

Article

Game-Theoretic Models of Coopetition in Cournot Oligopoly

Guennady Ougolnitsky ^{1,*}  and Alexey Korolev ² 

¹ I.I. Vorovich Institute for Mathematics, Mechanics, and Computer Science, Southern Federal University, 344090 Rostov-on-Don, Russia

² Department of Mathematics, Higher School of Economics, St. Petersburg Branch, 190121 St. Petersburg, Russia; danitschi@gmail.com

* Correspondence: gaugolnickiy@sfnu.ru; Tel.: +7-918-558-47-07

Abstract: Coopetition means that in economic interactions, both competition and cooperation are presented in the same time. We built and investigated analytically and numerically game theoretic models of coopetition in normal form and in the form of characteristic function. The basic model in normal form reflects competition between firms in Cournot oligopoly and their cooperation in mutually profitable activities such as marketing, R&D, and environmental protection. Each firm divides its resource between competition and cooperation. In the model in normal form we study Nash and Stackelberg settings and compare the results. In cooperative setting we consider Neumann–Morgenstern, Petrosyan–Zaccour, and Gromova–Petrosyan versions of characteristic functions and calculate the respective Shapley values. The payoffs in all cases are compared, and the respective conclusions about the relative efficiency of different ways of organization for separate agents and the whole society are made.

Keywords: coopetition; Cournot oligopoly; games in normal form; games in the form of characteristic function

MSC: 91A80



Citation: Ougolnitsky, G.; Korolev, A. Game-Theoretic Models of Coopetition in Cournot Oligopoly. *Stats* **2023**, *6*, 576–595. <https://doi.org/10.3390/stats6020037>

Academic Editor: Wei Zhu

Received: 6 April 2023

Revised: 24 April 2023

Accepted: 28 April 2023

Published: 4 May 2023



Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

A notion of coopetition was proposed in the monograph by Brandenburger and Nalebuff [1]. Its idea consists in that almost all economic interactions contain both elements of competition and cooperation. For example, two firms may compete for their clients but join their efforts in marketing or R&D. Moreover, temporary price cartels may be advantageous for all participants. Firms can share the accounts of their clients. In general, coopetition is well combined with modern ESG-trends.

A connection between coopetition and game theory was established by Okura and Carfi [2]. Carfi and his coauthors published many papers on game theoretic models of coopetition; for example, Carfi and Donato have investigated cooperative games for sustainability of global feeding and climate change [3]. Very interesting analysis of cooperation in games in normal form is presented by Kalai and Kalai [4].

Strategies of coopetition are also analyzed in Cygler et al. [5], Gnyawali et al. [6], Ritala [7], Shvindina [8], and Walley [9]. Different applications are considered in Czakon and Dana [10], Rodrigues et al. [11], and Sharma, M. G. and Singh [12]. Game theoretic aspects of coopetition are studied in Heiets et al. [13], Le Roy et al. [14,15], and Ozkan-Canbolat et al. [16].

Another important idea is that there are different ways of economic and social organization (competition, cooperation, hierarchy), and they differ essentially from the point of view of the whole society and individuals. A very convenient model for the comparative analysis of different methods of organization of the economic and other active agents is Cournot oligopoly (see Maskin and Tirole [17]; Geras'kin [18,19]; Algazin and Algazina [20,21]). Xiao et al. [22,23] studied a Cournot duopoly with bounded rationality and

investigated the respective equilibria. Raoufinia et al. [24] analyzed open-loop and closed-loop solutions in a Cournot duopoly game with advertizing. Al-Khedhairi [25] considered a non-trivial Cournot duopoly model based on fractal differential equations. Julien [26] investigated a Cournot oligopoly with several Stackelberg leaders and followers. Zouhar and Zouharova [27] have compared Cournot and Stackelberg equilibria.

Several classes of games may be used for the analysis of coopetition. Games in normal form are very well known [28,29]. Games in the form of characteristic function (cooperative games) are also well known [30]. However, it should be noticed that different characteristic functions may be used in them. The most known model is a superadditive Neumann–Morgenstern characteristic function where a coalition plays a zero-sum game with its anti-coalition [31]. However, this assumption is not natural in many economic contexts. That is why Petrosyan and Zaccour proposed another characteristic function [32]. Unfortunately, this function may not be superadditive. Thus, Gromova and Petrosyan proposed the third characteristic function and proved its superadditivity [33]. Certainly, other characteristic functions can exist.

Korolev and Ougolnitsky analyzed cooperative game theoretic models of Cournot oligopoly [34]. Korolev et al. investigated dynamic models of Cournot oligopoly [35]. Ougolnitsky and Usov studied differential game-theoretic models of Cournot oligopoly with consideration of the green effect [36]. In all those papers a comparative analysis of the ways of organization of economic agents is made.

The main idea of the paper consists in comparison of the different ways of organization of economic agents such as competition, cooperation, and hierarchical control for the original coopetition models of Cournot oligopoly. The contribution of this paper and its novelty is the following:

- Game theoretic models of coopetition on the base of Cournot oligopoly are built and investigated analytically and numerically;
- A comparative analysis of selfish agents behavior (respectively, a game in normal form), their hierarchical organization (respectively, Stackelberg games), and cooperation (respectively, optimization problem) is conducted;
- We made the same analysis for the game theoretic models in the form of characteristic functions proposed by von Neumann and Morgenstern, Petrosyan and Zaccour, and Gromova and Petrosyan.

In Section 2, we build and investigate coopetition models in games in normal form of Cournot oligopoly for selfish behavior of players, their hierarchical organization, and cooperation. In Section 3, a similar work is carried out for game theoretic models in the form of characteristic function in different versions. In Section 4 we consider a Stackelberg oligopoly when the first firm becomes a leader. In Section 5 we compare the results. Section 6 concludes.

2. Coopetition in Games in Normal Form

The model of coopetition has the following form:

$$g_i(x) = (a - c_i - \bar{x})x_i + \frac{1}{n} \sum_{j=1}^n b_j (r_j - x_j)^{a_j} \rightarrow \max, \quad (1)$$

$$0 \leq x_i \leq r_i \leq \frac{a}{n}, c_i < \frac{a}{n}, i \in N. \quad (2)$$

Here $N = \{1, 2, \dots, n\}$ —a set of agents (firms); r_i —an amount of resource of the i -th agent; x_i —a share of the i -th agent resource allocated to the production as in standard Cournot oligopoly (output volume); c_i —production unit cost; a —demand parameter; $r_i - x_i$ —a share of the i -th agent resource invested in cooperation (for example, marketing, R&D, or environmental protection) profitable for all agents; $0 < a_i \leq 1, b_i > 0$ —coefficients of efficiency of cooperation; $x = (x_1, x_2, \dots, x_n)$; $\bar{x} = x_1 + x_2 + \dots + x_n$.

Thus, the first term in (1) reflects firms' competition, and the second term describes their payoffs from cooperation. The problem is that the firms should divide their resources between competition and cooperation, and the optimal ratio is to be determined [37].

The interests of the grand coalition are given by the criterion

$$\bar{g}(x) = \sum_{i=1}^n g_i(x) = (a - \bar{x})\bar{x} - \sum_{i=1}^n c_i x_i + \sum_{j=1}^n b_j (r_j - x_j)^{\alpha_j} \rightarrow \max \quad (3)$$

subject to (2).

In the case of centralized control a special agent (Principal) is introduced. The Principal can exert an administrative influence (compulsion) or economic influence (impulsion) on other agents. The Principal's interests are described by the cooperative criterion

$$g_0(x) = \sum_{i=1}^n b_i (r_i - x_i)^{\alpha_i} \rightarrow \max \quad (4)$$

In the case of administrative control (compulsion) the Principal bounds from above production investments of the agents. Namely, restrictions (2) take the form

$$0 \leq x_i \leq q_i. \quad (5)$$

This incurs Principal's administrative costs reflecting by a convex function decreasing with each argument $C(q_1, q_2, \dots, q_n)$, $C(q_1, q_2, \dots, q_n) = 0$. Then Principal's optimization problem is (4) with constraints

$$C(q_1, q_2, \dots, q_n) \leq R \quad (6)$$

where R is Principal's resource, $C(q_1, q_2, \dots, q_n) = \frac{\sum_{i=1}^n (r_i - q_i)}{\sum_{i=1}^n q_i}$.

In the case of economic control (impulsion) the Principal reports to each agent a control mechanism $r_i(x)$ that stimulates cooperation. Then Principal's optimization problem is (4) with constraints

$$\sum_{i=1}^n r_i \leq R. \quad (7)$$

In other words, we consider three ways of organization of the agents' interaction.

1. A selfish behavior of the agents without the Principal: a game in normal form (1)–(2).
2. A cooperative behavior of the agents: an optimization problems (2)–(3).
3. A hierarchical control with Principal: a Stackelberg game (4)–(6), (1)–(2), or (4), (7), (1)–(2).

In the cooperative setup we can identify cooperation and centralized control. We keep this logic in Section 3 for games in the form of characteristic function.

