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A Model for the Dynamics of Stable Gas Bubbles in Viscoelastic Fluids Based on Bubble Volume Variation

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Abstract

We present a novel formulation of the Rayleigh–Plesset equation to describe stable gas bubble dynamics in viscoelastic media, using bubble volume variation, rather than radius, as the primary variable of the resulting nonlinear ordinary differential equation. This formulation incorporates the linear Kelvin–Voigt model as the constitutive relation for the surrounding fluid, capturing both viscous and elastic contributions, to track the oscillations of a gas bubble subjected to an ultrasonic field over time. The proposed model is solved numerically, subjected to a convergence analysis, and validated by comparisons with theoretical and experimental results from the literature. We systematically investigate the nonlinear oscillations of a single spherical gas bubble in various viscoelastic environments, each modeled with varying levels of rheological complexity. The influence of medium properties, specifically shear elasticity and viscosity, is examined in detail across both linear and nonlinear regimes. This work improves our understanding of stable cavitation dynamics by emphasizing key differences from Newtonian fluid behavior, resonance frequency, phase shifts, and oscillation damping. Elasticity has a pronounced effect in low-viscosity media, whereas viscosity emerges as the dominant factor modulating the amplitude of oscillations in both the linear and nonlinear regimes. The model equation developed here provides a robust tool for analyzing how viscoelastic properties affect bubble dynamics, contributing to improved prediction and control of stable cavitation phenomena in complex media.



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1. Introduction

Acoustic cavitation, the formation, growth, and oscillation of gas bubbles under the influence of an acoustic field, is a fundamental phenomenon in a wide range of scientific, biomedical, and industrial applications. Technologies such as ultrasound imaging, targeted drug delivery [1], sonochemistry [2], material processing, and marine engineering applications [3,4] rely on the controlled behavior of bubbles to achieve both efficacy and safety.

Bubble dynamics has traditionally been studied in Newtonian fluids, such as water, for which there are well-established theoretical and numerical models [5]. However, real-world applications, including cavitation-enhanced therapies, food processing, cavitation erosion, and oceanic microbubble scattering, often involve bubbles oscillating in non-Newtonian media. This has driven a growing interest in understanding the dynamics

of bubbles in these fluids, particularly with the advent of new materials, the inclusion of polymeric additives in many fluids [6], and the expansion of biomedical technologies into soft media. These fluids include polymeric liquids, gels, colloidal suspensions, and, notably, viscoelastic biological tissues such as cells and tissues, which exhibit mechanical behavior lying between that of an ideal fluid and a solid. These materials are characterized by viscosity and elasticity. For example, Jamburidze et al. [7] characterized the resonant behavior of ultrasound in isolated microbubbles embedded in agarose gels, commonly used as tissue-mimicking phantoms for biomedical applications. They found that the resonance frequency of the bubbles increased with the shear modulus of the medium. This implies that imaging and therapeutic ultrasound protocols must be optimized on the basis of the type of tissue in which the bubbles are embedded.

Moreover, substituting Newtonian liquids with viscoelastic media, such as polymeric gels, in experimental setups has proven to be an effective approach for overcoming challenges such as uniformity of bubble sizes and bubble retention [8]. For example, gas-seeded polymeric gels have been used as analog models for oceanic bubble clouds in geophysical acoustics research [9], providing a more controlled platform for exploring complex acoustic interactions.

Viscoelasticity and its impact on the dynamics of spherical bubble are the main focus of this paper. Several studies have investigated bubble behavior in viscoelastic media, leading to models based on extensions of the Rayleigh–Plesset equation, accounting for viscosity, elasticity, and compressibility [10]. The early work by Fogler & Goddard [11] combined the Maxwell model with the Rayleigh–Plesset equation to study bubble collapse in viscoelastic fluids, finding that elasticity can retard collapse in certain parameter ranges. Tanasawa & Yang [12] applied the three-parameter Oldroyd model to bubble oscillations, showing that elasticity reduces viscous damping compared to pure fluids. Allen & Roy [13] used the Maxwell and Jeffreys models to study bubble oscillations, revealing that elasticity alters the oscillation phase and harmonic structure. Yang & Church [14] extended the Keller–Miksis equation by incorporating the Kelvin–Voigt model, showing that elasticity increases the threshold pressure for inertial bubble oscillations. Hua & Johnsen [15] coupled the standard linear solid model with the compressible Keller–Miksis equation, and they found that nonlinear effects are significant when the stress tensor of the medium is prominent, affecting bubble dynamics. Warnez & Johnsen [16] simulated the bubble dynamics in various viscoelastic media, finding that relaxation time increases bubble growth. Both Zilonova et al. [17] and Filonets et al. [18] employed the Gilmore–Akulichev–Zener model to study bubble dynamics in viscoelastic soft tissues. Zilonova et al. focused on high-intensity focal ultrasound thermal therapy, showing that elasticity and viscosity dampen oscillations, with relaxation time affecting bubble behavior depending on ultrasound frequencies. Similarly, Filonets et al. studied the inertial cavitation threshold in viscoelastic soft tissues using a dual-frequency signal. Finally, Murakami [19] experimentally observed and theoretically analyzed the oscillations of a spherical bubble in a gelatin gel under ultrasound irradiation. They compared the finite-amplitude oscillations they found with nonlinear Rayleigh–Plesset calculations, suggesting the need to include gel elasticity in calculations to accurately reproduce nonlinear bubble dynamics.

The study of bubble dynamics in viscoelastic media primarily involves integrating different constitutive viscoelastic models into the equations that govern bubble dynamics. This approach aims to generate models applicable to various tissues and conditions, allowing for accurate predictions of bubble behavior. The correct choice of constitutive equations is crucial for describing the rheological properties of the surrounding medium and capturing the complex interactions between the bubble and its environment. Several linear viscoelastic models, such as the Maxwell, Kelvin–Voigt, and Zener models, are

commonly used in the literature to simulate bubble oscillations in viscoelastic materials. They are effective when the deformations within the material are relatively small.

However, in many practical scenarios, especially under large-amplitude acoustic forcing, viscoelastic media respond nonlinearly, significantly altering bubble dynamics. Nonlinear viscoelastic models capture these effects and predict complex phenomena such as aperiodic oscillations, modulation, and chaos under intense excitation [20]. Fractional viscoelastic models, on the other hand, are receiving increasing attention for their ability to capture power law behaviors in creep, relaxation, and frequency-dependent responses [21–23]. These features, commonly observed in biological tissues, gels, and polymers, are difficult to reproduce with classical models. Fractional models rely on fractional calculus, which generalizes differentiation and integration to non-integer orders, offering a flexible framework for viscoelastic characterization.

The most popular nonlinear equation used to describe the behavior of a gas bubble in an acoustic field is the Rayleigh–Plesset equation [24,25]. Most models are based on extensions of this equation to account for phenomena such as fluid compressibility, viscoelasticity, thermal effects, multi-bubble interactions, and mass and heat transfer. Other factors, such as bubble migration and phase transitions, are also recognized as important in multi-period bubble dynamics [26,27]. In particular, recent studies on large-scale air-gun bubbles have highlighted the role of bubble migration, showing that translational motion can significantly influence the dynamic properties of bubbles and the signatures of pressure waves [3,4]. At the same time, although it is widely accepted that the noncondensable gas within the bubble undergoes adiabatic compression, experimental evidence shows that the vapor content in transient cavitation bubbles contributes directly to energy loss through phase transitions [27].

Most differential equations developed in this context adopt the bubble's radius as the dependent variable (radius framework) [28] and successfully describe the nonlinear oscillations and collapse of the bubble. Alternatively, models based on the bubble volume as the dependent variable also exist. A classic second-order approximation was first derived by Zabolotskaya and Soluyan in the 1960s–1970s [29,30], followed by a third-order approximation in the 1990s [31]. These models describe bubble behavior under the assumption that volume variations remain small relative to the initial volume of the bubble, making them suitable for moderate-amplitude oscillations. A fourth-order approximation was later introduced within the Rayleigh–Plesset framework [32], enhancing the accuracy of the bubble response under acoustic forcing by including a fourth-order term in the adiabatic gas law in the development of the volume-frame equation. Reformulating the Rayleigh–Plesset equation in terms of bubble volume variation yields a physically equivalent model that retains the same dynamics as the classical radius-based form, offering a complementary perspective for analyzing bubble behavior [33]. Any variations in their predictions can be ascribed to factors such as numerical precision or differences in the truncation of asymptotic expansions, rather than differences in the fundamental physics.

In this work, we adopt the bubble volume framework. One of the main advantages of our approach is that it can capture the stable cavitation phenomenon in quite a simple and pleasant way, which is enough for the purposes of our studies. The use of bubble volume variations for the description of bubble dynamics in viscoelastic media vs. traditional models based on the bubble radius follows the perspective of our works carried out for years with respect to the case of bubbles in liquids. The consideration of variations in bubble volume, with time as the dependent variable of the differential equation, with respect to the dynamics of bubbles in a liquid, as said above, was firstly adopted by Zabalotskaya et al. [29,30]. Their formulation defined a new framework that allowed them to obtain an analytical solution of both the volume variation and acoustic pressure up to the

second term by applying a perturbation method. In their papers, they described the virtues of their approach. In particular, the simplicity of the ordinary differential equation allowed them to obtain the above-mentioned analytical solution, thus enabling the definition of the main characteristics of bubbly liquids, such as attenuation and dispersion curves, through analytical expressions. This was possible because of the clear mutual interaction of both variables, acoustic pressure and volume variation, which perfectly captures the mutual interaction of both physical fields (bubble oscillations and acoustic field), and this turned out to be an adequate way of modeling stable cavitation in liquids (without collapse). Since this observation, this approach had not been used much, and Leighton in [33] specifically mentioned the lack of numerical solutions to Zabalotskaya's model. It must be noted here that this strategy was the one chosen by Blackstock and Hamilton (in their book entitled *Nonlinear Acoustics* [34]) to describe the dispersion and nonlinear phenomena occurring in bubbly liquids. For several years, we worked on the numerical solution of the models obtained through the application of the bubble volume variation approach. Our results have allowed us to understand the behavior and effects derived from the propagation of nonlinear ultrasound in bubbly liquids [35–38]. The objective here is to extend this approach to the case of bubbles in viscoelastic media, which we believe will be useful, as was the case for bubbles in liquids. This work is the first within this framework to analyze the behavior of one single bubble.

To our knowledge, no previous model has described the dynamics of bubbles in viscoelastic media using a volume-based formulation of the Rayleigh–Plesset equation. Motivated by this gap and the aforementioned studies, we propose a new model that couples the Rayleigh–Plesset equation with the Kelvin–Voigt viscoelastic constitutive law, formulated in terms of bubble volume variations. This approach extends previous works in Newtonian fluids by incorporating the effects of elasticity. The remainder of the paper is organized as follows: Section 2 presents the derivation of the model; in Section 3, we validate the performance of the proposed model, present some numerical simulations, and analyze the impact of the viscoelasticity of the medium on bubble dynamics. The main conclusions of this work are given in Section 4.

2. Materials and Methods

In this section, we derive a generalized Rayleigh–Plesset equation for bubble dynamics in an arbitrary viscoelastic medium, formulated in terms of bubble volume variations. Building on this framework, we introduce the Kelvin–Voigt constitutive relation and derive the corresponding governing equation that incorporates the linear viscoelastic response of the medium.

