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A Novel Exponentiated Pareto Exponential Distribution with Applications in Environmental and Financial Datasets

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Abstract

Environmental and financial datasets often display complex distributional characteristics, including heavy tails, high skewness and the presence of extreme observations. Traditional probability models such as the exponential, gamma or log-normal distributions may not adequately capture these behaviours particularly when modelling extreme events such as rainfall, pollution levels, stock returns or loss severities. By integrating the characteristics of Pareto and exponential distributions into an exponentiated framework that can describe datasets arising from environmental and finance fields, this study presents a novel three-parameter exponentiated Pareto exponential distributions using the exponentiated Pareto family of distributions with classical exponential distribution as the baseline model. This novel model extends the classical exponential distribution with the addition of extra shape parameters which simultaneously regulate the centre and tail behaviours of the new model. The statistical and mathematical characteristics of the proposed distribution are determined and studied. The maximum likelihood estimate approach is used in a conducted simulation exercise, and the estimator's efficiency is evaluated as seen from the results. The practical applicability of the model is illustrated with four real-life datasets utilising model adequacy and goodness-of-fit measurements such as log–likelihood, Akaike information criteria and Bayesian information criteria. The data reveal that the proposed model gives a better fit than the models chosen as comparators, making the EPE distribution useful and robust in environmental and financial fields of study.

Keywords: extreme events; Pareto-type model; risk assessment; statistical models

1. Introduction

Environmental and financial datasets often display complex distributional characteristics, including heavy tails, high skewness and the presence of extreme observations. Traditional probability models such as the exponential, gamma or log-normal distributions may not adequately capture these behaviours particularly when modelling extreme events such as rainfall, pollution levels, stock returns or loss severities [1,2]. As a result, there has been growing interest in developing flexible statistical distributions capable of modelling both the bulk and tail of the data more effectively.

In recent years, exponentiated and generalised families of distributions have become popular due to their ability to introduce additional shape parameters that enhance flexibility [3,4]. Pareto-type models, known for their heavy-tailed properties, have also been widely applied in extreme modelling of events in environmental science and finance [4,5]. Motivated by the advantages of these families, this study introduces a novel exponentiated

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Pareto exponential distribution (EPED) that combines Pareto-induced heavy tails with exponential decay and an exponentiation mechanism to improve shape adaptability. The proposed model is evaluated using environmental and financial datasets to demonstrate its modelling capability. By offering enhanced flexibility and tractability, the EPED provides a promising alternative for risk assessment, prediction and statistical modelling in both fields.

The concept of exponentiating a baseline distribution to enhance flexibility was first introduced by [3,6], in which the exponentiated–exponential family was derived. This innovation allowed researchers to capture both increasing and decreasing hazard rates in reliability data. Reference [4] provided a method for adding parameters to classical distributions, suggesting a more generic expansion methodology. Reference [7] made additional progress by examining several exponentiated type distributions and showcasing their enhanced flexibility in simulating skewed datasets.

Generalised families of distributions such as the T-X family by [8] and beta-generated families by [9] have gained popularity for their mathematical tractability and superior fitting performance. These families allow statisticians to generate new distributions tailored to data with complex behaviour. Reference [10] proposed the Topp–Leone–G family of distributions. The pdf and cdf of the new family were given as the weighted sum of exponentiated G distribution. The exponential distribution was extended as a special case of the new family of distributions. The Topp–Leone exponential distribution was applied to real life datasets and the results indicated that the data were adequately fitted by Topp–Leone exponential distribution. Reference [11] first proposed the exponentiated Weibull distribution to analyse bathtub failure data. A new class of exponentiated generalised distribution that extends the exponentiated G class was proposed by [12]. The family became popular after the published works by [13,14] and [3] on the exponentiated–exponential distribution. The exponentiated Weibull distribution was presented by [11] and was considered a good distribution. The cdf of the Kumaraswamy–G family using the distribution function pioneered by [15] was proposed by [16]. The beta-generated family of distribution was proposed by [17] by using the beta distribution as a generator. This family of distributions can be described as a generalization of the distribution of order statistics [18]. Based on the idea of T-X family pioneered by [8], Fréchet Topp–Leone–G family was proposed by [19] and Some of its mathematical properties were studied. Some of the recently developed families of probability distributions are the ones by [20–31].

Similarly, Ref. [32] introduced the exponentiated Pareto–G (EP-G) family of distributions, with probability density function (pdf) and cumulative distribution function (cdf) given as follows:

$$F(x;\pi,\kappa,\varpi) = \left[1-(1+H(x;\varpi))^{-\pi}\right]^{\kappa} \tag{1}$$

and

$$f(x;\pi,\kappa,\varpi) = \pi\kappa h(x;\varpi)(1+H(x;\varpi))^{-\pi-1}\left[1-(1+H(x;\varpi))^{-\pi}\right]^{\kappa-1}. \tag{2}$$

$$0 \le x \le \infty.$$

where $\pi, \kappa > 0$ are the shape parameters and $\varpi > 0$ is a vector of parameters depending on the considered baseline distribution.

The exponentiated Pareto–G family is adopted due to its ability to generate highly flexible distributions with heavy-tail behaviour, diverse hazard rate shapes, tractable mathematical properties, and superior goodness-of-fit compared to classical lifetime models.

The Pareto distribution and its extensions are widely known for modelling heavy-tailed phenomena such as income inequality, flood levels and financial losses [5,33]. The generalised Pareto distribution has become central in extreme value theory and is frequently used for modelling excesses over thresholds in hydrology and finance [1,34].

Pareto-based models have also been widely employed in risk management, where large deviations and rare events play a crucial role [35]. The combination of Pareto properties with other distributions has shown promise in creating hybrid models capable of capturing both central tendencies and extreme outcomes effectively.

Environmental datasets such as rainfall intensity, pollutant concentration, temperature anomalies and drought indicators often exhibit skewed or heavy-tailed distributions [36]. Extreme value models are commonly applied to estimate the probability of rare but impactful events such as floods, storms, or heatwaves [1].

Financial data are similarly characterised by volatility, skewness and fat tails. Daily returns, exchange rates and loss severities frequently deviate from the assumptions of normality [37]. Heavy-tailed models such as the generalised Pareto distribution, Student-t and mixture distributions have been widely used to capture the stochastic nature of financial markets [2,38]. Pareto based models have been particularly important in quantifying risk measures such as Value at Risk (VaR) and expected shortfalls [35].

Given the similarity in distributional patterns between environmental and financial data, a flexible distribution such as the proposed EPED is expected to perform well in both areas.

Environmental and financial sectors require modelling tools to understand and predict extreme events. Traditional models often underestimate the probability of extreme outcomes such as heavy rainfall, severe pollution peaks, large stock market crashes or catastrophic insurance losses leading to insufficient preparedness or poor risk assessment.

The motivation for proposing the EPED stems from the following needs:

i. Greater flexibility in modelling datasets with varying skewness and tail behaviour;
ii. Improved tail modelling, especially where extreme events have major consequences;
iii. A unified distribution that performs well in both environmental and financial contexts;
iv. Better risk qualification for decision making in climate science, environmental policy and finance.

Because the EPED incorporated both exponential and Pareto characteristics while adding shape flexibility through exponentiation, it is well suited to datasets where traditional models fail.

This research work is aimed at developing a novel EPED and applying it to modelling real-world datasets arising from the environmental and financial field of research.

The significance of this study is that the EPED enriches statistical literature by introducing a new hybrid model that combines advantages of exponential, Pareto and exponentiated families, and by doing so, it provides a flexible model that can capture a wide range of shapes and tail behaviours.

Accurate modelling of environmental and financial data is crucial for policy development, risk assessment and forecasting. The EPED enhances modelling accuracy for both moderate and extreme events.

The EPED has a cross-disciplinary approach, as the model can be used across two major fields such as environmental science and finance, demonstrating its adaptability and robustness.

