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An Archimedean Copulas-Based Approach for *m*-Consecutive-*k*-Out-of-*n*: *F* Systems with Exchangeable Components

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Abstract: It is evident that several real-life applications, such as telecommunication systems, call for the establishment of consecutive-type networks. Moreover, some of them require more complex connectors than the ones that exist already in the literature. Thereof, in the present work we provide a signature-based study of a reliability network consisting of identical m-consecutive-k-out-of-n: F structures with exchangeable components. The dependency of the components of each system is modeled with the aid of well-known Archimedean copulas. Exact formulae for determining the expected lifetime of the underlying reliability scheme are provided under different Archimedean copulas-based assumptions. Several numerical results are carried out to shed light on the performance of the resulting consecutive-type design. Some thoughts on extending the present study to more complex consecutive-type reliability structures are also discussed.

Keywords: m-consecutive-k-out-of-n: F systems; Clayton copula model; Gumbel–Hougaard copula model; maximal signature; exchangeable components

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1. Introduction

Over the past four decades, a noticeable research effort has been steered towards the study of a particular family of reliability models, which are known as consecutive-type structures. Several real-life applications, such as telecommunication networks, oil pipeline systems or complex infrared-detecting systems, can be well-handled with the aid of the aforementioned reliability designs.

Generally speaking, a great collection of consecutive-type systems appear in the existing literature and can be viewed as generalizations of the well-known consecutive-k-out-of-n: F structure. The particular system consists of n linearly (or circularly) ordered components and stops its operation if, and only if, at least k consecutive components fail (see, e.g., [1–3]). On the other hand, its generalizations may offer more flexible operation principles. For example, it is common for the practitioner to face problems related to two different failure criteria. For such cases, structures whose operation stops due to more than one reason are more suitable. The interested reader is referred to the (n,f,k) and $<n,f,k>$ structures (see, e.g., [4,5]), the constrained-(k,d)-out-of-n systems (see [6,7]) or the consecutive-k_1 and k_2-out-of-n: F structures (see [8]).

On the other hand, enhancement of the performance of the underlying structures can be achieved by adding the so-called cold-standby redundancy therein. The aforementioned cold-standby sparing has been adopted by several authors in order to deal with optimization problems (see, e.g., [9–12]). Moreover, the dilemma of whether the cold standby redundancy is more effective when it is applied at system level or at component level has attracted the attention of some researchers (see, e.g., [13–16]). For additional surveys concerning the cost optimization or optimal allocation in reliability structures, the interested reader is referred to [17–19].

In addition, several signature-based reliability studies of k-out-of-n or consecutive-k-out-of-n systems appear in the existing literature. For instance, we refer to the works [20–22]. Moreover, some more complex reliability structures have also been investigated by several authors (see, e.g., [23–30]).

It is evident that in many real-life engineering applications, the components of the systems are dependent (or at least exchangeable). That is why, throughout the course of the present article, we implement well-known copulas for modeling the dependence of the components of the underlying structures. It is noteworthy that copulas have been proved to be useful tools for studying the joint distribution of random lifetimes of the components of a reliability model (see, e.g., [31–38]).

In Section 2, we describe the general framework of constructing a reliability network consisting of ξ independent subsystems. Each subsystem consists of n exchangeable components, while it follows the design of the well-known m-consecutive-k-out-of-n: F structures. An illustrative example is also presented for making the architecture of the resulting reliability network clearer.

In Section 3, we prove the main results of the paper, which refer to the expected lifetime of the resulting reliability network under different Archimedean copulas-based models. More precisely, the Clayton copula model and the Gumbel–Hougaard copula model are considered, and the mean time to failure of the resulting networks is determined via explicit formulae.

In Section 4, we provide several numerical results for the performance of the resulting network consisting of ξ m-consecutive-k-out-of-n: F substructures under different choices for its design parameters. Finally, the discussion section summarizes the contribution of the present paper, while some practical concluding remarks are also highlighted.

2. The Copulas-Based Framework for the Proposed Network Consisting of ξ m-Consecutive-k-Out-of-n: F Subsystems

In the present section, we give an account of the general framework for constructing the proposed reliability network. More specifically, the network consists of ξ independent subsystems, where ξ is a positive and integer-valued parameter. We assume that each subsystem consisting of n exchangeable components, follows the design of the well-known m-consecutive-k-out-of-n: F structures. Under the exchangeability assumption, the components have identical distributions, but they are not necessarily independent, as they may affect one another (see, e.g., [38]).

