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New Methods for Multivariate Normal Moments

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Abstract: Multivariate normal moments are foundational for statistical methods. The derivation and simplification of these moments are critical for the accuracy of various statistical estimates and analyses. Normal moments are the building blocks of the Hermite polynomials, which in turn are the building blocks of the Edgeworth expansions for the distribution of parameter estimates. Isserlis (1918) gave the bivariate normal moments and two special cases of trivariate moments. Beyond that, convenient expressions for multivariate variate normal moments are still not available. We compare three methods for obtaining them, the most powerful being the differential method. We give simpler formulas for the bivariate moment than that of Isserlis, and explicit expressions for the general moments of dimensions 3 and 4.

Keywords: multivariate normal; moments; Hermite polynomials; Isserlis; Soper

1. Introduction

The p-variate normal distribution and its moments play a central role in the Edgeworth expansions for the distribution and density of the standardized vector sample mean, and more generally for a wide class of vector estimates based on a sample of size n. When $p = 1$, for the density and distribution of an estimate satisfying artifical assumptions, see [1], and for a standard estimate with these artifical assumptions removed, see [2]. When $p > 1$, for the density and distribution of a sample mean, see (19.17) and (20.48) of [3], and for its extensions to general standard estimates, see [4,5]. The first r terms of these Edgeworth expansions need most of the multivariate Hermite polynomials and normal moments of order $3r$.

Suppose that $X \sim \mathcal{N}_p(\mathbf{0}, V)$, the p-dimensional normal distribution with mean $\mathbf{0} \in N^p$ and covariance V. For $N = \{0, 1, 2, \cdots\}$, $n \in N^p$, and $x \in R^p$, set

$$|n| = \sum_{j=1}^{p} n_j,\ n! = \Pi_{j=1}^{p} n_j!,\ x^n = x_1^{n_1} \cdots x_p^{n_p},\ V_{r_1 \cdots r_p} = E\, X_{r_1} \cdots X_{r_p}, \tag{1}$$

$$\mu_n = \mu_{n_1 \cdots n_p} = V_{1^{n_1} \ldots p^{n_p}} = E\, X^n = E\, X_1^{n_1} \cdots X_p^{n_p} \tag{2}$$

$= q_{1^{n_1} \ldots p^{n_p}}$ in the notation of [6]. Since $-X$ has the same distribution as X, $\mu_n = 0$ if its *order* , $|n|$, is odd. Also,

$$\begin{aligned} &p = V = 1 \Rightarrow \mu_{2n} = \nu_{2n} \text{ where} \\ &\nu_{2n} = 1\,3 \cdots (2n-1) = (2n)!/(2^n n!) = E\, N^n \text{ for } N \sim \mathcal{N}_1(0,1): \\ &\nu_2 = 1,\ \nu_4 = 3,\ \nu_6 = 15,\ \nu_8 = 105,\ \nu_{10} = 945,\ \nu_{12} = 10,395,\ \nu_{14} = 13,5135. \end{aligned} \tag{3}$$

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Edgeworth expansions for the distribution and density of an estimate, need the moments and Hermite polynomials of $Y = V^{-1}X \sim \mathcal{N}_p(0, V^{-1})$, that is, with $V = (V_{jk})$ replaced by $V^{-1} = (V^{jk})$.

Reference [6] showed that the general $2r$th moment of order $2r$ is

$$V_{12\cdots,2r} = E\ X_1X_2\cdots X_{2r} = \sum^{\nu_{2r}} V_{j_1j_2}\cdots V_{j_{2r-1}j_{2r}}, \tag{4}$$

where summation is over all ν_{2n} permutations $j_1, \cdots, j_{2r}$ of $1, 2, \cdots, 2r$ giving distinct terms. This is a special case of the formula for a multivariate moment in terms of the cumulants of a *general* random vector $X \in R^p$,

$$E\ X_{i_1}X_{i_2}\cdots X_{i_r} = B^{i_1i_2\cdots i_r}(\kappa) = \sum_{k=1}^{r} B_k^{i_1i_2\cdots i_r}(\kappa)$$

with $i_1, i_2, \cdots, i_r$ replaced by $1, 2, \cdots, 2r$ where $B^{i_1i_2\cdots i_r}(\kappa)$ and $B_k^{i_1i_2\cdots i_r}(\kappa)$ are the multivariate complete and partial exponential Bell polynomials, as given in (3.1) of [7].

Section 2 briefly outlines the methods used in Sections 3 and 5–8. Section 4 summarises the historical results of [6,8] for bivariate moments. Section 9 shows that these results are easily extended to moments of the multivariate normal with non-zero mean. Section 10 discusses the results of Sections 3 and 5–9. Section 11 gives conclusions and suggests future directions.

$$\text{Set } [r]_k = r!/(r-k)! = r\cdots(r-k+1),\ k = 0, 1, \ldots \text{ and } 1/k! = 0,\ k = -1, -2, \ldots \tag{5}$$

2. Methods

We give several methods for deriving normal moments, and a number of new results. Section 3 uses *the method of successive specialisation* using the step down rule (6). This soon becomes unwieldy without writing software.

Section 5 gives *the method of successive generalisation using quasi-differential operators*. This powerful method has been used to obtain multivariate moments in terms of multivariate cumulants from the univariate formulas: see 3.29 of [9]. But it can also be used to obtain multivariate moments from univariate moments, as we demonstrate here.

Section 6 gives *the multinomial method*, but in detail only for bivariate moments. Sections 7 and 8 give, for the first time, the general moments of dimensions 3 and 4. and illustrate how it can be extended to find moments with $p > 4$.

3. The Step Down Rule and the Method of Successive Specialisation

From (4) follows the new and more useful *step down* recurrence rule,

$$E\ X_1\ldots X_{2r} = \mu_{1^{2r}} = V_{12\cdots,2r} = \sum_{k=1}^{2r-1} V_{k,2r}V^{(k)}_{12\cdots,2r-1} \tag{6}$$

where $V^{(k)}_{12\cdots,2r-1}$ is $V_{12\cdots,2r}$ with k and $2r$ removed. This gives the normal moments of order $2r$ in terms of those of order $2r-2$. For example,

$$E\ X_1\ldots X_4 = \mu_{1111} = V_{1234} = \sum_{k=1}^{3} V_{k4}V^{(k)}_{123} = V_{14}V_{23} + V_{24}V_{13} + V_{34}V_{12},$$

$$E\ X_1\ldots X_6 = \mu_{1^6} = V_{12\cdots6} = \sum_{k=1}^{5} V_{k6}V^{(k)}_{12\cdots5}$$

$$= V_{16}V_{2345} + V_{26}V_{1345} + V_{36}V_{1245} + V_{46}V_{1235} + V_{56}V_{1234},$$

$$E\ X_1 \dots X_8 = \mu_{1^8} = V_{12\cdots 8} = \sum_{k=1}^{7} V_{k8} V^{(k)}_{12\cdots 7} = V_{18}V_{23\cdots 7} + \cdots V_{78}V_{12\cdots 6}.$$

By (6) with $2r$ replaced by 1, that is, with $X_{2r} = X_1$,

$$E\ X_1^2 X_2 \cdots X_{2r-1} = V_{1^2 23\cdots,2r-1} = \mu_{21^{2r-2}} = \sum_{k=1}^{2r-1} V_{k1} V^{(k)}_{123\cdots,2r-1}. \tag{7}$$

For example, taking $r = 3$ then replacing 5 by 1 then 4 by 1, gives

$$E\ X_1^2 X_2 \cdots X_5 = V_{1^2 2345} = \mu_{21111} = V_{11}V_{2345} + \sum_{2345}^{4} V_{12}V_{1345}, \tag{8}$$

$$E\ X_1^3 X_2 X_3 X_4 = \mu_{3111} = V_{111234} = 3V_{11}V_{1234} + 6V_{12}V_{13}V_{14}, \tag{9}$$

$$E\ X_1^4 X_2 X_3 = V_{1^4 23} = \mu_{411} = 3V_{11}(4V_{12}V_{13} + V_{11}V_{23}), \tag{10}$$

where $\sum_{2345}^{4}$ in (8) sums over the 4 distinct terms obtained by permuting 2345. This shows that there is an error in $q_{1^4 23}$ p139 of [6]: his 12 should be 4. His other formulas on p139 pass **the** $X_i \equiv X_1$ **or** $V_{ij} \equiv 1$ **test**: under this condition the moments of order $2r$ are ν_{2r} of (3). This provides a useful check on moment formulas.

We now give all $\mu_{n_1 n_2 \cdots}$ of (2) of order $n = n_1 + n_2 + \cdots$ up to $n = 8$. These are obtained from the bottom up. For example $X_7 = X_1$ in (25) gives (24), and $X_7 = X_1$ in (24) gives (23).

Without loss of generality, we assume that

$$V_{jj} \equiv 1. \tag{11}$$

To emphasize this, when $V_{jj} \equiv 1$, we use $\rho_{12\cdots}$ for $V_{12\cdots}$ of (1), and $\rho_{1^{n_1}\cdots p^{n_p}}$ for $V_{1^{n_1}\cdots p^{n_p}}$.
Moments of order 2.

$$E\ X_1^2 = \mu_2 = \rho_{11} = 1,\ E\ X_1 X_2 = \mu_{11} = \rho_{12}.$$

Moments of order 4.

$$E\ X_1^4 = \mu_4 = \nu_4 = 3,\ E\ X_1^3 X_2 = \mu_{31} = 3\rho_{12},\ E\ X_1^2 X_2^2 = \mu_{22} = 1 + 2\rho_{12}^2,$$

$$E\ X_1^2 X_2 X_3 = \mu_{211} = \rho_{23} + 2\rho_{12}\rho_{13}, \tag{12}$$

$$E\ X_1 \cdots X_4 = \mu_{1111} = \rho_{1234} = \rho_{12}\rho_{34} + \rho_{13}\rho_{24} + \rho_{14}\rho_{23}. \tag{13}$$

Moments of order 6 in terms of moments of order 2 and 4.

$$E\ X_1^6 = \mu_6 = \nu_6 = 15,\ E\ X_1^5 X_2 = \mu_{51} = 15\rho_{12},$$

$$E\ X_1^4 X_2^2 = \rho_{1^4 2^2} = \mu_{42} = 3(1 + 4\rho_{12}^2),$$

$$E\ X_1^4 X_2 X_3 = \rho_{1^4 23} = \mu_{411} = 3(\rho_{23} + 4\rho_{12}\rho_{13}) \text{ by (10)}, \tag{14}$$

$$E\ X_1^3 X_2^3 = \rho_{1^3 2^3} = \mu_{33} = 3(3\rho_{12} + 2\rho_{12}^3),$$

$$E\ X_1^3 X_2^2 X_3 = \mu_{321} = 3(1 + 2\rho_{12}^2)\rho_{13} + 6\rho_{12}\rho_{23} \text{ by (16)}, \tag{15}$$

$$E\ X_1^3 X_2 X_3 X_4 = \mu_{3111} = 3(\rho_{12}\rho_{34} + \rho_{13}\rho_{24} + \rho_{14}\rho_{23} + 2\rho_{12}\rho_{13}\rho_{14}), \tag{16}$$

$$E\ X_1^2 X_2^2 X_3^2 = \mu_{222} = 1 + 2\rho_{12}^2 + 2\rho_{13}^2 + 2\rho_{23}^2 + 8\rho_{12}\rho_{13}\rho_{23}, \tag{17}$$

$$E\,X_1^2X_2^2X_3X_4 = \mu_{2211} = (\mu_{21^4} \text{ at } X_5 = X_2) = \rho_{2^234} + 2\rho_{12}\rho_{1234} + \sum_{34}^{2}\rho_{13}\rho_{12^24}$$

$$= \rho_{34}(1+2\rho_{12}^2) + 2\sum_{12}^{2}\rho_{13}\rho_{14} + 4\rho_{12}\sum_{34}^{2}\rho_{13}\rho_{24}, \tag{18}$$

$$E\,X_1^2X_2\cdots X_5 = \mu_{21^4} = \rho_{1^22345} = \rho_{2345} + \sum_{2345}^{4}\rho_{12}\rho_{1345} \text{ of (13) by (8)}$$

$$= \sum_{2345}^{3}\rho_{23}\rho_{45} + 2\sum_{2345}^{6}\rho_{12}\rho_{13}\rho_{45}, \tag{19}$$

$$E\,X_1\cdots X_6 = \mu_{1^6} = \rho_{123456} = \sum^{15}\rho_{12}\rho_{34}\rho_{56}.$$

Of these moments, μ_{3111}, μ_{2211}, μ_{21^4} of (16), (18), (19), are new. As noted, there is an error in $\mu_{411} = q_{1^423}$ of [6] but his $q_{1^\lambda 23}$ is correct.