2.1. Governing Equations

2.1.1. Bubble Dynamics in an Arbitrary Viscoelastic Medium Written in the Volume Variation Framework

The Rayleigh–Plesset equation describes the radial dynamics of a spherically symmetric gas bubble in an incompressible fluid and can be expressed in general forms that are valid for any rheological behavior of the surrounding medium [13]:

$$\rho \left(R \ddot{R} + \frac{3}{2} \dot{R}^2 \right) = P_b - P_\infty - \frac{2\sigma}{R} + 3 \int_R^\infty \frac{\tau_{rr}}{r} dr, \quad (1)$$

where $R(t)$ is the instantaneous radius of the bubble, ρ is the density of the fluid, σ is the surface tension, P_b is the gas pressure inside the bubble, P_∞ is the ambient pressure, and τ_{rr} is the radial component of the deviatoric stress tensor (in the radial direction r). Note that

the overdots denote time derivatives. Assuming an ideal gas inside the bubble, its pressure is modeled using a polytropic relation:

$$P_b = \left(P_{b0} + \frac{2\sigma}{R} \right) \left(\frac{R_0}{R} \right)^{3\gamma}, \quad (2)$$

where γ is the polytropic exponent, $P_{b0} = \rho_b c_b^2 / \gamma$ is the gas pressure at equilibrium, and R_0 is the radius of equilibrium bubble. Here, ρ_b and c_b denote the density and speed of sound of the gas at equilibrium, respectively. The ambient pressure is given by the following:

$$P_\infty = P_{b0} + p_a(t), \quad (3)$$

with $p_a(t)$ being the acoustic pressure that excites the bubble.

We begin with Equation (1) to derive a nonlinear ordinary differential equation for bubble volume variations. Neglecting surface tension, the equation becomes

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho} \left(P_b - P_\infty + 3 \int_R^\infty \frac{\tau_{rr}}{r} dr \right). \quad (4)$$

Let us define the instantaneous radius and volume of the bubble at time t as $R(t) = R_0 + r(t)$ and $V(t) = V_0 + v(t)$, where $r(t)$ and $v(t)$ represent the radius and volume variations of the bubble, respectively. The equilibrium volume is given by $V_0 = \frac{4\pi}{3} R_0^3$. Multiplying both sides of Equation (4) by $4\pi R_0$ and using the relation for P_∞ in Equation (3), we obtain

$$4\pi R_0 \left(R\ddot{R} + \frac{3}{2}\dot{R}^2 \right) = \frac{4\pi R_0}{\rho} (P_b - P_{b0}) - \frac{4\pi R_0}{\rho} p_a + \frac{4\pi R_0}{\rho} 3 \int_R^\infty \frac{\tau_{rr}}{r} dr. \quad (5)$$

Following the procedure outlined in [32], we expand the right-hand side of Equation (5) in terms of the bubble volume perturbation $v(t)$, obtaining a second-order nonlinear approximation. Here, $\omega_b^2 = 3\gamma P_{b0} / \rho R_0^2$ defines the Minnaert resonance frequency of the bubble in a Newtonian fluid [39], and $a = (\gamma + 1)\omega_b^2 / 2V_0$ is the second-order nonlinear coefficient, which arises from the nonlinearity of the adiabatic state equation of the gas in the bubble:

$$4\pi R_0 \left(R\ddot{R} + \frac{3}{2}\dot{R}^2 \right) = -\omega_b^2 v + av^2 - \frac{4\pi R_0}{\rho} p_a + \frac{4\pi R_0}{\rho} 3 \int_R^\infty \frac{\tau_{rr}}{r} dr. \quad (6)$$

To express the left-hand side of Equation (4) in terms of the volume variation, we make use of the following relations:

$$V = \frac{4}{3}\pi R^3, \quad \dot{V} = 4\pi R^2 \dot{R}, \quad \ddot{V} = 4\pi R(2\dot{R}^2 + R\ddot{R}). \quad (7)$$

Therefore, in terms of V ,

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{4\pi} \left(\frac{4\pi}{3V} \right)^{\frac{1}{3}} \left(\ddot{V} - \frac{1}{6} \frac{\dot{V}^2}{V} \right). \quad (8)$$

Since $V = V_0 + v$, we have $\dot{V} = \dot{v}$ and $\ddot{V} = \ddot{v}$. On the other hand, provided that $v \ll V_0$ (bubble oscillations assumed to be of moderate amplitude), which implies $V = V_0 + v \approx V_0$, we can approximate some terms through the linear exponent's Taylor series expansion:

$$V^{-1/3} = V_0^{-1/3} \left(1 + \frac{v}{V_0} \right)^{-1/3} \approx \frac{1}{V_0^{1/3}} - \frac{1}{3V_0^{4/3}} v. \quad (9)$$

Assuming the error of approximation of order $O(v/V_0)^2$, Equation (8) can be written as follows:

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{4\pi} \left(\frac{4\pi}{3} \right)^{\frac{1}{3}} \left(\frac{1}{V_0^{1/3}} - \frac{1}{3V_0^{4/3}}v \right) \left(\ddot{v} - \frac{\dot{v}^2}{6(V_0 + v)} \right). \quad (10)$$

As seen in the previous paragraph, we have $v \ll V_0$ and $V = V_0 + v \approx V_0$, which allows us to write

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 \approx \frac{1}{4\pi} \left(\frac{4\pi}{3} \right)^{\frac{1}{3}} \left(\frac{\ddot{v}}{V_0^{1/3}} - \frac{\dot{v}^2}{6V_0^{4/3}} - \frac{v\ddot{v}}{3V_0^{4/3}} + \frac{v\dot{v}^2}{18V_0^{7/3}} \right). \quad (11)$$

Assuming the error of approximation $V = V_0 + v \approx V_0$ and since $V_0 = 4\pi R_0^3/3$, the above expression becomes

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{4\pi R_0} \left(\ddot{v} - \frac{1}{6V_0}(\dot{v}^2 + 2v\ddot{v}) + \frac{v\dot{v}^2}{18V_0^2} \right), \quad (12)$$

and we neglect the third-order term to obtain

$$4\pi R_0 \left(R\ddot{R} + \frac{3}{2}\dot{R}^2 \right) = \ddot{v} - \frac{1}{6V_0}(\dot{v}^2 + 2v\ddot{v}). \quad (13)$$

Substituting Equation (13) into Equation (6) allows us to express Equation (4) entirely in terms of the volume variation v :

$$\ddot{v} - \frac{1}{6V_0}(\dot{v}^2 + 2v\ddot{v}) = -\omega_b^2 v + av^2 - \frac{4\pi R_0}{\rho} p_a + \frac{4\pi R_0}{\rho} 3 \int_R^\infty \frac{\tau_{rr}}{r} dr. \quad (14)$$

The nonlinear coefficient, accounting for the dynamic response of the bubble, is set as $b = 1/(6V_0)$, while the constant η is introduced to simplify the notation and defined as $\eta = 4\pi R_0/\rho$. We finally obtain the following new Rayleigh–Plesset differential equation of second-order approximation, which governs the nonlinear behavior of a bubble in an acoustic field in terms of bubble volume variations in an arbitrary viscoelastic medium:

$$\ddot{v} + \omega_b^2 v = b(\dot{v}^2 + 2v\ddot{v}) + av^2 - \eta p_a + 3\eta \int_R^\infty \frac{\tau_{rr}}{r} dr. \quad (15)$$

2.1.2. Bubble Dynamics in a Kelvin–Voigt Viscoelastic Medium Based on the Volume Variation Framework

Now we consider a homogeneous, isotropic viscoelastic medium. In order to determine the stress term τ_{rr} , it is necessary to choose a suitable viscoelastic model, i.e., a constitutive equation that relates the radial stress tensor to the spatial deformations in the fluid. The linear Kelvin–Voigt model [40] is chosen as a starting framework, as it is a simple yet effective model and has been shown to outperform other models, such as the Maxwell model, in capturing soft tissue dynamics [41]:

$$\tau_{rr} = 2G\gamma_{rr} + 2\mu\dot{\gamma}_{rr}, \quad (16)$$

where γ_{rr} is the strain, $\dot{\gamma}_{rr}$ is the strain rate in the radial direction, G is the shear modulus, and μ is the shear viscosity of the medium. As $G \rightarrow 0$, the elastic contribution disappears, leaving only a viscous stress response. In this case, the classical Rayleigh–Plesset equation

for a Newtonian fluid is recovered. Evaluating the integral in the right-hand side of Equation (1), by taking Equation (16) into account, yields [14] the following:

$$\rho \left(R\ddot{R} + \frac{3}{2}\dot{R}^2 \right) = P_b - P_\infty - \frac{2\sigma}{R} - \frac{4\mu\dot{R}}{R} - \frac{4G}{3} \frac{R^3 - R_0^3}{R^3}. \quad (17)$$

We follow a procedure analogous to that described in Section 2.1.1 to incorporate the additional viscous and elastic contributions into the bubble dynamics equation, formulated in terms of volume variations. Thus, Equation (17), once multiplied by $4\pi R_0$ and using the expression for the ambient pressure in Equation (3), can be rewritten as follows:

$$4\pi R_0 \left(R\ddot{R} + \frac{3}{2}\dot{R}^2 + \frac{4\mu\dot{R}}{\rho R} + \frac{4G}{3\rho} \frac{R^3 - R_0^3}{R^3} \right) = \frac{4\pi R_0}{\rho} (P_b - P_{b0}) - \frac{4\pi R_0}{\rho} p_a, \quad (18)$$

The first term on the right-hand side of Equation (18) can be expressed following the same procedure used in Equation (5) to Equation (6), leading to

$$4\pi R_0 \left(R\ddot{R} + \frac{3}{2}\dot{R}^2 + \frac{4\mu\dot{R}}{\rho R} + \frac{4G}{3\rho} \frac{R^3 - R_0^3}{R^3} \right) = -\omega_b^2 v + av^2 - \frac{4\pi R_0}{\rho} p_a. \quad (19)$$

Using the relations in Equation (7) and assuming $v \ll V_0$, i.e., $V = V_0 + v \approx V_0$, we can rewrite the new left-hand side in terms of V , retaining errors of order $O(v/V_0)^2$:

$$\begin{aligned} \frac{4\mu\dot{R}}{\rho R} + \frac{4G}{3\rho} \frac{R^3 - R_0^3}{R^3} &= \frac{4\mu}{3\rho} \frac{\dot{V}}{V} + \frac{4G}{3\rho} \frac{V - V_0}{V} \\ &= \frac{4\mu}{3\rho} \frac{\dot{v}}{V_0 + v} + \frac{4G}{3\rho} \frac{V_0 + v - V_0}{V_0 + v} \\ &\approx \frac{4\mu}{3\rho} \frac{\dot{v}}{V_0} + \frac{4G}{3\rho} \frac{v}{V_0}. \end{aligned} \quad (20)$$

Recalling that $V_0 = \frac{4\pi}{3} R_0^3$, we further simplify the following:

$$\frac{4\mu\dot{R}}{\rho R} + \frac{4G}{3\rho} \frac{R^3 - R_0^3}{R^3} = \frac{\mu}{\pi\rho R_0^3} \dot{v} + \frac{G}{\rho} \frac{v}{\pi R_0^3}. \quad (21)$$