First, consider a selfish behavior of the firms without Principal's influence. Notice that

$$\frac{\partial g_i(x)}{\partial x_i} = a - c_i - 2x_i - \sum_{j \neq i} x_j - \frac{\alpha_i b_i}{n} (r_i - x_i)^{\alpha_i - 1}$$

$$\frac{\partial^2 g_i(x)}{\partial x_i^2} = -2 + \frac{\alpha_i(\alpha_i - 1)b_i}{n} (r_i - x_i)^{\alpha_i - 2}$$

The first order condition is

$$\frac{\partial g_i(x)}{\partial x_i} = a - c_i - 2x_i - \sum_{j \neq i} x_j - \frac{\alpha_i b_i}{n} (r_i - x_i)^{\alpha_i - 1} = 0. \quad (8)$$

If the following condition holds

$$\frac{\alpha_i b_i}{r_i^{1-\alpha_i}} \leq n(a - c_i) \quad (9)$$

then the Equation (8) has in the segment $[0, r_i]$ the unique solution denoted by $x_i^*(x_{-i}, r_i)$. In this point a maximum of the payoff function $g_i(x)$ is attained, and $0 \leq x_i^*(r) < r_i$. Notice that $x_i^*(x_{-i}, r_i)$ increases with r_i and decreases with each $x_j, j \neq i$, i.e., with each $x_j \in x_{-i}$. Really,

$$(x_i)'_{x_j} = \frac{1}{-2 + (\alpha_i - 1) \frac{\alpha_i b_i}{n} (r_i - x_i)^{\alpha_i - 2}} < 0, \quad (10)$$

$$(x_i)'_{r_i} = \frac{(\alpha_i - 1) \frac{\alpha_i b_i}{n} (r_i - x_i)^{\alpha_i - 2}}{-2 + (\alpha_i - 1) \frac{\alpha_i b_i}{n} (r_i - x_i)^{\alpha_i - 2}} > 0.$$

Now consider a case of administrative control (compulsion) by the Principal that solves the problem (4) with constraint

$$\sum_{i=1}^n r_i - \sum_{i=1}^n q_i \leq R \sum_{i=1}^n q_i$$

In the point of extremum this constraint becomes a strict equality, i.e.,

$$\sum_{i=1}^n q_i = \frac{\sum_{i=1}^n r_i}{R + 1} = Q, \quad (11)$$

where the right hand side is denoted by Q for convenience.

Notice that an output volume x_i chosen by the i -th firm is equal to

$$x_i = \begin{cases} x_i^*(x_{-i}, r_i) & \text{if } x_i^*(x_{-i}, r_i) \leq q_i, \\ q_i & \text{if } x_i^*(x_{-i}, r_i) > q_i. \end{cases} \quad (12)$$

Consider first a case of a small enough Q such that a solution of the problem $b_1(r_1 - q_1)^{\alpha_1} + b_2(r_2 - q_2)^{\alpha_2} + \dots + b_n(r_n - q_n)^{\alpha_n} \rightarrow \max, q_1 + q_2 + \dots + q_n$ satisfies the condition

$$q_i \leq x_i^*(q_{-i}, r_i), \quad i = 1, 2, \dots, n.$$

Evidently, in this case it solves Principal's problems (4) and (11).

The Lagrange multipliers method implies that

$$\alpha_i b_i (r_i - q_i)^{\alpha_i - 1} = \lambda, \quad (13)$$

$$r_i - q_i = \left(\frac{\lambda}{\alpha_i b_i} \right)^{\frac{1}{\alpha_i - 1}}, \quad q_i = r_i - \left(\frac{\lambda}{\alpha_i b_i} \right)^{\frac{1}{\alpha_i - 1}}$$

Then

$$\begin{aligned} \sum_{i=1}^n q_i &= \sum_{i=1}^n r_i - \sum_{i=1}^n \lambda^{\frac{1}{\alpha_i - 1}} (\alpha_i b_i)^{\frac{1}{\alpha_i - 1}} \frac{1}{1 - \alpha_i} = Q = \frac{\sum_{i=1}^n r_i}{R + 1} \\ \sum_{i=1}^n r_i - \frac{\sum_{i=1}^n r_i}{R + 1} &= \frac{R}{R + 1} \sum_{i=1}^n r_i = \sum_{i=1}^n \lambda^{\frac{1}{\alpha_i - 1}} (\alpha_i b_i)^{\frac{1}{\alpha_i - 1}} \frac{1}{1 - \alpha_i}. \end{aligned}$$

If $\alpha_1 = \alpha_2 = \dots = \alpha_n = \alpha, b_1 = b_2 = \dots = b_n = b$ then an explicit form of the solution is:

$$\sum_{i=1}^n r_i - \frac{\sum_{i=1}^n r_i}{R+1} = \frac{R}{R+1} \sum_{i=1}^n r_i = \sum_{i=1}^n \lambda^{\frac{1}{\alpha-1}} (\alpha_i b_i) \frac{1}{1-\alpha_i}.$$

$$\left(\frac{\lambda}{\alpha b}\right)^{\frac{1}{\alpha-1}} = \frac{R \sum_{i=1}^n r_i}{n(R+1)}, q_i = r_i - \frac{R}{n(R+1)} \sum_{i=1}^n r_i,$$

$$g_0 = \sum_{i=1}^n b_i \left(\frac{R \sum_{j=1}^n r_j}{(R+1)n}\right)^\alpha = nb \left(\frac{R \sum_{j=1}^n r_j}{(R+1)n}\right)^\alpha.$$

Now consider a case when Q is as big as for one firm the inequality in the first alternative in (12) holds as the strict equality:

$$x_{i_0}^*(x_{-i_0}, r_{i_0}) = q_{i_0}. \quad (14)$$

Then Principal increases q_{i_0} (because the firm i_0 cannot use it all the same), and decreases in the same time by the equal amount the sum of all other values q_j (equal to x_j):

$$\sum_{j=1, j \neq i_0}^n q_j \quad (15)$$

If the output volume $x_j, j \neq i_0$ decreases then $x_{i_0}^*(x_{-i_0}, r_{i_0})$ increases according to (10). Is it possible that the inequality in the second alternative in (12) be satisfied for another non-optimal distribution of terms in Q ? No, it is impossible. According to (10), when q_{i_0} increases on a small amount dq_{i_0} , and a summary value (14) decreases on the same amount then the optimal value of the output of i -th firm i_0 will increase on a smaller value:

$$dx_{i_0}^*(x_{-i_0}, r_{i_0}) = \frac{dx_{i_0}}{2 + (1 - \alpha_{i_0}) \frac{\alpha_{i_0} b_{i_0}}{n} (r_{i_0} - x_{i_0})^{\alpha_{i_0} - 2}} < dx_{i_0} = dq_{i_0}.$$

Thus, in this case Principal's optimal strategy is the following: to assign the whole output volume Q to the firm q_{i_0} while all other firms can participate only in cooperative activity: $q_j = 0, j \neq i_0$.

Let us describe Principal's optimal strategy of compulsion in general.

1. Suppose that Q is big enough, i.e., in the set of firms $N = \{1, 2, \dots, n\}$ exists a non-empty subset $I \subseteq N$ such as for any firm $i \in I$ holds $Q \geq x_i^*(x_{-i}, r_i)$, where $x_{-i} = (0, 0, \dots, 0)$. In other words, $x_i^*(x_{-i}, r_i)$ is a solution of the Equation (8) for the case when all firms except the i -th one participate only in cooperative activity:

$$a - c_i - 2x_i - \frac{\alpha_i b_i}{n} (r_i - x_i)^{\alpha_i - 1} = 0$$

Then the Principal chooses in the set I such a firm i^* that maximizes a value

$$b_i (r_i - x_i^*(x_{-i}, r_i))^{\alpha_i} + \sum_{j \in I, j \neq i} b_j r_j^{\alpha_j},$$

of the total payoff when only one firm can invest in the individual production, i.e.,

$$i^* = \arg \max_i \left\{ b_i (r_i - x_i^*(x_{-i}, r_i))^{\alpha_i} + \sum_{j \in I, j \neq i} b_j r_j^{\alpha_j} \right\}$$

The Principal should permit to the firm i^* a right to invest to the individual production any amount of resource up to Q while for all other firms the individual production is forbidden at all:

$$q_{i^*} = Q, q_j = 0, j \neq i^*$$

2. Now suppose that the set I mentioned above is empty, i.e., Q is not as big as before. Then optimal values q_i are determined from the system of Equation (13), $i \in N$. Particularly, if $\alpha_1 = \alpha_2 = \dots = \alpha_n = \alpha$ then

$$q_i = r_i - \frac{R}{n(R+1)} \sum_{i=1}^n r_i, i = 1, 2, \dots, n$$

Example 1. Consider a case of administrative control (compulsion). Assume that

$$a = 10, n = 3, c_1 = 3, c_2 = 2.5, c_3 = 2, r_1 = r_2 = r_3 = 3.3$$

$$\alpha_1 = \alpha_2 = \alpha_3 = 0.5; b_1 = b_2 = b_3 = 6; R = 5; Q = 1.65$$

Then the values $x_1^*(x_{-1}, r_1), x_2^*(x_{-2}, r_2), x_3^*(x_{-3}, r_3)$ are determined from the system of Equation (8) that takes the form:

$$\begin{cases} 7 - 2x_1 - x_2 - x_3 - \frac{1}{\sqrt{3.3-x_1}} = 0, \\ 7.5 - x_1 - 2x_2 - x_3 - \frac{1}{\sqrt{3.3-x_2}} = 0, \\ 8 - x_1 - x_2 - 2x_3 - \frac{1}{\sqrt{3.3-x_3}} = 0. \end{cases}$$