Moments of order 8 in terms of lower moments. Again we use ν_{2n} of (3) and $\mu_{n_1n_2\ldots} = \rho_{1^{n_1}2^{n_2}\ldots} = E\,X_1^{n_1}X_2^{n_2}\ldots$.

$$\mu_8 = \rho_{1^8} = \nu_8 = 105,\ \mu_{71} = \rho_{1^72} = 105\rho_{12},\ \mu_{62} = \rho_{1^62^2} = 15(1+6\rho_{12}^2),$$

$$\mu_{611} = (\mu_{51^3} \text{ at } X_4 = X_2) = \rho_{1^623} = 15(\rho_{23} + 6\rho_{12}\rho_{13}), \tag{20}$$

$$\mu_{53} = \rho_{1^52^3} = 15(3\rho_{12} + 4\rho_{12}^3),$$

$$\mu_{521} = (\mu_{51^3} \text{ at } X_4 = X_2) = \rho_{1^52^23} = 15(\rho_{13} + 2\rho_{12}\rho_{23} + 4\rho_{12}^2\rho_{13}), \tag{21}$$

$$\mu_{51^3} = (\mu_{41^4} \text{ at } X_5 = X_1) = \rho_{1^5234} = 4\rho_{1^3234} + \sum_{234}^{3}\rho_{12}\rho_{1^434} \text{ of (16),(14),(92),}$$

$$\mu_{44} = \rho_{1^42^4} = (\mu_{431} \text{ at } X_3 = X_2) = 3(3 + 24\rho_{12}^2 + 8\rho_{12}^4),$$

$$\mu_{431} = \rho_{1^42^33} = (\mu_{4211} \text{ at } X_4 = X_2)$$

$$= 3\rho_{1^22^33} + 3\rho_{12}\rho_{1^32^23} + \rho_{13}\rho_{1^32^3} \text{ of (75) and (8)}$$

$$= 9\rho_{12}(1 + 4\rho_{13} + 4\rho_{12}\rho_{23}) + 24\rho_{12}^3\rho_{13} \text{ by (79),}$$

$$\mu_{422} = (\mu_{4211} \text{ at } X_4 = X_2) = 3\rho_{1^22^33} + 3\rho_{12}\rho_{1^32^23} + \rho_{13}\rho_{1^32^3}: \text{ see (78).}$$

$$\mu_{4211} = \rho_{1^42^234} = (\mu_{41^4} \text{ at } X_5 = X_2) = 3\rho_{1^22^234} + 2\rho_{12}\rho_{1^3234}$$

$$+ \sum_{34}^{2}\rho_{13}\rho_{1^32^24} \text{ of (18), (16) and (8): see (94), (89);}$$

$$\mu_{41^4} = \rho_{1^42345} = (\mu_{31^5} \text{ at } X_6 = X_1)$$

$$= 3\rho_{112345} + \sum_{2345}^{4}\rho_{12}\rho_{1^3345} \text{ of (19) and (16),} \tag{22}$$

$$\mu_{332} = \rho_{1^32^33^2} = (\mu_{3311} \text{ at } X_4 = X_3) = 2\rho_{12^33^2} + 3\rho_{12}\rho_{1^22^23^2} + 2\rho_{13}\rho_{1^22^33}: \text{ see (76).}$$

$$\mu_{3311} = \rho_{1^32^334} = (\mu_{32111} \text{ at } X_5 = X_2) = 2\rho_{12^334} + 3\rho_{12}\rho_{1^22^234} + \sum_{34}^{2}\rho_{13}\rho_{1^22^34}:$$

see (96).

$$\mu_{3221} = \rho_{1^32^23^24} = (\mu_{321^3} \text{ at } X_5 = X_4) = 2\rho_{12^234^2} + 2\rho_{12}\rho_{1^2234^2} + \rho_{13}\rho_{1^22^24^2}$$

$$+ 2\rho_{14}\rho_{1^22^234}: \text{ see (99).}$$

$$\mu_{32111} = \rho_{1^32^2345} = (\mu_{31^5} \text{ at } X_6 = X_2) = 2\rho_{12^2345} + 2\rho_{12}\rho_{1^22345} + \sum_{345}^{3}\rho_{13}\rho_{1^22^245},$$

$$\mu_{31^5} = (\mu_{21^6} \text{ at } X_7 = X_1) = \rho_{1^3 2\cdots 6} = 2\rho_{1\cdots 6} + \sum_{2\cdots 6}^{5} \rho_{12}\rho_{113456}, \tag{23}$$

$$\mu_{2^4} = \rho_{1^2 2^2 3^2 4^2} = (\mu_{2^3 1^2} \text{ at } X_5 = X_4) = \rho_{3^2 4^2} + 2\sum_{12}^{2}\sum_{12}^{2}\rho_{1344} + 2\rho_{12}\rho_{123^2 4^2}$$

$$+ 2\rho_{14}(\rho_{24} + 2\rho_{23}\rho_{34}) + 2\rho_{13}\rho_{2344} + 2\rho_{12}\sum_{34}^{2}\rho_{13}\rho_{2344} : \text{ see (105).}$$

$$\mu_{2^3 1^2} = \rho_{1^2 2^2 3^2 45} = (\mu_{2^2 1^4} \text{ at } X_6 = X_3)$$

$$= \rho_{3^2 45} + \sum_{12}^{2}(2\rho_{13}\rho_{1345} + \sum_{45}^{2}\rho_{14}\rho_{1335}) + 2\rho_{12}\rho_{123^2 45} + \sum_{45}^{2}\rho_{14}(\rho_{25} + 2\rho_{23}\rho_{35})$$

$$+ 2\rho_{13}\rho_{2345} + \rho_{12}(2\rho_{13}\rho_{2345} + \sum_{45}^{2}\rho_{14}\rho_{2335}),$$

$$\mu_{2^2 1^4} = \rho_{1^2 2^2 3\cdots 6} = (\mu_{21^6} \text{ at } X_7 = X_2) = \rho_{2^2 3\cdots 6} + 2\rho_{12}\rho_{1\cdots 6} + \sum_{3\cdots 6}^{4}\rho_{13}(\rho_{1456}$$

$$+ \sum_{1456}^{4}\rho_{12}\rho_{2456}) = \rho_{3\cdots 6} + \sum_{12}^{2}\sum_{3\cdots 6}^{4}\rho_{13}\rho_{1456} + 2\rho_{12}\rho_{1\cdots 6} + \sum_{3\cdots 6}^{12}\rho_{13}\rho_{24}\rho_{1256} + \rho_{12}\,\alpha_2$$

$$\text{where } \alpha_2 = \sum_{3\cdots 6}^{4}\rho_{13}\rho_{2456} = \sum_{3\cdots 6}^{12}\rho_{13}\rho_{24}\rho_{56},$$

$$\mu_{21^6} = \rho_{1^2 234567} = \rho_{234567} + \sum_{2\cdots 7}^{6}\rho_{12}\rho_{134567} \text{ by (7)}, \tag{24}$$

$$\mu_{1^8} = \sum^{105}\rho_{12}\rho_{34}\rho_{56}\rho_{78} = \sum_{k=1}^{7}\rho_{k8}\rho^{(k)}_{12\cdots 7} = \rho_{18}\rho_{2\cdots 7} + \cdots + \rho_{78}\rho_{12\cdots 6}, \tag{25}$$

where $\rho^{(k)}_{12\cdots 7}$ is $\rho_{12\cdots 7}$ with k removed. These iterative expressions for μ_{51^3}, μ_{41^4}, μ_{31^5}, $\mu_{2^2 1^4}$, μ_{21^6} are new. However, without software, it is easy to make an error with these substitutions.

Note 1. *This relation giving moments in terms of the covariance is a special case of the following. For* any *random variable $X \in R$ with rth cumulant κ_r, $r \geq 1$, its rth moment is given by*

$$m_r = B_r(\kappa) = \sum_{k=1}^{r} B_{rk}(\kappa),\ r \geq 1,\ \textit{where } \kappa = (\kappa_1, \kappa_2, \cdots)$$

where $B_{rk}(\kappa)$ is the partial exponential Bell polynomial defined by (36) below. For any random vector $X \in R^p$ with rth cumulant κ_r, $r \in N^p$, this becomes

$$m_r = B_r(\kappa) = \sum_{k=1}^{|r|} B_{rk}(\kappa) \textit{ where } |r| = \sum_{k=1}^{p} r_k$$

and $B_{rk}(\kappa)$ is the multivariate partial exponential Bell polynomial. It may be written down from the univariate form. See [4,7,8].

4. Bivariate Moments of Soper [8] and Isserlis [6]

Refs. [6,10] gave the general bivariate normal moment with correlation $\rho = \rho_{12}$ when $V_{jj} \equiv 1$ in terms of ν_{2n} of (3), as

$$\mu_{rs} = E\ X_1^r X_2^s = \sum_{0 \leq k \leq s/2} m_{rsk}(1-\rho^2)^k \rho^{s-2k} \text{ where } m_{rsk} = \binom{s}{2k}\nu_{2k}\nu_{r+s-2k}. \tag{26}$$

Putting $s = 0, 1, 2, 3, 4, 5$ then expanding $(1-\rho^2)^k$ and simplifying, gives

$$\mu_{r0} = E\ X_1^r = \nu_r,\ \mu_{r1} = E\ X_1^r X_2 = \nu_{r+1}\rho, \tag{27}$$
$$\mu_{r2} = E\ X_1^r X_2^2 = \nu_r(1+r\rho^2),\ \mu_{r3} = E\ X_1^r X_2^3 = \nu_{r+1}[3\rho + (r-1)\rho^3], \tag{28}$$
$$\mu_{r4} = E\ X_1^r X_2^4 = \nu_r[3 + 6r\rho^2 + r(r-2)\rho^4], \tag{29}$$
$$\mu_{r5} = E\ X_1^r X_2^5 = \nu_{r+1}[15\rho + 10(r-1)\rho^3 + (r-1)(r-3)\rho^5]. \tag{30}$$