By better capturing tail risk, the EPED may support improved disaster preparedness, financial stress testing and insurance pricing.

This model can also be extended, compounded or generalised providing a basis for future theoretical advances.

The exponential distribution's memorylessness and mathematical simplicity make it a basic building block in applied probability [39]. In reliability, survival analysis, queueing, and basic risk models where a constant failure/arrival rate is suitable, it is still utilised as a baseline model; new studies of parametric lifetime models summarise these traditional applications and maintain its canonical status [40].

The non-constant hazard behaviour observed in numerous datasets is frequently not captured by the single-parameter exponential. A substantial amount of recent work that expands the exponential family by adding shape or mixing parameters such as generalised, exponentiated, transmuted, and T-X-generated families has been inspired by this flaw. These extensions, which are frequently used for lifetime and reliability data, increase flexibility for modelling rising, falling, bathtub, or unimodal hazard rates. Because they maintain tractability while adding a shape parameter, the exponentiated–exponential and several other exponentiated/integrated variations are widely utilised. Applications in the modern era go beyond traditional dependability. The exponential is used as a baseline in accelerated failure-time and cure-rate models and as a building block for more intricate censoring/competing-risks setups in survival analysis and biomedical statistics. Recent methodological papers highlight the exponential's role in scalable inference (including Bayesian approaches) for truncated and censored data [41].

Many studies use generalised exponential variations (typically via compound or mixture constructions) to incorporate skewness and heavy tails in environmental and financial modelling. For dependence modelling and tail risk measurements (VaR, ES), where closed-form or semi-closed form expressions are useful, researchers such as [42] also used exponential-based copulas and exponential mixtures in which an extended exponential distribution was derived and applied to a financial dataset. The cdf and pdf of the classical exponential distribution is given by

$$H(x;\eta) = 1 - e^{-\eta x} \tag{3}$$

and

$$\begin{gathered} h(x,\eta) = \eta e^{-\eta x} \\ x \geq 0, \eta > 0. \end{gathered} \tag{4}$$

However, the need to expand the scope of the exponential distribution has become increasingly evident in modern statistical modelling to address its shortcomings.

The novelty of the proposed model is:

- A new EPED is introduced, providing a flexible four-parameter model that generalises several classical lifetime and Pareto-type models.
- The proposed model exhibits rich hazard rate behaviour, including increasing, decreasing, bathtub, and unimodal shapes, making it suitable for complex survival patterns in environmental and financial data.
- Closed-form expressions for key distributional and actuarial functions are derived, enabling efficient statistical inference, simulation, and risk analysis.
- Empirical applications show that the new model outperforms competing distributions in terms of goodness-of-fit and tail representation for real environmental and financial datasets.

2. Materials and Methods

This section derives the EPED by utilising the EP-G family of distributions proposed by [32]. The expressions for the cdf and pdf of the EPED are presented. Additionally, graphical representations of the cdf and pdf are provided, highlighting their unique and distinct shapes.

The EPED with cdf expressed in the following equation is obtained by substituting the cdf specified in Equation (3) into the cdf specified in Equation (1) as follows:

$$F(x;\pi,\kappa,\eta) = \left[1 - \left(1 + 1 - e^{-\eta x}\right)^{-\pi}\right]^{\kappa}$$

$$F(x;\pi,\kappa,\eta) = \left[1 - \left[2 - e^{-\eta x}\right]^{-\pi}\right]^{\kappa}. \tag{5}$$

Equation (5) is differentiated with respect to x to obtain the pdf of the EPED, which is as follows:

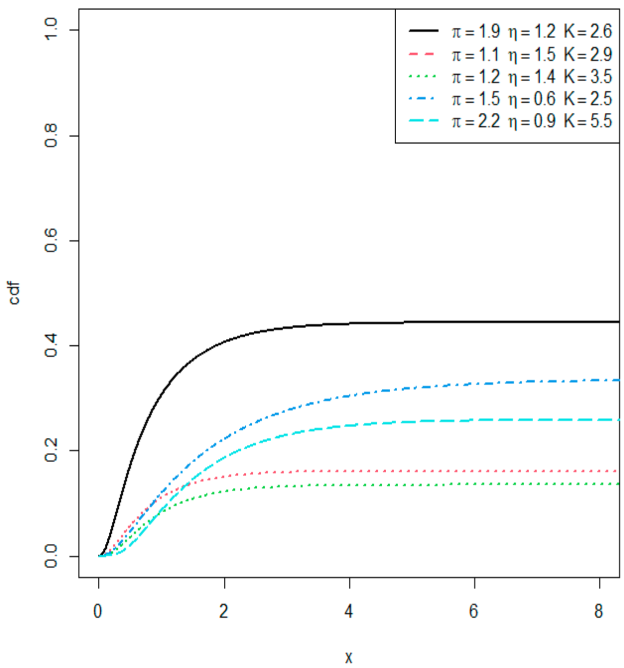
$$\frac{\partial F(x;\pi,\kappa,\eta)}{\partial x} = \pi\kappa\eta e^{-\eta x}\left(2 - e^{-\eta x}\right)^{-\pi-1}\left[1 - \left(2 - e^{-\eta x}\right)^{-\pi}\right]^{\kappa-1}$$

Hence, the pdf of the EPED is given as

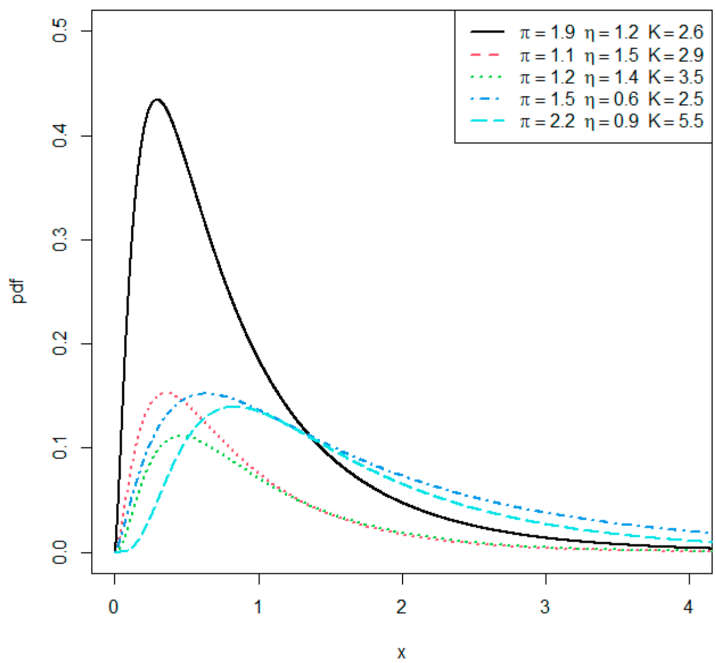
$$f(x;\pi,\kappa,\eta) = \pi\kappa\eta e^{-\eta x}\left(2 - e^{-\eta x}\right)^{-\pi-1}\left[1 - \left(2 - e^{-\eta x}\right)^{-\pi}\right]^{\kappa-1} \tag{6}$$

$$0 \le x \le \infty,\ \pi,\kappa > 0 \text{ and } \eta > 0.$$

The EPED cdf is shown in Figure 1a. The EPED can be used to describe both mild and severe environmental occurrences because it flexibly incorporates the variety of tail behaviours seen in environmental extremes. The EPED pdf is shown in Figure 1b. Compared to traditional exponential or normal models, the pdf offers a more realistic framework for tail-risk assessment by accommodating heavy-tailed loss behaviour frequently seen in financial markets.



(**a**)



(**b**)

Figure 1. Plots of the (**a**) cdf and (**b**) pdf of the EPE distribution with varying parameter values.

3. Results

This section presents the results obtained from the new model. These results include the linear representation of the density, the properties of the models, order statistics, actuarial measure, simulation study and applications in real-life datasets. The properties that have been derived and studied include moments, moment-generating function, quantile function, survival function, hazard rate function, reverse hazard function, cumulative hazard rate, and odds function.