Generally speaking, the m-consecutive-k-out-of-n: F system is a natural generalization of the classical m-out-of-n: F system and the consecutive-k-out-of-n: F system; it consists of n linearly ordered components such that the system fails if, and only if, there are at least m non-overlapping runs of k consecutive failed components ($n \geq m \cdot k$). The general architecture of the proposed network ($NET(\xi,m,k,n)$ hereafter) is illustrated in Figure 1.

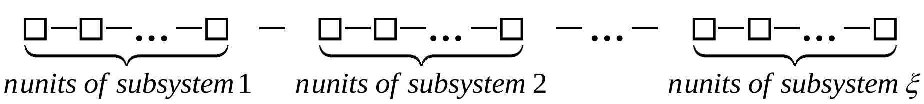

Figure 1. The general architecture of the network $NET(\xi,m,k,n)$.

Note that the symbol □ corresponds to a working unit of a subsystem within the network. For instance, let us assume that four independent m-consecutive-k-out-of-n: F subsystems with seven exchangeable components are connected to each other. We also consider that all of them share a common design, namely the remaining parameters m and k are equal to two and three, respectively. The resulting network $NET(4,2,3,7)$ is given in Figure 2.

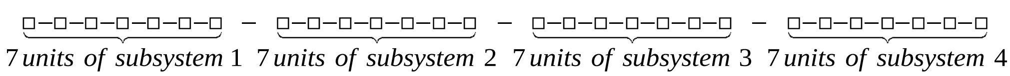

Figure 2. The network $NET(4,2,3,7)$.

The proposed network stops its operation when an appropriately chosen failure criterion is met. Depending on how the network is built, its overall failure shall be observed at the time when the corresponding stopping criterion is fulfilled. For instance, let us next consider the case where the abovementioned independent subsystems formulate a parallel network. In simple words, we next assume that four m-consecutive-k-out-of-n: F structures are serially connected to each other. Under such an assumption, the overall network fails if and only if all subsystems stop their operation, e.g., all four 2-consecutive-3-out-of-7: F subsystems fail. The following figure displays two different scenarios, under which the operation of the series network mentioned before has already stopped.

Kindly note that the symbol $\bullet$ corresponds to a failed unit of a subsystem within the network. Based on Figure 3, it is quite clear that each subsystem appearing in the network, does not operate anymore. Indeed, one can readily observe that at each subsystem there are two non-overlapping runs of three consecutive failed components. Therefore, the operation of each subsystem has already stopped. It goes without saying that once the practitioner chooses to build the network alternatively, the failure criterion shall follow the alternative architecture.

$$\underbrace{\bullet{-}\bullet{-}\bullet{-}\square{-}\bullet{-}\bullet{-}\bullet}_{7\, units\ of\ subsystem\ 1} - \underbrace{\square{-}\bullet{-}\bullet{-}\bullet{-}\bullet{-}\bullet{-}\bullet}_{7\, units\ of\ subsystem\ 2} - \underbrace{\bullet{-}\bullet{-}\bullet{-}\bullet{-}\bullet{-}\bullet{-}\square}_{7\, units\ of\ subsystem\ 3} - \underbrace{\bullet{-}\bullet{-}\bullet{-}\bullet{-}\bullet{-}\bullet{-}\square}_{7\, units\ of\ subsystem\ 4}$$

$$\underbrace{\bullet{-}\bullet{-}\bullet{-}\bullet{-}\bullet{-}\bullet{-}\square}_{7\, units\ of\ subsystem\ 1} - \underbrace{\bullet{-}\bullet{-}\bullet{-}\square{-}\bullet{-}\bullet{-}\bullet}_{7\, units\ of\ subsystem\ 2} - \underbrace{\bullet{-}\bullet{-}\bullet{-}\square{-}\bullet{-}\bullet{-}\bullet}_{7\, units\ of\ subsystem\ 3} - \underbrace{\square{-}\bullet{-}\bullet{-}\bullet{-}\bullet{-}\bullet{-}\bullet}_{7\, units\ of\ subsystem\ 4}$$

Figure 3. Failure scenarios for the series network NET(4,2,3,7).

Kindly note that the design parameters ξ,m,k,n of the proposed network are integer-valued and free of unit, since all of them correspond to a number of components.

