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A Class of Enhanced Nonparametric Control Schemes Based on Order Statistics and Runs

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Abstract: In this article, we establish a new class of nonparametric Shewhart-type control charts based on order statistics with signaling runs-type rules. The proposed charts offer to the practitioner the opportunity to reach, as close as possible, a pre-specified level of performance by determining appropriately their design parameters. Special monitoring schemes, already established in the literature, are ascertained to be members of the proposed class. In addition, several new nonparametric control charts that belong to the family are introduced and studied in some detail. Exact formulae for the variance of the run length distribution and the average run length (ARL) for the proposed monitoring schemes are also derived. A numerical investigation is carried out and demonstrates that the proposed schemes acquire competitive performance in detecting the shift of the underlying distribution. Although the large number of design parameters is quite hard to handle, the numerical results presented throughout the lines of the present manuscript provide practical guidance for the implementation of the proposed charts.

Keywords: average run length; nonparametric control schemes; Lehmann alternatives; runs-type signaling rules; statistical process control

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1. Introduction

Statistical process control is broadly applied to monitor the quality of a production process, where a natural variability occurs in any case. Control charts assist the practitioners to single out assignable sources so that the state of statistical control can be fulfilled. In the case of observing a displeasing shift in the process, a control chart should detect it as quickly as possible and produce an out-of-control signal.

Most of the monitoring schemes are distribution-based procedures, even though this presumption is not always realized in practice. To overcome this obstacle and yet keep the primary formation of the traditional control charts, several nonparametric (or distribution-free) monitoring schemes have been proposed in the literature, such as the Shewhart-type, the Cumulative (CUSUM) or the Exponentially Weighted Moving Average (EWMA) schemes. Shewhart-type control charts were introduced in early work [1] and since then several modifications (such as the nonparametric) have been established and studied in detail. Some recent advances on the specific class of monitoring schemes appear in [2–5].

On the other hand, an up-to-date overview of distribution-free CUSUM and EWMA charts is provided in [6,7]. For a thorough study and some interesting perspectives on Nonparametric Statistical Process Control, the interested reader is referred to [8–10].

It is widely accepted that Shewhart-type control schemes perform well, especially under large shifts of the underlying distribution. In this direction, several distribution-free control schemes based on order statistics have been proposed in the literature. For example, [11] introduced two-sided nonparametric monitoring schemes with control limits determined by the aid of a reference sample from a presumably in-control population. In their framework, the decision of whether the process is in-control or out-of-control relies on the median of each sequentially observed test sample from the production process. The above-mentioned median scheme has also been studied in [12], where a broader class of distribution-free monitoring schemes based on future sample quantiles was introduced. Taking this direction of research forward, [13,14] took into account the location of the specific order statistics of the test sample, as well as the test data that are placed between the lower and the upper control limit to arrive at the status of the underlying process. Some recent advances on the topic of nonparametric Shewhart-type control schemes can be found in [15–17].

In the present article, we introduce a new class of nonparametric Shewhart-type control chart based on order statistics with signaling runs-type rules. More specifically, we apply the framework established by either [13] or [14] and, in any case, the resulting chart is enhanced by adding well-known simple runs-rules. The motivation for establishing the proposed control charts is to provide some new nonparametric alternatives for monitoring a process. Our main goal, which has been successfully achieved, was to improve the in- and out-of-control performance of some existing control charts based on order statistics. In Section 2, the setup of the proposed monitoring schemes is presented in detail, while explicit formulae are derived in Section 3 for determining the average and the variance of the corresponding run length. In Section 4, several numerical comparisons shed light on the efficacy of the proposed charts in comparison to competitive nonparametric control schemes. Finally, the Discussion section summarizes the contribution of the present article, while some potential directions are articulated for future research work.

2. The k-of-k DR Nonparametric Control Charts Based on Order Statistics and Runs

In this section, we establish a new family of monitoring schemes based on order statistics which utilizes the well-known runs-type rule introduced by [18] (DR runs rules, hereafter). The control limits of the proposed charts are based on reference observations drawn from the in-control process. The resulting schemes are built by following the general framework introduced by either [13] or [14] and enhancing, in any case, its performance with the aid of the above-mentioned DR runs-rules.