Solving this system of equations numerically we receive

$$x_1^*(x_{-1}, r_1) = 1.270759,$$

$$x_2^*(x_{-2}, r_2) = 1.685694$$

$$x_3^*(x_{-3}, r_3) = 2.070793$$

The values of q_1, q_2, q_3 are equal to

$$q_i = r_i - \frac{R}{n(R+1)} \sum_{i=1}^n r_i = 3.3 - \frac{5}{3.6} 9.9 = 0.55, i = 1, 2, 3$$

Since $q_i < x_i^*(x_{-i}, r_i), i = 1, 2, 3$, this is the solution (see (12)):

$$x_1 = 0.55, x_2 = 0.55, x_3 = 0.55$$

Now consider a case of economic control (impulsion). The Principal solves the problems (4) and (7). It is evident that in a point of maximum the inequality (7) is satisfied as a strict equality. The Lagrange multipliers method gives

$$\alpha_i b_i (r_i - x_i^*(x_{-i}, r_i))^{\alpha_i - 1} = \lambda \quad (16)$$

where $x_i^*(x_{-i}, r_i)$ is the unique solution of (8) for the chosen r_i :

$$a - c_i - 2x_i^*(x_{-i}, r_i) - \sum_{j \neq i} x_j^*(x_{-j}, r_j) - \frac{\alpha_i b_i}{n} (r_i - x_i^*(x_{-i}, r_i))^{\alpha_i - 1} = 0$$

From (16) we receive

$$r_i - x_i^*(x_{-i}, r_i) = \left(\frac{\lambda}{\alpha_i b_i} \right)^{\frac{1}{\alpha_i - 1}}, \quad (17)$$

$$x_i^*(x_{-i}, r_i) = r_i - \left(\frac{\lambda}{\alpha_i b_i} \right)^{\frac{1}{\alpha_i - 1}}. \quad (18)$$

Then

$$\sum_{i=1}^n r_i - \sum_{i=1}^n x_i^*(x_{-i}, r_i) = \sum_{i=1}^n \left(\frac{\lambda}{\alpha_i b_i} \right)^{\frac{1}{\alpha_i - 1}}$$

or

$$R - \sum_{i=1}^n \left(\frac{\lambda}{\alpha_i b_i} \right)^{\frac{1}{\alpha_i - 1}} = \sum_{i=1}^n x_i^*(x_{-i}, r_i). \quad (19)$$

From (19) we can determine λ , and from (17) and (18) we find r_i and $x_i^*(x_{-i}, r_i)$, $i = 1, 2, \dots, n$. Particularly, if $\alpha_1 = \alpha_2 = \dots = \alpha_n = \alpha$ then from (19) we receive

$$\lambda = \alpha \left(\sum_{i=1}^n b_i^{\frac{1}{1-\alpha}} \right)^{1-\alpha} \left[R - \sum_{i=1}^n x_i^*(x_{-i}, r_i) \right]^{\alpha-1}$$

If, besides, $b_1 = b_2 = \dots = b_n = b$ then the expression takes the form

$$\lambda = \alpha b \left(\frac{R - \bar{x}}{n} \right)^{\alpha-1}, \quad \bar{x} = \sum_{i=1}^n x_i^*(x_{-i}, r_i)$$

Substitution of (16) to (8) implies

$$a - c_i - x_i - \bar{x} = \frac{\lambda}{n}. \quad (20)$$

In the case $\alpha_1 = \alpha_2 = \dots = \alpha_n = \alpha$, $b_1 = b_2 = \dots = b_n = b$ we can simply find $\bar{x}$, adding term by term the Equation (20) for $i = 1, 2, \dots, n$ that gives the equation

$$na - \bar{c} - \bar{x} - n\bar{x} = \lambda = \alpha b \left(\frac{R - \bar{x}}{n} \right)^{\alpha-1}, \quad (21)$$

where $\bar{c} = \sum_{i=1}^n c_i$.

Example 2. Consider a case of impulsion. Assume that

$$a = 10; n = 3; c_1 = 3; c_2 = 2.5; c_3 = 2; r_1 = r_2 = r_3 = 3.3;$$

$$\alpha_1 = \alpha_2 = \alpha_3 = 0.5; b_1 = b_2 = b_3 = 6; R = 9.9.$$

A value of R has changed because in the case of impulsion its sense differs from the case of compulsion. The Equation (21) takes the form:

$$22.5 - 4\bar{x} - \frac{3\sqrt{3}}{\sqrt{9.9 - \bar{x}}} = 0$$

Solving it numerically we find $\bar{x} = 5.035987$. Then

$$\lambda = \alpha b \left(\frac{R - \bar{x}}{n} \right)^{\alpha-1} = 0.5 \cdot 6 \left(\frac{9.9 - 5.035987}{3} \right)^{-0.5} = 2.356050$$

According to (20),

$$x_i = a - \bar{x}_i - \frac{\lambda}{n} - c_i, \rightarrow x_1 = 1.178663, x_2 = 1.678663, x_3 = 2.178663,$$

and, according to (17),

$$r_i = x_i + \left(\frac{\lambda}{ab} \right)^{\frac{1}{a-1}}, \rightarrow r_1 = 2.8, r_2 = 3.3, r_3 = 3.8.$$

3. Coopetition in Games in the Form of Characteristic Function

Remind that in a cooperative game with a set of players $N = \{1, \dots, n\}$ the characteristic function is a map $v : 2^N \rightarrow R$, and its value $v(K)$ is called a characteristic of the coalition $K \subseteq N$. It is assumed as a rule that a characteristic function is superadditive:

$$v(K \cup L) \geq v(K) + v(L), K \cap L = \emptyset. \text{ Moreover, } v(\emptyset) = 0.$$

The most well-known characteristic function was proposed by von Neumann and Morgenstern [31] in the form

$$v^{NM}(K) = val(K, N \setminus K) = \sup_{x_i, i \in K} \inf_{x_j, j \in N \setminus K} \sum_{i \in K} u_i(x_1, \dots, x_n)$$

Here $u_i(x_1, \dots, x_n)$ is a payoff function of the player i in a normal form game between the agents from the set $N = \{1, \dots, n\}$, $val(K, N \setminus K)$ is a value of the coalition K in a zero-sum game with the anti-coalition $N \setminus K$. It is proved that this characteristic function is superadditive.

Petrosyan and Zaccour [32] proposed the following form of characteristic function:

$$v^{PZ}(K) = \sup_{x_i, i \in K} \sum_{i \in K} u_i(x_K, x_{N \setminus K}^{NE})$$

Here x_K is a set of strategies of the players from a coalition K , $x_{N \setminus K}^{NE}$ is a set of strategies of the players from the anti-coalition that belong to a Nash equilibrium in a normal form game between the agents from the set $N = \{1, \dots, n\}$.

However, this function may be not superadditive. So, Gromova and Petrosyan [33] proposed the following characteristic function

$$v^{PG}(K) = \inf_{x_j, j \in N \setminus K} \sum_{i \in K} u_i(x_K^C, x_{N \setminus K}),$$

where x_K^C is a set of strategies of the players from a coalition K that belong to the cooperative solution of a normal form game between the agents from the set $N = \{1, \dots, n\}$, and proved that this characteristic function is superadditive. Notice that for all three characteristic functions

$$v^{NM}(N) = v^{PZ}(N) = v^{PG}(N) = \sup_{x_1, \dots, x_n} \sum_{i \in N} u_i(x_1, \dots, x_n)$$

There are different optimality principles that define the solutions for cooperative games [30]. In this paper we consider only Shapley value because it always exists and is unique. The Shapley value can be calculated by the formula received by Shapley:

$$\Phi_i(v) = \sum_{i \in K} \gamma_n(k) [v(K) - v(K \setminus \{i\})], i \in N,$$

where $\Phi_i(v)$ is the i -th component of the Shapley value, $\gamma_n(k) = \frac{(n-k)!(k-1)!}{n!}$, $k = |K|$.

All payoff functions in this section are continuous on a closed n -dimensional parallelepiped. Therefore, they attain their maximum in a point of this parallelepiped. It is clear that all these functions are strictly concave. It is also clear that a point of maximum can not belong to a face of that parallelepiped where $x_i = r_i$ for some i . However, the maximum point is not necessarily an inner one. The maximum may be attained on a face where $x_i = 0$ for some i . For example, it holds when $\alpha_i = 1$ for any i and in the same time $c_i + b_i \neq c_j + b_j$ for some i and j . Then the solution coincides with the solution received in [34,35] with substitution of c_i by $c_i + b_i$. In this section we will assume that $0 < \alpha_i < 1$ for any i . The fact that the system of equations received from the first order conditions has no solutions or the solution contains negative values for some x_i will indicate situations when the maximum is attained on a face of the parallelepiped.