Since the components are assumed to be exchangeable, our next step is to model their dependence with the aid of appropriate Archimedean copulas. Generally speaking, the copulas are useful tools for determining the joint distribution of random variables. Additionally, copulas provide a conventional framework for analyzing Markov processes and establishing scale-free measures of dependence between pairs of random variables. From one point of view, copulas are functions that join (or couple) multivariate distribution functions to their one-dimensional marginal distribution functions. In what follows, we shall apply well-known copula models for studying the lifetime of the resulting reliability network consisting of subsystems with exchangeable components. Having at hand the copula that seems to be appropriate for modeling the components' dependence, one may investigate the network's mean lifetime until it fails.

Let us first denote using $T_1^{(i)}, T_2^{(i)}, \ldots, T_n^{(i)}$, the lifetimes of the components of the i-th subsystem of the network $NET(\xi,m,k,n)$ defined earlier. We also denote using $G_j^{(i)}(t_{i,j}) = P\left(T_j^{(i)} \leq t_{i,j}\right)$, the cumulative distribution function of the variable $T_j^{(i)}, i = 1,2,\ldots,\xi$ and $j = 1,2,\ldots,n$, while $H(t_{i,1},\ldots,t_{i,n}) = P\left(T_1^{(i)} \leq t_{i,1}, T_2^{(i)} \leq t_{i,2}, \ldots, T_n^{(i)} \leq t_{i,n}\right)$ corresponds to the joint distribution function of the lifetimes of the components of the i-th subsystem.

Each vector $(t_{i,1}, t_{i,2}, \ldots, t_{i,n})$ of real numbers leads to a point $\left(G_1^{(i)}(t_{i,1}), G_2^{(i)}(t_{i,2}), \ldots, G_n^{(i)}(t_{i,n})\right)$ in the unit region $[0,1] \times [0,1] \ldots \times [0,1]$, while these ordered coordinates correspond to a number $H(t_{i,1},\ldots,t_{i,n})$ in [0,1]. The aforementioned correspondence, which assigns the value of the joint distribution function to each ordered vector of values of the individual distribution functions, is actually the copula function C.

Given that the components of each subsystem are exchangeable, we have that $G_j^{(i)}(t) = G^{(i)}(t),\ i = 1,2,\ldots,\xi$ and $j = 1,2,\ldots,n$. Therefore, if $C(u_1,\ldots,u_n)$ is the copula function related to $H(t_{i,1}, t_{i,2}, \ldots, t_{i,n})$, the following holds true

$$H(t_{i,1},\ldots,t_{i,n}) = C\left(G^{(i)}(t_{i,1}), G^{(i)}(t_{i,2}), \ldots, G^{(i)}(t_{i,n})\right). \tag{1}$$

Moreover, if the ξ subsystems of the resulting network are considered to be independent and identical, then the index i in the above formula is not necessary anymore. It is clear that the corresponding subsystems' lifetimes $S_1, S_2, \ldots, S_\xi$ are independent and identically distributed random variables, and thereof equality (1) can be simplified as follows

$$H(t_1, t_2, \ldots, t_n) = C(G(t_1), G(t_2), \ldots, G(t_n)). \tag{2}$$

Throughout the course of the present work, we shall consider two different copula functions for modeling the dependence of the components in each subsystem. All multivariate models implemented in the present manuscript, are members of the well-known class of Archimedean copulas (see, e.g., [32] and references therein). More precisely, we shall next consider the following multivariate Archimedean copulas:

- The Clayton family of n-copulas.

  The generator function of the bivariate Clayton copulas is given by

$$\varphi_\theta(t) = t^{-\theta} - 1. \tag{3}$$

For $\theta > 0$ and $n \geq 2$, the copula function of the multivariate Clayton class of n-copulas is expressed as

$$C_\theta^n(u_1, u_2, \ldots, u_n) = \left(u_1^{-\theta} + u_1^{-\theta} + \ldots + u_1^{-\theta} - n + 1\right)^{-1/\theta}. \tag{4}$$

- The Gumbel–Hougaard family of n-copulas.