Special cases are

$$\mu_{11} = \rho,\ \mu_{31} = 3\rho,\ \mu_{51} = 15\rho,\ \mu_{71} = 105\rho,\ \mu_{91} = 945\rho,\ \mu_{11,1} = 10395\rho.$$
$$\mu_{22} = 1 + 2\rho^2,\ \mu_{42} = 3(1+4\rho^2),\ \mu_{62} = 15(1+6\rho^2), \mu_{82} = 105(1+8\rho^2),$$
$$\mu_{10,2} = 945(1+10\rho^2),\ \mu_{13} = 3\rho,\ \mu_{33} = 3(3\rho + 2\rho^3),\ \mu_{53} = 15(3\rho + 4\rho^3),$$
$$\mu_{73} = 315(\rho + 2\rho^3),\ \mu_{93} = 945(3\rho + 8\rho^3).$$
$$\mu_{04} = 3,\ \mu_{24} = 3 + 12\rho^2,\ \mu_{44} = 3(3 + 24\rho^2 + 8\rho^4),$$
$$\mu_{64} = 45(1 + 12\rho^2 + 8\rho^4),\ \mu_{84} = 315(1 + 16\rho^2 + 16\rho^4).$$

By swapping r and s, this includes all bivariate moments up to order 10. (26) is not a symmetric rule. Setting $Y_1 = X_2$ and $Y_2 = X_1$ we see that (11) $\Rightarrow\ \mu_{rs} = \mu_{sr}$ giving its *dual*

$$\mu_{rs} = \sum_{0 \le k \le r/2} m_{srk}(1-\rho^2)^k \rho^{r-2k}. \tag{31}$$

This has about $r/2$ terms while (26) has about $s/2$ terms if $s > r$. The same is true if we expand $(1-\rho^2)^k$ to write (26) as

$$\mu_{rs} = \sum_{-1 \le n \le s/2} M_{rsn}\rho^{s-2n} \text{ where } M_{rsn} = \sum_{0 \le i \le s/2} [m_{rsk}(-1)^k \binom{k+1}{i+1}]_{k=n+i+1}. \tag{32}$$

The method becomes increasingly cumbersome as s increases. In Section 4 we give a new and simpler formula for the bivariate moment μ_{rs}.

5. Successive Generalisation Using Quasi-Differential Operators

An alternative to deriving particular cases from the general moment, is to derive bivariate moments from univariate moments, then trivariate moments from bivariate moments, and so on, by the method of 3.29 of [9], given there to obtain multivariate moments in terms of multivariate cumulants. But equally well, it applies to relations among moments. To increase consistency with the notation there, we set

$$\mu(ij) = V_{ij},\ \mu(1^{n_1} \cdots p^{n_p}) = E\ X_1^{n_1} \cdots X_p^{n_p} = \mu_n \text{ of (2)}. \tag{33}$$

We do not assume that $V_{jj} \equiv 1$. Define the operator $\partial(1.2)$ by

$$\partial(1.2)\ \mu(1^r 2^s 3^t \dots) = r\mu(1^{r-1}2^{s+1}3^t \dots).$$
$$\text{So } \partial(1.2)^k\ \mu(1^r 2^s \dots) = [r]_k\ \mu(1^{r-k}2^{s+k} \dots) \text{ for } [r]_k \text{ of (5)}. \tag{34}$$

Define $\partial(i.j)$ similarly. We show how to obtain $\mu_{n_1 \cdots n_{p+1}}$ by applying $\partial(j.p+1)^m$ to $\mu_{n_1 \cdots n_p}$ for any $j = 1, \dots, p$. Applying $\partial(1.2)$ to

$$\mu(1^{2r}) = E\ X_1^{2r} = \nu_{2r}\mu(1^2)^r,$$

gives

$$2r\mu(1^{2r-1}2) = \nu_{2r}\, r\mu(1^2)^{r-1}\, 2\mu(12), \tag{35}$$

or dividing by $2r$,

$$\mu(1^{2r-1}2) = \nu_{2r}\, \mu(1^2)^{r-1}\mu(12), \text{ that is, } E\, X_1^{2r-1}X_2 = \nu_{2r}V_{11}^{r-1}\, V_{12},$$

proving (27). Applying this operator, a second, third, and fourth time give (28)–(30). We can apply the operator k times using Faa di Bruno's rule for differentiating $h(x) = f(g(x))$, [4c] p. 137 of [11]: for $k \geq 0$, its kth derivative is

$$h_{.k}(x) = \sum_{j=0}^{k} B_{kj}\, f_{.j}(g(x))$$

where $B_{kj} = B_{kj}(g)$ is the partial exponential Bell polynomial in $g = (g_1, g_2, \cdots)$ and $g_k = g_{.k}(x)$. He tables them on pp. 307–308. $B_{kj}(g)$ is defined by

$$(\sum_{k=1}^{\infty} g_k t^k/k!)^j/j! = \sum_{k=j}^{\infty} B_{kj}t^k/k!. \text{ So } B_{k0} = I(k=0). \tag{36}$$

We apply this with

$$\begin{aligned} &d/dx = \partial(1.2),\ g(x) = \mu(1^2),\ f(g) = g^r,\ h(x) = \mu(1^2)^r. \\ &\text{So } f_{.j}(g(x)) = [r]_j\mu(1^2)^{r-j},\ g_1 = 2\mu(12),\ g_2 = 2\mu(2^2),\ g_k = 0 \text{ for } k \geq 3, \end{aligned}$$

and $B_{kj}/k!$ is the coefficient of t^k in $[2\mu(12)t + \mu(2^2)t^2]^j/j!$. So

$$B_{kj} = [k]_j[2\mu(12)]^{2j-k}\mu(2^2)^{k-j}/(2j-k)!,\ k/2 \leq j \leq k.$$

So by (34) we obtain for $k \geq 1$,

$$\begin{aligned} &\mu(1^{2r-k}2^k)2^r r!/(2r-k)! = [2r]_k\mu(1^{2r-k}2^k)/\nu_{2r} \\ &= \partial(1.2)^k\mu(1^{2r})/\nu_{2r} = \partial(1.2)^k\mu(1^2)^r = \sum_{j=1}^{k} B_{kj}[r]_j\mu(1^2)^{r-j} \qquad (37) \\ &= \sum_{k/2\leq j\leq min(k,r)} [r]_j[k]_j\mu(1^2)^{r-j}[2\mu(12)]^{2j-k}\mu(2^2)^{k-j}/(2j-k)! = H_{rk}^{12}, \text{ say,} \qquad (38) \end{aligned}$$

Dividing by $r!k!$ gives

Theorem 1. *At $s = 2r-k$, $\mu_{sk} = \mu(1^s2^k) = E\, X_1^sX_2^k$ is given by*

$$\mu_{sk}\, 2^r/s!k! = \sum_{k/2\leq j\leq min(k,r)} B_{r-j}C_{2j-k}D_{k-j} \text{ at } 2r = s+k \text{ where} \tag{39}$$

$$B_r = \mu(1^2)^r/r!,\ C_r = C_r^{12} = [2\mu(12)]^r/r!,\ D_r = \mu(2^2)^r/r!, \tag{40}$$

$$\mu(12) = V_{12} = \rho_{12}\sigma_1\sigma_2 \text{ for } \sigma_j^2 = \mu(j^2) = V_{jj}.$$

This is our first new formula for the bivariate normal moments.

Corollary 1. *At $s = 2r - k$, if $V_{jj} \equiv 1$, then $\mu_{sk}/\nu_{2r} = H^{12}_{rk}/[2r]_k$*

$$\text{where } H^{12}_{rk} = \sum_{k/2 \le j \le min(k,r)} [k]_j [r]_j \, (2\rho_{12})^{2j-k}/(2j-k)!. \tag{41}$$

$$\text{So, } \mu_{sk}\, 2^r/s! = \sum_{k/2 \le j \le min(k,r)} [k]_j (2\rho_{12})^{2j-k}/(2j-k)!(r-j)!. \tag{42}$$

This is our second new formula for the bivariate normal moments. So if $V_{jj} \equiv 1$,

$$H^{12}_{r0} = 1,\ H^{12}_{r1} = 2r\rho_{12},\ H^{12}_{r2} = 2r + 4[r]_2\rho^2_{12},$$
$$H^{12}_{r3} = 12[r]_2\rho_{12} + 8[r]_3\rho^3_{12},\ H^{12}_{r4}/4 = 3[r]_2 + 12[r]_3\rho^2_{12} + 4[r]_4\rho^4_{12}. \tag{43}$$

Swapping s and k in (39) and (42) give equivalent formulas. (42) is simpler than Soper and Isserlis's (26) as that is a polynomial in both ρ_{12} and $1 - \rho^2_{12}$. Putting $\rho_{12} = 1$ in Corollary 1 gives the new identity $[H^{12}_{rk}]_{\rho_{12}=1} =$

$$\sum_{k/2 \le j \le min(k,r)} [k]_j [r]_j \, 2^{2j-k}/(2j-k)! = [2r]_k \text{ for } 0 \le k \le 2r. \tag{44}$$

6. The Multinomial Method

We now obtain (27)–(30) by another new method. Given

$$t \in R^p,\ t'X \sim \mathcal{N}_1(0, v_t) \text{ where } v_t = t'Vt. \text{ So } E\,(t'X)^{2r} = \nu_{2r}\, v_t^r. \tag{45}$$

By the multinomial theorem for $|n|$ and $n!$ of (1),

$$(t'X)^{2r} = \sum_{|n|=2r} \binom{2r}{n} t^n X^n \text{ where } \binom{2r}{n} = (2r)!/n!.$$

$$\text{So, } \nu_{2r}\, v_t^r = E\,(t'X)^{2r} = \sum_{|n|=2r} \binom{2r}{n} t^n \mu_n, \text{ and } \mu_n = E_n F_n \text{ where}$$

$$E_n = \nu_{2r}/\binom{2r}{n} = n!/2^r r!, \text{ and } F_n = \text{coeff}(t^n) \text{ in } v_t^r \text{ when } |n| = 2r, \tag{46}$$

F_n can be obtained from a second application of the multinomial theorem. Suppose that $V_{jj} \equiv 1$. So

$$\rho_{jk} = E\, X_j X_k \text{ and } v_t = 2 \sum_{1 \le j < k \le p} \rho_{jk} t_j t_k + \sum_{j=1}^{p} t_j^2. \tag{47}$$

For $c = (c_{jk})$ a symmetric $p \times p$ matrix of integers in N and $a \in N^p$, set

$$\rho^c = \Pi_{1 \le j < k \le p}\, \rho_{jk}^{c_{jk}},\ c_j = \sum_{k \ne j} c_{jk}.$$

By (47), for $a, c \in N^p$, the coefficient of t^n in v_t^r is

$$F_n = \sum \{ \binom{r}{ac} 2^{|c|} \rho^c :\ c_j + 2a_j = n_k,\ k = 1, \cdots, p \},\ |n| = 2r \tag{48}$$

where $\binom{r}{ac} = r!/a!c!$, $a! = \Pi_{j=1}^p a_j!$ and $c! = \Pi_{1 \le j < k \le p} c_{jk}!$.

The multinomial method for bivariate moments.

Set $\rho = \rho_{12},\ c = c_{12}$. So $v_t = 2\rho t_1 t_2 + t_1^2 + t_2^2$ and the coefficient of t^n in v_t^r needed for (46) is

$$\begin{aligned} &\mu_n/E_n = 2^r r!\mu_n/n! = F_n \\ &= \sum\{\binom{r}{ca_1a_2}(2\rho)^c :\ c+2a_1 = n_1,\ c+2a_2 = n_2\} \text{ where} \\ &2r = |n|, \text{ and } \binom{r}{ca_1a_2} = r!/(c!a_1!a_2!). \end{aligned} \tag{49}$$

Taking $n = (2r-i, i)$ for $i \le 5$ gives (27)–(30).