The first order conditions for (3) have the form:

$$a - c_i - 2\bar{x} - \alpha_i b_i (r_i - x_i)^{\alpha_i - 1} = 0, \quad i = 1, 2, \dots, n. \quad (22)$$

Denote for convenience

$$y_i = r_i - x_i.$$

Then (22) takes the form

$$a - c_i - 2R + 2\bar{y} - \alpha_i b_i y_i^{\alpha_i - 1} = 0, \quad i = 1, 2, \dots, n, \quad (23)$$

where $R = r_1 + r_2 + \dots + r_n$, and from (23) we receive

$$c_i + \alpha_i b_i y_i^{\alpha_i - 1} = c_j + \alpha_j b_j y_j^{\alpha_j - 1}, \quad i = 1, 2, \dots, n, j = 1, 2, \dots, n$$

and

$$y_j = \left(\frac{c_i - c_j}{\alpha_j b_j} + \frac{\alpha_i b_i}{\alpha_j b_j} y_i^{\alpha_i - 1} \right)^{\frac{1}{\alpha_j - 1}}. \quad (24)$$

Substituting (24) in (22) we receive an equation for finding y_i :

$$2 \sum_{j=1}^n \left(\frac{c_i - c_j}{\alpha_j b_j} + \frac{\alpha_i b_i}{\alpha_j b_j} y_i^{\alpha_i - 1} \right)^{\frac{1}{\alpha_j - 1}} - \alpha_i b_i y_i^{\alpha_i - 1} - c_i = 2R - a. \quad (25)$$

Example 3. Set $a = 25$, $n = 3$, $c_1 = 6$, $c_2 = 7$, $c_3 = 8$, $r_1 = 8$, $r_2 = 7$, $r_3 = 6$, $R = 21$, $\alpha_1 = 0.7$, $\alpha_2 = 0.6$, $\alpha_3 = 0.5$, $b_1 = 15$, $b_2 = 14$, $b_3 = 13$.

Solving the Equation (25) for $i = 1$, we find the value $y_1 = 7.327059$ and, respectively, $x_1 = 0.6729411$. Given y_1 we find from (24) the values of y_2 and y_3 , and then $x_2 = 2.899981$, $x_3 = 3.038524$. Substitution of the found values to (3) gives the payoff of grand coalition, or cooperative payoff: $v^{GC} = 188.4149$. The payoffs of separate players in the case of cooperation and uniform distribution of the total payoff are the following:

$$u_1^{GC} = 62.80495, u_2^{GC} = 62.80495, u_3^{GC} = 62.80495.$$

3.1. VonNeumann–Morgenstern Characteristic Function

Let us calculate a value of Neumann–Morgenstern characteristic function for a coalition of two players $\{i, j\}$. The third player's strategy is then $x_k = r_k$. Remind that we consider three-players games. The problem of the considered coalition is

$$\begin{aligned} g(x_i, x_j) = g_i(x) + g_j(x) = (a - \bar{x})(x_i + x_j) - (c_i x_i + c_j x_j) \\ + \frac{2}{3} (b_i (r_i - x_i)^{\alpha_i} + b_j (r_j - x_j)^{\alpha_j}) \rightarrow \max, \quad \bar{x} = x_i + x_j + r_k \end{aligned} \quad (26)$$

with constraints $0 \leq x_i \leq r_i \leq \frac{a}{n}, 0 \leq x_j \leq r_j \leq \frac{a}{n}, c_i < \frac{a}{n}, c_j < \frac{a}{n}$.

The first order conditions for the problem (26) are:

$$a - c_i - 2(x_i + x_j) - r_k - \frac{2\alpha_i b_i}{3}(r_i - x_i)^{\alpha_i - 1} = 0, \quad (27)$$

$$a - c_j - 2(x_i + x_j) - r_k - \frac{2\alpha_j b_j}{3}(r_j - x_j)^{\alpha_j - 1} = 0. \quad (28)$$

Denote for convenience $y_i = r_i - x_i$.

Condition (27) takes the form

$$a - c_i - 2r_i - 2r_j + 2(y_i + y_j) - r_k - \frac{2\alpha_i b_i}{3}y_i^{\alpha_i - 1} = 0, \quad (29)$$

and (28) takes a similar form. Their comparison gives

$$c_i + \frac{2\alpha_i b_i}{3}y_i^{\alpha_i - 1} = c_j + \frac{2\alpha_j b_j}{3}y_j^{\alpha_j - 1}$$

and we receive an expression

$$y_j = \left(\frac{3(c_i - c_j)}{2\alpha_j b_j} + \frac{\alpha_i b_i}{\alpha_j b_j} y_i^{\alpha_i - 1} \right)^{\frac{1}{\alpha_j - 1}}. \quad (30)$$

Its substitution in (29) gives

$$2 \left(y_i + \left(\frac{3(c_i - c_j)}{2\alpha_j b_j} + \frac{\alpha_i b_i}{\alpha_j b_j} y_i^{\alpha_i - 1} \right)^{\frac{1}{\alpha_j - 1}} \right) - \frac{2\alpha_i b_i}{3} y_i^{\alpha_i - 1} - c_i = 2R - r_k - a. \quad (31)$$

Solving the Equation (31) numerically, we find the value of y_i and, respectively, x_i . From (30) we determine y_j , and, respectively, x_j . For Example 3, choosing different pairs of integer numbers from 1 to 3 as i and j , we find optimal strategies of the players x_i and x_j and the value of Neumann–Morgenstern function for possible coalitions of two players:

$$x^{NM12}(1) = 1.934159, x^{NM12}(2) = 2.527858, x^{NM12}(3) = 6,$$

$$x^{NM13}(1) = 2.132335, x^{NM13}(2) = 7, x^{NM13}(3) = 1.80927,$$

$$x^{NM23}(1) = 8, x^{NM23}(2) = 2.219906, x^{NM23}(3) = 1.282537,$$

$$v^{NM}(1, 2) = 93.81595, v^{NM}(1, 3) = 80.39471, v^{NM}(2, 3) = 64.16009.$$

Let us calculate a value of Neumann–Morgenstern characteristic function for a singleton $\{i\}$. The strategies of two other players in this case are: $x_j = r_j, x_k = r_k$. The problem of a singleton coalition is

$$g(x_i) = g_i(x) = (a - \bar{x})x_i - c_i x_i + \frac{1}{3}b_i(r_i - x_i)^{\alpha_i} \rightarrow \max, \bar{x} = x_i + r_j + r_k \quad (32)$$

with constraints $0 \leq x_i \leq r_i \leq \frac{a}{n}, c_i < \frac{a}{n}$.

Using the first order conditions for the problem (32), we receive an equation

$$2x_i + \frac{\alpha_i b_i}{3}(r_i - x_i)^{\alpha_i - 1} = a - c_i - (r_j + r_k). \quad (33)$$

Solving the Equation (33) numerically, we find the optimal value of x_i . For the Example 3, choosing different integer numbers from 1 to 3 as i , we find optimal strategies of the player x_i and the value of Neumann–Morgenstern function for possible singleton coalitions:

$$x^{NM1}(1) = 1.97875, x^{NM1}(2) = 7, x^{NM1}(3) = 6,$$

$$x^{NM2}(1) = 8, x^{NM2}(2) = 1.302025, x^{NM2}(3) = 6,$$

$$x^{NM3}(1) = 8, x^{NM3}(2) = 7, x^{NM3}(3) = 0.5365237,$$

$$v^{NM}(1) = 25.5262, v^{NM}(2) = 16.76964, v^{NM}(3) = 10.91396$$

Let us calculate the components of Shapley value for this characteristic function:

$$\begin{aligned}\Phi_1(v^{NM}) &= \frac{1}{3}v^{NM}(1) + \frac{1}{6}(v^{NM}(1,2) - v^{NM}(2)) + \\ &+ \frac{1}{6}(v^{NM}(1,3) - v^{NM}(3)) + \frac{1}{3}(v^{NM}(1,2,3) - v^{NM}(2,3)), \\ \Phi_2(v^{NM}) &= \frac{1}{3}v^{NM}(2) + \frac{1}{6}(v^{NM}(1,2) - v^{NM}(1)) + \\ &+ \frac{1}{6}(v^{NM}(2,3) - v^{NM}(3)) + \frac{1}{3}(v^{NM}(1,2,3) - v^{NM}(1,3)), \\ \Phi_3(v^{NM}) &= \frac{1}{3}v^{NM}(3) + \frac{1}{6}(v^{NM}(1,3) - v^{NM}(1)) + \\ &+ \frac{1}{6}(v^{NM}(2,3) - v^{NM}(2)) + \frac{1}{3}(v^{NM}(1,2,3) - v^{NM}(1,2))\end{aligned}$$

Thus, we receive

$$\Phi_1(v^{NM}) = 74.34816, \Phi_2(v^{NM}) = 61.85258, \Phi_3(v^{NM}) = 52.21411.$$

Consider an independent behavior of the players when each of them solves the problems (1)–(2). The first order conditions give the system of equations

$$\begin{cases} \frac{\partial g_1}{\partial x_1} = a - c_1 - 2x_1 - x_2 - x_3 - \frac{\alpha_1 b_1}{3}(r_1 - x_1)^{\alpha_1 - 1} = 0, \\ \frac{\partial g_2}{\partial x_2} = a - c_2 - x_1 - 2x_2 - x_3 - \frac{\alpha_2 b_2}{3}(r_2 - x_2)^{\alpha_2 - 1} = 0, \\ \frac{\partial g_3}{\partial x_3} = a - c_3 - x_1 - x_2 - 2x_3 - \frac{\alpha_3 b_3}{3}(r_3 - x_3)^{\alpha_3 - 1} = 0. \end{cases} \quad (34)$$

Solving the system of Equation (34) numerically, we find equilibrium strategies and payoffs of the players

$$x_1^{NE} = 4.497760, x_2^{NE} = 4.077521, x_3^{NE} = 3.523900,$$

$$u_1^{NE} = 58.76089, u_2^{NE} = 51.78338, u_3^{NE} = 44.99266.$$

3.2. Petrosyan–Zaccour Characteristic Function

Let us calculate a value of Petrosyan–Zaccour characteristic function for a coalition of two players $\{i, j\}$. The third player's strategy is then: $x_k = x_k^{NE}$.