  The generator function of the bivariate Gumbel–Hougaard copulas is given by

$$\varphi_\theta(t) = (-lnt)^\theta,\ \theta \geq 1. \tag{5}$$

For $\theta \geq 1$ and $n \geq 2$, the copula function of the multivariate Gumbel–Hougaard class of n-copulas is written as

$$C_\theta^n(u_1, u_2, \ldots, u_n) = \exp(-[(-lnu_1)^\theta + (-lnu_2)^\theta + \ldots + (-lnu_n)^\theta]^{\frac{1}{\theta}}). \tag{6}$$

In what follows, we shall also refer to the reliability function of a coherent system, namely the probability that the system works at a specific time point. In particular, let us denote using S the lifetime of a coherent system with n exchangeable components, while $T_1, T_2, \ldots, T_n$ correspond to the lifetimes of its components. If $T_{i:i}, i = 1, 2, \ldots, n$ is the i-th ordered component's lifetime, then the reliability function of the coherent system is expressed as

$$P(S > s) = \textstyle\sum_{i=1}^n \beta_i P(T_{i:i} > s) = \sum_{i=1}^n \beta_i (1 - P(T_1 \leq s, T_2 \leq s, \ldots, T_n \leq s)), \tag{7}$$

where $\beta_i,\ i = 1, 2, \ldots, n$ satisfy the condition $\sum_{i=1}^n \beta_i = 1$. Note that the vector $(\beta_1, \beta_2, \ldots, \beta_n)$ is the maximal signature of the coherent system (see, e.g., [39–41]).

Since the resulting network consists of substructures of a specific type, namely m-consecutive-k-out-of-n: F structures, it is clear that some limitations are now present. For example, if we assume that the subsystems of a real-life network do not fit well to the ones mentioned before, then the proposed network cannot be applied. Moreover, an additional limitation is generated by the fact that the dependence of the components of each substructure is assumed to be modeled either with the Clayton or Gumbel–Hougaard family of n-copulas. Needless to say that once this dependence cannot be modeled by the aforementioned copulas, then the present study needs to be extended. In other words, the proposed reliability network manages to cover several real-life situations, which call for an m-consecutive-k-out-of-n: F-type structure.

3. Main Results

In this section, we establish the theoretical results of the present paper. The proposed network $NET(\xi,m,k,n)$ is under consideration. In other words, we assume once again that ξ independent m-consecutive-k-out-of-n: F structures with exchangeable components are connected to each other.

Two different scenarios are taken into account for the design of the resulting network $NET(\xi,m,k,n)$. More specifically, we consider the case of a series network consisting of ξ independent subsystems and also a parallel one. For both cases mentioned above, the expected lifetime (or alternatively the mean time to failure (MTTF)) of the resulting reliability scheme is determined with the aid of explicit integral expressions.

The next proposition offers closed formulae for calculating the MTTFs of the proposed consecutive-type networks under different distributional assumptions. Three Archimedean copulas mentioned in the previous section are used for modeling the dependence between the components of each subsystem in the underlying network. According to the first scenario, the appropriate copula model comes from the so-called Clayton family.

Proposition 1. *Let us consider a network $NET(\xi,m,k,n)$, consisting of ξ independent m-consecutive-k-out-of-n: F structures with exchangeable components that have the absolutely continuous joint cumulative distribution function H. We denote using T_{NET}, the lifetime of the network; using $T_1^{(i)}, T_2^{(i)}, \ldots, T_n^{(i)}$, the lifetimes of the components of the i-th subsystem of the network; and using $S_1, S_2, \ldots, S_\xi$, the corresponding subsystems' lifetimes. If the multivariate copula function, which is related toH, belongs to the Clayton family, then the following holds true:*

(i) *Given that the ξ subsystems formulate a series network, then the MTTF of the resulting reliability scheme is given by*

$$E(T_{NET}) = \int_0^\infty \prod_{i=1}^{\xi} \left(1 - \sum_{j=m\cdot k}^{n} (j\cdot(G(t))^{-\theta} - j + 1)^{-\frac{1}{\theta}} \cdot\left(\sum_{s=m}^{[j/k]} \sum_{q=s\cdot k}^{j} \binom{n-q}{n-j}\cdot\binom{s+n-q}{s}\cdot C(q-s\cdot k, k, n-q+1)\right)\right)dt, \tag{8}$$

where G is the marginal cumulative distribution function of H, while

$$\binom{a}{b} = \frac{a!}{b!(a-b)!},\ C(a,b,c) = \sum_{s=0}^{c} (-1)^s \binom{c}{s}\cdot\binom{a+c-1-s\cdot b}{a-s\cdot b}. \tag{9}$$

(ii) *Given that the ξ subsystems formulate a parallel network, then the MTTF of the resulting reliability scheme is given by*

$$E(T_{NET}) = \int_0^\infty (1 - \prod_{i=1}^{\xi}(\sum_{j=m\cdot k}^{n} (j\cdot(G(t))^{-\theta} - j + 1)^{-\frac{1}{\theta}} \times(\sum_{s=m}^{[j/k]} \sum_{q=s\cdot k}^{j} \binom{n-q}{n-j}\cdot\binom{s+n-q}{s}\cdot C(q-s\cdot k,k,n-q+1)))dt, \tag{10}$$

where G is the marginal cumulative distribution function of H and the quantities $\binom{a}{b}$, $C(a,b,c)$are defined in (9).