The proposed monitoring schemes rely on a reference sample of size m, namely m observations, say $X_1, X_2, \ldots, X_m$ with cumulative distribution function F, which have been drawn from the process when it is in-control. In fact, the control limits of the proposed control charts coincide with appropriately chosen observations of the corresponding ordered sample $X_{1:m}, X_{2:m}, \ldots, X_{m:m}$. Test samples of size n, say $Y_1, Y_2, \ldots, Y_n$ with cumulative distribution function G, are then picked out independently of each other (and of the reference sample) and our target is to decide whether the process is still in-control or not. In statistical terms, we are looking for a possible shift in the underlying distribution from F to G.

In the proposed framework, the information which is investigated from each test sample relies on specific observations of the corresponding ordered test sample. In other words, some observations from the ordered sample $Y_{1:n}, Y_{2:n}, \ldots, Y_{n:n}$ will play the role of monitoring statistics. In addition, we attempt to boost the performance of the proposed scheme by applying the DR runs rules. It is known that when a k-of-k DR runs rule is activated, an out-of-control signal is produced whenever k consecutive plotting points: (1) all fall on or above the upper control limit, (2) all fall on or below the lower control limit, (3) one falls on or above the upper control limit and the remaining fall on or below the lower control limit, (4) one falls on or below the lower control limit and the remaining fall on or above the upper control limit.

The general framework of the class of the proposed enhanced control charts follows the next steps.

Step 1. Draw a reference sample of size m from the in-control process.
Step 2. Determine the control limits of the monitoring scheme by the aid of the corresponding reference ordered observations.
Step 3. Draw independent test samples of size n from the process.
Step 4. Pick up appropriately chosen ordered test observations as monitoring statistics.

Step 5. Activate a k-of-k DR runs-type rule.
Step 6. Declare whether the process is in- or out-of control, by combining the plotting statistics and the DR runs rule defined in the previous steps.

The step-by-step constructing procedure is summarized in the following flowchart (Figure 1).

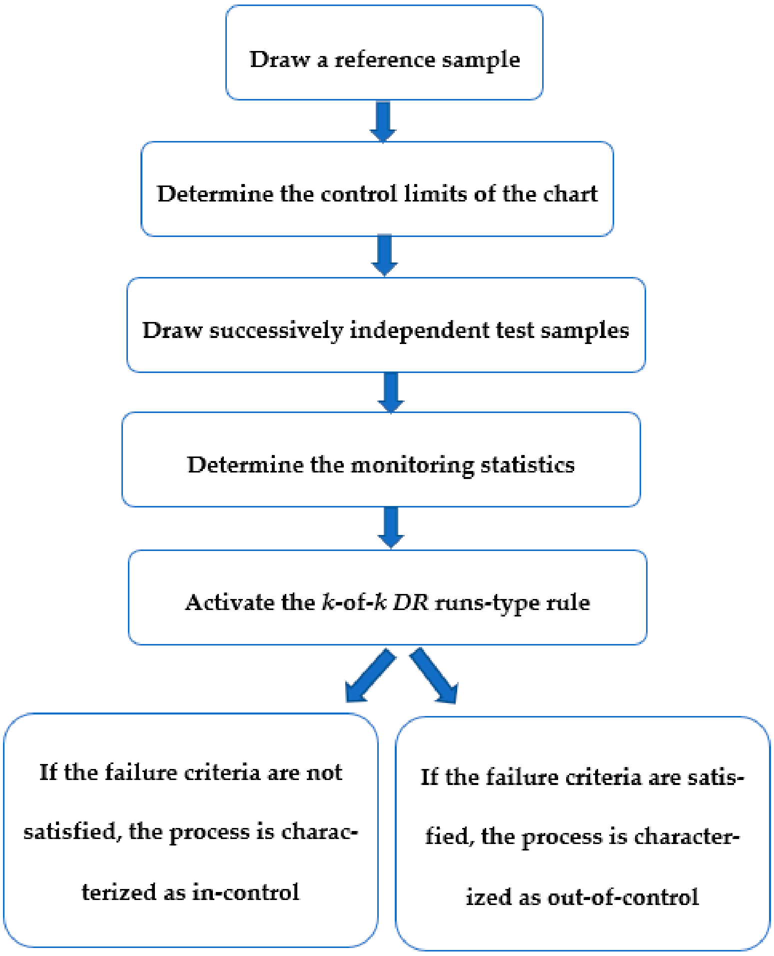

Figure 1. Flowchart of the constructing procedure for the proposed k-out-of-k DR monitoring schemes.