3.1. Linear Representation of the Density

This subsection presents the density expansion of the EPED as a linear representation of the EPED. The general binomial is expressed as

$$(1-\wp)^{\vartheta}=\sum_{i=0}^{\infty}\frac{(-1)^{i}\Gamma(\vartheta)}{i!\Gamma(\vartheta-i)}\wp^{i}. \tag{7}$$

For any real or complex parameter, ϑ, the binomial expansion is

$$(1-\wp)^{\vartheta}=\sum_{j=0}^{\infty}\binom{\alpha}{j}\wp^{j},\ |\wp|<1.$$

When $\alpha=\vartheta-1$ and binomial coefficients are expressed using Gamma functions,

$$\binom{\vartheta-1}{j}=\frac{\Gamma(\vartheta)}{j!\Gamma(\vartheta-j)}.$$

It allows complicated expressions such as $(1-\wp)^{\vartheta-1}$ to be rewritten as an infinite sum. Using Equation (7) on the pdf of EPED, the density expansion of the EPED is given as

$$f(x;\pi,\kappa,\eta)=\pi\kappa\eta\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}\Gamma(\kappa)\Gamma(-\pi(i+1))}{i!j!\Gamma(\kappa-i)\Gamma(-\pi(i+1)-j)}\left[e^{-\eta x}\right]^{j+1}. \tag{8}$$

3.2. Properties of the EPED

This section deals with the statistical and mathematical properties of the EPED highlighted above.

3.2.1. Moments

The shape and characteristics of a pdf are described by moments and are used to estimate the mean, variance skewness and kurtosis. The rth moment of X is obtained by

$$\mu_r'=\int_{-\infty}^{\infty}x^r f(x)dx. \tag{9}$$

Using Equation (9), the rth moments for the EPED is obtained as follows:

$$\mu_r'=\pi\kappa\eta\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}\Gamma(\kappa)\Gamma(-\pi(i+1))}{i!j!\Gamma(\kappa-i)\Gamma(-\pi(i+1)-j)}\int_{0}^{\infty}x^r\left[e^{-\eta x}\right]^{b+1}dx. \tag{10}$$

On solving the integral part in Equation (10), the moments of the EPED are obtained as follows:

$$\mu_r'=\pi\kappa\eta^{-r}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}\Gamma(\kappa)\Gamma(-\pi(i+1))}{i!j!\Gamma(\kappa-i)\Gamma(-\pi(i+1)-j)}\frac{\Gamma(r+1)}{(j+1)^{r+1}}. \tag{11}$$

When $r=1$ in Equation (11), the mean of the EPED is obtained.

3.2.2. Moment-Generating Function (mgf)

The mgf is a strong tool in probability theory used to examine random variables. They turn a random variable into a function that simplifies the calculation of crucial features, such as the mean, variance, skewness, and kurtosis. The mgf $M_x(t) = E(e^{tx})$ of a random variable X is defined as

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx. \tag{12}$$

The mgf of the EPED is given as

$$M_x(t) = \pi\kappa\eta^{-m} \sum_{i,j,m=0}^{\infty} \frac{(-1)^{i+j}\Gamma(\kappa)\Gamma(-\pi(i+1))t^m}{i!j!m!\Gamma(\kappa-i)\Gamma(-\pi(i+1)-j)} \frac{\Gamma(m+1)}{(j+1)^{m+1}}. \tag{13}$$

3.2.3. Quantile Function (qf)

The qf of a probability distribution is crucial in statistical analysis. It computes the value in the cdf of a distribution inversely at a given probability, and it serves as a cornerstone in statistical modelling. The EPED has a qf given as follows:

$$M(q) = \frac{1}{\eta}\left\{-\log\left[2 - \left[1 - q^{\frac{1}{k}}\right]^{\frac{1}{-\pi}}\right]\right\}. \tag{14}$$

To obtain the median of the EPED, we set $q = 0.5$ in Equation (14).

3.2.4. Survival Function (sf)

The chance that a person or thing will live longer than a given period is represented by the sf. It gives an overview of the population's experience with survival. The sf is mathematically defined as

$$s(x) = 1 - F(x). \tag{15}$$

The sf of the EPED is given as

$$s(x) = 1 - \left[1 - \left[2 - e^{-\eta x}\right]^{-\pi}\right]^{\kappa} \tag{16}$$

3.2.5. Hazard Rate Function (hrf)

The hrf reflects the risk of an event occurring at a particular period, given that the individual has survived up to that time. It helps identify periods of high or low risk, and it provides insight into the likelihood of failure or event occurrence across time. The mathematical representation of the hrf is

$$\Lambda(x) = \frac{f(x)}{s(x)} \tag{17}$$

The hrf of the EPED is given as

$$\Lambda(x) = \frac{\pi\kappa\eta e^{-\eta x}(2 - e^{-\eta x})^{-\pi-1}\left[1 - (2 - e^{-\eta x})^{-\pi}\right]^{\kappa-1}}{1 - \left[1 - \left[2 - e^{-\eta x}\right]^{-\pi}\right]^{\kappa}} \tag{18}$$

3.2.6. Reverse Hazard Function (rhf)

The rhf of the EPED is given as

$$\Re(x) = \frac{\pi\kappa\eta e^{-\eta x}(2 - e^{-\eta x})^{-\pi-1}\left[1 - (2 - e^{-\eta x})^{-\pi}\right]^{\kappa-1}}{\left[1 - \left[2 - e^{-\eta x}\right]^{-\pi}\right]^{\kappa}} \tag{19}$$

3.2.7. Cumulative Hazard Rate (chr)

The chr is expressed by using the general expression as follows:

$$\Phi(x) = -lin(s(x)) \tag{20}$$

The chr of the EPED is given as

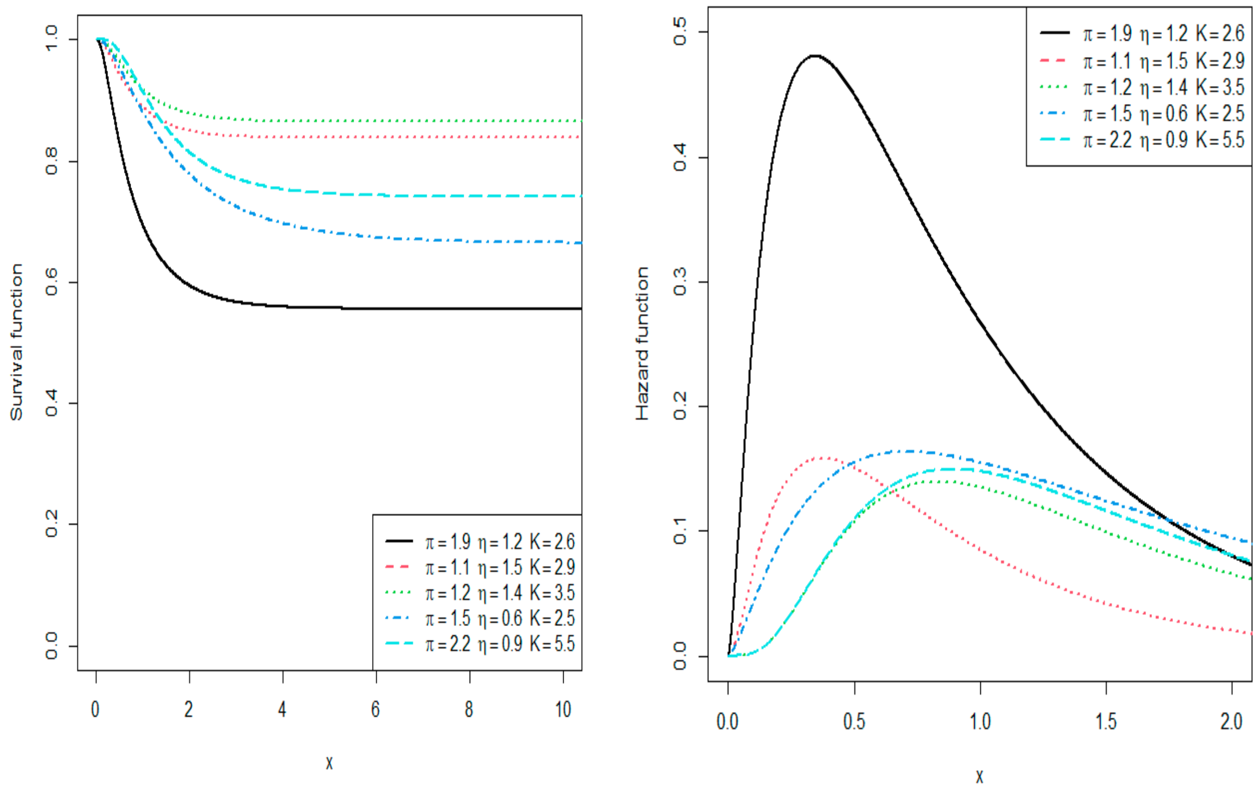
$$\Phi(x) = -lin\left[1 - \left[1 - \left[2 - e^{-\eta x}\right]^{-\pi}\right]^{\kappa}\right] \tag{21}$$