Proof.

(i) The MTTF of the series network $NET(\xi,m,k,n)$ can be computed with the aid of the following formula (see, e.g., [14–16])

$$E(T_{NET}) = \int_0^\infty P(T_{NET} > t)dt = \int_0^\infty \prod_{i=1}^{\xi} P(S_i > t)dt, \tag{11}$$

where $S_i, i = 1, 2, \ldots, \xi$ is the lifetime of the i-th subsystem of the network. The probability appearing in the above integral expression, can be determined by observing that (see Formula (7))

$$P(S_i \leq t) = \sum_{j=1}^{n} \beta_j P\left(T_{j:j}^{(i)} \leq t\right), \tag{12}$$

where $\beta_j,\ j = 1, 2, \ldots, n$ are the coordinates of the maximal signature vector of each subsystem within the network and

$$T_{j:j}^{(i)} = \max\left(T_1^{(i)}, T_2^{(i)}, \ldots, T_j^{(i)}\right),\ j = 1, 2, \ldots, n \text{ and } i = 1, 2, \ldots, \xi. \tag{13}$$

In addition, the following holds true

$$\begin{aligned} P\left(T_{j:j}^{(i)} \leq t\right) &= P\left(T_1^{(i)} \leq t, T_2^{(i)} \leq t, \ldots, T_j^{(i)} \leq t\right) \\ &= C_\theta^n\left(\underbrace{G(t), G(t), \ldots G(t)}_{j}, \underbrace{1, 1, \ldots 1}_{n-j}\right), \end{aligned} \tag{14}$$

where

$$G(t) = P\left(T_j^{(i)} \leq t\right),\ j = 1, 2, \ldots, n \text{ and } i = 1, 2, \ldots, \xi. \tag{15}$$

Recalling the copula function of the multivariate Clayton class (see Formula (4)), the above expression is rewritten as

$$P\left(T_{j:j}^{(i)} \leq t\right) = (j \cdot (G(t))^{-\theta} - j + 1)^{-\frac{1}{\theta}},\ j = 1, 2, \ldots, n \text{ and } i = 1, 2, \ldots, \xi. \tag{16}$$

Since the subsystems are considered to be independent m-consecutive-k-out-of-n: F structures with exchangeable components, their maximal signature vector is determined with the aid of the following formula (see [33])

$$\beta_j = \sum_{s=m}^{[j/k]} \sum_{q=s\cdot k}^{j} \binom{n-q}{n-j} \cdot \binom{s+n-q}{s} \cdot C(q - s \cdot k, k, n-q+1), \tag{17}$$

where the quantities $\binom{a}{b}$, $C(a, b, c)$ are defined in (9).

By combining Formulas (11)–(17), the desired result is readily deduced.

(ii) The MTTF of the parallel network $NET(\xi,m,k,n)$ can be determined as

$$E(T_{NET}) = \int_0^\infty P(T_{NET} > t)dt = \int_0^\infty (1 - \prod_{i=1}^{\xi}(1 - P(S_i > t)))dt. \tag{18}$$

Following a parallel argumentation with the one implemented at part (i), we next substitute Formulas (12), (16) and (17) in (18) and the expression we are chasing for is concluded after some straightforward manipulations. □

The following proposition deals with the case where the dependence of the components in each substructure of the network is modeled with the aid of the so-called Gumbel–Hougaard multivariate copulas.