The problem of the considered coalition is

$$\begin{aligned} g(x_i, x_j) &= g_i(x) + g_j(x) \\ &= (a - c_i - x_i - x_j - x_k^{NE})x_i + (a - c_i - x_i - x_j - x_k^{NE})x_j \\ &\quad + \frac{2}{3}b_i(r_i - x_i)^{\alpha_i} + \frac{2}{3}b_j(r_j - x_j)^{\alpha_j} + \frac{2}{3}b_k(r_k - x_k^{NE})^{\alpha_k} \longrightarrow \max \end{aligned} \quad (35)$$

with constraints $0 \leq x_i \leq r_i \leq \frac{a}{n}, 0 \leq x_j \leq r_j \leq \frac{a}{n}, c_i < \frac{a}{n}, c_j < \frac{a}{n}$.

The first order conditions for the problem (35) are

$$\frac{\partial g}{\partial x_i} = a - c_i - 2(x_i + x_j) - x_k^{NE} - \frac{2\alpha_i b_i}{3}(r_i - x_i)^{\alpha_i - 1} = 0, \quad (36)$$

$$\frac{\partial g}{\partial x_j} = a - c_j - 2(x_i + x_j) - x_k^{NE} - \frac{2\alpha_j b_j}{3}(r_j - x_j)^{\alpha_j - 1} = 0. \quad (37)$$

Denote again for convenience

$$y_i = r_i - x_i$$

Condition (36) takes the form

$$a - c_i - 2(r_i + r_j) + 2(y_i + y_j) - x_k^{NE} - \frac{2\alpha_i b_i}{3}(y_i)^{\alpha_i - 1} = 0 \quad (38)$$

and (37) has a similar form. Comparison of the conditions gives

$$y_j = \left(\frac{3(c_i - c_j)}{2\alpha_j b_j} + \frac{\alpha_i b_i}{\alpha_j b_j} y_i^{\alpha_i - 1} \right)^{\frac{1}{\alpha_j - 1}} \quad (39)$$

Substituting (39) in (38), we receive an equation for y_i :

$$\begin{aligned} 2 \left(y_i + \left(\frac{3(c_i - c_j)}{2\alpha_j b_j} + \frac{\alpha_i b_i}{\alpha_j b_j} y_i^{\alpha_i - 1} \right)^{\frac{1}{\alpha_j - 1}} \right) - \frac{2\alpha_i b_i}{3} y_i^{\alpha_i - 1} - c_i \\ = 2(r_i + r_j) - a + x_k^{NE}. \end{aligned} \quad (40)$$

Solving the Equation (40) numerically, we find the value of y_i and, respectively, x_i . From (39) we determine y_j , and therefore x_j . For the Example 3, choosing different pairs of integer numbers from 1 to 3 as i and j , we find optimal strategies of the players x_i and x_j and the value of Petrosyan–Zaccour function for possible coalitions of two players:

$$x^{PZ12}(1) = 2.607805, x^{PZ12}(2) = 3.019001, x^{PZ12}(3) = 3.5239,$$

$$x^{PZ13}(1) = 2.891508, x^{PZ13}(2) = 4.077521, x^{PZ13}(3) = 2.423976,$$

$$x^{PZ23}(1) = 4.49776, x^{PZ23}(2) = 2.98485, x^{PZ23}(3) = 2.160523,$$

$$v^{PZ}(1, 2) = 119.9462, v^{PZ}(1, 3) = 111.6875, v^{PZ}(2, 3) = 103.3577.$$

Let us calculate a value of Petrosyan–Zaccour characteristic function for a singleton $\{i\}$. The strategies of two other players in this case are: $x_j = x_j^{NE}, x_k = x_k^{NE}$. The problem of a singleton coalition is

$$\begin{aligned} g(x_i) &= (a - x_i - x_j^{NE} - x_k^{NE})x_i - c_i x_i + \frac{1}{3}b_i(r_i - x_i)^{\alpha_i} + \frac{1}{3}b_j(r_j - x_k^{NE})^{\alpha_j} \\ &\quad + \frac{1}{3}b_k(r_k - x_k^{NE})^{\alpha_k} \longrightarrow \max \end{aligned} \quad (41)$$

with constraints $0 \leq x_i \leq r_i \leq \frac{a}{n}, c_i < \frac{a}{n}$.

The first order conditions for the problem (41) give the equation

$$2x_i + \frac{\alpha_i b_i}{3} (r_i - x_i)^{\alpha_i - 1} = a - c_i - (x_j^{NE} + x_k^{NE}). \quad (42)$$

Solving the Equation (42) numerically, we find the optimal value of x_i . For the Example 3, choosing different integer numbers from 1 to 3 as i , we find optimal strategies of the player x_i and the value of Petrosyan–Zaccour function for possible singleton coalitions:

$$x^{PZ1}(1) = 4.49776, x^{PZ1}(2) = 4.077521, x^{PZ1}(3) = 3.5239,$$

$$x^{PZ2}(1) = 4.49776, x^{PZ2}(2) = 4.077521, x^{PZ2}(3) = 3.5239,$$

$$x^{PZ3}(1) = 4.49776, x^{PZ3}(2) = 4.077521, x^{PZ3}(3) = 3.5239,$$

$$v^{PZ}(1) = 58.76089, v^{PZ}(2) = 51.78338, v^{PZ}(3) = 44.99266.$$

Let us calculate the components of Shapley value in this game:

$$\begin{aligned} \Phi_1(v^{PZ}) &= \frac{1}{3}v^{PZ}(1) + \frac{1}{6}(v^{PZ}(1,2) - v^{PZ}(2)) + \\ &+ \frac{1}{6}(v^{PZ}(1,3) - v^{PZ}(3)) + \frac{1}{3}(v^{PZ}(1,2,3) - v^{PZ}(2,3)), \\ \Phi_2(v^{PZ}) &= \frac{1}{3}v^{PZ}(2) + \frac{1}{6}(v^{PZ}(1,2) - v^{PZ}(1)) + \\ &+ \frac{1}{6}(v^{PZ}(2,3) - v^{PZ}(3)) + \frac{1}{3}(v^{PZ}(1,2,3) - v^{PZ}(1,3)), \\ \Phi_3(v^{PZ}) &= \frac{1}{3}v^{PZ}(3) + \frac{1}{6}(v^{PZ}(1,3) - v^{PZ}(1)) + \\ &+ \frac{1}{6}(v^{PZ}(2,3) - v^{PZ}(2)) + \frac{1}{3}(v^{PZ}(1,2,3) - v^{PZ}(1,2)). \end{aligned}$$

Then

$$\Phi_1(v^{PZ}) = 70.41563, \Phi_2(v^{PZ}) = 62.76197, \Phi_3(v^{PZ}) = 55.23725.$$

3.3. Gromova–Petrosyan Characteristic Function

Let us calculate the value of Gromova–Petrosyan characteristic function for a coalition of two players $\{i, j\}$. Strategies of both players are the same as in the grand coalition, and the strategy of the third player is: $x_k = r_k$. The coalitional payoff is

$$\begin{aligned} g(x_i, x_j) &= (a - c_i - x_i^{GC} - x_j^{GC} - r_k)x_i^{GC} + (a - c_j - x_i^{GC} - x_j^{GC} - r_k)x_j^{GC} \\ &+ \frac{2}{3}b_i(r_i - x_i^{GC})^{\alpha_i} + \frac{2}{3}b_j(r_j - x_j^{GC})^{\alpha_j} \rightarrow \max. \end{aligned}$$

For the Example 3, choosing different pairs of integer numbers from 1 to 3 as i and j , we find the values of Gromova–Petrosyan function for possible coalitions of two players:

$$v^{PG}(1,2) = 92.85824, v^{PG}(1,3) = 79.9135, v^{PG}(2,3) = 57.75757.$$

Let us calculate the value of Gromova–Petrosyan characteristic function for a singleton coalition $\{i\}$. Its strategy is the same as it was in the grand coalition. The strategies of two other players are: $x_j = r_j, x_k = r_k$. The payoff of a singleton coalition is

$$g(x_i) = \left(a - c_i - x_i^{GC} - r_j - r_k\right) x_i^{GC} + \frac{1}{3} b_i \left(r_i - x_i^{GC}\right)^{\alpha_i} \rightarrow \max.$$