By substituting this along with the previous result in Equation (13) into the left-hand side of Equation (18), we obtain

$$\begin{aligned} 4\pi R_0 \left(R\ddot{R} + \frac{3}{2}\dot{R}^2 + \frac{4\mu\dot{R}}{\rho R} + \frac{4G}{3\rho} \frac{R^3 - R_0^3}{R^3} \right) &= \\ &= \ddot{v} - \frac{1}{6V_0} (\dot{v}^2 + 2v\ddot{v}) + \frac{4\mu}{\rho R_0^2} \dot{v} + \frac{4G}{\rho R_0^2} v. \end{aligned} \quad (22)$$

As a result, Equation (19) can now be entirely reformulated in terms of the bubble volume variation v as

$$\ddot{v} - \frac{1}{6V_0} (\dot{v}^2 + 2v\ddot{v}) + \frac{4\mu}{\rho R_0^2} \dot{v} + \frac{4G}{\rho R_0^2} v = -\omega_b^2 v + av^2 - \frac{4\pi R_0}{\rho} p_a. \quad (23)$$

We now define the new elastic coefficient:

$$\overline{G} = \frac{4G}{\rho R_0^2}. \quad (24)$$

We introduce the viscous damping coefficient $\delta = 4\mu/\rho\omega_b R_0^2$ [32], which accounts for energy dissipation due to the medium's viscosity, together with the nonlinear coefficient b and the constant η . We then derive a novel second-order approximation Rayleigh–Plesset-type equation governing the nonlinear dynamics of a bubble in a viscoelastic medium using the Kelvin–Voigt formulation:

$$\ddot{v} + \delta\omega_b\dot{v} + (\omega_b^2 + \overline{G})v = b(\dot{v}^2 + 2v\ddot{v}) + av^2 - \eta p_a. \quad (25)$$

In addition to the approximations made for this derivation, for the description of the bubble's oscillations due to an acoustic field, we assume some physical restrictions: The bubble's radius is small compared to the wavelength of the acoustic field; the bubble has a constant gas content, without vapor; the spatial conditions within the bubble are uniform; thermal damping is neglected; the nonlinearity of the problem is only due to the bubble's behavior; the bubble does not radiate sound itself. Moreover, in this work, we neglect buoyancy effects (i.e., no body force is considered), as well as Bjerknes forces and bubble migration.

Under these assumptions, surface tension was neglected in this study to isolate the role of viscoelasticity. Resonance and oscillation amplitudes are dominated by tissue elasticity rather than surface effects [14], as previous works have shown that surface tension has minimal influence on bubble dynamics in tissue-like media [1,17], making this simplification appropriate for the conditions investigated here.

Figure 1 presents a comparison of the radius and volume variation frameworks, including the resulting equations and the main assumptions behind each formulation. An additional advantage of the volume variation approach is that the resulting model can be solved more easily, as both variables involved (acoustic pressure and bubble volume variation) remain smooth (continuous and differentiable), avoiding singular terms in the numerical scheme. Traditional radius-based models are simplified here by introducing the volume variation variable; specifically, the nonlinear term $(R^3 - R_0^3)/R^3$ in Equation (17) is replaced by the linear terms $\delta\omega_b\dot{v}$ and $\overline{G}v$ in Equation (25), yielding smoother expressions and ensuring well-behaved model dynamics.

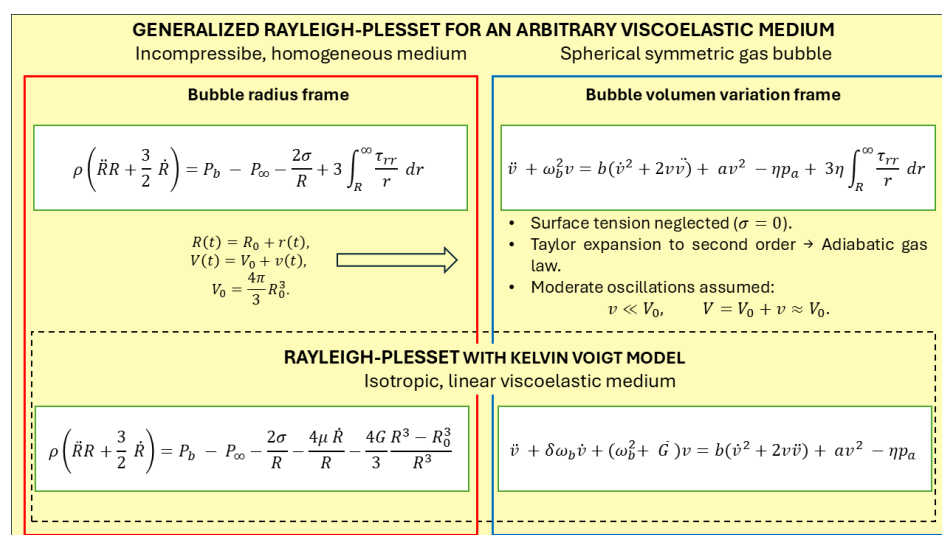


Figure 1. Box summary of radius and volume-based formulations and their main assumptions.

This model will be solved in Section 2.2 and used in Section 3.

2.2. Numerical Solution of the Bubble Equation

We solve the differential system formed by Equation (25) and the Cauchy conditions, which state the natural rest of the bubble at the onset from $t = 0$ to the lifetime of the experiment, $t = T$:

$$\begin{cases} \ddot{v} + \delta\omega_b \dot{v} + (\omega_b^2 + \overline{G})v = b(\dot{v}^2 + 2v\ddot{v}) + av^2 - \eta p_a, & 0 < t < T, \\ v(0) = 0, \\ \dot{v}(0) = 0, \end{cases} \quad (26)$$

Here, the acoustic excitation of the bubble is defined as a continuous signal at the driven frequency f and set at $p_a(t) = p_0 \sin(\omega t)$, where $\omega = 2\pi f$. The second-order ordinary differential equation is reduced to a first-order system of two coupled ordinary differential equations by setting the new variable $u = \dot{v}$, which yields the following:

$$\begin{cases} \dot{v} = u, & 0 < t < T, \\ \dot{u} + \delta\omega_b u + (\omega_b^2 + \overline{G})v = b(u^2 + 2v\dot{u}) + av^2 - \eta p_a, & 0 < t < T, \\ v(0) = 0, \\ u(0) = 0. \end{cases} \quad (27)$$

The resulting system is then solved using the fourth-order Runge–Kutta method [42], and numerical solutions are obtained for the instantaneous volume variations in the bubble under ultrasonic excitation. In the following, we use a time step of $\tau = 1 \times 10^{-9}$ s to ensure the convergence of the numerical solution (see Section 3.2.2).

3. Results

This section presents the main results obtained with the proposed model described in Equation (25). We begin by validating the formulation through the prediction of the theoretical resonance frequency of bubbles in viscoelastic media and through comparisons with experimental observations. We then assess the utility of our model by analyzing the variation in bubble volume for several representative soft materials. Next, we examine the effect of medium elasticity on bubble dynamics, focusing on the influence of the newly introduced elastic term. Finally, we explore the broader impact of viscoelasticity on bubble behavior.

3.1. Validation of the Model

To validate our model, Equation (25), two complementary studies are performed. First, the predicted resonance frequency of a single bubble is compared with established theoretical results [40]. Specifically, we verify whether the model accurately reproduces the expected resonance frequency of bubble oscillations under linear conditions. The resonance frequency of gas bubbles in different media is computed as a function of their initial radius. Next, using the proposed model, a frequency sweep is performed to identify the excitation frequency at which the bubble exhibits its maximal response, defined as the frequency that induces the largest volume variation. This analysis is carried out for two distinct sets of media with varying elasticity and viscosity values to confirm that the model correctly captures these variations in Section 3.1.1. Subsequently, we employ a set of representative soft media to validate the model against realistic material properties and behavior in Section 3.1.2.

Second, an experimental validation is performed using the data reported in [7]. In Section 3.1.3, resonance curves obtained from the model are compared with experimental

measurements. This allows us to verify that the model not only reproduces theoretical predictions but also accurately reflects observed bubble dynamics in realistic conditions.

3.1.1. Effect of Elasticity and Viscosity on Bubble Resonance

As a first step, we validate Equation (25) of the model against the theoretical expression of the natural frequency [40], which is given by the following:

$$\omega_0^2 = \frac{1}{\rho R_0^2} \left[3\gamma P_{b0} + \frac{2\sigma}{R_0} (3\gamma - 1) + 4G \right]. \quad (28)$$

When surface tension σ and elasticity G are neglected, this expression reduces to the classical Minnaert frequency that was used to obtain Equation (6).

For the given media listed in Table 1, characterized by their shear elasticity G , we solve the differential system (27) by considering an ultrasonic perturbation of infinitesimal amplitude $p_0 = 1$ mPa. For simplicity, biological soft tissues are modeled as viscoelastic liquids [20,43–46]. In this study, standardized values of $\rho = 1000$ kg/m³ for density and $\mu = 0.001$ Pa·s for viscosity are used for all biological tissues. Although this simplified scenario does not fully reflect the actual properties of biological tissues [47], it is used to validate the inclusion of shear elasticity in the proposed model.

Table 1. Shear modulus G of the considered biological soft media [43,47].

Medium (Soft Biological Tissue)	Shear Modulus G (kPa)
Without shear elasticity	0
Fat	3.3
Liver	4.3
Muscle	6.7
Glandular breast	11

Figure 2 shows the variation in the linear resonance frequency $f_0 = \omega_0/2\pi$ as a function of the initial bubble radius R_0 for gas bubbles embedded in soft media with increasing shear modulus G (Table 1). Theoretical predictions (solid lines) based on Equation (28) are compared with the results obtained using the proposed model Equation (25) (star symbols).

The results are all consistent, confirming that for bubbles with an initial radius greater than 1 μm , an increase in the shear modulus G leads to a rise in the resonance frequency of the bubble, as previously reported in the literature [14,40,43]. For example, for a bubble with $R_0 = 1$ μm , Figure 2 displays the natural frequency f_0 for each medium considered, illustrating how the resonance frequency shifts with an increase in the shear modulus G . This upward trend reflects the role of medium elasticity in constraining the bubble's radial oscillations, thus increasing the resonance frequency of the system. Previous studies investigated the effects of larger variations in the shear modulus, specifically $G = 0, 0.5, 1$, and 1.5 MPa, and clearly demonstrated that the natural frequency increases noticeably with increasing rigidity G due to the increased stiffness of the system imparted by the solid-like elasticity of the surrounding medium. In contrast, in our case, we select real soft materials with smaller differences in the shear modulus between media, resulting in a more moderate change in resonance frequency.