Example 1. *Take* $n = (2r-5, 5)$. *Then* $(c, a_2) = (1,2),\ (3,1)$ *or* $(5,0)$. *So* F_n *sums over* $(c, a_1, a_2) = (1, r-3, 2),\ (3, r-4, 1),\ (5, r-5, 0)$ *giving*

$$F_n = \sum_{j=3}^{5} [r]_j \alpha_j \text{ where } \alpha_3 = \rho,\ \alpha_4 = 4\rho^3/3,\ \alpha_5 = 4\rho^5/15.$$

So since $E_n = (2r-5)!5!/2^r r!$, *by (49),* $r = 3,4,5,6$ *give*

$$\mu_{15} = 15\rho,\ \mu_{35} = 8(3\rho + 4\rho^3),\ \mu_{55} = 15(15\rho + 40\rho^3 + 8\rho^5),\ \mu_{75} = 105(5\rho + 20\rho^3 + 8\rho^5).$$

Example 2. *Take* $n = (2r-6, 6)$. *Then* $(c, a_2) = (0,3), (2,2), (4,1)$ *or* $(6,0)$. *So* F_n *sums over* $(c, a_1, a_2) = (c, a_2) = (0, r-3, 3), (2, r-4, 2), (4, r-5, 1), (6, r-6, 0)$*:*

$$F_n = \sum_{j=3}^{6} [r]_j \alpha_j \text{ where } \alpha_3 = 1/6,\ \alpha_4 = \rho^2,\ \alpha_5 = \rho^4/3,\ \alpha_6 = 4\rho^6/45.$$

By (49), $E_n = (2r-6)!6!/2^r r!$, *so* $r = 3,4,5,6,7$ *give* $\mu_{06}, \mu_{26}, \mu_{46}$ *above, and*

$$\mu_{66} = 45(5 + 90\rho^2 + 120\rho^4 + 16\rho^6),\ \mu_{86} = 5.7.9(5 + 120\rho^2 + 240\rho^4 + 64\rho^6).$$

Moments with $p > 2$ can be dealt with similarly, but the operator method of Section 6 is easier.

7. Trivariate Moments

Here, we give μ_{rst} for arbitrary r, s, t. Define B_r, C_r, D_r as in (40). Assuming $V_{jj} \equiv 1$, ref. [6] gave (17) and the following two trivariate moments. q_I is his notation.

$$\mu_{2r,11} = E\ X_1^{2r} X_2 X_3 = q_{1^{2r}23} = \nu_{2r}(\rho_{23} + 2r\rho_{12}\rho_{13}), \tag{50}$$

$$\begin{aligned} &\mu_{2r-1,21} = q_{1^{2r-1}2^23} = \nu_{2r}[\rho_{13} + 2\rho_{12}\rho_{23} + 2(r-1)\rho_{12}^2\rho_{13}] \\ &= \rho_{13}a_{2r-1,2} + \rho_{23}b_{2r-1,2} \text{ where } a_{2r-1,2} = \nu_{2r}[1 + 2(r-1)\rho_{12}^2],\ b_{2r-1,2} = 2\nu_{2r}\rho_{12}. \end{aligned} \tag{51}$$

So $\mu_{211}, \mu_{411}, \mu_{611}, \mu_{321}, \mu_{521}$ are given by (12), (10), (20), (15), (21).

Trivariate moments using the differential operator method.

We now extend the operator $\partial(1.2)$ of (34) to $\partial(i.j)$. So,

$$\partial(1.3)\ \mu(1^r 2^s 3^t 4^u \cdots) = r\mu(1^{r-1} 2^s 3^{t+1} 4^u \cdots).$$

That is, $\partial(1.3)\ \mu_{rstu\ldots} = r\mu_{r-1,s,t+1,u\ldots}$. By (34), for B_r, C_r, D_r of (40),

$$\partial(1.3)\ B_r = 2B_{r-1}\ \mu(13),\ \partial(1.3)\ C_r = 2C_{r-1}\ \mu(23). \tag{52}$$

$$\text{So, } \partial(1.3)\ \mu(1^2)^r = 2r\mu(1^2)^{r-1}\mu(13),$$

$$\partial(1.3)\ \mu(12)^r = r\mu(12)^{r-1}\mu(23),\ \partial(1.3)^e\ \mu(12)^r = [r]_e\mu(12)^{r-e}\mu(23)^e. \tag{53}$$

Applying $\partial(1.3)$ to (39) gives

Theorem 2. *For* $2r = s + k + 1$, $\mu_{sk1} = \mu(1^s2^k3)$ *is given by*

$$\mu_{sk1}\ 2^{r-1}/s!k! = \sum [B_{r-j-1}C_J\mu(13) + B_{r-j}C_{J-1}\mu(23)]_{J=2j-k}\ D_{k-j} \tag{54}$$

summed over $k/2 \le j \le min(k, r)$.

For example, (54) with $k = 0$ and (60) with $k = 1$, reduce to (27) with r changed to $2r - 1$:

$$\mu_{2r-1,1} = \mu(1^{2r-1}2) = \nu_{2r}\mu(1^2)^{r-1}\mu(12).$$

Corollary 2. *When* $V_{jj} \equiv 1$, *for* $2r = s + k + 1$,

$$\mu_{sk1}\ 2^{r-1}/s! = \sum_{k/2\le j\le min(k,r)} [k]_j 2^J [R\rho_{13}\rho_{12}^J + J\rho_{23}\rho_{12}^{J-1}/2]/J!R! \tag{55}$$

at $J = 2j - k, R = r - j$.

For $k = 1$ and 2, (55) gives Isserlis's (50) and (51). But otherwise (55) is new.

Corollary 3. *For* $\rho = \rho_{12}$,

$$\mu_{sk1} = \rho_{13}a_{sk} + \rho_{23}b_{sk}\ \textit{where} \tag{56}$$

$$a_{2r,3}/\nu_{2r} = 3(1 + 2r\rho^2),\ b_{2r,3}/\nu_{2r} = 2r[3\rho + 2(r-1)\rho^3]. \tag{57}$$

$$\textit{So for } \mu_{211},\ a_{21} = 2\rho,\ b_{21} = 1.\ \textit{For } \mu_{321},\ a_{32} = 3(1 + 2\rho^2),\ b_{32} = 6\rho.$$

$$\textit{For } \mu_{431},\ a_{43} = 12(3\rho + 2\rho^3),\ b_{43} = 3(1 + 4\rho^2). \tag{58}$$

$$\textit{For } \mu_{541},\ a_{54} = 15(3 + 24\rho^2 + 8\rho^4),\ b_{54} = 60(3\rho + 4\rho^3).$$

$$\textit{For } \mu_{631},\ a_{63} = 45(1 + 6\rho^2),\ b_{63} = 90(3\rho + 4\rho^3).$$

$$\textit{For } \mu_{651},\ a_{65} = 90(15\rho + 40\rho^3 + 8\rho^5),\ b_{65} = 225(1 + 12\rho^2 + 8\rho^4).$$

$$\textit{For } \mu_{741},\ a_{74} = 315(1 + 12\rho^2 + 8\rho^4),\ b_{74} = 1260(\rho + 2\rho^3).$$

$$\textit{For } \mu_{831},\ a_{83}/\nu_8 = 3(1 + 8\rho^2),\ b_{83}/\nu_8 = 24(\rho + 2\rho^3).$$

By (34),

$$\partial(2.3)\ \mu(1^r2^s) = s\mu(1^r2^{s-1}3),\ \text{ that is, } \partial(2.3)\ \mu_{rs} = s\ \mu_{r,s-1,1},$$

$$\text{so } \partial(2.3)\ C_r = 2C_{r-1}\mu(13),\ \partial(2.3)D_r = 2D_{r-1}\mu(23). \tag{59}$$

Applying $\partial(2.3)$ to (39) gives an alternative to Theorem 2:

Theorem 3. *For* $2r = s + k$, $\mu_{s,k-1,1} = \mu(1^s2^{k-1}3)$ *is given by*

$$\mu_{s,k-1,1}2^{r-1}/s!(k-1)! = \sum_{k/2\le j\le min(k,r)} B_{r-j}[C_{J-1}D_K\mu(13) + C_JD_{K-1}\mu(23)] \tag{60}$$

$$\textit{at } J = 2j - k, K = k - j. \tag{61}$$

Corollary 4. *When* $V_{jj} \equiv 1$, *for* $2r = s + k$, *and* J, K *of (61),*

$$\mu_{s,k-1,1}\, 2^{r-1}/s!(k-1)! = \sum_{k/2 \le j \le min(k,r)} [J\rho_{13}(2\rho_{12})^{J-1} + K\rho_{23}(2\rho_{12})^{J}]/J!K!(r-j)!.$$

By (59), applying $\partial(2.3)$ to (60) gives

Theorem 4. *For* $2r = s + k$, *and* J, K *of (61),*

$$\mu_{s,k-2,2}\, 2^{r-1}/s!(k-2)! = \sum_{k/2 \le j \le min(k,r)} B_{r-j}\, [2C_{J-2}D_K\mu(13)^2 + C_{J-1}D_{K-1}4\mu(13)\mu(23) + C_J D_{K-2}2\mu(23)^2 + C_J D_{K-1}\mu(33)]. \tag{62}$$

Corollary 5. *When* $V_{jj} \equiv 1$, *for* $2r = s + k$ *and* J, K *of (61),*

$$\mu_{s,k-2,2}\, 2^{r-1}/s!(k-2)! = \sum_{k/2 \le j \le k} [2(2\rho_{12})^{J-2}\rho_{13}^2/(J-2)!K! + 4\rho_{13}\rho_{23}(2\rho_{12})^{J-1}/(J-1)!(K-1)! + \{1/(K-1)! + 2\rho_{23}^2/(K-2)!\}\,(2\rho_{12})^J/J!]/(r-j)!. \tag{63}$$

Here are some examples. $k = 2$ gives the first equation in (28) with r changed to $2r - 2$ and X_2 replaced by X_3. (62) with $k = 3$ gives (51) with r changed to $2r - 3$ and X_2 replaced by X_3. (63) with $k = 4$ gives for $2r = s + 4$,

$$\mu_{2r-4,22} = \nu_{2r-4}A_r \text{ where } A_r = a + (2r-4)b + 4(2r-4)(2r-6)\,c, \tag{64}$$
$$a = 1 + 2\rho_{23}^2,\ b = \rho_{12}^2 + \rho_{13}^2 + 4\rho_{12}\rho_{13}\rho_{23},\ c = \rho_{12}^2\rho_{13}^2.$$
So, $\mu_{222} = a + 2b$ as in (17), $\mu_{422} = 3(a + 4b + 8c)$, $\mu_{622} = 15(a + 6b + 24c)$,
$\mu_{822} = 105(a + 8b + 48c)$, $\mu_{10,22} = 945(a + 10b + 80c)$, $\mu_{12,22} = \nu_{12}(a + 12b + 120c)$.

(63) with $k = 5$ gives for $s = 2r - 5$,

$$\mu_{s32}\, 2^{r-1}/s!3! = \sum_{j=3}^{5} a_j/(r-j)! \text{ where } a_3 = 4\rho_{13}\rho_{23} + 2(1 + 2\rho_{23}^2)\rho_{12}, \tag{65}$$
$$a_4 = 4\rho_{12}\rho_{13}^2 + 8\rho_{13}\rho_{23}\rho_{12}^2,\ a_5 = 8\rho_{12}^2\rho_{13}^2.$$
So, $2\mu_{332}/9 = a_3 + a_4$, $\mu_{532}/45 = a_3/2 + a_4 + a_5$, $\mu_{732}/945 = a_3/6 + a_4/2 + a_5$.