For the Example 3, choosing different integer numbers from 1 to 3 as i , we find the values of Gromova–Petrosyan function for possible singleton coalitions:

$$v^{PG}(1) = 23.74156, v^{PG}(2) = 14.07127, v^{PG}(3) = 4.301626.$$

Calculate the components of Shapley value in this game:

$$\begin{aligned} \Phi_1(v^{PG}) &= \frac{1}{3} v^{PG}(1) + \frac{1}{6} (v^{PG}(1, 2) - v^{PG}(2)) \\ &+ \frac{1}{6} (v^{PG}(1, 3) - v^{PG}(3)) + \frac{1}{3} (v^{PG}(1, 2, 3) - v^{PG}(2, 3)), \\ \Phi_2(v^{PG}) &= \frac{1}{3} v^{PG}(2) + \frac{1}{6} (v^{PG}(1, 2) - v^{PG}(1)) \\ &+ \frac{1}{6} (v^{PG}(2, 3) - v^{PG}(3)) + \frac{1}{3} (v^{PG}(1, 2, 3) - v^{PG}(1, 3)), \\ \Phi_3(v^{PG}) &= \frac{1}{3} v^{PG}(3) + \frac{1}{6} (v^{PG}(1, 3) - v^{PG}(1)) \\ &+ \frac{1}{6} (v^{PG}(2, 3) - v^{PG}(2)) + \frac{1}{3} (v^{PG}(1, 2, 3) - v^{PG}(1, 2)). \end{aligned}$$

We receive

$$\Phi_1(v^{PG}) = 77.19942, \Phi_2(v^{PG}) = 61.28631, \Phi_3(v^{PG}) = 49.92912.$$

Example 4. Set $a = 45, n = 3, c_1 = 10, c_2 = 11, c_3 = 12, r_1 = 14, r_2 = 12, r_3 = 12, R = 38, \alpha_1 = 0.8, \alpha_2 = 0.7, \alpha_3 = 0.7, b_1 = 15, b_2 = 14, b_3 = 13$.

The same actions as before give:

$$x_1^{GC} = 1.04, x_2^{GC} = 7.37, x_3^{GC} = 5.49, u^{GC} = 480.57,$$

$$u_1^{GC} = 160.19, u_2^{GC} = 160.19, u_3^{GC} = 160.19,$$

$$x^{NM12}(1) = 2.56, x^{NM12}(2) = 6.48, x^{NM12}(3) = 12,$$

$$x^{NM13}(1) = 5.36, x^{NM13}(2) = 12, x^{NM13}(3) = 3.54,$$

$$x^{NM23}(1) = 14, x^{NM23}(2) = 6.76, x^{NM23}(3) = 1.25,$$

$$v^{NM}(1, 2) = 220.84, v^{NM}(1, 3) = 213.18, v^{NM}(2, 3) = 170.24,$$

$$x^{NM1}(1) = 4.24, x^{NM1}(2) = 12, x^{NM1}(3) = 12,$$

$$x^{NM2}(1) = 14, x^{NM2}(2) = 3.15, x^{NM2}(3) = 12,$$

$$x^{NM3}(1) = 8, x^{NM3}(2) = 7, x^{NM3}(3) = 32.25,$$

$$v^{NM}(1) = 14, v^{NM}(2) = 12, v^{NM}(3) = 2.72.$$

$$\Phi_1(v^{NM}) = 184.15, \Phi_2(v^{NM}) = 151.25, \Phi_3(v^{NM}) = 145.17,$$

$$x_1^{NE} = 8.39, x_2^{NE} = 8.06, x_3^{NE} = 7.32,$$

$$u_1^{NE} = 139.04, u_2^{NE} = 127.25, u_3^{NE} = 112.33,$$

$$x^{PZ12}(1) = 4.17, x^{PZ12}(2) = 7.14, x^{PZ12}(3) = 7.32,$$

$$x^{PZ13}(1) = 6.29, x^{PZ13}(2) = 8.06, x^{PZ13}(3) = 4.52,$$

$$x^{PZ23}(1) = 8.39, x^{PZ23}(2) = 7.50, x^{PZ23}(3) = 3.22,$$

$$v^{PZ}(1, 2) = 294.03, v^{PZ}(1, 3) = 276.36, v^{PZ}(2, 3) = 262.48,$$

$$x^{PZ1}(1) = 8.39, x^{PZ1}(2) = 8.06, x^{PZ1}(3) = 7.32,$$

$$x^{PZ2}(1) = 8.39, x^{PZ2}(2) = 8.06, x^{PZ2}(3) = 7.32,$$

$$x^{PZ3}(1) = 8.39, x^{PZ3}(2) = 8.06, x^{PZ3}(3) = 7.32,$$

$$v^{PZ}(1) = 139.04, v^{PZ}(2) = 127.25, v^{PZ}(3) = 112.33,$$

$$\Phi_1(v^{PZ}) = 174.18, \Phi_2(v^{PZ}) = 161.34, \Phi_3(v^{PZ}) = 145.05,$$

$$v^{PG}(1, 2) = 220.26, v^{PG}(1, 3) = 206.39, v^{PG}(2, 3) = 145.71,$$

$$v^{PG}(1) = 49.17, v^{PG}(2) = 18.27, v^{PG}(3) = 24.34,$$

$$\Phi_1(v^{PG}) = 192.02, \Phi_2(v^{PG}) = 146.23, \Phi_3(v^{PG}) = 142.33.$$

4. Hierarchical Model (Stackelberg Oligopoly)

In this model each i -th firm solves the problems (1)–(2). The first firm is a Stackelberg leader, and all other firms are followers. In other words, all firms except the first one take its strategy as given and maximize their payoffs as functions of the first firm's strategy. The first firm anticipates it and substitute the received functions in its payoff function, and then maximizes its payoff by the strategy as a function of best responses of other firms. At last, we can calculate the final values of the best responses and obtain a Stackelberg equilibrium. From the first order conditions for the second and the third firms we receive:

$$\begin{cases} \frac{\partial g_2}{\partial x_2} = a - c_2 - x_1 - 2x_2 - x_3 - \frac{\alpha_2 b_2}{3}(r_2 - x_2)^{\alpha_2 - 1} = 0, \\ \frac{\partial g_3}{\partial x_3} = a - c_3 - x_1 - x_2 - 2x_3 - \frac{\alpha_3 b_3}{3}(r_3 - x_3)^{\alpha_3 - 1} = 0. \end{cases}$$

We cannot express analytically x_2 and x_3 by x_1 but we can express their derivatives on x_1 as derivatives of an implicit function. We have

$$A = \begin{pmatrix} \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \\ \frac{\partial g_3}{\partial x_2} & \frac{\partial g_3}{\partial x_3} \end{pmatrix} = \begin{pmatrix} -2 + \frac{\alpha_3(\alpha_3-1)b_3(r_3-x_3)^{\alpha_3-2}}{3} & -1 \\ -1 & -2 + \frac{\alpha_2(\alpha_2-1)b_2(r_2-x_2)^{\alpha_2-2}}{3} \end{pmatrix}$$

where a determinannt of the matrix A is equal to

$$\Delta = 3 - \frac{2}{3} \left[\alpha_2(\alpha_2-1)b_2(r_2-x_2)^{\alpha_2-2} + \alpha_3(\alpha_3-1)b_3(r_3-x_3)^{\alpha_3-2} \right] + \frac{\alpha_2(\alpha_2-1)\alpha_3(\alpha_3-1)b_2b_3(r_2-x_2)^{\alpha_2-2}(r_3-x_3)^{\alpha_3-2}}{9}. \quad (43)$$

Therefore,

$$A^{-1} = \frac{1}{\Delta} \begin{pmatrix} -2 + \frac{\alpha_3(\alpha_3-1)b_3(r_3-x_3)^{\alpha_3-2}}{3} & 1 \\ 1 & -2 + \frac{\alpha_2(\alpha_2-1)b_2(r_2-x_2)^{\alpha_2-2}}{3} \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{dx_2}{dx_1} \\ \frac{dx_3}{dx_1} \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} \frac{\alpha_3(\alpha_3-1)b_3(r_3-x_3)^{\alpha_3-2}}{3} - 1 \\ \frac{\alpha_2(\alpha_2-1)b_2(r_2-x_2)^{\alpha_2-2}}{3} - 1 \end{pmatrix}. \quad (44)$$

Then

$$\begin{aligned} \frac{dg_1}{dx_1} = & a - c_1 - 2x_1 - x_2(x_1) - x_3(x_1) - \frac{dx_2(x_1)}{dx_1}x_1 - \frac{dx_3(x_1)}{dx_1}x_1 \\ & - \frac{\alpha_1b_1}{3}(r_1-x_1)^{\alpha_1-1} - \frac{\alpha_2b_2}{3}(r_2-x_2)^{\alpha_2-1}\frac{dx_2}{dx_1} \\ & - \frac{\alpha_3b_3}{3}(r_3-x_3)^{\alpha_3-1}\frac{dx_3}{dx_1} \end{aligned}$$

where $\frac{dx_2(x_1)}{dx_1}$ and $\frac{dx_3(x_1)}{dx_1}$ are expressed by (44). We receive a system of three equations:

$$\begin{cases} a - c_1 - 2x_1 - x_2(x_1) - x_3(x_1) - \frac{dx_2(x_1)}{dx_1}x_1 - \frac{dx_3(x_1)}{dx_1}x_1 - \frac{\alpha_1b_1}{3}(r_1-x_1)^{\alpha_1-1} - \\ \quad - \frac{\alpha_2b_2}{3}(r_2-x_2)^{\alpha_2-1}\frac{dx_2}{dx_1} - \frac{\alpha_3b_3}{3}(r_3-x_3)^{\alpha_3-1}\frac{dx_3}{dx_1} = 0, \\ a - c_2 - x_1 - 2x_2 - x_3 - \frac{\alpha_2b_2}{3}(r_2-x_2)^{\alpha_2-1} = 0, \\ a - c_3 - x_1 - x_2 - 2x_3 - \frac{\alpha_3b_3}{3}(r_3-x_3)^{\alpha_3-1} = 0, \end{cases} \quad (45)$$

Where $\frac{dx_2(x_1)}{dx_1}$ and $\frac{dx_3(x_1)}{dx_1}$ are determined by (44), and Δ is determined by (43).