3.2.8. Odds Function

The odds function of the EPED is given as

$$\prod(x) = \frac{\left[1 - \left[2 - e^{-\eta x}\right]^{-\pi}\right]^{\kappa}}{1 - \left[1 - \left[2 - e^{-\eta x}\right]^{-\pi}\right]^{\kappa}} \tag{22}$$

The curve in Figure 2a shows the survival function curves of the EPED. The curves demonstrate rapid early decay and a sharp first drop in survival probability, suggesting that while very large events become less likely as x increases, moderate environmental catastrophes happen frequently. Additionally, these curves show a longer persistence of extreme occurrences and a slower decay, which is compatible with climates or places that experience protracted rainfall, extended droughts, or chronic pollution episodes.



(**a**) (**b**)

Figure 2. Plots of the (**a**) survival function and (**b**) hazard function of the EPED with varying parameter values.

In financial applications, higher survival probabilities reflect heavy-tailed loss distributions, where extreme losses occur with non-negligible probability, which is common during financial crises or in insurance portfolios exposed to catastrophic risks. Lower survival probabilities indicate a lower likelihood of very large losses, typical of stable markets or well-diversified portfolios. Expected shortfall and Value at Risk (VaR) increase with heavier tails.

The hazard rate function of the EPED for various parameter combinations is shown in Figure 2b. Every curve has a unimodal (upside-down bathtub) shape, meaning that the risk first rises, peaks, and then falls. Systems where risk increases, peaks, and then decreases are represented by this form. The lifecycle of environmental hazards, from commencement to peak intensity and ultimate decline, is accurately represented by the EPED. Particularly throughout the phases of crisis build-up and resolution, the unimodal hazard structure of the EPE distribution is in good agreement with actual financial risk dynamics.

3.3. Distribution of Order Statistics

In several fields, such as inference and quality control, order statistics have proved crucial. They are employed in sorting the data used to deal with the application of this ordered data and their related functions. Order statistics are used for numerous reasons, including reliability, outlier detection, quality control, disaster planning, statistical inference, health-related data analysis and goodness-of-fit assessment. Thus, the distribution order statistics is given as follows:

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)} \sum_{i=0}^{n-r} (-1)^i F(x)^{r+i-1} f(x). \tag{23}$$

Substituting the cdf and pdf of the EPED into Equation (23), we have

$$f_{r:n}(x) = \frac{\pi\kappa\eta}{B(r, n-r+1)} \sum_{i=0}^{n-r} (-1)^i \left[\left[1 - \left[2 - e^{-\eta x} \right]^{-\pi} \right]^{\kappa} \right]^{r+i-1} e^{-\eta x} \left(2 - e^{-\eta x} \right)^{-\pi-1} \left[1 - \left(2 - e^{-\eta x} \right)^{-\pi} \right]^{\kappa-1}. \tag{24}$$

On expanding and simplifying Equation (24), the rth order statistic for the EPED is obtained as follows:

$$f_{r:n}(x) = \frac{\pi\kappa\eta}{B(r, n-r+1)} \sum_{i=0}^{n-r} \sum_{j,l=0}^{\infty} (-1)^{i+j+l} \frac{\Gamma(\kappa(r+i))\Gamma(-\pi(j+i))}{j!l!\Gamma(\kappa(r+i)-j)\Gamma(-\pi(j+i)-l)} e^{-\eta x}. \tag{25}$$

To obtain the pdf of the minimum order statistic for the EPED, we set $r = 1$ in Equation (25) as follows:

$$f_{1:n}(x) = \frac{\pi\kappa\eta}{B(1, n)} \sum_{i=0}^{n-1} \sum_{j,l=0}^{\infty} (-1)^{i+j+l} \frac{\Gamma(\kappa(1+i))\Gamma(-\pi(j+i))}{j!l!\Gamma(\kappa(1+i)-j)\Gamma(-\pi(j+i)-l)} e^{-\eta x}. \tag{26}$$

Also, to obtain the pdf of the maximum order statistic for the EPED, we set $r = n$ in Equation (25) as follows:

$$f_{n:n}(x) = \frac{\pi\kappa\eta}{B(n, 1)} \sum_{j,l=0}^{\infty} (-1)^{j+l} \frac{\Gamma(\kappa(n))\Gamma(-\pi(j))}{j!l!\Gamma(\kappa(n)-j)\Gamma(-\pi(j)-l)} e^{-\eta x}. \tag{27}$$

3.4. Value at Risk (VaR) Measure

Risk measurement is the process of assessing and calculating the possible loss connected to a choice, course of action, or investment. To maximise return while accepting a calculated risk, it seeks to prioritise the seriousness of possible outcomes of any activity and arrange resource allocation accordingly. A popular risk metric called Value at Risk (VaR) calculates the maximum possible loss of a random variable (such as portfolio loss

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or return) over a set time horizon at a particular confidence level. The VaR is presented in quantile form as

$$VaR_p = Q_X(p) \tag{28}$$

Then, the VaR for the EPED is given by

$$VaR_p = \frac{1}{\eta}\left\{-\log\left[2-\left[1-p^{\frac{1}{\kappa}}\right]^{\frac{1}{-\pi}}\right]\right\} \tag{29}$$

VaR_p is the pth-quantile of the loss distribution and represents a loss threshold not exceeded by probability p.

3.5. Asymptotic Behaviour of EPED

The asymptotic behaviour of the EPED for $x \to 0$ and $x \to \infty$ is

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[\pi\kappa\eta e^{-\eta x}\left(2-e^{-\eta x}\right)^{-\pi-1}\left[1-\left(2-e^{-\eta x}\right)^{-\pi}\right]^{\kappa-1}\right] = 0 \tag{30}$$

and

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty}\left[\pi\kappa\eta e^{-\eta x}\left(2-e^{-\eta x}\right)^{-\pi-1}\left[1-\left(2-e^{-\eta x}\right)^{-\pi}\right]^{\kappa-1}\right] = 0 \tag{31}$$

These two results in Equations (30) and (31) confirm that the EPED has a mode.