(63) with $k = 6$ gives for $s = 2r - 6$,

$$\mu_{s42}\, 2^{r-1}/s!4! = \sum_{j=3}^{6} a_j/(r-j)! \text{ where} \tag{66}$$
$$a_3 = 1/2 + 2\rho_{23}^2,\ a_4 = \rho_{13}^2 + 8\rho_{12}\rho_{13}\rho_{23} + 2\rho_{12}^2 + 2\rho_{23}^2,$$
$$a_5 = 4\rho_{12}^2\rho_{13}^2 + 16\rho_{12}^3\rho_{13}\rho_{23}/3 + 1 + 2\rho_{12}^4/3,\ a_6 = 4\rho_{12}^4\rho_{13}^2/3.$$
So, $\mu_{442}/36 = a_3/2 + a_4 + a_5$, $\mu_{642}/540 = a_3/6 + a_4/2 + a_5 + a_6$,
$\mu_{842}/15120 = a_3/24 + a_4/6 + a_5/2 + a_6$.

(63) with $k = 6$ gives for $s = 2r - 7$,

$$\mu_{s52}\, 2^{r-1}/s!5! = \sum_{j=4}^{7} a_j/(r-j)! \text{ where} \tag{67}$$

$$a_4 = 2\rho_{13}\rho_{23} + (1+4\rho_{23}^2)\rho_{12},$$
$$a_5 = 2\rho_{12}\rho_{13}^2 + 8\rho_{12}^2\rho_{13}\rho_{23} + 4\rho_{12}^3(1+2\rho_{23}^2)/3,$$
$$a_6 = 8\rho_{12}^3\rho_{13}(\rho_{13}+2\rho_{23})/3 + 4\rho_{12}^5/15,\ a_7 = 8\rho_{12}^5\rho_{13}^2/15.$$
$$\text{So, } \mu_{552}/450 = \sum_{j=4}^{6} a_j/(6-j)!,\ \mu_{752}/9450 = \sum_{j=4}^{7} a_j/(7-j)!,\ \mu_{952}/340200 = \sum_{j=4}^{7} a_j/(8-j)!.$$

Now apply $\partial(1.3)^t$ to (39) with $u = s - t$ using Faa di Bruno's rule of (37) with 2 changed to 3, and Leibniz' rule for the tth derivative of a product. So,

Theorem 5. *For B_r, C_r, D_r of (40) and $2r = u + k + t$,*

$$\mu_{ukt}\, 2^r/u!k! = \sum_{k/2\le j\le min(k,r)} D_{k-j}\partial(1.3)^t[B_{r-j}C_{2j-k}], \tag{68}$$

$$where\ \partial(1.3)^t[B_hC_J] = \sum_{d+e=t}\binom{t}{d} B_{h.3d}C_{J.3e} \tag{69}$$

$$B_{h.3d} = \partial(1.3)^d B_h = H_{hd}^{13}/h!,\ C_{J.3e} = \partial(1.3)^e C_J = C_{J-e}[2\mu(23)]^e, \tag{70}$$

and H_{hd}^{13} is H_{rk}^{12} of (38) with $2, r, k$ replaced by $3, h, d$. That is,

$$\mu_{ukt}\, 2^r/u! = \sum_{k/2\le j\le min(k,r)} [k]_j \mu(2^2)^{k-j} I_{r-j,2j-k,t}^{123}\, 2^{2j-k}/(r-j)!(2j-k)!,$$

$$where\ I_{hJt}^{123} = \partial(1.3)^t\, \mu(1^2)^h\mu(12)^J = \sum_{d+e=t}\binom{t}{d}[J]_e\, H_{hd}^{13}\, \mu(12)^{J-e}\mu(23)^e. \tag{71}$$

(70) follows from (53). (71) follows from

$$\partial(1.3)^e\mu(12)^J = [J]_e\mu(12)^{J-e}.$$

This is our first formula for the general trivariate moment. Example 3 will need

$$I_{h0t}^{123} = H_{ht}^{13},\ I_{h1t}^{123} = H_{ht}^{13}\mu(12) + tH_{h,t-1}^{13}\mu(23),$$
$$I_{h2t}^{123} = H_{ht}^{13}\mu(12)^2 + 2tH_{h,t-1}^{13}\mu(12)\mu(23) + [t]_2H_{h,t-2}^{13}\mu(23)^2. \tag{72}$$
$$\text{Set } L_{hJt} = h!\ \text{RHS}(69) = \sum_{d+e=t}\binom{t}{d} H_{hd}^{13}(2\rho_{12})^g(2\rho_{23})^e/g!\ \text{at } g = J-e.$$

Corollary 6. *When $V_{jj} \equiv 1$, for $2r = u + k + t$ and H_{rk}^{12} of (41),*

$$\mu_{ukt}\, 2^r/u! = \sum_{k/2\le j\le min(k,r)} [k]_j\, L_{r-j,2j-k,t}/(r-j)! \tag{73}$$

$$where\ L_{hJt} = 2^J\sum_{e=0}^{J}[t]_e\rho_{12}^g\rho_{23}^e H_{h,t-e}^{13}/g!e!\ at\ g = J-e. \tag{74}$$

$$\text{So } L_{h0t} = H_{ht}^{13},\ L_{h1t} = 2\rho_{12}H_{ht}^{13} + 2t\rho_{23}H_{h,t-1}^{13},$$
$$L_{h2t} = 2\rho_{12}^2H_{ht}^{13} + 4t\rho_{12}\rho_{23}H_{h,t-1}^{13} + 2[t]_2\rho_{23}^2H_{h,t-2}^{13},$$

$$L_{h3t} = 4\rho_{12}^3 H_{ht}^{13}/3 + 4t\rho_{12}^2\rho_{23}H_{h,t-1}^{13} + 4[t]_2\rho_{12}\rho_{23}^2 H_{h,t-2}^{13} + 4[t]_3\rho_{23}^3 H_{h,t-3}^{13}/3,$$
$$L_{hJ1} = 2^J[2h\rho_{12}^J\rho_{13}/J! + \rho_{12}^{J-1}\rho_{23}/(J-1)!.$$

Putting $\tilde{L}_{hJt} = L_{hJt}$ when $\rho_{ij} \equiv 1$, gives the new identity

$$[2r]_{k+t} = \sum_{k/2\le j\le min(k,r)} [k]_j[r]_j\ \tilde{L}_{r-j,2j-k,t}/(r-j)!$$
$$\text{where } \tilde{L}_{hJt} = 2^J \sum_{e=0}^{min(J,t)} [t]_e[2h]_{t-e}/(J-e)!e!.$$

Some examples: $k = 3$ gives for $u + t = 2r - 3$,

$\mu_{u3t}\ (r-2)!2^r/u!3! = L_{r-2,1t} + (r-2)L_{r-3,3t}$.

$t = 3, u = 2r-6, h = r-2 \Rightarrow \mu_{u33}\ (r-2)!2^r/u!3! = L_{h13} + hL_{h-1,33}$.

Set $a = \rho_{12}\rho_{13}$, and $b = \rho_{12}^2 + \rho_{13}^2$.

$$\text{For } h = r-2,\ \mu_{2r-6,33}\ h!2^r/(2r-6)!3! = \sum_{j=0}^{3} c_j\ \rho_{23}^j \text{ where} \tag{75}$$

$c_0/16 = 3[h]_2\ a/2 + [h]_3\ ab + 2[h]_4\ a^3/3,$

$c_1/12 = h + 2[h]_2\ b + 4[h]_3\ a^2,\ c_2 = 48[h]_2\ a,\ c_3 = 8h.$

$$\text{So, } r = 4, h = 2 \text{ gives } 8\mu_{233}/3 = \sum_{j=0}^{3} c_j\rho_{23}^j \text{ where} \tag{76}$$

$c_0 = 48a,\ c_1 = 24(1+2b),\ c_2 = 96a,\ c_3 = 16.$

$$r = 5, u = 4, h = 3 \text{ gives } 4\mu_{433}/3 = \sum_{j=0}^{3} c_j\rho_{23}^j \text{ where}$$

$c_0 = 48(3a + 2ab),\ c_1 = 36(1 + 4b + 8a^2),\ c_2 = 288a,\ c_3 = 24.$

$$r = u = 6, h = 4 \text{ gives } 16\mu_{633}/45 = \sum_{j=0}^{3} c_j\rho_{23}^j \text{ where}$$

$c_0/32 = 9a + 12ab + 8a^3,\ c_1/48 = 1 + 6b + 24a^2,\ c_2 = 576a,\ c_3 = 32.$

$$r = 7, u = 8, h = 5 \text{ gives } 4\mu_{833}/63 = \sum_{j=0}^{3} c_j\rho_{23}^j \text{ where}$$

$c_0 = 160(3a + 6ab + 8a^3),\ c_1 = 60(1 + 8b + 48a^2),\ c_2 = 960a,\ c_3 = 40.$

By Corollary 6 with $k = 4$,

$$\text{for } u + t = 2r - 4,\ \mu_{u4t}2^r/u!4! = \sum_{j=2}^{4} L_{r-j,2j-4,t}/(r-j)!(4-j)!. \tag{77}$$

$$\text{So, } \mu_{242}/3 = 1 + 2\rho_{13}^2 + 4\rho_{12}^2 + 4\rho_{23}^2 + 16\rho_{12}\rho_{13}\rho_{23} + 8\rho_{12}^2\rho_{23}^2, \tag{78}$$

$$\mu_{341}/9 = \rho_{13} + 4\rho_{12}(\rho_{23} + \rho_{12}\rho_{13}) + 8\rho_{12}^3\rho_{23}/3, \tag{79}$$

$L_{143} = 16(\rho_{12}\rho_{23}^3 + \rho_{12}^3\rho_{23} + 3\rho_{12}^3\rho_{23}^2\rho_{13},$

$\mu_{343}/9 = 3\rho_{13} + 2\rho_{13}^3 + 12[\rho_{12}^2\rho_{13} + \rho_{12}\rho_{23}(1 + 2\rho_{13}^2) + \rho_{13}\rho_{23}^2]$
$+ 8[\rho_{12}\rho_{23}^3 + \rho_{12}^3\rho_{23} + 3\rho_{12}^2\rho_{23}^2\rho_{13}],$

$\mu_{444}/9 = 3 + 24b + 72a + 8c + 96d + 128ab + 192a^2$ where now

$a = \Pi_{123}^3\ \rho_{12},\ b = \sum_{123}^{3}\rho_{12}^2,\ c = \sum_{123}^{3}\rho_{12}^4,\ d = \sum_{123}^{3}\rho_{12}^2\rho_{13}^2 = (b^2 - c)/2.$

An alternative to Theorem 5 is given by applying $\partial(2.3)^t$ to (39) with $u = k - t$.

Trivariate moments by the multinomial method of Section 6.