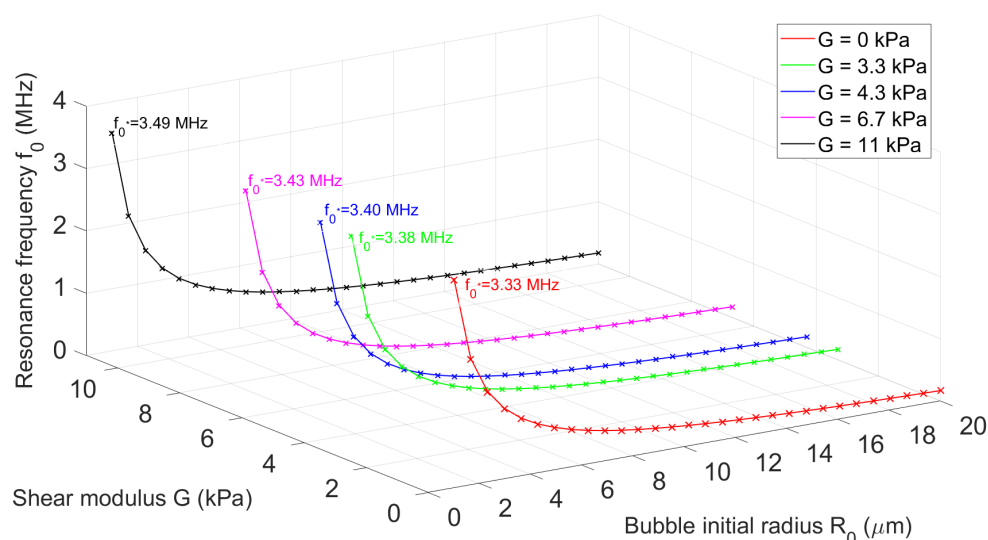


Figure 2. Natural frequency f_0 as a function of the initial bubble radius R_0 for different viscoelastic soft biological tissues. Theoretical values computed from Equation (28) (solid lines) are compared with predictions from the proposed model (Equation (25)) (star symbols).

Nonetheless, the results validate our model and confirm that the assumption that the medium behaves purely as a Newtonian fluid, while neglecting its elastic properties, can lead to substantial inaccuracies in the predicted resonance frequency.

Secondly, we consider the resonance frequency, which differs from the natural frequency due to damping effects, particularly viscous dissipation, given by the following [40,43]:

$$\omega_{0*}^2 = \frac{1}{\rho R_0^2} \left[3\gamma P_{b0} + \frac{2\sigma}{R_0} (3\gamma - 1) + 4G \right] - 2 \left(\frac{2\mu}{\rho R_0^2} \right)^2. \quad (29)$$

Note that this expression becomes non-real for certain combinations of G , μ , and $R_0 < 1 \mu\text{m}$, limiting the applicability of the model in stiff or highly viscous media when excitation occurs at the resonance frequency. To validate our model (Equation (25)) using this expression, we consider the media listed in Table 2, which include various aqueous glycerol solutions [48] with increasing viscosity. For this analysis, we assume $G = 0$ and a constant density of $\rho = 1000 \text{ kg/m}^3$.

Table 2. Shear viscosity μ of considered aqueous glycerol solutions [48].

Medium	Viscosity μ (mPa · s)
GLY00	0.00
GLY04	1.13
GLY25	1.89
GLY35	2.67
GLY47	4.15

Figure 3 shows the variation in the linear resonance frequency $f_0^* = \omega_{0*}^*/2\pi$ as a function of the initial bubble radius R_0 for different viscoelastic media (Table 2). Theoretical predictions (solid lines) based on Equation (29) are compared with the results obtained using the proposed model (Equation (25)) (star symbols). As shown in this figure, the model's results closely follow the solid lines, offering further validation of the proposed model equation.

For a bubble with a fixed initial radius, the resonance frequency decreases as the fluid viscosity increases. This trend is observed for $R_0 = 1 \mu\text{m}$, where the resonance frequency drops from $f_0^* = 3.42 \text{ MHz}$ in GLY00 to about $f_0^* = 2.75 \text{ MHz}$ in the most viscous solution considered. These results demonstrate that viscosity plays a non-negligible role in determining the natural frequency, even in moderately viscous fluids, due to its dissipative effect on bubble oscillations. This increased damping shifts the resonance frequency to lower values. Consequently, assuming that the fluid has inviscid behavior can lead to an overestimation of the natural frequency of the bubble, particularly in viscoelastic or biological media where viscous effects cannot be neglected.

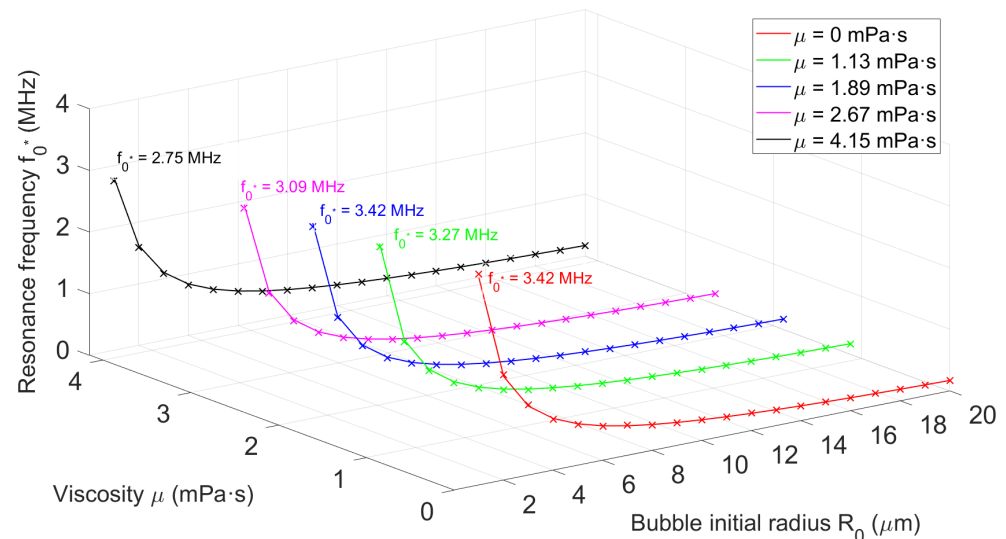


Figure 3. Resonance frequency f_0^* as a function of the initial bubble radius R_0 for different aqueous glycerol solutions. Theoretical values computed from Equation (29) (solid lines) are compared with predictions from the proposed model (Equation (25)) (star symbols).

3.1.2. Bubble Resonance in Representative Soft Viscoelastic Media

To assess the accuracy of the proposed model, we now compare its predictions with theoretical results for a set of representative soft media. Figure 4 presents how the linear resonance frequency $f_0^* = \omega_0^*/2\pi$ varies with the initial bubble radius R_0 for each medium listed in Table 3, characterized by their density ρ , shear elasticity G , and viscosity μ . The numerical results of our model (depicted as star symbols) show clear agreement with the theoretical predictions (solid lines) obtained from Equation (29), which confirms the precision of our model in capturing resonance behavior.

Table 3. Rheological properties of the considered viscoelastic media. Water is included for comparison purposes.

Medium	Density ρ (kg/m ³)	Viscosity μ (mPa · s)	Shear Modulus G (kPa)
Water [14]	1000	1.4	0
Agarose 0.5% gel [7]	1168	144	10
6 wt% gelatin gel [19]	1020	18.3	4
Liver [18,49]	1100	9	40

The results highlight the critical role of the viscoelastic properties of the surrounding medium in shaping the resonant behavior of the bubble.

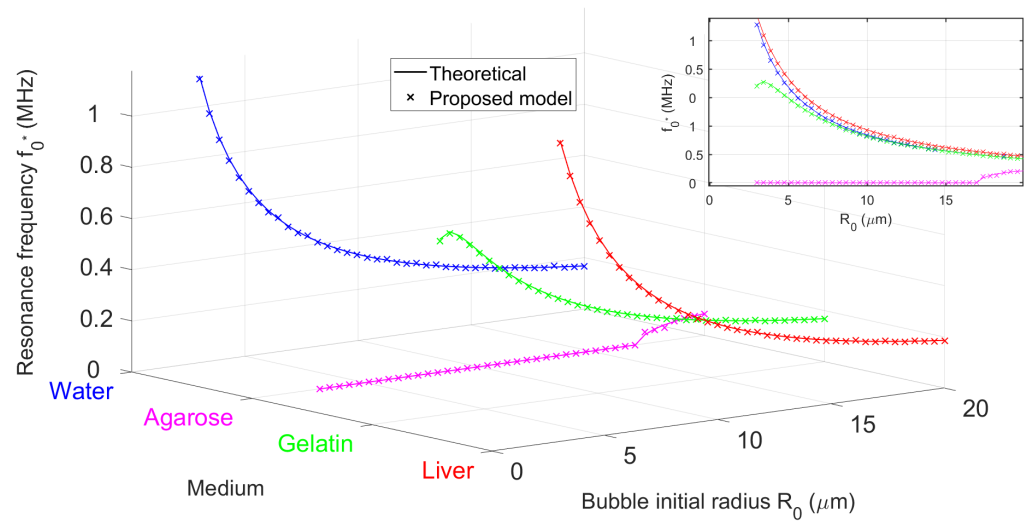


Figure 4. Resonance frequency f_0^* as a function of the initial bubble radius R_0 for the viscoelastic media listed in Table 3. Theoretical values computed from Equation (29) (solid lines) are compared with predictions from the proposed model Equation (25) (star symbols).

3.1.3. Experimental Validation

To further validate the proposed model, we compare our predictions with the experimental data reported in [7], where the resonant behavior of bubbles in viscoelastic gels was characterized using acoustic spectroscopy. In that study, bubbles were driven by increasing frequencies in the range of $f = 10\text{--}50\text{ kHz}$ with steps of 1 kHz. Optical recordings were analyzed frame by frame to extract the bubble radius $R(t)$, and the oscillation amplitude was quantified as $\Delta R = (R_{\text{max}} - R_{\text{min}})/2$, with R_{max} and R_{min} being the maximum and minimum radii. Repeating this procedure for each frequency allowed the reconstruction of experimental resonance curves. Using the same parameters, we simulate bubble oscillations in terms of volume variation and then transform the results into a radius variable for direct comparison. Figure 5 shows the resonance curves obtained with the proposed model in each gel for bubbles with identical equilibrium radii ($R_0 = 180, 178$, and $171\text{ }\mu\text{m}$ in 0.5%, 1%, and 2% gels, respectively). Close agreement is observed between our simulations and the experimental measurements reported in Figure 4 of [7]. This confirms that the present formulation is capable of capturing the main physical mechanisms governing bubble dynamics in viscoelastic media. The observed discrepancies can be attributed to numerical factors, such as precision requirements or differences arising from the truncation of higher-order terms in asymptotic expansions.

3.2. Bubble Volume Dynamics over Time

3.2.1. Bubble Volume Variation in Representative Soft Media

To illustrate the capabilities of the model, we examine the behavior of an air bubble with an initial radius of $R_0 = 4.5\text{ }\mu\text{m}$ excited by an ultrasonic wave at its natural frequency $f = f_0$. The different media analyzed in this study are those listed in Table 3.

Figure 6 presents the variations in the bubble volume obtained from the proposed model (Equation (25)) as a function of time under two excitation conditions: (a) an infinitesimal source amplitude $p_0 = 1\text{ mPa}$ and (b) a high finite source amplitude $p_0 = 5\text{ kPa}$.