3.6. Parameter Estimation

In this subsection, the unknown parameters of the EPED are estimated using the method of maximum likelihood estimation (mle). Given that $x_1, x_2, \ldots, x_n$, the log–likelihood of the EPED is given as

$$\ell(\kappa,\pi,\eta) = \sum_{i=1}^{n}\log f(x_i;\kappa,\pi,\eta) \tag{32}$$

$$\ell(\kappa,\pi,\eta) = \sum_{i=1}^{n}\log\left\{\pi\kappa\eta e^{-\eta x}\left(2-e^{-\eta x}\right)^{-\pi-1}\left[1-\left(2-e^{-\eta x}\right)^{-\pi}\right]^{\kappa-1}\right\}. \tag{33}$$

$$\ell(\kappa,\pi,\eta) = n\log\pi + n\log\kappa + n\log\eta - \eta\sum_{i=1}^{n}x_i - (\pi+1)\sum_{i=1}^{n}\log\left(2-e^{-\eta x_i}\right) + (\kappa-1)\sum_{i=1}^{n}\log\left[1-\left(2-e^{-\eta x}\right)^{-\pi}\right]. \tag{34}$$

The parameters are estimated by differentiating Equation (34) with respect to each parameter and equating the results to zero.

$$\frac{\partial l(\Theta)}{\partial\pi} = \frac{n}{\pi} - \sum_{i=1}^{n}\log\left(2-e^{-\eta x_i}\right) + (\kappa-1)\sum_{i=1}^{n}\frac{\left(2-e^{-\eta x_i}\right)^{-\pi}\log\left(2-e^{-\eta x_i}\right)}{\left\{1-\left(2-e^{-\eta x_i}\right)^{-\pi}\right\}} = 0 \tag{35}$$

$$\frac{\partial l(\Theta)}{\partial\kappa} = \frac{n}{\kappa} + \sum_{i=1}^{n}\log\left\{1-\left(2-e^{-\eta x_i}\right)^{-\pi}\right\} = 0 \tag{36}$$

$$\frac{\partial l(\Theta)}{\partial\eta} = \frac{n}{\eta} - \sum_{i=1}^{n}x_i - (\pi-1)\sum_{i=1}^{n}\frac{x_i e^{-\eta x_i}}{\left(2-e^{-\eta x_i}\right)} + (\kappa-1)\sum_{i=1}^{n}\frac{x_i e^{-\eta x_i}\left(2-e^{-\eta x_i}\right)^{-\pi-1}}{\left\{1-\left(2-e^{-\eta x_i}\right)^{-\pi}\right\}} = 0. \tag{37}$$

where $\Theta = (\kappa, \pi, \eta)$.

Since the equations are non-linear, an iterative technique will be used to solve them numerically using R software version 5901.9.2.0. The log–likelihood function has a nonlinear structure and there are no closed-form solutions. Numerical optimization is used to estimate the parameters. To improve convergence stability and enforce parameter limitations, the R implementation of the BFGS algorithm is utilised. Gradient norms and Hessian diagnostics were used to evaluate convergence, and several initial values are considered to reduce the possibility of local maxima. Switch from a gradient-based approach (BFGS) to a derivative-free approach (Nelder-Mead) if the Hessian is singular or the algorithm fails to converge.

3.7. Simulation Study and Applications

This subsection deals with simulation study and applications of the new model to real-life datasets together with its comparators.

3.7.1. Simulation Study

The qf of the EPED is used to generate samples used in the simulation process. To generate accurate samples, and to get perfect results, both large and small samples are considered as $n = 20, 50, 100, 250, 500, 1000$, with the parameter scenario $\pi = 0.5, \eta = 0.7$, $\kappa = 0.4$ for set I and $\pi = 1.5, \eta = 1.7, \kappa = 1.2$ for set II. The mle method is used to determine the mean of the estimated parameters, the bias and root mean square error (RMSE). This is done to determine the efficiency and consistency of the mle method.

The optimization process was started at the actual parameter values used to produce the data for every Monte Carlo replication. This strategy was used to separate the estimators' finite-sample behaviour from any possible impacts of inadequate initialization. Convergence was proclaimed when the optimiser provided a convergence code equal to zero, and the maximum number of iterations was set at 1000.

The observed Hessian matrix of the negative log–likelihood was calculated numerically after convergence. The appropriate standard errors were determined from the diagonal elements of the covariance matrix of the estimators, which was constructed as the inverse of the Hessian matrix. Inferential summaries did not include replications that produced a singular or non-positive-definite Hessian matrix since they were considered numerically unstable. The optimization results, such as successful convergence, non-convergence, boundary solutions (parameter estimates near the lower bounds), and irregular Hessian behaviour, were used to categorise each simulation run. For the purpose of calculating bias, RMSE, and coverage probability, only replications that met both convergence and Hessian regularity requirements were kept.

Out of 1000 Monte Carlo replications, 987 (98.7%) converged successfully. Boundary solutions were observed in 9 (0.9%) cases, while 11 (1.1%) runs resulted in a singular or non-positive–definite Hessian matrix. In total, 976 (97.6%) replications satisfied both convergence and regularity conditions and were retained for inferential analysis. These results indicate that the proposed maximum likelihood estimation procedure is numerically stable and reliable for moderate sample sizes.

Tables 1 and 2 display the simulation results for the EPED. The simulation results show that the maximum likelihood estimation (mle) approach yields an accurate and dependable estimate of the unknown parameters. This is because the final estimates are consistent and close to the true values of the parameters. Furthermore, as the table demonstrates, as the sample size grows, there is a decrease in the RMSEs and biases, making the mle consistent when estimating unknown parameters.

Table 1. Simulation results for parameters: $\pi = 0.5, \eta = 0.7, \kappa = 0.4$.

		Set I		
Parameters	**N**	**Mle**	**Bias**	**RMSE**
	20	0.8483	0.3483	0.7828
	50	0.6055	0.1055	0.1177
	100	0.5372	0.0372	0.0313
π	250	0.5240	0.0240	0.0162
	500	0.5052	0.0052	0.0135
	1000	0.5028	0.0028	0.0133
	20	1.3831	0.6831	0.9902
	50	1.2953	0.5953	0.8382
	100	1.2403	0.5403	0.8287
η	250	1.1174	0.4174	0.6375
	500	1.0849	0.3849	0.4244
	1000	1.0477	0.3477	0.3013
	20	0.4720	0.0720	0.2376
	50	0.4619	0.0619	0.2134
	100	0.4181	0.0181	0.1472
κ	250	0.4035	0.0035	0.0511
	500	0.4014	0.0014	0.0323
	1000	0.4002	0.0002	0.0131

Table 2. Simulation results for parameters: $\pi = 1.5, \eta = 1.7, \kappa = 1.2$.

		Set II		
Parameters	**N**	**Mle**	**Bias**	**RMSE**
	20	1.6887	0.1887	1.8102
	50	1.6428	0.1428	1.6278
	100	1.5349	0.0349	0.3010
π	250	1.5316	0.0316	0.0507
	500	1.5109	0.0109	0.0314
	1000	1.5033	0.0033	0.0173
	20	1.9370	0.2370	1.1973
	50	1.8932	0.1932	0.8729
	100	1.7438	0.0438	0.4319
η	250	1.7257	0.0257	0.2446
	500	1.7129	0.0129	0.2132
	1000	1.7010	0.0010	0.0465
	20	1.5786	0.3786	0.5798
	50	1.5614	0.3614	0.5468
	100	1.4256	0.2256	0.2047
κ	250	1.2354	0.0354	0.1250
	500	1.2170	0.0170	0.0141
	1000	1.2023	0.0023	0.0025

3.7.2. Applications in Real-Life Datasets

In this section, four real-life datasets relating to environmental and finance area of disciplines are applied to the EPED to assess its fit and flexibility. The EPED is compared with some other competing distributions such as Lomax distribution by [43], Weibull (W) distribution, exponential (E) distribution, Lognormal distribution, and Gamma distribution. To get the best model that fits the considered datasets, the Bayesian information criterion (BIC) and the Akaike information criterion (AIC) is used and the distribution that best describes the dataset is the one with the smallest value of the AIC and BIC. Also, a goodness-of-fit statistic such as the Cramér–von Mises (CvM) goodness-of-fit test with its p-value is applied. A higher p-value and a smaller CvM statistic show a better fit and are indicators of the best model. The datasets used for the analysis are presented as follows:

Dataset I

Dataset I comprises 58 observations on monthly measures of unemployment insurance as reported by [44]. This variable thus measures the monthly frequency at which unemployment insurance checks are issued to former federal government employees. The data are 0.052, 0.033, 0.039, 0.050, 0.029, 0.052, 0.060, 0.032, 0.057, 0.064, 0.061, 0.064, 0.041, 0.036, 0.050, 0.053, 0.061, 0.068, 0.060, 0.050, 0.064, 0.057, 0.061, 0.059, 0.069, 0.070, 0.137, 0.170, 0.100, 0.090, 0.222, 0.109, 0.068, 0.063, 0.056, 0.090, 0.074, 0.095, 0.114, 0.133, 0.066, 0.075, 0.072, 0.054, 0.057, 0.052, 0.066, 0.069, 0.083, 0.044, 0.060, 0.080, 0.058, 0.080, 0.080, 0.052, 0.065, and 0.073.