We now apply (48) with $p = 3$. Set $c = (c_{12}, c_{13}, c_{23})$. Then

$$\rho^c = \rho_{12}^{c_{12}} \rho_{13}^{c_{13}} \rho_{23}^{c_{23}},\ c_1 = c_{12} + c_{13},\ c_2 = c_{12} + c_{23},\ c_3 = c_{13} + c_{23}.$$

Example 3. *Take* $n = (2r-2, 1, 1)$. *Then* $a_2 = a_3 = 0$ *and either* $c = (0,0,1)$ *or* $(1,1,0)$, *so that* F_n *sums over either* $a = (r-1,0,0),\ c = (0,0,1)$ *or* $a = (r-2,0,0),\ c = (1,1,0)$ *giving*

$$F_n = 2r\rho_{23} + 4[r]_2\rho_{12}\rho_{13},\ E_n = \nu_{2r}/[2r]_2 = \nu_{2r-2}/(2r),$$
$$\mu_n/\nu_{2r-2} = \rho_{23} + 2(r-1)\alpha \text{ where } \alpha = \rho_{12}\rho_{13}, \text{ as in (50).}$$
$$\text{So, } \mu_{211} = \rho_{23} + 2\alpha,\ \mu_{411}/3 = \rho_{23} + 4\alpha,\ \mu_{611}/15 = \rho_{23} + 6\alpha,$$
$$\mu_{811}/105 = \rho_{23} + 8\alpha,\ \mu_{10,11}/945 = \rho_{23} + 10\alpha,\ \mu_{12,11}/10395 = \rho_{23} + 12\alpha.$$

Example 4. *Take* $n = (2r-3, 2, 1)$. *Then* F_n *sums over either* $c = (0,1,0), a = (r-2,1,0)$ *or* $c = (1,0,1), a = (r-2,0,0)$ *or* $c = (2,1,0), a = (r-3,0,0)$ *giving*

$$F_n/2[r]_2 = \alpha_0 + (r-2)\alpha_1 \text{ where } \alpha_0 = \rho_{13} + 2\rho_{12}\rho_{23},\ \alpha_1 = 2\rho_{12}^2\rho_{23},$$
$$E_n = \nu_{2r}/D \text{ where } D = \binom{2r}{2r-3,2,1} = [2r]_3/2 = 2(2r-1)[r]_2 \Rightarrow$$
$$E_n = \nu_{2r-2}/2[r]_2,\ \mu_n/\nu_{2r-2} = \alpha_0 + (r-2)\alpha_1 :$$
$$\mu_{121} = \alpha_0 \text{ as in (12)},\ \mu_{321}/3 = \alpha_0 + \alpha_1 \text{ as in (15)},$$
$$\mu_{521} = \alpha_0 + 2\alpha_1 \text{ as in (21)},\ \mu_{721} = \alpha_0 + 3\alpha_1,\ \mu_{921} = \alpha_0 + 4\alpha_1.$$

This agrees with (50) with $\lambda = 2r - 3$.

Example 5. *Take* $n = (2r-4, 3, 1)$. $2a_3 + c_{13} + c_{23} = 1$ *so* $a_3 = 0$ *and* $(c_{13}, c_{23}) = (01)$ *(Case 1) or* (10) *(Case 2). Case 1:* $2a_2 + c_{12} + c_{23} = 3$ *so* $a_2 = 0$ *and* $(c_{12}, c_{23}) = (21)$ *(I) say, or* (30), *(II) say. If (I) then* $\binom{r}{ac} = \binom{r}{r-3,00201} = [r]_3/2,\ (2\rho)^c = 2^3\rho_{12}^2\rho_{23}$. *If (II) then* $\binom{r}{ac} = \binom{r}{r-2,10001} = [r]_2,\ (2\rho)^c = 2\rho_{23}$. *Case 2:* $2a_2 + c_{12} + c_{23} = 3$ *so* $a_2 = 0$ *and* $(c_{12}, c_{23}) = (30)$ *(I) say, or* (10), *(II) say. If (I) then* $\binom{r}{ac} = \binom{r}{r-4,00310} = [r]_4/6,\ (2\rho)^c = 2^4\rho_{12}^3\rho_{13}$. *If (II) then* $\binom{r}{ac} = \binom{r}{r-3,10110} = [r]_3,\ (2\rho)^c = 4\rho_{12}\rho_{13}$. *Also* $E_n = \nu_{2r}/\binom{2r}{2r-4,31} = 3\nu_{2r-4}/2[r]_2$. *So finally,*

$$\mu_n/\nu_{2r-4} = \sum_{j=0}^{2}[r-2]_j\alpha_j \text{ for } \alpha_0 = 3\rho_{23},\ \alpha_1 = 6\rho_{12}(\rho_{13} + \rho_{12}\rho_{23}),\ \alpha_2 = 4\rho_{12}^3\rho_{13}.$$

For example $r = 2, \cdots, 8$ *give (27), (15), (58),*

$$\mu_{631}/15 = \alpha_0 + 3\alpha_1 + 6\alpha_2,\ \mu_{831}/105 = \alpha_0 + 4\alpha_1 + 12\alpha_2,\ \mu_{10,31}/945 = \alpha_0 + 5\alpha_1 + 20\alpha_2,$$
$$\mu_{12,31}/10395 = \alpha_0 + 6\alpha_1 + 30\alpha_2.$$

Example 6. *Take* $n = (2r-4, 2, 2)$. *Then,* F_n *sums over* $a = (r-2,0,0),\ c = (0,0,2),\ a = (r-3,0,0),\ c = (1,1,1),\ a = (r-4,0,0),\ c = (2,2,0),\ a = (r-3,0,1),\ c = (2,0,0),\ a = (r-3,1,0),\ c = (0,2,0),\ a = (r-2,1,1),\ c = (0,0,0)$, *giving*

$$F_n/[r]_2 = \sum_{j=0}^{2}[r-2]_j\alpha_j \text{ where } \alpha_0 = 1+2\rho_{23}^2,\ \alpha_1 = 2\sum_{j=1}^{2}\rho_{1j}^2 + 8\rho_{12}\rho_{13}\rho_{23},\ \alpha_2 = 4\rho_{12}^2\rho_{13},$$

$$E_n = \nu_{2r-4}/[r]_2 \Rightarrow \mu_n/\nu_{2r-4} = \sum_{j=0}^{2}[r-2]_j\alpha_j.$$

$$\text{So, } \mu_{022} = \alpha_0,\ \mu_{222} = \alpha_0+\alpha_1,\ \mu_{422}/3 = \alpha_0+2\alpha_1+4\alpha_2,\ \mu_{622}/15 = \alpha_0+3\alpha_1+6\alpha_2,$$
$$\mu_{822}/105 = \alpha_0+4\alpha_1+12\alpha_2,\ \mu_{10,22}/945 = \alpha_0+5\alpha_1+20\alpha_2,\ \mu_{12,22}/10395 = \alpha_0+6\alpha_1+30\alpha_2.$$

Moments with $p > 3$ can be obtained by either of the last two methods.

8. Moments of Dimension 4

We use H_{rk}^{12} of (38) and (41). Recall that by (70), for $C_J = [2\mu(12)]^J/J!, \partial(1.3)^e C_J = C_{J-e}[2\mu(23)]^e$. That is,

$$\partial(1.3)^e\mu(12)^J = [J]_e\mu(12)^{J-e}\mu(23)^e.$$

By (34), applying $\partial(1.4)^v$ to (68) and setting $w = u - v$ gives

Theorem 6. *For $B_r, C_r^{12} = C_r, D_r$ of (40) and I_{hJt}^{123} of (71),*

$$\mu_{wktv}\,2^r/w!k! = \sum_{k/2\le j\le min(k,r)} D_{k-j}E_{r-j,2j-k,tv}/(r-j)! \text{ at } 2r = w+k+t+v,$$

$$E_{hJtv} = h!\partial(1.4)^v RHS(69) = \sum_{0\le e_1\le min(J,t)}\binom{t}{e_1}F_{hJ,t-e_1,e_1v}, \tag{80}$$

$$F_{hJd_1e_1v} = h!\partial(1.4)^v B_{h.3d_1}C_{J.3e_1} = \sum_{0\le e_2\le min(J-e_1,v)}\binom{v}{e_2}S_{hd_1,v-e_2}T_{Je_1e_2}, \tag{81}$$

$$S_{hd_1d_2} = h!\,\partial(1.4)^{d_2}B_{h.3d_1} = \partial(1.4)^{d_2}H_{hd_1}^{13}$$
$$= \sum_{d_1/2\le j\le min(d_1,h)}[h]_j[d_1]_j\ I_{h-j,2j-d_1,d_2}^{134}\mu(3^2)^{d_1-j}2^{2j-d_1}/(2j-d_1)!,$$

$$T_{Je_1e_2} = \partial(1.4)^{e_2}C_{J.3e_1} = T'\ [2\mu(23)]^{e_1}, T' = \partial(1.4)^{e_2}C_{J-e_1} = [2\mu(24)]^{e_2}C_{J-e_1-e_2}.$$

Corollary 7. *For $2r = w+k+t+v$, $V_{jj} \equiv 1$, E_{hJtv} of (80) and (81), and I_{hJt}^{123} of (71),*

$$\mu_{wktv}\,2^r/w! = \sum_{k/2\le j\le min(k,r)}[k]_j\ E_{r-j,2j-k,tv}/(r-j)! \text{ where}$$

$$S_{hd_1d_2} = \sum_{d_1/2\le j\le min(d_1,h)}[h]_j[d_1]_j\ I_{h-j,2j-d_1,d_2}^{134}2^{2j-d_1}/(2j-d_1)!, \tag{82}$$

$$T_{Je_1e_2} = (2\rho_{23})^{e_1}(2\rho_{24})^{e_2}C_{J-e_1-e_2} = 2^J\rho_{23}^{e_1}\rho_{24}^{e_2}\rho_{12}^{J-e_1-e_2}/(J-e_1-e_2)!. \tag{83}$$

For Examples 7 and 8, and H_{rk}^{12} of (38), we need

$$S_{h0d_2} = I_{h0d_2}^{134} = H_{hd_2}^{14},\ S_{h1d_2}/2h = I_{h-1,1d_2}^{134} = H_{h-1,d_2}^{14}\rho_{13} + d_2H_{h-1,d_2-1}^{14}\rho_{34},$$
$$S_{h2d_2}/2h = H_{h-1,d_2}^{14} + 2(h-1)I_{h-2,2d_2}^{134}. \tag{84}$$
$$\text{So } S_{h00} = 1,\ S_{h01} = 2h\rho_{14},\ S_{h02} = H_{h2}^{14} = 2h + 4[h]_2\rho_{14}^2,\ S_{h10} = 2h\rho_{13},$$
$$S_{h11} = 2h\rho_{34} + 4[h]_2\rho_{13}\rho_{14},\ S_{h12} = 4[h]_2\ (\rho_{13}+2\rho_{14}\rho_{34}) + 8[h]_3\rho_{13}\rho_{14}^2.$$
$$S_{h20} = 2h + 4[h]_2\rho_{13}^2,\ S_{h21} = 4[h]_2\rho_{14.3}^{(2)} + 8[h]_3\rho_{13}^2\rho_{14}, \tag{85}$$
$$S_{h22}/4 = [h]_2(1+2\rho_{34}^2) + 2[h]_3(\rho_{13}^2+\rho_{14}^2+4\rho_{13}\rho_{14}\rho_{34}) + 4[h]_4\rho_{13}^2\rho_{14}^2, \tag{86}$$

$$S_{h23}/8 = 3[h]_3(\rho_{14} + 2\rho_{13}\rho_{34} + 2\rho_{14}\rho_{34}^2) + 2[h]_4(\rho_{14}^3 + 3\rho_{13}^2\rho_{14} + 6\rho_{13}\rho_{14}^2\rho_{34}) + 4[h]_5\rho_{13}^2\rho_{14}^3,$$
$$T_{Je_10} = (2\rho_{23})^{e_1}C_{J-e_1},\ T_{Je_11} = (2\rho_{23})^{e_1}(2\rho_{24})C_{J-e_1-1},$$
$$T_{J00} = C_J,\ T_{J01} = 2\rho_{24}C_{J-1},\ T_{J10} = 2\rho_{23}C_{J-1},\ T_{J11} = 4\rho_{23}\rho_{24}C_{J-2},$$
$$T_{J20} = 2^J\rho_{23}^2\rho_{12}^{J-2}/(J-2)!,\ T_{J21} = 2^J\rho_{23}^2\rho_{24}\rho_{12}^{J-3}/(J-3)!.$$
$$I_{h23}^{134} = H_{h3}^{14}\rho_{13}^2 + 6H_{h2}^{14}\rho_{13}\rho_{34} + 6H_{h1}^{14}\rho_{34}^2.$$