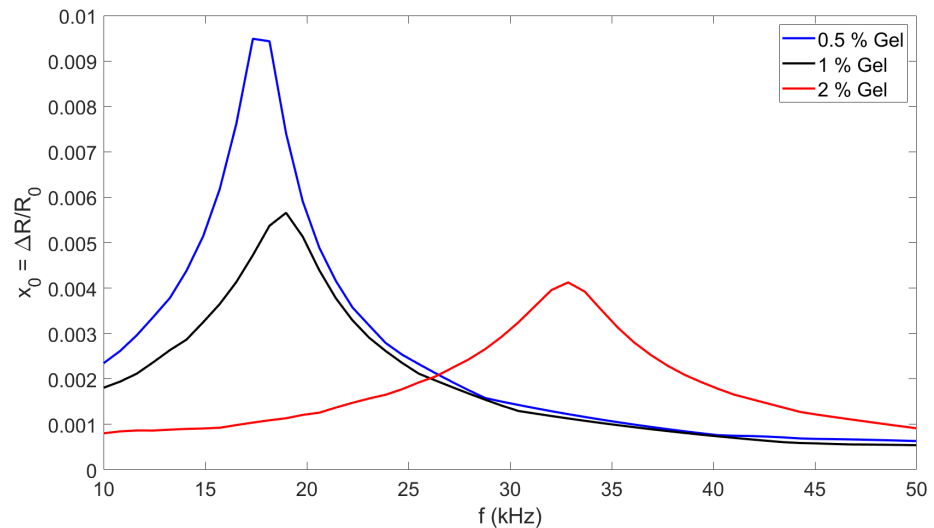


Figure 5. Resonance curves in 0.5%, 1%, and 2% gels obtained with the proposed model expressed in terms of bubble radii. Solid lines show the numerical predictions, which allow validation with experimental data.

For infinitesimal amplitudes (Figure 6a), the system exhibits a linear response characterized by a vertically symmetric sinusoidal waveform. A substantial difference is observed in the amplitude of the volume variation across different media, with water exhibiting the largest oscillations because of the absence of viscoelastic resistance.

However, under high finite-amplitude excitation (Figure 6b), the response becomes nonlinear in the purely liquid medium (water). The signal is visibly distorted and asymmetric with respect to the $v = 0$ axis, indicating nonlinear behavior. In the case of viscoelastic media, the response shows a more restrained increase in amplitude, and the overall behavior remains within the linear regime despite the higher excitation. This confirms that the viscoelastic properties of the medium significantly influence the dynamics of bubble oscillations. In particular, it mitigates certain nonlinear components of the bubble's oscillation. Moreover, the results show that not only an increase in the shear modulus (rigidity) but also an increase in viscosity leads to a notable reduction in the amplitude of bubble oscillations, in agreement with previous studies such as [17,50,51].

The model captures qualitative differences in bubble dynamics by identifying distinct oscillation patterns across media with markedly different rheological properties. Despite the simultaneous variation in elasticity, viscosity, and density, the individual contribution of each property, particularly elasticity, remains unclear, which motivates the following section.

3.2.2. Effect of Medium Elasticity on Bubble Dynamics

A more detailed parametric study is conducted with our proposed model Equation (25) to isolate the specific effects of elasticity on the dynamics of bubbles. We analyze the oscillatory behavior of a bubble with an equilibrium radius of $R_0 = 4.5 \mu\text{m}$, subjected to an infinitesimal acoustic amplitude at the source ($p_0 = 1 \text{ mPa}$) on the one hand and to a finite acoustic amplitude at the source ($p_0 = 5 \text{ kPa}$) on the other hand. The shear modulus is varied within the range of $G = 0\text{--}100 \text{ kPa}$, covering values representative of soft biological tissues [52]: $G = 0$, $G = 50$, and $G = 100 \text{ kPa}$. The viscosity and density are set to $\mu = 0.0014 \text{ Pa} \cdot \text{s}$ and $\rho = 1000 \text{ kg/m}^3$, corresponding to water as the host medium.

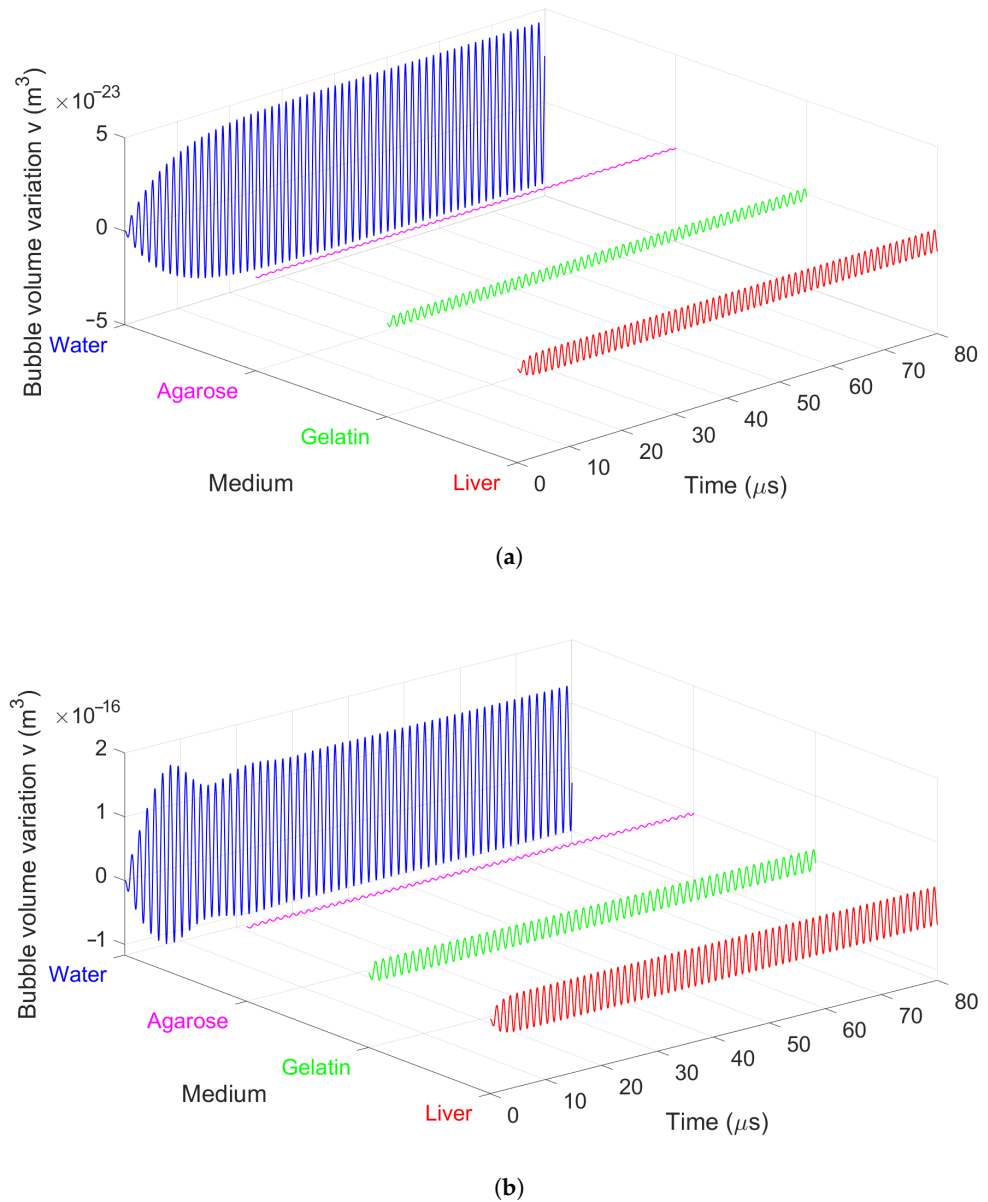


Figure 6. Bubble volume variation obtained from the proposed model (Equation (25)) in the considered soft media listed in Table 3. $f = f_0$. (a) Infinitesimal amplitude $p_0 = 1$ mPa (linear regime); (b) finite amplitude $p_0 = 5$ kPa (nonlinear regime).

The temporal evolution of the bubble volume variation obtained from the proposed model in Equation (25) is illustrated in the following figures, for infinitesimal and finite amplitudes, to evaluate how elasticity influences expansion behavior. To better capture both transient and steady-state dynamics, the initial and final segments of the simulation are also shown. For comparison, results for water ($G = 0$), representing the Newtonian medium case, are included as a reference.

In the linear regime (Figures 7 and 8a,b), the oscillations exhibit sinusoidal-like behavior and are progressively damped as elasticity increases, indicating a strong stabilizing influence. Transients decay rapidly, and a quasi-steady periodic state is reached, with amplitudes notably reduced in stiffer media. Additionally, the results reveal subtle yet important differences in the oscillation phase between bubbles oscillating in a viscoelastic medium and those in a Newtonian fluid. A slight phase shift is observed, with peak oscillations in the viscoelastic case occurring slightly earlier than those in the Newtonian case. This shift becomes more noticeable as the elasticity of the medium increases. This

phase advancement arises from the elastic component of the Kelvin–Voigt model, which introduces a restorative force that accelerates the return of the bubble to equilibrium, thus altering the timing of the oscillatory response. Whereas the amplitude behavior reflects energy dissipation and nonlinear behavior, the phase shift reveals the memory effects inherent in viscoelasticity.

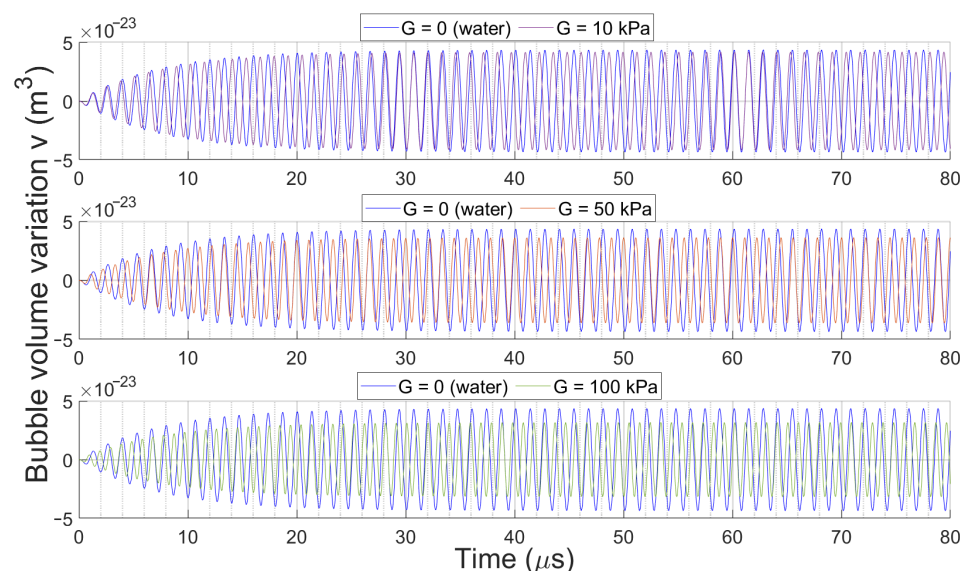


Figure 7. Bubble volume variation curves vs. time from the proposed model (Equation (25)) for different shear moduli G compared with a Newtonian fluid (water). Common values for viscosity, $\mu = 0.0014 \text{ Pa} \cdot \text{s}$, and density, $\rho = 1000 \text{ kg/m}^3$. Infinitesimal amplitude $p_0 = 1 \text{ mPa}$ (linear regime), $0 \leq t \leq T$.

In contrast, in the nonlinear regime (Figures 9 and 10a,b), the bubble exhibits larger amplitudes, and elasticity modulates the nonlinear amplification effect. The initial stage (Figure 10a) shows significant transient growth, with bubbles reaching higher expansion rates in the water media. By the final stage of the simulation, periodic oscillations persist, but their amplitude and waveform vary with G , revealing a complex interplay between elasticity and nonlinear resonance. Elasticity mitigates the large-amplitude oscillations associated with nonlinear resonance. This attenuation at constant pressure amplitude indicates that viscoelasticity can elevate cavitation thresholds, highlighting its relevance for predicting and controlling bioeffects in non-Newtonian media. The differences between Newtonian and viscoelastic responses are more pronounced here, underscoring the importance of rheological effects in finite-amplitude acoustic forcing.