Dataset II

This dataset represents insurance and financial services, including the different kinds of insurance that resident insurance companies offer to non-residents and vice versa, as well as the services of financial intermediaries and auxiliary services that residents and non-residents exchange. The data cover the period from 1974 to 2021 and were reported by [45]. The dataset is presented as follows: 3.3761087, 3.7916736, 1.8623592, 2.5312656, 1.8700584, 2.6127494, 2.2765957, 2.2220998, 2.2375761, 2.3314328, 1.9051169, 4.9536694, 4.3726093, 5.9473222, 4.9426842, 7.0728029, 6.1907114, 5.0430431, 5.6950926, 2.6111077, 3.2495827, 4.6814338, 3.5658858, 2.1803091, 2.1144394, 2.5407750, 0.4822188, 1.4132725, 2.1065506, 2.2378263, 2.0468322, 2.1193183, 2.2995888, 2.0718469, 0.8246948, 10.7223221, 1.9822990, 3.7373134, 4.7968947, 3.7395944, 3.9088539, 24.7991060, 11.3768892, 9.9587088, and 5.6085853.

Dataset III

Dataset III represents the snow accumulation in inches in the Raleigh–Durham airport, North Carolina, from 1948 to 2000. The dataset was previously reported and analysed by [46]. The dataset involves 63 observations, which are listed as follows: 1.0, 2.5, 1.2, 1.2, 4.1, 9.0, 3.0, 1.0, 1.4, 2.0, 3.0, 1.7, 1.2, 1.2, 1.1, 1.5, 5.0, 1.6, 2.0, 0.1, 0.4, 0.8, 3.7, 1.3, 3.8, 0.1, 0.1, 0.2, 2.0, 7.6, 0.1, 1.8, 0.5, 0.5, 0.5, 1.1, 1.4, 1.0, 1.0, 0.7, 5.7, 0.4, 0.3, 1.8, 0.4, 1.0, 1.2, 2.6, 1.0, 5.0, 1.7, 2.4, 0.1, 0.5, 7.1, 0.2, 0.7, 0.1, 2.7, 2.9, 0.4, 2.0, and 20.3.

Data IV

Dataset IV is obtained from [47]. The dataset consists of thirty successive values of March precipitation (in inches) in Minneapolis/St Paul. The dataset is listed as follows: 0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, and 2.05.

4. Discussion

Tables 3–6 contain descriptive statistics for datasets I, II, III and IV. From both tables, it is observed that the four datasets contain positively skewed data (skewness is estimated to be 2.4356 for dataset I, 3.4730 for dataset II, 4.0102 for dataset III and 1.0867 for dataset VI) with a heavier tail (kurtosis is estimated to be 10.6221 for dataset I, 17.6943 for dataset II, 23.2503 for dataset III and 4.2069 for dataset IV). With the skewness and heavy tails noticed from the datasets, as well as the addition of extra parameters to the classical exponential distribution through exponentiation, the new model can capture the extreme values, which makes the new models suitable for modelling extreme events as found in the field of environmental sciences and finance.

Table 3. Descriptive statistics based on dataset I.

N	Min.	Max.	Q1	Q2	Q3	Skewness	Kurtosis	Variance	SD	Mean
58	0.0290	0.2220	0.0533	0.0635	0.0748	2.4356	10.6221	0.0011	0.0326	0.0707

Table 4. Descriptive statistics based on dataset II.

N	Min.	Max.	Q1	Q2	Q3	Skewness	Kurtosis	Variance	SD	Mean
45	0.4822	24.799	2.1193	2.6127	4.9427	3.4730	17.6943	15.6702	3.9586	4.1869

Table 5. Descriptive statistics based on dataset III.

N	Min.	Max.	Q1	Q2	Q3	Skewness	Kurtosis	Variance	SD	Mean
63	0.100	20.300	0.500	1.200	2.450	4.0102	23.2503	8.9413	2.9902	2.1250

Table 6. Descriptive statistics based on dataset IV.

N	Min.	Max.	Q1	Q2	Q3	Skewness	Kurtosis	Variance	SD	Mean
30	0.3200	4.7500	0.9150	1.4700	2.0870	1.0867	4.2069	1.0012	1.0006	1.6750

Tables 7–10 present the results of the analysis using the datasets. Table 4 shows the results from the analysis of dataset I, and as seen from the table, the EPED has the lowest AIC and BIC values, making it the best fit for dataset I. Table 6 shows the results from the analysis of dataset II, and as seen from the table, the EPED has the lowest AIC and BIC, making it the best fit for dataset II. Table 8 shows the results from the analysis of dataset III, and as seen from the table, the EPED has the lowest AIC and BIC, making it the best fit for dataset III. Table 10 shows the results from the analysis of dataset IV, and as seen from the table, the EPED has the lowest AIC and BIC, making it the best fit for dataset IV. Finally, these results are also supported in Figures 3–6, respectively.

Table 7. The mle values, negative log–likelihood, AIC and BIC for the EPED and its comparators for dataset I.

Models	$\hat{\pi}$	$\hat{\eta}$	$\hat{\kappa}$	AIC	BIC	CvM	*p*-Value
EPED	19.2728	4.2645	41.7668	−263.6777	−257.4964	0.1393	0.4249
(SE)	(7.7543)	(2.9721)	(30.0985)				
Lomax	0.0114	124.8595	-	−187.3283	−183.2075	2.4229	9.015×10^{-7}
(SE)	(0.0129)	(143.0684)	-				
Weibull	2.2499	0.0798	-	−238.4385	−234.3176	0.6038	0.0216
(SE)	(0.1954)	(0.0050)					
Exponential	-	14.1498	-	−189.3652	−187.3048	2.4166	9.342×10^{-7}
(SE)	-	(1.8580)					
Lognormal	2.7267	0.3735	-	−261.854	−257.2331	0.2041	0.2600
(SE)	(0.0490)	(0.0347)					
Gamma	6.6547	94.1564		−254.7316	−250.6108	0.3237	0.1159
(SE)	(1.2060)	(17.7243)					

Table 8. The mle values, negative log–likelihood, AIC and BIC for the EPED and its comparators for dataset II.

Models	$\hat{\pi}$	$\hat{\eta}$	$\hat{\kappa}$	AIC	BIC	CvM	p-Value
EPED	11.2240	0.0735	5.4202	203.4209	208.8409	0.1261	0.4728
(SE)	(3.3458)	(0.0367)	(2.1082)				
Lomax	0.0029	83.5861	-	222.9554	226.5687	0.6384	0.0176
(SE)	(0.0007)	(2.3636)	-				
Weibull	4.6045	1.3192	-	216.6120	220.2253	0.3422	0.1029
(SE)	(0.5533)	(0.1324)	-				
Exponential	-	0.2389	-	220.8778	222.6835	0.6307	0.0184
(SE)	-	(0.0356)	-				
Lognornal	1.1757	0.6814	-	205.8920	209.5053	0.1483	0.3961
(SE)	(0.1016)	(0.0718)	-				
Gamma	2.1028	0.5022	-	211.1145	214.7279	0.2719	0.1626
(SE)	(0.4130)	(0.1113)	-				

Table 9. The mle values, negative log–likelihood, AIC and BIC for the EPED and its comparators for dataset III.