Example 7. *Take* $v = 1, 2r-1 = w+k+t$, $S_{hd_1d_2}$ *of (82), and* $T_{Je_1e_2}$ *of (83). If* $V_{jj} \equiv 1$, *then*

$$\mu_{wkt1}\, 2^r/w! = \sum_{k/2\le j\le min(k,r)} [k]_j\, E_{r-j,2j-k,t1}/(r-j)!\ where \tag{87}$$
$$E_{hJt1} = \sum_{0\le e_1\le min(J,t)} \binom{t}{e_1} F_{hJ,t-e_1,e_11},\ F_{hJd_1e_11} = S_{hd_10}T_{Je_11} + S_{hd_11}T_{Je_10}.$$

Now take $t = 1, 2r-2 = w+k$. *Then, by (43) and (84),*

$$\mu_{wk11}\, 2^r/w! = \sum_{k/2\le j\le min(k,r)} [k]_j\, E_{r-j,2j-k,11}/(r-j)!\ where \tag{88}$$
$$E_{hJ11} = F_{hJ011} + F_{hJ101},$$
$$F_{hJ011} = S_{h00}T_{J11} + S_{h01}T_{J10},\ F_{hJ101} = S_{h10}T_{J01} + S_{h11}T_{J00},$$
$$So\ E_{hJ11} = 4\rho_{23}\rho_{24}C_{J-2} + 4hc\ C_{J-1} + (2h\rho_{34} + 4[h]_2\rho_{13}\rho_{14})C_J$$
$$where\ c = \rho_{13}\rho_{24} + \rho_{14}\rho_{23}. \tag{89}$$
$$So\ for\ k = 1,\ w = 2r-3,\ \mu_{w111}\, 2^r/w! = E_{r-1,111}/(r-1)!\ where \tag{90}$$
$$E_{h111} = 4hb + 8[h]_2a,\ a = \rho_{12}\rho_{13}\rho_{14},\ b = \rho_{12}\rho_{34} + \rho_{13}\rho_{24} + \rho_{14}\rho_{23}: \tag{91}$$
$$\mu_{1111} = b \Rightarrow (13),\ \mu_{3111} = 3(b+2a) \Rightarrow (15),\ \mu_{5111} = 15(b+4a), \tag{92}$$
$$\mu_{7111} = 105(b+6a),\ \mu_{2r+1,111} = \nu_{2r+2}\,(b+2ra).$$
$$For\ k = 2,\ w = 2r-4,\ \mu_{w211}\, 2^{r-1}/w! = \sum_{j=1}^{2} E_{r-j,2j-2,11}/(r-j)!, \tag{93}$$
$$where\ E_{h011} = 2h\rho_{34} + 4[h]_2\rho_{13}\rho_{24},$$
$$E_{h211} = 4\rho_{23}\rho_{24} + 8h\rho_{12}c + 4\rho_{12}^2(h\rho_{34} + 2[h]_2\rho_{13}\rho_{14}):$$
$$\mu_{0211} = \rho_{34.2}^{(2)} \Rightarrow (12),\ and\ for\ c\ of\ (89),$$
$$\mu_{2211} = \rho_{34} + 2\rho_{13}\rho_{14} + 2\rho_{23}\rho_{24} + 4\rho_{12}c + 2\rho_{12}^2\rho_{34} \Rightarrow (18),$$
$$\mu_{4211}/3 = \rho_{34} + 4\rho_{13}\rho_{14} + 2\rho_{23}\rho_{24} + 8\rho_{12}c + 4\rho_{12}^2\rho_{34.1}^{(2)}, \tag{94}$$
$$\mu_{6211}/15 = \rho_{34} + 6\rho_{13}\rho_{14} + 2\rho_{23}\rho_{24} + 12\rho_{12}c + 6\rho_{12}^2\rho_{34.1}^{(4)},$$
$$where\ \rho_{ij.k}^{(r)} = \rho_{ij} + r\rho_{ik}\rho_{jk}.$$
$$For\ k = 3, w = 2r-5, \mu_{w311}\, 2^{r-1}/3w! = \sum_{j=2}^{3} E_{r-j,2j-3,11}/(r-j)! \tag{95}$$
$$where\ E_{h311} = 8\rho_{12}\rho_{23}\rho_{24} + 8h\rho_{12}^2c + 8\rho_{12}^3(h\rho_{34} + 2[h]_2\rho_{13}\rho_{14})/3:$$
$$r = 3 \Rightarrow \mu_{1311} = 3b + 6\rho_{21}\rho_{23}\rho_{24} \Rightarrow (16),$$
$$r = 4 \Rightarrow \mu_{3311}/3 = 3b + 6\rho_{12}(\rho_{13}\rho_{14} + \rho_{23}\rho_{24}) + 6\rho_{12}^2c + 2\rho_{12}^3\rho_{34}, \tag{96}$$
$$r = 5 \Rightarrow \mu_{5311}/15 = 3b + 12a + 6\rho_{12}\rho_{23}\rho_{24} + 12\rho_{12}^2c + 4\rho_{12}^3\rho_{34.1}^{(2)},$$
$$r = 6 \Rightarrow \mu_{7311}/315 = b + 6a + 2\rho_{21}\rho_{23}\rho_{24} + 6\rho_{12}^2c + 2\rho_{12}^3\rho_{34.1}^{(4)}.$$
$$r = 7 \Rightarrow \mu_{9311}/945 = 3(b+6a) + 6\rho_{12}\rho_{23}\rho_{24} + 12\rho_{12}^2c + 8\rho_{12}^3\rho_{34.1}^{(6)}.$$

$$r = 8 \Rightarrow 4\mu_{11,311}/10395 = 12(b + 6a) + 24\rho_{12}\rho_{23}\rho_{24} + 120\rho_{12}^2 c + 120\rho_{12}^3\rho_{34.1}^{(8)}.$$

Now take $t = 2, 2r - 3 = w + k$. *Then, by (43) and (84),*

$$\mu_{wk21}\, 2^r/w! = \sum_{k/2 \le j \le min(k,r)} [k]_j\, E_{r-j,2j-k,21}/(r-j)! \text{ where} \tag{97}$$

$$E_{hJ21} = F_{hJ201} + 2F_{hJ111} + F_{hJ021},\ F_{hJ201} = S_{h20}T_{J01} + S_{h21}T_{J00},$$

$$F_{hJ111} = S_{h10}T_{J11} + S_{h11}T_{J10},\ F_{hJ021} = S_{h00}T_{J21} + S_{h01}T_{J20}.$$

$$k = 2,\ w = 2r - 5 \Rightarrow \mu_{w221}\, 2^{r-1}/w! = E_{r-1,021}/(r-1)! + E_{r-2,221}/(r-2)!$$

where $E_{h021} = S_{h21}$ *of (85),* $E_{h221} = F_{h2201} + 2F_{h2111} + F_{h2021}$,

$$F_{h2201} = S_{h20}T_{201} + S_{h21}T_{200},\ F_{h2111} = S_{h10}T_{211} + S_{h11}T_{210},\ F_{h2021} = S_{h01}T_{220}.$$

$$r = 3 \Rightarrow \mu_{1221} = \rho_{14}(1 + 2\rho_{23}^2) + 2\rho_{12}\rho_{24} + 2\rho_{13}\rho_{14}$$
$$+ 4\rho_{23}(\rho_{12}\rho_{34} + 2\rho_{13}\rho_{24}) \Rightarrow (18). \tag{98}$$

$$r = 4 \Rightarrow \mu_{3221} = \sum_{i=1}^{4} a_i \text{ where } a_1 = 3\rho_{14},\ a_2 = 6\rho_{12}\rho_{24} + 6\rho_{13}\rho_{34}, \tag{99}$$

$$a_3 = 6\rho_{14}(\rho_{12}^2 + \rho_{13}^2 + \rho_{23}^2) + 12\rho_{23}(\rho_{13}\rho_{24} + \rho_{12}\rho_{34}),$$

$$a_4 = 12\rho_{12}^2\rho_{13}\rho_{34} + 12\rho_{12}\rho_{13}^2\rho_{24} + 24\rho_{12}\rho_{13}\rho_{14}\rho_{23}.$$

$$r = 5 \Rightarrow 2\mu_{5221}/5 = E_{4021}/8 + E_{3221}/2,\ E_{4021} = S_{421} = 48(\rho_{14.3}^{(2)} + 4\rho_{13}^2\rho_{14}),$$

by (85), $E_{3221} = S_{320}T_{201} + S_{321}T_{200} + 2(S_{310}T_{211} + S_{311}T_{210}) + S_{301}T_{220}$,

$$\Rightarrow \mu_{5221}/15 = \sum_{i=1}^{5} a_i \text{ where } a_1 = \rho_{14},\ a_2 = 2\rho_{12}\rho_{24} + 2\rho_{13}\rho_{34},$$

$$a_3 = 4\rho_{12}\rho_{23}\rho_{34} + 4(\rho_{12}^2 + \rho_{13}^2)\rho_{14} + 4\rho_{13}\rho_{23}\rho_{24} + 2\rho_{14}\rho_{23}^2,$$

$$a_4 = 16\rho_{12}\rho_{13}\rho_{14}\rho_{23} + 8\sum_{23}^{2}\rho_{12}\rho_{13}^2\rho_{24},\ a_5 = 8\rho_{12}^2\rho_{13}^2\rho_{14}.$$

Now take $t = 3, 2r - 7 = w + k$. *By (87),*

$$\mu_{wk31}\, 2^r/w! = \sum_{k/2 \le j \le min(k,r)} [k]_j\, E_{r-j,2j-k,31}/(r-j)!,$$

$$w = 2r - 8 \Rightarrow \mu_{w131}\, 2^r/w! = E_{r-1,131}/(r-1)!,$$

$$w = 2r - 9 \Rightarrow \mu_{w231}\, 2^{r-1}/w! = E_{r-1,031}/(r-1)! + E_{r-2,231}/(r-2)!,$$

$$w = 2r - 10 \Rightarrow \mu_{w331}\, 2^{r-1}/3w! = E_{r-2,131}/(r-2)! + E_{r-3,331}/(r-3)!,$$

$$w = 2r - 11 \Rightarrow \mu_{w431}\, 2^{r-3}/3w! = E_{r-2,031}/2(r-2)! + E_{r-3,231}/(r-3)! + E_{r-4,431}/(r-4)!.$$

The reader can now easily work out special cases.