Before discussing the main results for the implementation of the proposed control schemes, we should first provide some details about their monitoring statistics. Throughout the present manuscript, we consider two different ideas for building the monitoring statistics. According to the first (see, e.g., [13]), two specific order statistics, say $X_{a:m}$, $X_{b:m}$, are used as control limits (say LCL, UCL), where $1 \leq a < b \leq m$. Afterwards, for the w-th test sample $Y_1^w, Y_2^w, \ldots, Y_n^w$ which is drawn ($w = 1, 2, \ldots$), the j-th order statistic $Y_{j:n}^w$ is chosen and made use of along with the statistic $R^w = R\left(Y_1^w, Y_2^w, \ldots, Y_n^w; X_{a:m}, X_{b:m}\right)$, which corresponds to the number of observations of the w-th test sample that lie between LCL and UCL. The process is declared to be in-control if the following conditions hold true

$$X_{a:m} \leq Y_{j:n}^w \leq X_{b:m} \text{ and } R^w \geq r \tag{1}$$

where r is a positive integer. Under the proposed monitoring scheme based on (1) (C_1^k-chart, hereafter), the process is characterized as out-of-control if the abovementioned set of conditions is violated for k consecutive test samples.

On the other hand, our second idea for constructing the monitoring schemes of the new class, relies on four ordered reference observations, say $X_{a:m}, X_{b:m}, X_{c:m}, X_{d:m}$ with $1 \leq a < b < c < d \leq m$. From the w-th test sample $Y_1^w, Y_2^w, \ldots, Y_n^w$ ($w = 1, 2, \ldots$), two specific order statistic, say $Y_{i:n}^w, Y_{j:n}^w$ with $i < j$, are picked up and made use of along with the statistics $R_1^w = R\left(Y_1^w, Y_2^w, \ldots, Y_n^w; X_{a:m}, X_{b:m}\right)$ and $R_2^w = R\left(Y_1^w, Y_2^w, \ldots, Y_n^w; X_{c:m}, X_{d:m}\right)$, which correspond to the number of test observations between $(X_{a:m}, X_{b:m})$ and $(X_{c:m}, X_{d:m})$, respectively. The process is declared to be in-control if the following conditions hold true

$$X_{a:m} \leq Y_{i:n}^w \leq X_{b:m}, X_{c:m} \leq Y_{j:n}^w \leq X_{d:m} \;\; R_1^w \geq r_1 \text{ and } R_2^w \geq r_2 \tag{2}$$

where r_1, r_2 are positive integer-valued parameters. Under the proposed monitoring scheme based on (2) (C_2^k-chart, hereafter), the process is characterized once again as out-of-control if the abovementioned set of conditions is violated for k consecutive test samples. For more details about the general framework of the abovementioned monitoring scheme, the interested reader is referred to [14] and Section 3 therein.

It is worth mentioning that several control charts which have been already established in the literature, can be considered as members of the new class introduced in the present manuscript. For instance, the monitoring scheme proposed by

- Ref. [13] can be viewed as a C_1^1-chart
- Ref. [19] can be viewed as a C_1^2-chart
- Ref. [14] can be viewed as a C_2^1-chart
- Ref. [20] can be viewed as a C_2^3-chart.

3. Main Results for the Proposed *k*-of-*k* DR Nonparametric Control Charts

In the present section, we shall provide some general results for the proposed distribution-free control charts. More precisely, we investigate two crucial characteristics of the run length of the proposed k-of-k DR monitoring schemes, namely the average run length and the corresponding variance of both C_1^k-charts and C_2^k-charts.

The following proposition provides explicit expressions for the average run length and the corresponding variance of the proposed k-of-k DR monitoring schemes in the general case. It is straightforward that these expressions can be implemented for studying any member of either the class of C_1^k- or the C_2^k-charts, respectively.

