaternions and rotations in semi Euclidean space E_2^4. *J. Korean Math. Soc.* **2007**, *44*, 1313–1327. [CrossRef]
53. Schray, J. The General classical solution of the superparticle. *Class. Quant. Grav.* **1996**, *13*, 27–38. [CrossRef]