We now carry out a convergence study of our model to verify whether the parameters used in the simulations of this section ensure the convergence of the numerical solution. To this end, we vary the time step τ in the range from $\tau = 1 \times 10^{-7}$ to $\tau = 1 \times 10^{-11}$ in the simulations considered here. Figure 11 shows the relative error with respect to the reference solution obtained with the smallest time step for linear (Figure 11a) and nonlinear (Figure 11b) cases. This study clearly indicates that the chosen time step lies within the convergence range of the model. Similar analyses have been performed for the configurations in the other sections, confirming that convergence is achieved in all simulations presented in this work.

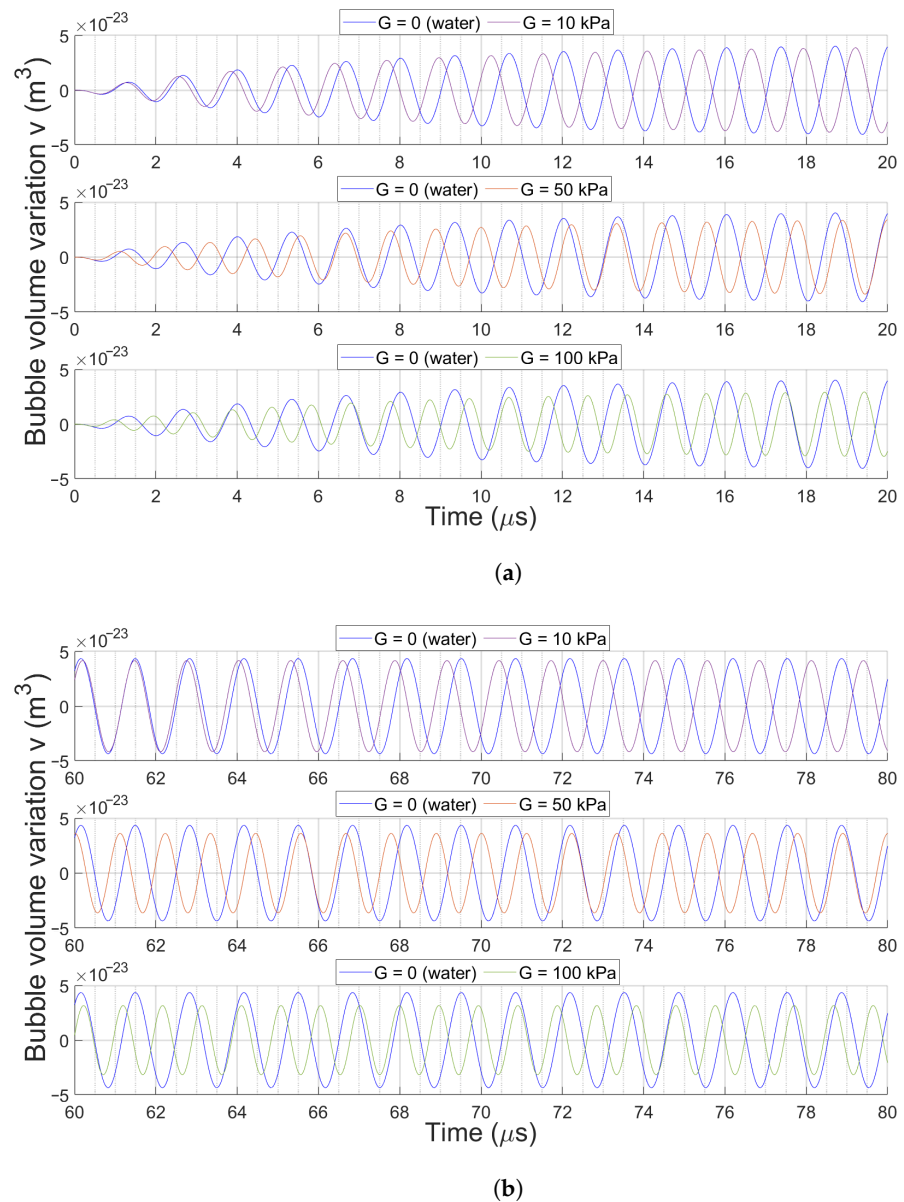


Figure 8. Bubble volume variation curves vs. time from the proposed model (Equation (25)) for different shear moduli G compared with a Newtonian fluid (water). Common values for viscosity, $\mu = 0.0014 \text{ Pa} \cdot \text{s}$, and density, $\rho = 1000 \text{ kg/m}^3$. Infinitesimal amplitude $p_0 = 1 \text{ mPa}$ (linear regime). (a) $0 \leq t \leq 20 \mu\text{s}$; (b) $60 \mu\text{s} \leq t \leq 80 \mu\text{s}$.

3.2.3. Bubble Behavior Across Shear Elasticity–Viscosity Parameter Space

Having examined the behavior of bubbles in well-characterized media and explored the individual effect of elasticity, we now turn to a more application-oriented analysis. This section investigates how the combined viscoelastic properties of soft media influence bubble oscillations, with the aim of optimizing the response depending on the type of tissue or gel in which the bubbles are embedded to achieve the desired effects. To this end, the shear modulus and viscosity are varied in the ranges of $G = 0\text{--}100 \text{ kPa}$ (as in Section 3.2.2) and $\mu = 1\text{--}10 \text{ mPa} \cdot \text{s}$, respectively. Direct dynamic studies on the brain, liver, or kidney are limited, so gels such as agarose or collagen are frequently used [53]. Agarose is highly versatile, with stiffness tunable from values typical of brain tissue ($G = 0.1\text{--}10 \text{ kPa}$) to those of stiffer tissues like the human aorta ($G = 100 \text{ kPa}$).

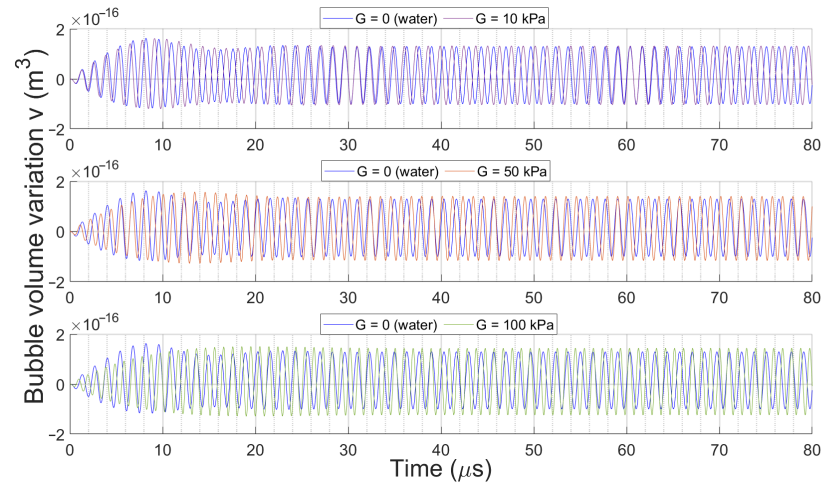
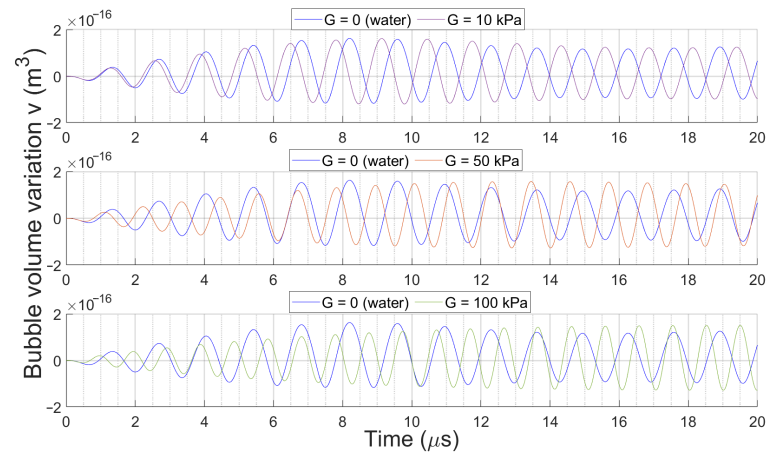
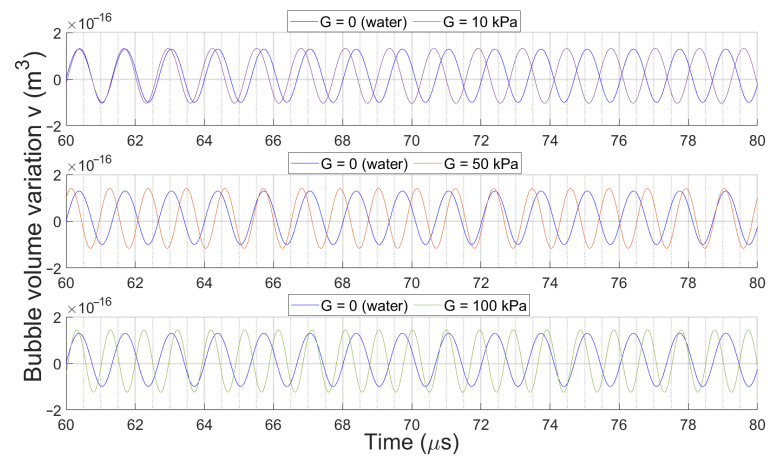


Figure 9. Bubble volume variation curves vs. time from the proposed model (Equation (25)) for different shear modulii G compared with a Newtonian fluid (water). Common values for viscosity, $\mu = 0.0014 \text{ Pa} \cdot \text{s}$, and density, $\rho = 1000 \text{ kg/m}^3$. Finite amplitude $p_0 = 5 \text{ kPa}$ (nonlinear regime), $0 \leq t \leq T$.



(a)



(b)

Figure 10. Bubble volume variation curves vs. time from the proposed model (Equation (25)) for different shear modulii G compared with a Newtonian fluid (water), with $\mu = 0.0014 \text{ Pa} \cdot \text{s}$ and $\rho = 1000 \text{ kg/m}^3$. Finite amplitude $p_0 = 5 \text{ kPa}$ (nonlinear regime). (a) $0 \leq t \leq 20 \mu\text{s}$; (b) $60 \mu\text{s} \leq t \leq 80 \mu\text{s}$.