Proposition 1. (i) *The unconditional Average Run Length and the unconditional Variance of the Run Length of the C_1^k-chart is given by*

$$ARL_1^{(k)} = \int_0^1 \int_0^t \frac{1 - p_1^k}{(1 - p_1) p_1^k} f_{a,b}(s,t) ds dt \tag{3}$$

and

$$Var_1^{(k)} = \int_0^1 \int_0^t \frac{1 - (2k+1)(1 - p_1) p_1^k - p_1^{2k+1}}{((1 - p_1) p_1^k)^2} f_{a,b}(s,t) ds dt \tag{4}$$

respectively, while $p_1 = 1 - q_1\left(GF^{-1}(s), GF^{-1}(t); r\right)$ and

$$q_1(v, w; r) = \sum_{g=0}^{n-1} \sum_{h=\max(r-g-1,0)}^{n-g-1} \frac{n!}{(j-g-1)!(g+h+1)!(n-j-h)!} \times v^{j-g-1} (w - v)^{g+h+1} (1 - w)^{n-j-h} \tag{5}$$

and

$$f_{a,b}(s,t) = \frac{m!}{(a-1)!(b-a-1)!(m-b)!} s^{a-1} (t-s)^{b-a-1} (1-t)^{m-b}, 0 < s < t < 1 \tag{6}$$

(ii) *The unconditional Average Run Length and the unconditional Variance of the Run Length of the C_2^k-chart is given by*

$$ARL_2^{(k)} = \int_0^1 \int_0^{t_2} \int_0^{s_2} \int_0^{t_1} \frac{1-p_2^k}{(1-p_2)p_2^k} f_{a,b,c,d}(s_1,t_1,s_2,t_2) ds_1 dt_1 ds_2 dt_2 \tag{7}$$

and

$$ar_2^{(k)} = \int_0^1 \int_0^{t_2} \int_0^{s_2} \int_0^{t_1} \frac{1-(2k+1)(1-p_2)p_2^k - p_2^{2k+1}}{((1-p_2)p_2^k)^2} \times f_{a,b,c,d}(s_1,t_1,s_2,t_2) ds_1 dt_1 ds_2 dt_2 \tag{8}$$

respectively, while $p_2 = 1 - q_2\left(GF^{-1}(s_1), GF^{-1}(t_1), GF^{-1}(s_2), GF^{-1}(t_2); r_1, r_2\right)$ *and*

$$\begin{aligned} q_2(v,w,t,z;r_1,r_2) &= \sum_{c_1=0}^{n-2} \sum_{c_2=\max(0,r_1-c_1-1)}^{\min(n-c_1-2,j-i-1)} \sum_{c_3=0}^{\min(n-c_1-c_2-2,j-i-1-c_2)} \sum_{c_4=\max(0,r_2+c_2+c_3-j+i)}^{n-c_1-j+i-1} n! \\ &\times \frac{1}{(i-c_1-1)!(c_1+c_2+1)!c_3!(j-i-c_2-c_3+c_4)!(n-j-c_4)!} \\ &\times v^{i-c_1-1}(w-v)^{c_1+c_2+1}(t-w)^{c_3}(z-t)^{j-i-c_2-c_3+c_4}(1-z)^{n-j-c_4} \end{aligned} \tag{9}$$

and

$$\begin{aligned} f_{a,b,c,d}(s_1,t_1,s_2,t_2) = &\frac{m!}{(a-1)!(b-a-1)!(c-b-1)!(d-c-1)!(m-d)!} \\ &\times s_1^{a-1}\left(t_1-s_1\right)^{b-a-1}\left(s_2-t_1\right)^{c-b-1}\left(t_2-s_2\right)^{d-c-1}\left(1-t_2\right)^{m-d}, \\ &0 < s_1 < t_1 < s_2 < t_2 < 1. \end{aligned} \tag{10}$$

Proof. Let us denote by T_k the waiting time until an out-of-control signal is produced by either a C_1^k- or a C_2^k-control chart. In other words, T_k corresponds to the run length of the underlying monitoring scheme. Given $X_{a:m} = x, X_{b:m} = y$, random variable T_k follows a geometric distribution of order k with success probability p. Therefore, the conditional expected value and variance of the run length T_k can be determined by the aid of the following formulae (see, e.g., [21])

$$E(T_k | X_{a:m} = x, X_{b:m} = y) = \frac{1-p^k}{(1-p)p^k} \tag{11}$$

and

$$Var(T_k | X_{a:m} = x, X_{b:m} = y) = \frac{1-(2k+1)(1-p)p^k - p^{2k+1}}{((1-p)p^k)^2} \tag{12}$$

respectively.