Alternatively, we can take different values of x_1 as given, find the respective values of x_2 and x_3 that satisfy the second and the third equations of the system (45), and by means of Newton method, chord method, or a hybrid method calculate the value of x_1 that satisfy the first equation in (45). In any case, for the input data from Example 3 we find:

$$x_1^{st} = 7.36, x_2^{st} = 3.18, x_3^{st} = 2.64, u_1^{st} = 64.87, u_2^{st} = 37.40, u_3^{st} = 32.14.$$

Similarly, for the data from Example 4 we receive:

$$x_1^{st} = 14.00, x_2^{st} = 6.29, x_3^{st} = 5.49, u_1^{st} = 160.99, u_2^{st} = 83.57, u_3^{st} = 71.55.$$

5. Comparative Analysis

Let us compare two situations: when there is a combination of the competitive and cooperative behavior, and when there is only a competition. In other words, there are two questions.

1. What does change cooperation in the production of a public good in comparison with a standard competitive behavior of oligopolists?
2. What is a difference between a cooperative non-market activity in comparison with a standard homogeneous Stackelberg oligopoly in the conditions of quantitative competition and strategic substitutability?

In our previous paper [34] in the latter situation we have introduced an indicator of the average sensibility of demand in relation to the marginal costs

$\lambda = \frac{a}{c}$, where a is a demand parameter, c —marginal costs. In our consideration $c < \frac{a}{n^2}$, and because in the examples $n = 3$ then $\lambda < 9$. We considered three firms: the first one is the most efficient (a Stackelberg leader in the case of hierarchy), the second one is less efficient, and the third one is the least efficient. In the case of hierarchy the second and the third firms were the followers. We received the following preference system.

1. For the most efficient firm (a Stackelberg leader):

$$u_1^{st} > \Phi_1(v^{NM}) > \Phi_1(v^{PG}) > u_1^{GC}$$

If $9 < \lambda < \frac{34+\sqrt{1120}}{6} \cong 11,24$ then $u_1^{st} > \Phi_1(v^{NM}) > \Phi_1(v^{PG}) = u_1^{NE} = u_1^{GC}$.

If $\lambda = \frac{34+\sqrt{1120}}{6}$ then $u_1^{st} > \Phi_1(v^{NM}) > \Phi_1(v^{PG}) = u_1^{NE} = u_1^{GC}$.

If $\lambda > \frac{34+\sqrt{1120}}{6}$ then $u_1^{st} > \Phi_1(v^{NM}) > \Phi_1(v^{PG}) > u_1^{GC} > u_1^{NE}$.

2. For a less efficient firm (one of the followers):

If $9 < \lambda < \frac{34+\sqrt{1120}}{6} \cong 11,24$ then $u_2^{GC} > \Phi_2(v^{PG}) > \Phi_2(v^{NM}) > u_2^{NE} > u_2^{st}$.

If $\lambda = \frac{34+\sqrt{1120}}{6}$ then $u_2^{GC} > \Phi_2(v^{PG}) = \Phi_2(v^{NM}) > u_2^{NE} > u_2^{st}$.

If $\lambda > \frac{34+\sqrt{1120}}{6}$ then $u_2^{GC} > \Phi_2(v^{NM}) > \Phi_2(v^{PG}) > u_2^{NE} > u_2^{st}$.

3. For the least efficient firm (another follower):

$$\Phi_3(v^{PG}) > u_3^{GC} > \Phi_3(v^{NM}) > u_3^{NE} > u_3^{st}$$

4. For the whole society:

$$v^{NM}(N) = v^{PG}(N) = \bar{u}^{GC} > \sum_{i=1}^3 u_i^{NE} > \sum_{i=1}^3 u_i^{st} \quad (46)$$

In this paper in the two examples of combination of the competitive and cooperative behavior we received the following results (in each column the payoffs are given in descending order).

Example 3:

$$\Phi_1(v^{PG}) = 77.20 \quad u_2^{GC} = 62.80 \quad u_3^{GC} = 62.80$$

$$\Phi_1(v^{NM}) = 74.35 \quad \Phi_2(v^{PG}) = 62.76 \quad \Phi_3(v^{PG}) = 55.24$$

$$\Phi_1(v^{PG}) = 70.42 \quad \Phi_2(v^{NM}) = 61.85 \quad \Phi_3(v^{NM}) = 52.21$$

$$u_1^{st} = 64.87 \quad \Phi_2(v^{PG}) = 61.29 \quad \Phi_3(v^{PG}) = 49.93$$

$$u_1^{GC} = 62.80 \quad u_2^{NE} = 51.78 \quad u_3^{NE} = 44.99$$

$$u_1^{NE} = 58.76 \quad u_2^{st} = 37.40 \quad u_3^{st} = 32.14$$

Example 4:

$$\begin{aligned}
\Phi_1(v^{PG}) &= 192.02 \quad \Phi_2(v^{PZ}) = 161.34 \quad u_3^{GC} = 160.19 \\
\Phi_1(v^{NM}) &= 184.15 \quad u_2^{GC} = 160.19 \quad \Phi_3(v^{NM}) = 145.17 \\
\Phi_1(v^{PZ}) &= 174.18 \quad \Phi_2(v^{NM}) = 151.25 \quad \Phi_3(v^{PZ}) = 145.05 \\
u_1^{st} &= 160.99 \quad \Phi_2(v^{PG}) = 146.23 \quad \Phi_3(v^{PG}) = 142.33 \\
u_1^{GC} &= 160.19 \quad u_2^{NE} = 127.25 \quad u_3^{NE} = 112.33 \\
u_1^{NE} &= 139.04 \quad u_2^{st} = 83.57 \quad u_3^{st} = 71.55
\end{aligned}$$

Now let us return to the first question. A difference between the cases of cooperation and pure competition is quite small.

In the case of pure competition the components of Shapley value (for different characteristic functions) for the most efficient firm are greater than its cooperative payoff. In other words, in the case of cooperation the most efficient firm has a payoff which is objectively smaller than it deserves by its role in the cooperation. A less efficient firm has a greater payoff than its reward according to Shapley value (for different characteristic functions). At last, the least efficient firm has a payoff approximately equal to its components of Shapley value. The payoffs of non-efficient firms in the case competition are smaller than their cooperative payoffs. As for the most efficient firm, its payoff for the big values of indicator λ is also smaller than its cooperative payoff. If λ is small then the competitive payoff of the most efficient firm exceeds its cooperative payoff.

In the case of cooperation the cooperative payoff of the most efficient firm is also smaller than its components of Shapley value for any characteristic function. Thus; in the case of cooperation the cooperative contribution of the most efficient firm is also underestimated. The cooperative payoffs of less efficient firms are approximately equal (for the mean efficient firms) or strictly less (for the least efficient firms) than the respective components of Shapley value. In the case of cooperation a competitive payoff of a firm is always smaller than its cooperative payoff independently of its efficiency.

Now let us discuss the second question (Stackelberg oligopoly). Evidently, for the whole society the best way of organization is cooperation because inequality (46) holds both for competition and for cooperation. A pure competition for the whole society is not the worst but also not at all the best way of economic organization. The worst variant for the society is a hierarchy.

However, for the most efficient firm (a Stackelberg leader) its hierarchical payoff is the greatest one in comparison with any other way of organization independently of the parameter λ . For less efficient firms which are Stackelberg followers their payoffs are less than their cooperative payoffs, much less than the respective components of Shapley value, and even less than their payoffs in the case of pure competition.

In the case of cooperation the payoff of the most efficient firm (a Stackelberg leader) is only slightly (in one example on 3.3%, in another on 0.5%) greater than its cooperative payoff. However, it is always less than the respective components of Shapley value (for any characteristic function). For less efficient firms which are Stackelberg followers their payoffs, as in the case of pure competition, are less than their cooperative payoffs, less than the respective components of Shapley value, and even less than their payoffs in the case of pure competition.

Thus, in the case of transition from a pure competition to the coopetition (combination of competition and cooperation) not only roles of Stackelberg followers but also the role of Stackelberg leader become non-attractive for the firms.

6. Conclusions

Let us briefly summarize again the results of comparative analysis.

A share of the most efficient firm in the case of cooperation and uniform distribution of the cooperative payoff is greater than its selfish payoff. However, it is a bit smaller than its payoff in a hierarchical game when this firm is a Stackelberg leader. For all three characteristic functions, a component of Shapley value responding to the most efficient firm is always greater than its selfish payoff, its payoff when it is a Stackelberg leader or its payoff in the case of cooperation and uniform distribution of the cooperative payoff between all players.