Models	$\hat{\pi}$	$\hat{\eta}$	$\hat{\kappa}$	AIC	BIC	CvM	p-Value
EPED	9.5169	0.0791	1.2639	220.9671	227.3965	0.0500	0.8782
(SE)	(3.6508)	(0.0436)	(0.2650)				
Lomax	0.1003	4.5099	-	223.6932	227.9795	0.0828	0.6774
(SE)	(0.1026)	(2.7980)	-				
Weibull	0.5332	0.9026	-	223.6552	227.9415	0.1053	0.5607
(SE)	(0.4705)	(0.2967)	-				
Exponential	-	0.4705	-	222.9988	225.1419	0.1489	0.3938
(SE)	-	(0.0593)	-				
Lognormal	0.5504	1.1937	-	223.6936	227.9799	0.1156	0.5149
(SE)	(0.1504)	(0.1063)	-				
Gamma	0.9190	0.4324	-	224.6968	228.9831	0.1254	0.4753
(SE)	(0.1431)	(0.0881)	-				

Table 10. The mle values, negative log–likelihood, AIC and BIC for the EPED and its comparators for dataset IV.

Models	$\hat{\pi}$	$\hat{\eta}$	$\hat{\kappa}$	AIC	BIC	CvM	p-Value
EPED	103.2612	0.0117	3.5507	82.2429	86.4465	0.0161	0.9995
(SE)	(64.8924)	(0.0102)	(1.1402)				
Lomax	0.0034	17.4932	-	95.0622	97.8646	0.4470	0.0538
(SE)	(0.0013)	(11.5832)	-				
Weibull	0.3154	1.8092	-	83.9970	86.7994	0.0207	0.9969
(SE)	(0.2220)	(0.2491)	-				
Exponential	-	0.5969	-	92.9488	94.3500	0.4539	0.0516
(SE)		(0.1090)	-				
Lognormal	0.3374	0.6226	-	83.7520	86.5544	0.0311	0.9904
(SE)	(0.1137)	(0.0804)	-				
Gamma	2.9585	1.7663	-	83.7068	86.5092	0.0139	0.9992
(SE)	(0.7250)	(0.4717)	-				

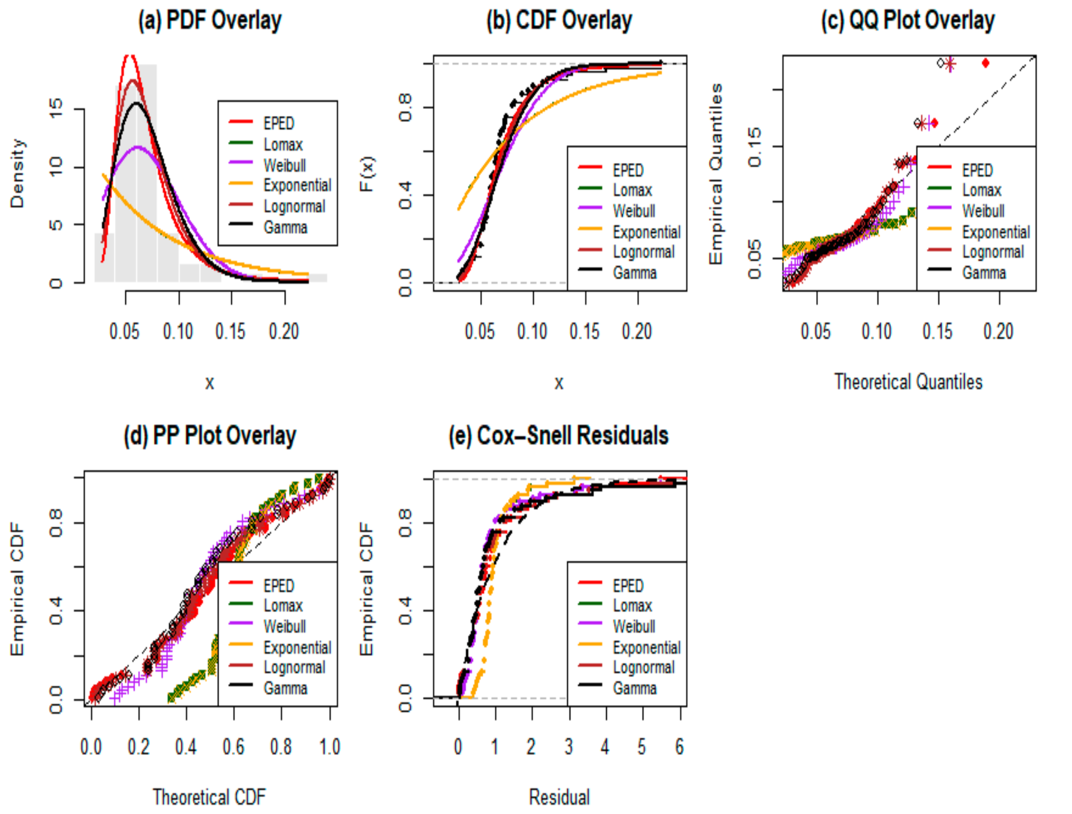

Figure 3. Multi-panel overlay diagnostic plots for the fitted models: (**a**) PDF, (**b**) CDF, (**c**) QQ plot, (**d**) PP plot, and (**e**) Cox–Snell residuals for dataset I.

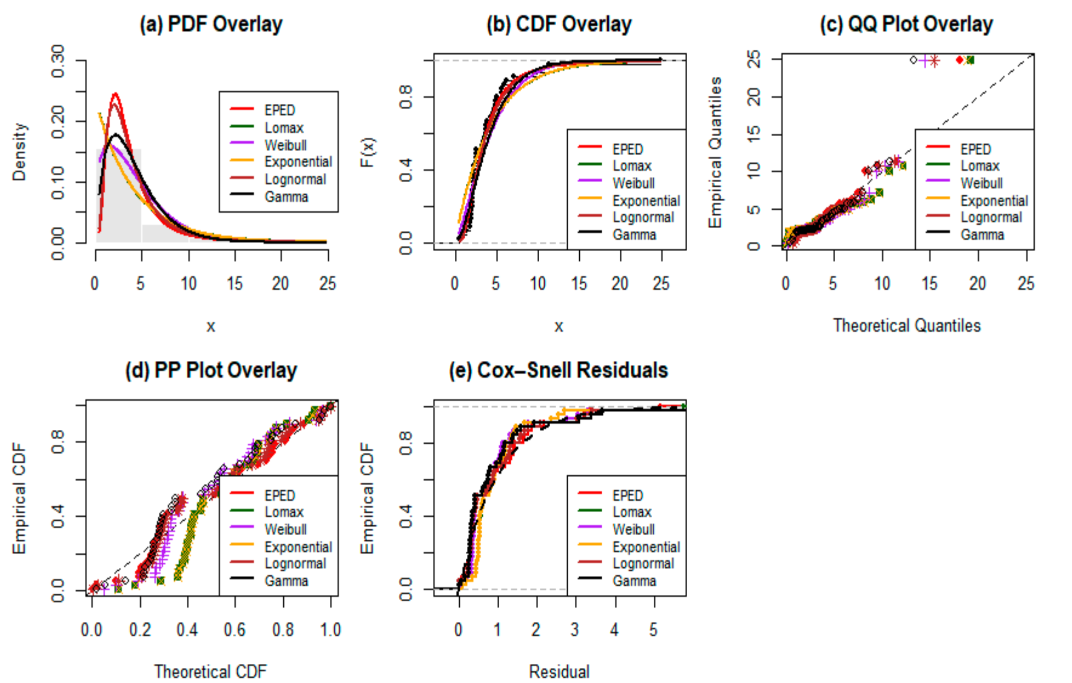

Figure 4. Multi-panel overlay diagnostic plots for the fitted models: (**a**) PDF, (**b**) CDF, (**c**) QQ plot, (**d**) PP plot, and (**e**) Cox–Snell residuals for dataset II.