(i) When the C_1^k-chart is applied, success probability p of the aforementioned geometric distribution of order k coincides to the probability p_1 that the set of conditions stated in (1) is not satisfied. However, Ref. [13] proved that p_1 is determined by the aid of the following double sum

$$\sum_{g=0}^{n-1} \sum_{h=\max(r-g-1,0)}^{n-g-1} \frac{n! v^{j-g-1}(w-v)^{g+h+1}(1-w)^{n-j-h}}{(j-g-1)!(g+h+1)!(n-j-h)!}$$

We next combine the last expression with Equations (11) and (12) and the desired results is derived by averaging over the distribution of $X_{a:m}, X_{b:m}$.

(ii) On the other hand, under the C_2^k-chart, the success probability p coincides now to the probability p_2 that the set of conditions stated in (2) is not satisfied. Taking into advantage that p_2 can be expressed by the aid of the following sum

$$\sum_{c_1=0}^{n-2} \sum_{c_2=\max(0,r_1-c_1-1)}^{\min(n-c_1-2,j-i-1)} \sum_{c_3=0}^{\min(n-c_1-c_2-2,j-i-1-c_2)} \sum_{c_4=\max(0,r_2+c_2+c_3-j+i)}^{n-c_1-j+i-1} n! \times \frac{v^{i-c_1-1}(w-v)^{c_1+c_2+1}(t-w)^{c_3}(z-t)^{j-i-c_2-c_3+c_4}(1-z)^{n-j-c_4}}{(i-c_1-1)!(c_1+c_2+1)!c_3!(j-i-c_2-c_3+c_4)!(n-j-c_4)}$$

(see, e.g., [14]), Formulae (11) and (12) lead effortlessly to the results we are seeking. □

Next, we consider three different values of the design parameter k. More specifically, we develop some results for the new class of nonparametric control charts under the choices $k = 2, 3, 4$. The following corollary offers explicit expressions for the average run length and the corresponding variance of the proposed 2-*of*-2 *DR* monitoring schemes.

Corollary 1. (i) *The unconditional Average Run Length and the unconditional Variance of the* C_1^2-*chart is given by*

$$ARL_1^{(2)} = \int_0^1 \int_0^t \frac{1+p_1}{p_1^2} f_{a,b}(s,t) ds dt \tag{13}$$

and

$$Var_1^{(2)} = \int_0^1 \int_0^t \frac{(1-p_1)(p_1^2+3p_1+1)}{p_1^4} f_{a,b}(s,t) ds dt \tag{14}$$

respectively, while p_1 *and* $f_{a,b}(s,t)$ *are given in (5) and (6).*

(ii) *The unconditional Average Run Length and the unconditional Variance of the* C_2^2-*chart is given by*

$$ARL_2^{(2)} = \int_0^1 \int_0^{t_2} \int_0^{s_2} \int_0^{t_1} \frac{1+p_2}{p_2^2} f_{a,b,c,d}(s_1,t_1,s_2,t_2) ds_1 dt_1 ds_2 dt_2 \tag{15}$$

and

$$Var_2^{(2)} = \int_0^1 \int_0^{t_2} \int_0^{s_2} \int_0^{t_1} \frac{(1-p_2)(p_2^2+3p_2+1)}{p_2^4} \times f_{a,b,c,d}(s_1,t_1,s_2,t_2) ds_1 dt_1 ds_2 dt_2 \tag{16}$$

respectively, while p_2 *and* $f_{a,b,c,d}(s_1,t_1,s_2,t_2)$ *are given in (9) and (10).*

Proof. The desired results are derived by the aid of Proposition 1 after simple algebraic maneuvering. □

It is worth mentioning that [19] delivered similar expressions with those appearing in (13) and (14). The following corollary offers explicit expressions for the average run length and the corresponding variance of the proposed *3-of-3 DR* monitoring schemes.