Proposition 2. *Let us consider, a network NET(ξ,m,k,n), consisting of ξ independent m-consecutive-k-out-of-n: F structures with exchangeable components having absolutely continuous joint cumulative distribution function H. We denote using T_{NET} the lifetime of the network, using $T_1^{(i)}, T_2^{(i)}, \ldots, T_n^{(i)}$ the lifetimes of the components of the i-th subsystem of the network and using $S_1, S_2, \ldots, S_\xi$ the corresponding subsystems' lifetimes. If the multivariate copula function, which is related to H, belongs to the Gumbel–Hougaard family, then the following holds true:*

(i) *Given that the ξ subsystems formulate a series network, then the MTTF of the resulting reliability scheme is given by*

$$E(T_{NET}) = \int_0^\infty \prod_{i=1}^{\xi} \left(1 - \sum_{j=m\cdot k}^{n} \exp(j^{1/\theta}\cdot \ln(G(t))\right.$$
$$\left.\cdot\left(\sum_{s=m}^{[j/k]} \sum_{q=s\cdot k}^{j} \binom{n-q}{n-j}\cdot\binom{s+n-q}{s}\cdot C(q-s\cdot k, k, n-q+1)\right)\right)dt, \quad (19)$$

where G is the marginal cumulative distribution function of H;the quantities $\binom{a}{b}$, $C(a,b,c)$*are defined in (9).*

(ii) *Given that the ξ subsystems formulate a parallel network, then the MTTF of the resulting reliability scheme is given by*

$$E(T_{NET}) = \int_0^\infty (1 - \prod_{i=1}^{\xi} (\sum_{j=m\cdot k}^{n} \exp(j^{1/\theta}\cdot \ln(G(t))$$
$$\times(\sum_{s=m}^{[j/k]} \sum_{q=s\cdot k}^{j} \binom{n-q}{n-j}\cdot\binom{s+n-q}{s}\cdot C(q-s\cdot k, k, n-q+1)))dt, \quad (20)$$

where G is the marginal cumulative distribution function of H and the quantities $\binom{a}{b}$, $C(a,b,c)$*are defined in (9).*

Proof.

(i) The MTTF $(E(T_{NET}))$ of the series network $NET(\xi,m,k,n)$ can be computed with the aid of (13). We first replace the copula function of the Gumbel–Hougaard family (see Formula (6)) in the expression (16) and the following is readily observed

$$P\left(T_{j:j}^{(i)} \le t\right) = \exp(j^{1/\theta}\cdot \ln(G(t)),\ j = 1,2,\ldots,n \text{ and } i = 1,2,\ldots,\xi. \quad (21)$$

We next substitute the above expression in Formula (12) and the expression for determining the MTTF of the series network under the Gumbel–Hougaard copula model, is clear.

(ii) The MTTF $(E(T_{NET}))$ of the parallel network $NET(\xi,m,k,n)$ can be determined via (20). Following a parallel argumentation with (i) partially implemented, we next substitute Formulas (12), (17) and (21) in (18) and the expression we are searching for is concluded after some straightforward manipulations. □

In addition, both the series and parallel network, which are studied in the previous propositions, include the free use of unit parameters θ,ξ,m,k,n for determining the appropriate design in each case.

As it clearly observed, the proof procedures of Propositions 1 and 2 follow a parallel argumentation in order to reach the desired result. It goes without saying, once the suitable copula-based model is appropriately chosen, the practitioner is able to calculate the corresponding expected lifetime of the resulting network with the aid of Propositions 1 and 2.

4. Numerical Results

In the present section, we carry out a numerical investigation to shed light on the behavior of the proposed reliability network. The numerical results and figural representations displayed throughout the next section are based on the mathematical outcomes which have been presented and proved in Section 3.

Let us assume that ξ independent 3-consecutive-2-out-of-n: F structures with exchangeable components are connected. The behavior of the resulting series network $NET(\xi,3,2,n)$ is studied for several values of the remaining parameters ξ,n,θ. In particular, Table 1 displays the MTTFs of the aforementioned networks for n = 6, 7, . . ., 10 and ξ = 2, 3, 4, under the assumption that components' lifetimes follow the exponential distribution with the parameter λ = 1, namely $G(t) = 1 - e^{-t}, t > 0$. The dependence of the components is modeled with the aid of all three copula families mentioned previously. The results can be used either for direct appraisal of the resulting reliability schemes or for comparative analysis of their performance among the cases considered.

Table 1. MTTF values of the network $NET(\xi,m,k,n)$ for several designs $(m = 3, k = 2, \lambda = 1)$.