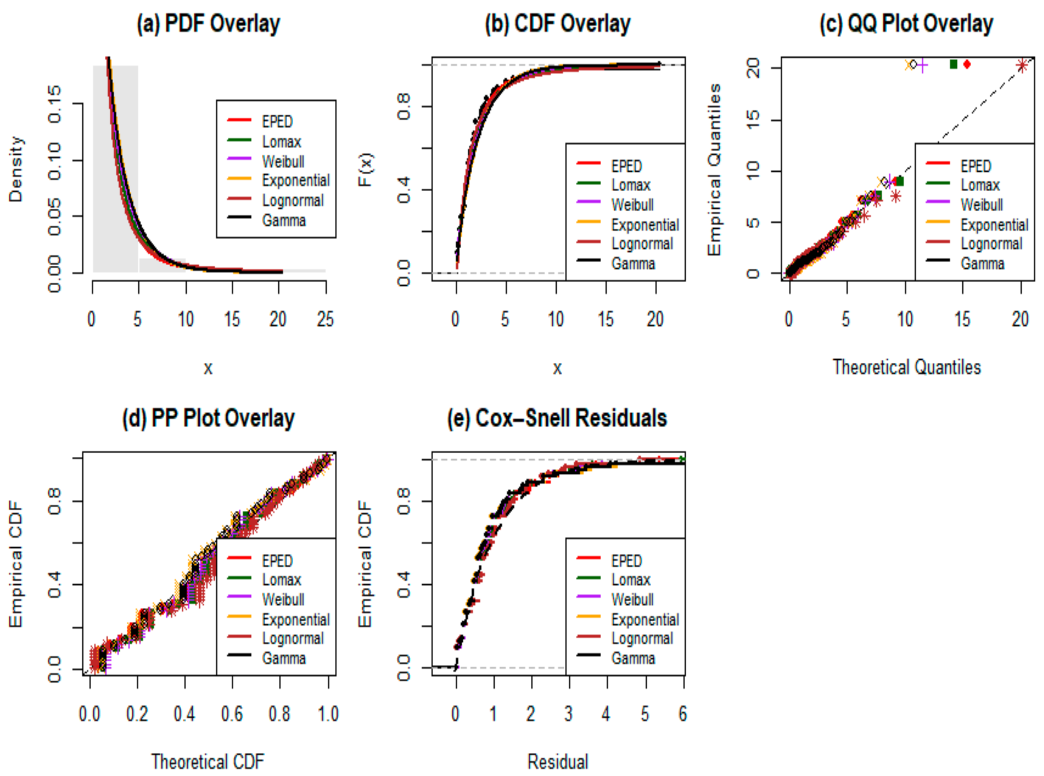

Figure 5. Multi-panel overlay diagnostic plots for the fitted models: (**a**) PDF, (**b**) CDF, (**c**) QQ plot, (**d**) PP plot, and (**e**) Cox–Snell residuals for dataset III.

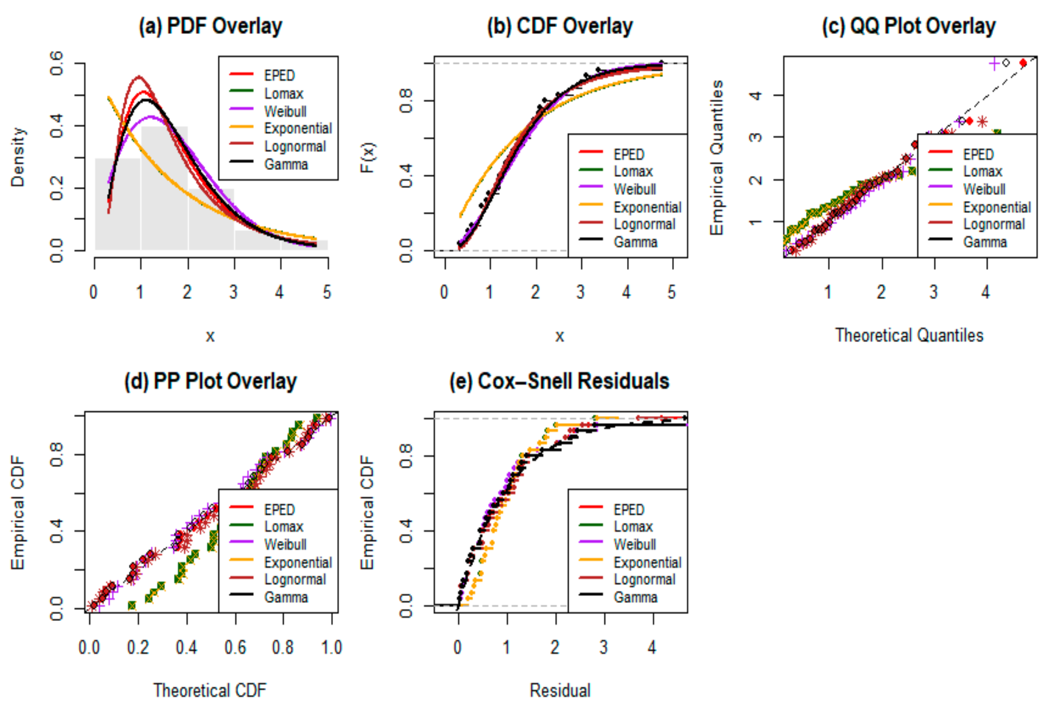

Figure 6. Multi-panel overlay diagnostic plots for the fitted models: (**a**) PDF, (**b**) CDF, (**c**) QQ plot, (**d**) PP plot, and (**e**) Cox–Snell residuals for dataset IV.

Figures 3–6 show graphical diagnostics, such as Cox–Snell residual plots, QQ and PP plots, and PDF and CDF overlays, which are used to evaluate the suitability of the fitted models. Close alignment with the empirical distribution, linearity in QQ/PP plots, and Cox–Snell residuals that follow the unit exponential distribution are all signs of a good match. An improved overall fit is indicated by the EPED's higher linearity in QQ and PP plots, tighter agreement with the empirical PDF and CDF, and Cox–Snell residuals that closely resemble the unit exponential distribution.

5. Conclusions

This research work introduces a new distribution called the exponentiated Pareto exponential distribution with real-life applications in the fields of environmental sciences and finance, thus increasing the flexibility of the classical exponential distribution. This study demonstrated its ability through pdf, cdf, sf and hrf plots to generate various distributional shapes, including increasing, decreasing, inverted bathtub-shaped, and unimodal hazard rates, making it suitable for diverse applications in environmental science and risk modelling. In this study, various statistical properties of the new distribution such as moments, moment-generating function, quantile function, survival function, hazard rate functions, reverse hazard rate, cumulative hazard rate, odds function, order statistics, Value at Risk measure and asymptotic behaviour of the new model were derived, studied and investigated. A simulation study was conducted to determine the parameter stability of the mle approach to evaluate its performance. Datasets relating to environmental sciences and finance with positive skewness and high kurtosis can all be modelled using the EPED, as seen from the pdf and hrf plots. The novel distribution was applied to four real-life datasets, and from the results, the performance metrics used, such as AIC, BIC, CvM and p-value, show that the EPED outperforms its competing distributions.

The proposed exponentiated Pareto exponential distribution has a number of drawbacks despite its proven adaptability and excellent empirical performance. It concentrates on univariate results, makes the assumption of independent observations, and might be susceptible to anomalies in the data. Future studies should improve robust and Bayesian estimate techniques; expand the model to bivariate, multivariate and time-dependent frameworks; and conduct more thorough benchmarking against other heavy-tailed distributions. These advancements will improve financial risk management and environmental risk assessment both theoretically and practically.

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