Corollary 2. (i) *The unconditional Average Run Length and the unconditional Variance of the* C_1^3-*chart is given by*

$$ARL_1^{(3)} = \int_0^1 \int_0^t \frac{p_1^2+p_1+1}{p_1^3} f_{a,b}(s,t) ds dt \tag{17}$$

and

$$Var_1^{(3)} = \int_0^1 \int_0^t \frac{(1-p_1)(p_1^4+3p_1^3+6p_1^2+3p_1+1)}{p_1^6} f_{a,b}(s,t) ds dt \tag{18}$$

respectively, while p_1 *and* $f_{a,b}(s,t)$ *are given in (5) and (6).*

(ii) The unconditional Average Run Length and the unconditional Variance of the C_2^3-*chart is given by*

$$ARL_2^{(3)} = \int_0^1 \int_0^{t_2} \int_0^{s_2} \int_0^{t_1} \frac{p_2^2 + p_2 + 1}{p_2^3} f_{a,b,c,d}(s_1, t_1, s_2, t_2) ds_1 dt_1 ds_2 dt_2 \quad (19)$$

and

$$Var_2^{(3)} = \int_0^1 \int_0^{t_2} \int_0^{s_2} \int_0^{t_1} \frac{(1 - p_2)(p_2^4 + 3p_2^3 + 6p_2^2 + 3p_2 + 1)}{p_2^6} \times f_{a,b,c,d}(s_1, t_1, s_2, t_2) ds_1 dt_1 ds_2 dt_2 \quad (20)$$

respectively, while p_2 and $f_{a,b,c,d}(s_1, t_1, s_2, t_2)$ are given in (9) and (10).

Proof. The desired results are derived by the aid of Proposition 1 after simple algebraic maneuvering. □

The following corollary offers explicit expressions for the average run length and the corresponding variance of the proposed *4-of-4 DR* monitoring schemes.

Corollary 3. (i) *The unconditional Average Run Length and the unconditional Variance of the C_1^4-chart is given by*

$$ARL_1^{(4)} = \int_0^1 \int_0^t \frac{(1 + p_1)(1 + p_1^2)}{p_1^4} f_{a,b}(s, t) ds dt \quad (21)$$

and

$$Var_1^{(4)} = \int_0^1 \int_0^t \frac{1 - p_1^4(p_1^5 - 9p_1 + 9)}{(1 - p_1)^2 p_1^8} f_{a,b}(s, t) ds dt \quad (22)$$

respectively, while p_1 and $f_{a,b}(s, t)$ are given in (5) and (6).

(ii) *The unconditional Average Run Length and the unconditional Variance of the C_2^4-chart is given by*

$$ARL_2^{(4)} = \int_0^1 \int_0^{t_2} \int_0^{s_2} \int_0^{t_1} \frac{(1 + p_2)(1 + p_2^2)}{p_2^4} f_{a,b,c,d}(s_1, t_1, s_2, t_2) ds_1 dt_1 ds_2 dt_2 \quad (23)$$

and

$$Var_2^{(4)} = \int_0^1 \int_0^{t_2} \int_0^{s_2} \int_0^{t_1} \frac{1 - p_2^4(p_2^5 - 9p_2 + 9)}{(1 - p_2)^2 p_2^8} \times f_{a,b,c,d}(s_1, t_1, s_2, t_2) ds_1 dt_1 ds_2 dt_2 \quad (24)$$

respectively, while p_2 and $f_{a,b,c,d}(s_1, t_1, s_2, t_2)$ are given in (9) and (10).

Proof. The desired results are derived by the aid of Proposition 1 after simple algebraic maneuvering. □

It is noticeable that the unconditional in-control Average Run Length and Variance of the in-control Run Length can be obtained by substituting $F = G$ in the formulae proved in Proposition 1. In Table 1, we present the in-control ARL's of the $C_1^2-, C_1^3 - C_1^4-$monitoring schemes for several values of design parameter. Note that all calculations were carried out with the aid of Corollaries 1,2 and 3.

Table 1. In-control Average Run Length of the C_1^2-, C_1^3- and C_1^4- charts for a given design.