		Clayton Copula		Gumbel–Hougaard Copula	
n	***ξ***	***θ***	***MTTF***	***θ***	***MTTF***
6	2	1	1.380130	1	1.796790
		2	1.143790	2	2.430460
	3	1	1.056160	1	1.535480
		2	0.813431	2	2.156320
	4	1	0.867392	1	1.535480
		2	0.629722	2	2.156320
7	2	1	1.113730	1	1.450640
		2	0.944208	2	2.062510
	3	1	0.837676	1	1.231230
		2	0.664296	2	1.827570
	4	1	0.679528	1	1.107220
		2	0.511124	2	1.692370
8	2	1	0.874650	1	1.156820
		2	0.761105	2	1.745730
	3	1	0.655515	1	0.998243
		2	0.535529	2	1.570740
	4	1	0.528980	1	0.905478
		2	0.411793	2	1.466350
9	2	1	0.765576	1	1.003020
		2	0.681833	2	1.574720
	3	1	0.569664	1	0.867208
		2	0.478560	2	1.421680
	4	1	0.457256	1	0.787787
		2	0.321452	2	1.330230
10	2	1	0.683293	1	0.887530
		2	0.621696	2	1.443990
	3	1	0.506140	1	0.770645
		2	0.425516	2	1.309540
	4	1	0.362310	1	0.701700
		2	0.261182	2	1.228410

Based on the numerical results presented in Table 1, we readily deduce that the MTTF of the resulting network (regardless of the copula model which has been applied).

- decreases as the parameter ξ increases
- decreases as the parameter n increases
- decreases as the parameter θ increases.

The corresponding results have also been produced under the assumption that the components' lifetimes follow exponential distribution with parameter the $\lambda > 1$. Based on Table 2, it is readily observed that similar conclusions to the ones stated previously can be drawn. More specifically, Table 2 displays the expected lifetime of the resulting network $NET(\xi, 3, 2, n)$ for $n = 6, 7, \ldots, 10$, $\xi = 2, 3, 4$, while the marginal cumulative distribution function of the joint distribution of the components' lifetimes is now given as $G(t) = 1 - e^{-t/1.2}, t > 0$ $(\lambda = 1.2)$.

It is also of some interest to investigate the behavior of the MTTF of the proposed network in terms of the distributional parameter θ.

More specifically, Figures 4 and 5 display the MTTF values that are achieved using the underlying network of different choices for the parameter θ. In all cases included in the following diagrams, the rest design parameters remain stable, so that some conclusions on the influence of θ upon the MTTF of the network can be drawn.

Table 2. MTTF values of the network $NET(\xi,m,k,n)$ for several designs $(m = 3, k = 2, \lambda = 1.2)$.

		Clayton Copula		Gumbel–Hougaard Copula	
n	***ξ***	***θ***	***MTTF***	***θ***	***MTTF***
6	2	1	1.656160	1	2.156150
		2	1.372550	2	2.916560
	3	1	1.26739	1	1.842570
		2	0.976117	2	2.587580
	4	1	1.040870	1	1.662250
		2	0.755646	2	2.395870
7	2	1	2.227470	1	1.740760
		2	1.888420	2	2.475020
	3	1	1.005121	1	1.477470
		2	0.797145	2	2.193220
	4	1	0.815422	1	1.328660
		2	0.613488	2	1.992311
8	2	1	1.049580	1	1.388180
		2	0.913326	2	2.094880
	3	1	0.786618	1	1.197890
		2	0.642624	2	1.885100
	4	1	0.634776	1	1.086570
		2	0.494148	2	1.576240
9	2	1	0.918691	1	1.203630
		2	0.818196	2	1.889660
	3	1	0.683592	1	1.040650
		2	0.574272	2	1.250000
	4	1	0.548722	1	0.945345
		2	0.385740	2	0.998672
10	2	1	0.819952	1	1.065040
		2	0.7460352	2	1.732830
	3	1	0.607368	1	0.9247740
		2	0.510619	2	1.4429821
	4	1	0.434772	1	0.8420390
		2	0.313418	2	1.2128762