Example 8. *Take* $v = 2$, $2r - 2 = w + k + t$. *If* $V_{jj} \equiv 1$, *then*

$$\mu_{wkt2}\, 2^r/w! = \sum_{k/2 \le j \le min(k,r)} [k]_j\, E_{r-j,2j-k,t2}/(r-j)! \text{ where} \tag{100}$$

$$E_{hJt2} = \sum_{0 \le e_1 \le min(J,t)} \binom{t}{e_1} F_{hJ,t-e_1,e_1 2},$$

$$F_{hJd_1e_12} = S_{hd_10}T_{Je_12} + 2S_{hd_11}T_{Je_11} + S_{hd_12}T_{Je_10}.$$

Now take $t = 2, 2r-4 = w+k$. *Then*

$$\mu_{wk22}\, 2^r/w! = \sum_{k/2\le j\le min(k,r)} [k]_j\, E_{r-j,2j-k,22}/(r-j)! \tag{101}$$

where $E_{hJ22} = F_{hJ022} + 2F_{hJ112} + F_{hJ202}$,

$F_{hJ022} = S_{h00}T_{J22} + 2S_{h01}T_{J21} + S_{h02}T_{J20}$, $F_{hJ112} = S_{h10}T_{J12} + 2S_{h11}T_{J11}$

$+ S_{h12}T_{J10}$, $F_{hJ202} = S_{h20}T_{J02} + 2S_{h21}T_{J01} + S_{h22}T_{J00}$.

So $k = 2, w = 2r-6 \Rightarrow$

$$\mu_{w222}\, 2^{r-1}/w! = E_{r-1,022}/(r-1)! + E_{r-2,222}/(r-2)!\ \textit{where} \tag{102}$$

$E_{h022} = S_{h22}$ *of (86),* $E_{h222} = 8\sum_{j=1}^{4}[h]_j\, a_j$ *where*

$a_1 = a_{12} + a_{13}$, $a_{12} = \rho_{23}^2 + \rho_{24}^2$, $a_{13} = 4\rho_{23}\rho_{24}\rho_{34}$,

$a_2 = \rho_{12}^2 + \sum_{i=3}^{4} a_{2i}$, $a_{23} = 4\rho_{12}\rho_{13}\rho_{23} + 4\rho_{12}\rho_{14}\rho_{24}$,

$a_{24} = 2\rho_{12}^2\rho_{34}^2 + 8\rho_{12}\rho_{13}\rho_{14}\rho_{34} + 8\rho_{12}\rho_{32}\rho_{24}\rho_{34} + 8\rho_{13}\rho_{14}\rho_{23}\rho_{24}$,

$a_3 = \sum_{i=4}^{5} a_{3i}$, $a_{34} = 2\rho_{12}^2(\rho_{13}^2 + \rho_{14}^2)$,

$$a_{35} = 8\rho_{12}\rho_{13}\rho_{14}(\rho_{12}\rho_{34} + \rho_{13}\rho_{24} + \rho_{14}\rho_{23}), \tag{103}$$

$a_4 = 4\rho_{12}^2\rho_{13}^2\rho_{14}^2$.

$$r = 3 \Rightarrow \mu_{0222} = 1 + 2\rho_{14}^2 + 2a_1 \Rightarrow \textit{(17)}, \tag{104}$$

$$r = 4 \Rightarrow \mu_{2^4} = 1 + 2\sum_{i=2}^{4} c_i\ \textit{where}\ c_2 = \sum(\rho_{ij}^2 :\ 1 \le i < j \le 4), \tag{105}$$

$c_3/4 = \rho_{12}\rho_{13}\rho_{23} + \rho_{12}\rho_{14}\rho_{24} + \rho_{13}\rho_{14}\rho_{34} + \rho_{23}\rho_{24}\rho_{34}$,

$c_4 = 2\sum_{1234}^{3}(\rho_{12}^2\rho_{34}^2 + 4\rho_{12}\rho_{13}\rho_{23}\rho_{34})$.

$r = 5,\ w = 4 \Rightarrow \mu_{4222} = S_{422}/16 + E_{3222}/4 = 3 + 2\sum_{i=2}^{5} d_i$ *where*

$d_2 = 2\rho_{12}^2 + 2\rho_{13}^2 + 2\rho_{14}^2 + \rho_{23}^2 + \rho_{24}^2 + \rho_{34}^2$,

$d_3/4 = 2\rho_{12}(\rho_{13}\rho_{23} + \rho_{14}\rho_{24}) + (2\rho_{13}\rho_{14} + \rho_{23}\rho_{24})\rho_{34}$,

$d_4/4 = \rho_{12}^2\rho_{13}^2 + a_{24}/2 + a_{34}/2 = \rho_{12}^2(\rho_{13}^2 + \rho_{14}^2 + \rho_{34}^2) + \rho_{13}^2\rho_{14}^2 + \rho_{14}^2\rho_{23}^2$

$+ 4\rho_{12}(\rho_{14}\rho_{23} + \rho_{13}\rho_{24})\rho_{34} + 4\rho_{13}\rho_{14}\rho_{23}\rho_{24}$, $d_5 = 2a_{35}$ *of (103).*

$r = w = 6 \Rightarrow \mu_{6222}/15 = 1 + 2\sum_{i=2}^{5} e_i$ *where*

$e_2 = 3\rho_{12}^2 + 3\rho_{13}^2 + 3\rho_{14}^2 + \rho_{23}^2 + \rho_{24}^2 + \rho_{34}^2$,

$e_3/4 = 3\rho_{12}\rho_{13}\rho_{23} + 3\rho_{12}\rho_{14}\rho_{24} + 3\rho_{13}\rho_{14}\rho_{34} + \rho_{23}\rho_{24}\rho_{34}$,

$e_4 = \rho_{12}^2(\rho_{13}^2 + \rho_{14}^2 + \rho_{34}^2) + a_{24}/2$, $e_5 = 6a_{35}$.

$$\textit{For}\ k = 3,\ w = 2r-7, \mu_{w322}\, 2^{r-1}/3w! = \sum_{j=2}^{3} E_{r-j,2j-3,22}/(r-j)! \tag{106}$$

where $E_{h122} = 2S_{h12}T_{110} + 2S_{h21}T_{101} + S_{h23}T_{100}$,

$E_{h322} = F_{h3022} + 2F_{h3112} + F_{h3202}$, $F_{h3022} = 2S_{h01}T_{321} + S_{h02}T_{320}$,

$F_{h3112} = S_{h10}T_{312} + 2S_{h11}T_{311} + S_{h12}T_{310}$,

$F_{h3202} = S_{h20}T_{302} + 2S_{h21}T_{301} + S_{h23}T_{300}$.

For example, $w = 3, r = 5 \Rightarrow \mu_{3322}/3 = E_{3122}/16 + E_{4322}/64$.

For μ_{wkt3}, put $v = 3$ in Corollary 7. The method can be continued for higher moments. For example, for moments of dimension 5, one route is to apply $\partial^e_{1.5}$ to Theorem 6.

9. Moments of $Z \sim \mathcal{N}_p(\mu, V)$

So far, we have given the moments of $X \sim \mathcal{N}_p(0, V)$. It is worthwhile adding some examples of the non-central moments of $Z = X + \mu$. Consider the case $p = 2$. Then,

$$m_{rs} = E\, Z_1^r Z_2^s = \sum_{j=0}^{r} \binom{r}{j} \mu_1^{r-j} \sum_{k=0}^{s} \binom{s}{k} \mu_2^{s-k}\, \mu_{jk}$$

for μ_{jk} of (26). For example,

$$m_{r2} = \sum_{j\ even} \binom{r}{j} \mu_1^{r-j} (\mu_2^2 \mu_{j0} + \mu_{j2}) + 2\mu_2 \sum_{j\ odd} \binom{r}{j} \mu_1^{r-j} \mu_{j1},$$
$$m_{33} = \mu_1^3(\mu_2^3 + 3\mu_2\mu_{02}) + 3\mu_1^2(3\mu_2^2\mu_{11} + \mu_{13}) + 3\mu_1(\mu_2^3\mu_{20} + 3\mu_2\mu_{22}) + 3\mu_2^2\mu_{31} + \mu_{33}.$$

So if $V_{jj} \equiv 1$ and $\rho = \rho_{12}$,

$$m_{r2} = \sum_{j\ even} \binom{r}{j} \mu_1^{r-j} \nu_j (\mu_2^2 + 1 + j\rho^2) + 2\rho\mu_2 \sum_{j\ odd} \binom{r}{j} \mu_1^{r-j} \nu_{j+1}:$$
$$m_{22} = \mu_1^2\mu_2^2 + 4\rho\mu_1\mu_2 + \mu_2^2 + 1 + 2\rho^2,$$
$$m_{32} = \mu_1^3(\mu_2^2 + 1) + 3\mu_1(\mu_2^2 + 1 + 2\rho^2) + 6\rho(\mu_1^2 + 1)\mu_2,$$
$$m_{42} = \mu_1^4(\mu_2^2 + 1) + 6\mu_1^2(\mu_2^2 + 1 + 2\rho^2) + 3(\mu_2^2 + 1 + 4\rho^2) + 8\rho(\mu_1^3 + \mu_1)\mu_2,$$
$$m_{33} = \mu_1^3(\mu_2^3 + 3\mu_2) + 9\mu_1^2\rho(\mu_2^2 + 1) + 3\mu_1[\mu_2^3 + 3\mu_2(1 + 2\rho^2)] + 9\rho\mu_2^2 + 9\rho + 6\rho^3.$$

10. Discussion

The calculation of multivariate normal moments is foundational for statistical methods. In Bayesian analysis, the calculation of multivariate normal moments can be used to construct more accurate prior and posterior distributions. In the field of machine learning, the information of moments can be used to improve feature selection, dimensionality reduction algorithms, and so on, to improve the performance and interpretability of the model. Yet, hitherto, little progress has been made on multivariate normal moments since Isserlis's (1918) formula in [6] for the general bivariate normal moment, and two special cases of trivariate moments.

Refs. [12–14] gave central moments by matrix differentiation of $E\, e^{t'X} = e^{t'Vt/2}$. However the results are in terms of vec, $\otimes$ and permutation matrices, so make interpretation difficult. Its differentiation with respect to t can be used to give moments in terms of multivariate Bell polynomials, but again these take some effort to understand. For applications to quantum mechanics and field theory, see page 9 of [15]. In electrical engineering and electronics, there are a vast number of applications of the *complex* normal and the complex Wishart. See for examples [16,17] and the many citations in [8].

We have given three methods for evaluating multivariate normal moments, giving explicit formulas for those of dimensions 2–4. This contribution is significant because existing methods have often been cumbersome and complex. Our bivariate formula of Corollary 1 is simpler than that of [6]. It introduces a function of the bivariate correlation ρ_{12}, H^{12}_{rk} of (41). Our trivariate formula of Corollary 6 builds on this function with the function L_{hJt} of (74), a function of ρ_{ij}, $1 \le i < j \le 3$. Our normal moments of dimension 4 in Theorem 6 and Corollary 7, lift the complexity of the formula one more step.

We hope that these formulas will be accessible to help practitioners in applying theoretical concepts without the burden of complex mathematical derivations. The explicit expressions for moments of dimensions 3 and 4, along with potential extensions for higher dimensions, represent a significant leap in the field, promoting broader applications in multivariate statistics. By comparing existing methodologies and presenting new results, this fills a critical gap in the literature on multivariate normal distributions, providing valuable insights based on earlier foundational work. The ability to accurately compute multivariate moments is crucial in various fields including finance, engineering, and social sciences, where joint distributions of multiple variables are analysed.

11. Conclusions and Future Directions

We leave it to others to take up the task of programming these results for use in practical case studies. This might build on the software in R for moments by [18].

The use of efficient algorithms and techniques are needed for high-dimensional, high-order cases. This could involve leveraging symmetry and sparsity to streamline computations, or investigating numerical and approximate methods to enhance computational efficiency and applicability.

As dimensionality and order rise, the computational effort of the proposed methods may escalate rapidly. In high-dimensional, high-order scenarios, the process can become highly intricate, necessitating computational software for assistance and limiting the methods' widespread practical application. However, multivariate analysis beyond dimension 4 is not common.

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