		Reference Sample Size m							
		50		**100**		**150**		**200**	
ARL_o	n	(a,b,j,r)	ARL_{in}	(a,b,j,r)	ARL_{in}	(a,b,j,r)	ARL_{in}	(a,b,j,r)	ARL_{in}
370	5	(6,45,2,2) (9,41,2,2) (11,38,2,2)	368.64 351.71 364.51	(10,91,2,2) (13,87,2,3) (22,98,2,3)	365.67 364.52 371.26	(12,122,2,2) (13,121,2,3) (20,119,2,3)	366.70 377.10 365.78	(18,187,2,2) (27,181,2,3) (34,173,2,3)	369.21 362.67 360.67
	11	(4,37,4,6) (5,33,5,6) (6,35,4,7)	378.61 372.34 362.10	(23,85,6,5) (33,87,6,5) (29,86,5,6)	369.64 366.63 367.58	(30,129,5,5) (32,127,5,6) (37,124,5,6)	361.71 375.42 367.91	(40,174,5,5) (42,170,5,6) (40,176,4,7)	359.21 374.24 362.29
	15	(18,46,8,7) (17,45,6,8) (12,33,6,8)	368.10 350.61 370.17	(14,74,8,7) (21,73,7,7) (33,87,6,7)	370.86 376.41 374.80	(26,118,7,7) (22,74,7,7) (34,90,6,7)	375.83 364.85 366.75	(28,152,8,7) (43,152,8,7) (54,167,7,8)	369.78 366.94 370.89
500	5	(6,48,2,2) (9,43,2,2) (12,43,2,2)	507.10 496.09 472.40	(8,82,2,2) (13,89,2,3) (21,97,2,3)	497.87 514.02 497.31	(11,122,2,2) (13,123,2,3) (21,123,2,3)	494.04 504.12 504.76	(16,175,2,2) (27,186,2,3) (35,180,2,3)	498.45 489.24 503.83
	11	(8,43,4,6) (15,41,5,5) (12,42,4,7)	512.97 473.40 509.58	(31,79,6,5) (33,88,6,5) (30,88,5,6)	470.09 493.29 470.89	(32,138,5,5) (30,126,5,6) (35,123,5,6)	505.59 504.22 504.88	(36,169,5,5) (39,167,6,6) (46,165,5,6)	478.11 491.04 504.88
	15	(16,46,7,7) (11,37,6,8) (9,31,5,8)	493.73 460.81 533.89	(23,93,6,7) (25,78,7,7) (34,92,6,7)	479.20 482.58 485.70	(26,119,7,7) (34,87,8,7) (22,70,6,7)	475.13 489.27 478.03	(37,163,7,7) (38,161,7,8) (47,161,7,8)	491.73 482.89 502.69

Each cell contains the in-control ARL-values attained for C_1^{2-}(upper entry), C_1^{3-}(middle entry) and C_1^{4-}(lower entry) chart respectively.

Having at hand the results displayed in Table 1, the practitioner can design a distribution-free control chart that attains a pre-specified in-control level of performance *(ARL_o)*. For instance, let us assume that a reference sample of size $m = 100$ is available and that we aim at constructing a monitoring scheme that achieves an in-control Average Run Length equal to 370 (approximately). Based on Table 1, our aim shall be fulfilled if we construct

- a C_1^4-chart with design parameters $a = 22, b = 98, n = 5, j = 2, r = 3$. In other words, the practitioner should select the 22nd and the 98th ordered reference observation as the control limits and work with test samples of size $n = 5$. Moreover, if the remaining parameters are determined as $j = 2, r = 3$, then the resulting C_1^4-chart achieves an in-control *ARL* equal to 371.26, or
- a C_1^3-chart with design parameters $a = 21, b = 73, n = 15, j = 7, r = 7$. In other words, the practitioner should select the 21st and the 73th ordered reference observation as the control limits and work with test samples of size $n = 15$. Moreover, if the remaining parameters are determined as $j = 7, r = 7$, then the resulting C_1^3-chart achieves an in-control ARL equal to 376.41, or
- a C_1^2-chart with design parameters $a = 36, b = 84, n = 11, j = 6, r = 5$. In other words, the practitioner should select the 36th and the 84th ordered reference observation as the control limits and work with test samples of size $n = 11$. Then, if the remaining parameters are determined as $j = 2, r = 3$, the resulting C_1^2-chart achieves an in-control ARL equal to 370.25.
- In addition, a similar numerical investigation has been carried out for the class of C_2^k-charts. More precisely, in Table 2, the in-control ARLs of the C_2^2-, $C_2^3 - C_2^4-$monitoring schemes are provided for several values of design parameters.