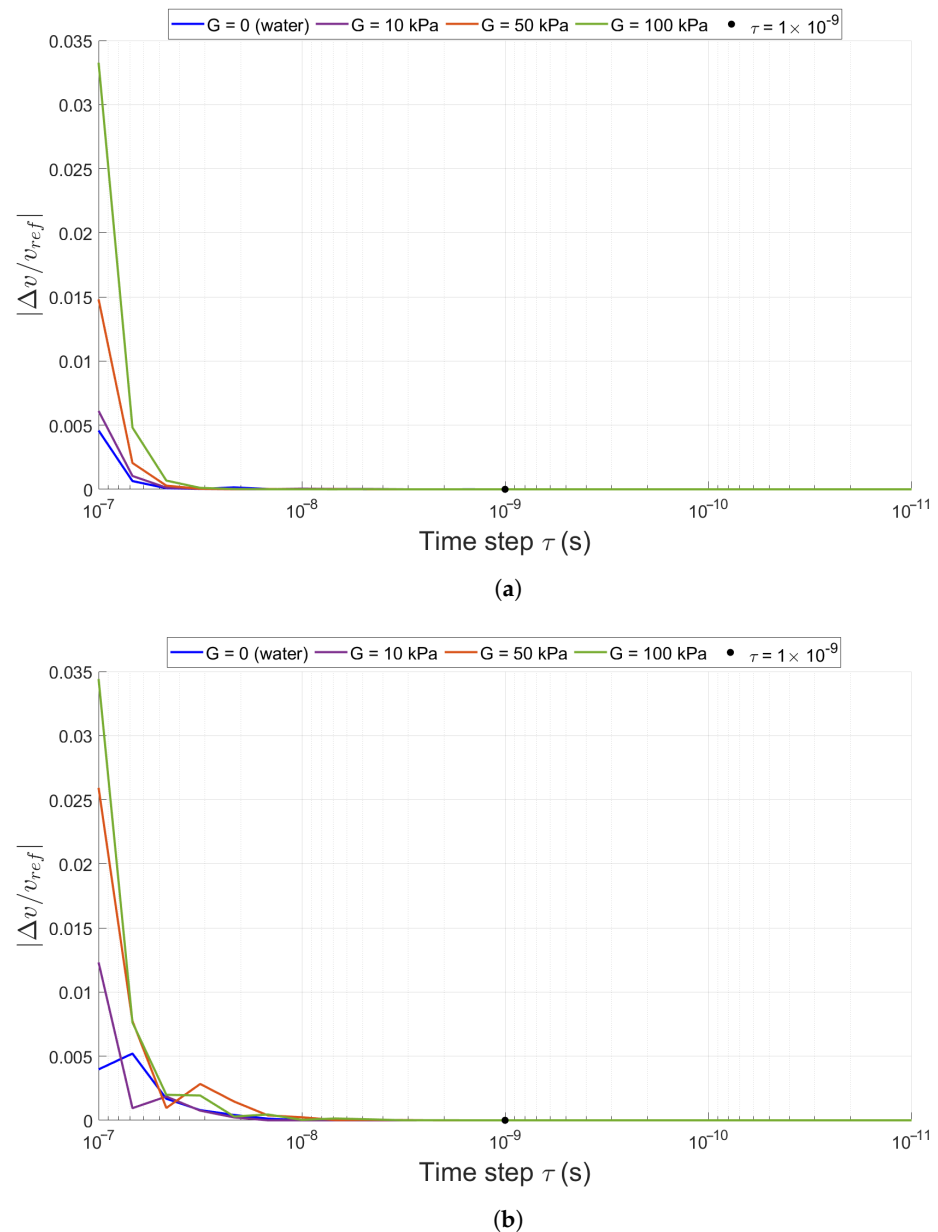


Figure 11. Relative error of the numerical solution with respect to the reference solution at the smallest time step (as a function of the time step τ) for (a) infinitesimal amplitude $p_0 = 1$ mPa (linear regime) and (b) finite amplitude $p_0 = 5$ kPa (nonlinear regime). The chosen time step $\tau = 1 \times 10^{-9}$ lies within the convergence range.

Figure 12 illustrates the dependence of bubble volume oscillations on the viscoelastic properties of the surrounding medium for a given excitation frequency $f = f_0$ by displaying the maximum variation in bubble volume obtained with the proposed model (Equation (25)) for different combinations of shear modulus G and viscosity μ values. It must be noted that $f = f_0$ is set for each pair of (G, μ) because it changes from one medium to the other. The horizontal axis on the right represents the shear modulus G , the horizontal axis on the left shows the viscosity μ , and the vertical axis, with the color scale, corresponds to the maximum variation in the volume of the bubble. The diagram (Figure 12a) corresponds to an infinitesimal excitation amplitude $p_0 = 1$ mPa (linear regime), whereas the diagram (Figure 12b) displays the response under a finite excitation amplitude $p_0 = 5$ kPa (nonlinear regime).

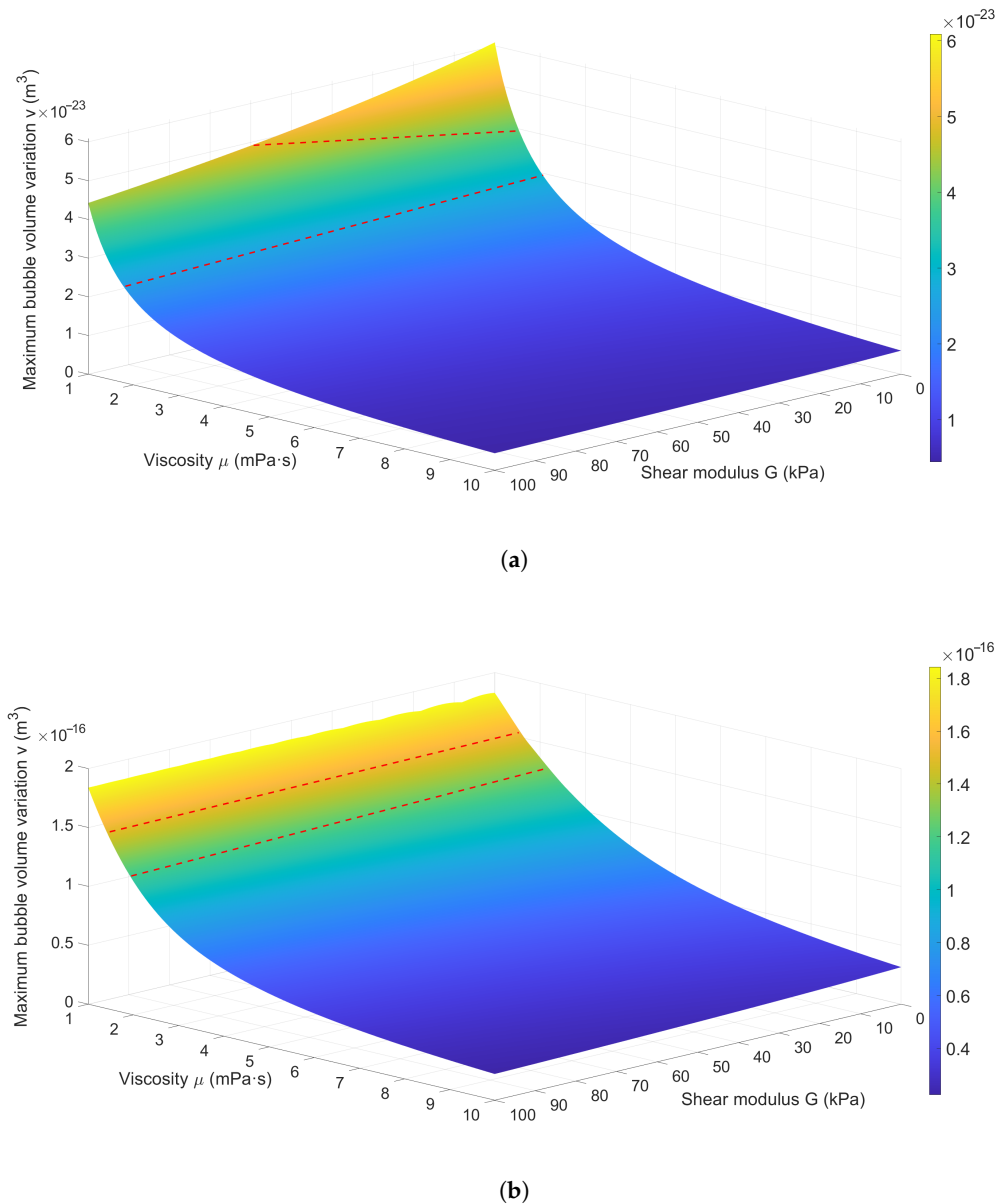


Figure 12. Maximum variation in bubble volume obtained from the proposed model Equation (25) for different combinations of shear modulus and viscosity. $f = f_0$. (a) Infinitesimal amplitude, $p_0 = 1 \text{ mPa}$ (linear regime); (b) finite amplitude, $p_0 = 5 \text{ kPa}$ (nonlinear regime). Dashed straight lines divide the domain in the region of interest described in the text.

In both cases, increasing either the shear modulus or the viscosity results in a reduction in the amplitude of bubble volume oscillations. This trend reflects the enhanced mechanical resistance and energy dissipation provided by stiffer and more viscous media, which suppress bubble expansion. The effect is particularly pronounced along the viscosity axis, where even moderate increases in μ significantly attenuate the bubble response.

In the linear regime, for low-viscosity media ($\mu \leq 2 \text{ mPa}\cdot\text{s}$), a noticeable drop in the amplitude of the oscillation occurs beyond a threshold shear modulus of approximately $G = 60 \text{ kPa}$, indicating that elasticity strongly suppresses bubble vibration when the stiffness is sufficiently high. Moreover, in the same linear regime, when viscosity exceeds approximately $\mu = 3 \text{ mPa}\cdot\text{s}$, viscous damping dominates the dynamics, making the influence of the shear modulus negligible. In contrast, in the nonlinear regime, the bubble exhibits significantly larger oscillation amplitudes across a broad parameter space, as expected, reflecting stronger volumetric deformation when the driving pressure is raised. Unlike the linear case, elasticity in low-viscosity media does not suppress bubble oscil-

lations, and the influence of viscosity becomes noticeable only for $\mu > 8 \text{ mPa} \cdot \text{s}$, where stabilization occurs.

Given the significant variation in mechanical properties across biological tissues and synthetic gels, the dynamic response of the bubble is highly sensitive to the viscoelastic characteristics of the surrounding medium. The parametric map in the G – μ space (shear modulus and viscosity) reveals regions (three in Figure 12a and three in Figure 12b, delimited by dashed lines) where oscillation is maximized or suppressed. In particular, areas can be identified where viscosity dominates over elasticity, or vice versa, providing insights into the relative influence of these two parameters.

This information enables the identification of regions of practical interest. For example, areas with strong bubble response may be targeted for drug delivery applications, where inertial cavitation improves transport and release mechanisms [54]. Note that our bubble model is valid for stable cavitation bubbles generated just before the onset of inertial cavitation. Minimizing tissue damage often requires operating in weak-response regimes [55], and stable cavitation in soft viscoelastic media also enables effective therapeutic interventions such as thrombolysis [56].

Building on this, our model finds interest in several application frameworks [57,58]. Applications of stable acoustic cavitation in gels and tissues include therapeutic methods like drug and gene delivery through microstreaming from oscillating bubbles to increase cell permeability, targeted heating of tissues through efficient energy deposition, and improvements in the structural properties of gels. Viscoelastic media (tissues and gels), in which gas bubbles are added, could be useful for designing new devices based on metamaterials that take advantage of the unique characteristics of bubbly media (dispersion, nonlinearity, and attenuation) and the stability of a bubble population in these media. This is one of the objectives of our studies on bubbly hydrogels [59,60]. Moreover, stable cavitation is a mandatory intermediate process for realizing inertial acoustic cavitation in all applications for which bubble collapse is required, which ranges from cleaning to sonochemistry [61,62].