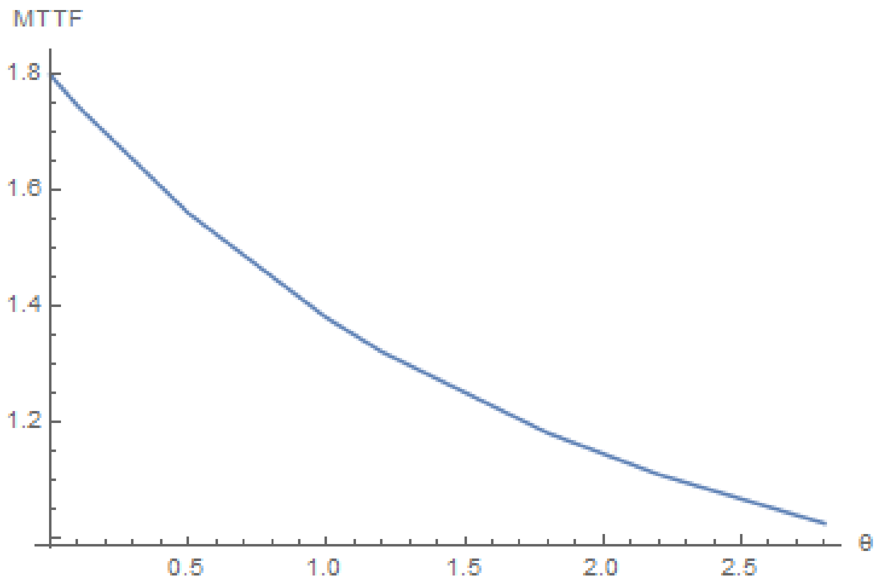

Figure 4. MTTFs of the network $NET(\xi,m,k,n)$ versus parameter θ under Clayton copula.

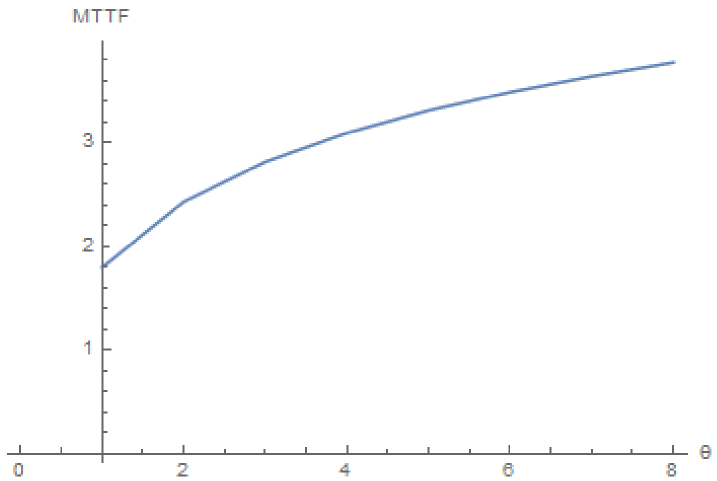

Figure 5. MTTFs of the network $NET(\xi,m,k,n)$ versus parameter θ under Gumbel–Hougaard copula.

Based on Figures 4 and 5, it is easily deduced that as the parameter θ gets smaller:

- The MTTF of the resulting network increases under the Clayton copula;
- The MTTF of the resulting network decreases under the Gumbel–Hougaard copula.

5. Symbols and Notations

$NET(\xi,m,k,n)$: The network that consists of ξ independent m-consecutive-k-out-of-n: F substructures.

$T_1^{(i)}, T_2^{(i)}, \ldots, T_n^{(i)}$: The lifetimes of the components of the i-th subsystem of the network $NET(\xi,m,k,n)$.

$G_j^{(i)}\left(t_{i,j}\right)$: The cumulative distribution function of the variable $T_j^{(i)}$.

$H\left(t_{i,1},\ldots,t_{i,n}\right)$: The joint distribution function of the lifetimes of the components of the i-th subsystem.

$C(u_1,\ldots,u_n)$: The copula function related to $H\left(t_{i,1},t_{i,2},\ldots,t_{i,n}\right)$.

$(\beta_1,\beta_2,\ldots,\beta_n)$: The maximal signature of a coherent system with n components.

T_{NET}: The lifetime of the network $NET(\xi,m,k,n)$.

$S_1,S_2,\ldots,S_{\xi}$: The corresponding subsystems' lifetimes of the network $NET(\xi,m,k,n)$. $E(T_{NET})$: Mean Time to Failure of the network $NET(\xi,m,k,n)$.

6. Discussion

In the present work, a reliability network consisting of m-consecutive-k-out-of-n: F structures with exchangeable components has been proposed and studied in some detail. Different copulas are used to model the dependence of the components of each substructure. It seems that the copulas which have been considered prompt the proposed network to perform quite differently. The main contribution of the manuscript refers to the determination of the expected lifetime of the resulting network, namely its MTTF. Based on the numerical results, we conclude that as parameter θ gets smaller, the MTTF does not behave with the same way under the two specific copulas considered. That leads to the conclusion that the copula model seems to play a significant role in the overall behavior of the network. For future scope, one may consider carrying out an analogous reliability study of networks consisting of different types of substructures or alternative copula models.

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