As for a less efficient firm, its cooperative payoff is greater than its selfish payoff, and almost two times greater than its payoff in the role of a follower on the Stackelberg game. A component of Shapley value for such firm is almost equal (abitless as a rule, and a bit greater in some cases) than its cooperative payoff but much greater than its selfish payoff, and almost two times greater than its payoff in the role of a follower on the Stackelberg game.

As for the least efficient firm, its cooperative payoff is much greater than its selfish payoff that in turn is much greater than its payoff in the role of a follower on the Stackelberg game. A component of Shapley value for such firm for any characteristic function is less than its cooperative payoff but much greater than its selfish payoff that in turn is much greater than its payoff in the role of a follower on the Stackelberg game.

In the future, we plan to study dynamic game theoretic models of coopetition in Cournot oligopoly. Moreover, the models with network structure both in normal form and in the form of characteristic function will be considered.

Author Contributions: Conceptualization, review, and methodology G.O.; investigation, A.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Branderburger, A.M.; Nalebuff, B.J. *Coopetition*; Currency Doubleday: New York, NY, USA, 1996.
2. Okura, M.; Carfi, D. Coopetition and Game Theory. *J. Appl. Econ. Sci.* **2014**, *9*, 458–469.
3. Carfi, D.; Donato, A. Coopetitive games for sustainability of global feeding and climate change: Recent developments. *J. Environ. Manag. Tour* **2018**, *9*, 200–215. [[CrossRef](#)] [[PubMed](#)]
4. Kalai, A.; Kalai, E. Cooperation in strategic games revisited. *Q. J. Econ.* **2013**, *128*, 917–966; Discussion Paper #1512. [[CrossRef](#)]
5. Cygler, J.; Sroka, W.; Solesvik, M.; Dębkowska, K. Benefits and Drawbacks of Coopetition: The Roles of Scope and Durability in Coopetitive Relationships. *Sustainability* **2018**, *10*, 2688. [[CrossRef](#)]
6. Gnyawali, D.R.; Madhavan, R.; He, J.; Bengtsson, M. The competition–cooperation paradox in inter-firm relationships: A conceptual framework. *Ind. Mark. Manag.* **2016**, *53*, 7–18. [[CrossRef](#)]
7. Ritala, P. Coopetition Strategy—When is it Successful? Empirical Evidence on Innovation and Market Performance. *Br. J. Manag.* **2001**, *23*, 307–324. [[CrossRef](#)]
8. Shvindina, H. Coopetition as an Emerging Trend in Research: Perspectives for Safety & Security. *Safety* **2019**, *5*, 61.
9. Walley, K. Coopetition—An Introduction to the Subject and an Agenda for Research. *Int. Stud. Manag. Organ.* **2007**, *37*, 11–31. [[CrossRef](#)]
10. Czakon, W.; Dana, L. Coopetition at work: How firms shaped the airline industry. *J. Soc. Manag.* **2013**, *11*, 32–61.
11. Rodrigues, F.; Souza, V.; Leitão, J. Strategic coopetition of global brands: A game theory approach to ‘Nike + iPod Sport Kit’ co-branding. *Int. J. Entrep. Ventur.* **2011**, *3*, 435. [[CrossRef](#)]

12. Sharma, M.G.; Singh, K.N. Servitization, coopetition, and sustainability: An operations perspective in aviation industry. *VIKALPA J. Decis. Mak.* **2017**, *42*, 145–152. [[CrossRef](#)]
13. Heiets, I.; Oleshko, T.; Leshchinsky, O. Game-Theoretic Principles of Decision Management Modeling Under the Coopetition. *Int. Game Theory Rev.* **2020**, *23*, 2050010. [[CrossRef](#)]
14. Le Roy, F.; Fernandez, A.; Chiambaretto, P. Managing coopetition in knowledge based industries. In *Global Opportunities for Entrepreneurial Growth: Coopetition and Knowledge Dynamics within and Across Firms*; Emerald Publishing Limited: Bingley, UK, 2017; pp. 187–198.
15. Le Roy, F.; Fernandez, A.; Chiambaretto, P. From strategizing coopetition to managing coopetition. In *The Routledge Companion to Coopetition Strategies*; Routledge: Abingdon, UK, 2018; pp. 36–46.
16. Ozkan-Canbolat, E.; Beraha, A.; Bas, A. Application of Evolutionary Game Theory to Strategic Innovation. *Procedia Soc. Behav. Sci.* **2016**, *235*, 685–693. [[CrossRef](#)]
17. Maskin, E.; Tirole, J. A Theory of Dynamic Oligopoly, III. Cournot Competition. *Eur. Econ. Rev.* **1987**, *31*, 947–968. [[CrossRef](#)]
18. Geraskin, M.I. The Properties of Conjectural Variations in the Nonlinear Stackelberg Oligopoly Model. *Autom. Remote Control* **2020**, *81*, 1051–1072. [[CrossRef](#)]
19. Geraskin, M.I. Approximate Calculation of Equilibria in the Nonlinear Stackelberg Oligopoly Model: A Linearization Based Approach. *Autom. Remote Control* **2020**, *81*, 1659–1678. [[CrossRef](#)]
20. Algazin, G.I.; Algazina, Y.G. Reflexion Reflexive Dynamics in the Cournot Oligopoly under Uncertainty. *Autom. Remote Control* **2020**, *81*, 287–301. [[CrossRef](#)]
21. Algazin, G.I.; Algazina, D.G. Reflexion processes and equilibrium in an oligopoly model with a leader. *Autom. Remote Control* **2020**, *81*, 1258–1270. [[CrossRef](#)]
22. Xiao, Y.; Peng, Y.; Lu, Q.; Wu, X. Dynamic investigations in a Stackelberg model with differentiated products and bounded rationality. *J. Comput. Appl. Math.* **2022**, *414*, 114409. [[CrossRef](#)]
23. Peng, Y.; Xiao, Y.; Lu, Q.; Wu, X.; Zhao, Y. Chaotic dynamics in Cournot duopoly model with bounded rationality based on relative profit delegation maximization. *Phys. AStat. Mech. Its Appl.* **2020**, *560*, 125174. [[CrossRef](#)]
24. Raoufinia, M.; Baradaran, V.; Shahrjerdi, R. A dynamic differential oligopoly game with sticky price and advertising: Open-loop and closed-loop solutions. *Kybernetes* **2018**, *48*, 586–611. [[CrossRef](#)]
25. Al-Khedhairi, A. Dynamical Study of Competition Cournot-like Duopoly Games Incorporating Fractional Order Derivatives and Seasonal Influences. *Int. J. Nonlinear Sci. Numer. Simul.* **2020**, *21*, 339–359. [[CrossRef](#)]
26. Julien, L.A. On noncooperative oligopoly equilibrium in the multiple leader–follower game. *Eur. J. Oper. Res.* **2017**, *256*, 650–662. [[CrossRef](#)]
27. Zouhar, J.; Zouharova, M. Stackelberg versus Cournot duopoly with asymmetric costs: Primary markups, entry deterrence, and a comparison of social welfare and industry profits. *Econ. Theory Bull.* **2019**, *8*, 89–96. [[CrossRef](#)]
28. Petrosjan, L.A.; Zenkevich, N.A. *Game Theory*; World Scientific Publishing: Singapore, 1996.
29. Fudenberg, D.; Tirole, J. *Game Theory*; MIT Press: Cambridge, MA, USA, 1991.
30. Peleg, B.; Sudholter, P. *Introduction to the Theory of Cooperative Games*; Springer: Berlin/Heidelberg, Germany, 2007.
31. Von Neumann, J.; Morgenstern, O. *Theory of Games and Economic Behavior*; Princeton University Press: Princeton, NJ, USA, 1953.
32. Petrosjan, L.; Zaccour, G. Time-consistent Shapley value allocation of pollution cost reduction. *J. Econ. Dyn. Control* **2003**, *27*, 381–398. [[CrossRef](#)]
33. Gromova, E.V.; Petrosyan, L.A. On an approach to constructing a characteristic function in cooperative differential games. *Autom. Remote Control* **2017**, *78*, 1680–1692. [[CrossRef](#)]
34. Korolev, A.V.; Ougolnitsky, G.A. Cooperative game-theoretic models of the Cournot oligopoly. *Int. Game Theory Rev.* **2022**, 2350004. [[CrossRef](#)]
35. Korolev, A.V.; Kotova, M.A.; Ugolnitsky, G.A. Comparison of Methods of Organization and Management Efficiency in Dynamic Models of Cournot Oligopoly. *J. Comput. Syst. Sci. Int.* **2023**, *62*, 34–56.
36. Ougolnitsky, G.A.; Usov, A.B. Differential Game-Theoretic Models of Cournot Oligopoly with Consideration of the Green Effect. *Games* **2023**, *14*, 14. [[CrossRef](#)]
37. Gorbaneva, O.I.; Ougolnitsky, G.A. Social and Private Interests Coordination Engines in Resource Allocation: System Compatibility, Corruption, and Regional Development. *Front. Games Dyn. Games Theory Appl. Numer. Methods* **2020**, *16*, 119–150.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.