Importantly, while these applications are diverse, the validity of the present model is restricted to conditions where $v \ll V_0$, i.e., bubble oscillations remain at a moderate amplitude. If the acoustic pressure p_0 exceeds a certain threshold, this assumption is no longer satisfied, and the model no longer accurately describes bubble dynamics. To verify that this condition is satisfied in the experiments shown in Figure 12, we defined $\epsilon = \max(v)/V_0$ and used $\epsilon = 0.5$ as an upper bound for moderate oscillations to determine the corresponding pressure limit p_0 in the range between 1 kPa and 10 kPa. In Figure 13, green points indicate conditions where a threshold is identified ($\epsilon > 0.5$) but exceeds the applied acoustic pressure ($p_0 = 5 \text{ kPa}$); red points indicate that the threshold is exceeded ($\epsilon > 0.5$); p_0 is below 5 kPa, making the model inapplicable; black points denote cases where no threshold could be identified. In this experiment, no red points were observed, which confirms that this condition is satisfied across the entire G – μ space explored and represents the characteristic combinations of parameters in gels and tissues relevant to the aforementioned applications.

This condition is maintained in all the results presented in this manuscript. This validation justifies the selection of a moderate yet finite pressure amplitude ($p_0 = 5 \text{ kPa}$), ensuring that bubble oscillations remain within the moderate range where the model is applicable. Additionally, while a linear Kelvin–Voigt model is employed, this choice is appropriate in the present context because the moderate excitation ensures that nonlinear viscoelastic effects remain negligible. Linear viscoelastic models have limitations, particularly under strong acoustic forcing or large deformations, where nonlinear behav-

ior becomes significant. However, for the range of pressures considered here, the linear approximation adequately captures the main physical mechanisms.

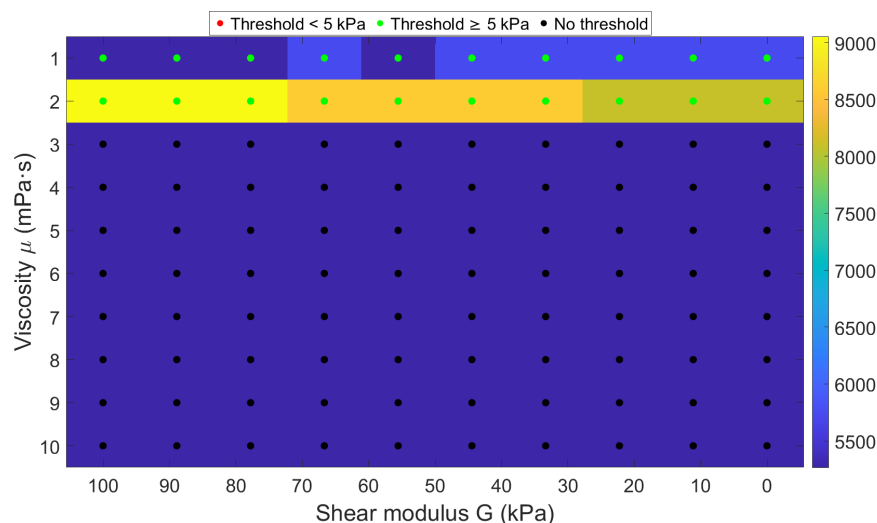


Figure 13. Validation of the moderate oscillation assumption ($\epsilon = \max(v)/V_0$) for the simulations shown in Figure 12. Green points: Threshold exceeds applied pressure ($p_0 = 5$ kPa); red points: threshold exceeded, model inapplicable; black points: threshold not determined.

Once the results are clarified, specific profiles are extracted from Figure 12 to study, in detail, the differences between linear and nonlinear regimes. Figure 14 shows how the normalized maximum amplitude of bubble volume variations varies with an increase in the shear modulus G of the medium for two significant values of viscosity μ under both excitation regimes. The volume response is normalized with respect to the maximum value at the lowest G , allowing comparisons between regimes.

For a medium with low viscosity, such as $\mu = 2$ mPa · s, the normalized bubble response decreases with an increase in shear modulus G in both linear and nonlinear regimes due to the stiffening effect of elasticity. However, the nonlinear regime exhibits a noticeably stronger reduction in oscillation amplitude, especially at low G . This indicates that elasticity exerts a stronger influence in mitigating large-amplitude volumetric oscillations, resulting in a clearer divergence between linear and nonlinear responses as G increases. On the other hand, for higher viscosity $\mu = 8$ mPa · s, both regimes exhibit a more gradual and similar decay in the amplitude of the oscillation with an increase in G , suggesting that elasticity does not play a leading role in reducing the response of the bubble when viscosity is raised.

Similarly, for two fixed values of the shear modulus G , we examine the evolution of the normalized maximum amplitude of bubble volume variations as viscosity μ increases. Figure 15 shows that, in both cases, $G = 10$ kPa and $G = 60$ kPa, the bubble response decreases with an increase in viscosity, reflecting the dissipative effect of viscosity on bubble oscillations. However, this decrease differs from the behavior observed when varying the shear modulus. Three distinct viscosity regions can be identified. In the low-viscosity range ($\mu \in [1, 2]$ mPa · s), the nonlinear regime exhibits a significantly higher normalized amplitude than the linear one, particularly at low elasticity. This suggests that nonlinear excitation is capable of enhancing the bubble response before viscous effects become dominant. In the intermediate-viscosity range ($\mu \in [2, 5]$ mPa · s), viscous damping becomes more influential, and the difference between linear and nonlinear responses begins to narrow, although nonlinear amplification remains visible. Finally, for high viscosity values ($\mu > 5$ mPa · s), both regimes display strong attenuation and converge to similar values, indicating that nonlinear effects are largely suppressed and the response is governed by viscous dissipation.

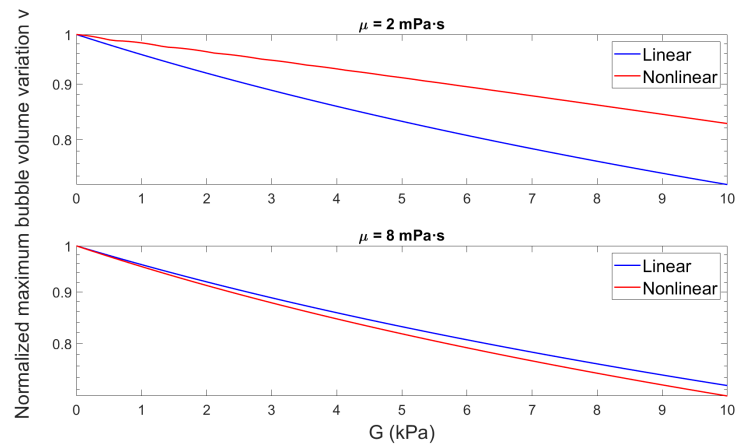


Figure 14. Normalized maximum variation in bubble volume (logarithmic scale on the y-axis) as a function of the shear modulus G under infinitesimal amplitude, $p_0 = 1$ mPa (linear regime), and finite amplitude, $p_0 = 5$ kPa (nonlinear regime), for selected viscosity values $\mu = 2$ mPa $\cdot$ s and $\mu = 8$ mPa $\cdot$ s.

At low elasticity ($G = 10$ kPa), the nonlinear regime maintains a higher normalized amplitude throughout the viscosity range, highlighting its ability to partially counteract damping. However, the increasing viscosity progressively reduces this difference. In contrast, at higher elasticity ($G = 60$ kPa), both regimes exhibit stronger attenuation and a more similar trend, with minimal nonlinear enhancement. This suggests that in media with high stiffness and viscosity, the oscillatory response of the bubble is significantly minimized, and nonlinear effects lose their ability to modify its dynamics.

Therefore, selecting appropriate values for the shear modulus G and viscosity μ , for fixed excitation amplitudes and frequencies, turns out to be a critical design criterion. These parametric diagrams thus serve as a practical tool for tailoring the properties of the host medium to achieve the desired cavitation responses in soft tissues, hydrogels, or engineered viscoelastic materials.

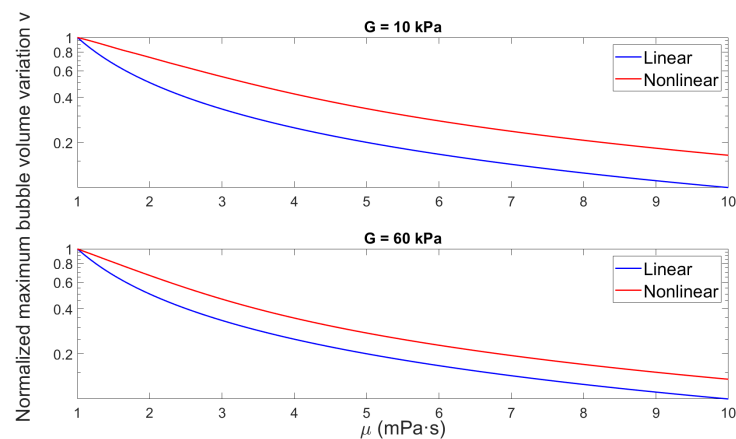


Figure 15. Normalized maximum variation in bubble volume (logarithmic scale on the y-axis) as a function of viscosity μ under infinitesimal amplitude, $p_0 = 1$ mPa (linear regime), and finite amplitude, $p_0 = 5$ kPa (nonlinear regime), for selected elasticities $G = 10$ kPa, and $G = 60$ kPa.

4. Conclusions

We proposed a nonlinear ordinary differential equation to model the nonlinear oscillations of a gas bubble subjected to an ultrasonic field, formulated in terms of bubble volume variations within a viscoelastic medium and valid in the stable cavitation framework. The model was developed by considering a second-order approximation of the adiabatic gas

law in the Rayleigh–Plesset equation that is valid for an arbitrary viscoelastic medium. The Kelvin–Voigt model was incorporated to account for the viscoelastic response of the surrounding medium, thus coupling classical nonlinear Rayleigh–Plesset dynamics with a linear viscoelastic constitutive relation. The proposed model was solved numerically and successfully exposed to a convergence analysis, and the results were compared with theoretical resonance curves and experimental data, showing close agreement and validating our formulation. As expected, the resonance frequency increases with the shear modulus because of enhanced stiffness, while increasing viscosity introduces damping, which slightly reduces the resonance frequency. This approach offers a consistent framework for analyzing how the viscoelastic properties of the medium, specifically shear elasticity, influence the volumetric dynamics of bubbles. This revealed significant deviations from Newtonian predictions: In particular, the oscillation phase shifts slightly due to elastic energy storage, i.e., the maximum expansion of the bubble occurs earlier in viscoelastic media. These effects become more pronounced with an increase in the shear modulus. By analyzing the bubble dynamics of the shear elasticity–viscosity parameter space, it is evident that both viscosity and elasticity play crucial roles in stable cavitation behavior. Elasticity has a pronounced effect in low-viscosity media, whereas viscosity emerges as the dominant factor modulating the amplitude of oscillations in both linear and nonlinear regimes. These results emphasize that accurately modeling bubble volume variations requires accounting for the full viscoelastic behavior of the medium, which is essential for predicting and controlling stable bubble dynamics. This work thus provides a robust tool for analyzing how viscoelastic properties affect bubble dynamics, contributing to the improved prediction and control of stable cavitation phenomena in complex media. Finally, while the present study adopts the linear Kelvin–Voigt model as a first approximation, future work will consider the Zener model. This will allow us to incorporate both elasticity and relaxation time, providing a more accurate representation of memory effects and delayed recovery in soft tissues. Additionally, the fourth-order approximation in the Rayleigh–Plesset model [32] will be explored to capture bubble dynamics in this context, which is expected to be particularly useful for driving frequencies near bubble resonance. Also, the bubble equation developed here will be considered to model the mutual nonlinear influence of ultrasound and populations of multiple bubbles in viscoelastic media.

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