Table 2. In-control Average Run Length of the C_2^2-, C_2^3- and C_2^4- charts for a given design.

		Reference Sample Size m			
		100		**200**	
ARL_o	n	(a,b,c,d,i,j,r_1,r_2)	ARL_{in}	(a,b,c,d,i,j,r_1,r_2)	ARL_{in}
		(6,43,55,92,5,21,1,1)	356.11	(9,83,103,183,5,21,1,1)	375.78
370	25	(8,43,53,87,6,21,2,1)	382.20	(8,71,108,176,5,21,1,1)	358.49
		(13,45,56,85,5,20,2,1)	364.56	(9,91,118,170,5,21,2,1)	367.35
		(7,44,53,90,6,25,2,1)	367.36	(9,74,108,178,6,24,1,1)	378.37
	30	(9,43,52,86,6,25,2,1)	381.07	(13,72,117,175,6,25,2,1)	370.24
		(12,42,56,81,5,20,2,1)	362.14	(14,70,119,170,6,25,2,1)	388.40
		(6,47,55,92,5,21,1,1)	491.42	(8,81,103,184,5,21,1,1)	511.35
500	25	(7,43,53,87,6,21,2,1)	487.87	(13,75,117,179,5,21,1,1)	499.30
		(12,42,56,85,5,20,2,1)	492.12	(6,97,115,170,5,21,2,1)	501.97
		(7,44,47,90,6,25,2,1)	489.71	(9,76,105,178,6,24,1,1)	500.44
	30	(8,42,53,86,6,25,2,1)	510.87	(13,78,117,175,6,25,2,1)	485.42
		(12,48,56,81,5,20,2,1)	496.10	(14,75,114,167,5,20,2,1)	500.47

Each cell contains the in-control ARL-values attained for C_2^{2-}(upper entry), C_2^{3-}(middle entry) and C_2^{4-}(lower entry) chart respectively.

Note that the performance of C_2^k-charts weakens when the test sample size is small. Therefore, it is recommended that, for the implementation of C_2^k-charts, the sample size (n) of the test samples drawn from the process should be equal to or more than 25. A similar comment has been stated by [14] for their monitoring framework.

Based on Table 2, one may investigate the in-control performance of the proposed nonparametric schemes. For instance, let us assume that, having at hand $m = 100$ reference observations, we aim at constructing a monitoring scheme with in-control Average Run Length equal to 500 (approximately). It seems that our requirement could be satisfied, if we construct

- a C_2^2-chart with design parameters $a = 6, b = 47, c = 55, d = 92, i = 5$, j = 21, n = 25, $r_1 = 1, r_2 = 1$ (with exact in-control ARL equal to 491.42) or
- a C_2^3-chart with design parameters $a = 8, b = 42, c = 53, d = 86, i = 6$, j = 25, n = 30, $r_1 = 2, r_2 = 1$ (with exact in-control ARL equal to 510.87) or
- a C_2^4-chart with design parameters $a = 12, b = 42, c = 56, d = 85, i = 5$, j = 20, n = 25, $r_1 = 2, r_2 = 1$ (with exact in-control ARL equal to 492.12).

The out-of-control performance could be evaluated via the corresponding *ARL* that the control chart attains. If the process shifts out-of-control, the out-of-control ARL of the proposed DR charts depends on both the in-control and out-of-control distributions F and G. If we assume that G belongs to the so-called Lehmann alternatives (see [22]), the out-of-control distribution function takes on the form $G = F^{\gamma}$ for some fixed, positive number $\gamma > 0$. Table 3 provides the out-of-control ARL values delivered by the C_1^2-, $C_1^3 - C_1^4-$ monitoring schemes under the Lehmann-type alternatives for $\gamma = 0.8$. The designs displayed in Table 3 are the same as those appearing in Table 1.

One may draw interesting conclusions based on the numerical results displayed in Table 3. For example, let us consider the same case study mentioned earlier, namely let us assume that the practitioner works with a reference sample of size $m = 100$ in order to reach an in-control ARL equal to 370. Then, under the Lehmann alternatives with parameter γ equal to 0.8, the C_1^2-, $C_1^3 - C_1^4-$monitoring schemes appearing in Tables 1 and 3 achieve an out-of-control *ARL* equal to 47.26, 91.17 and 50.57. respectively.

Table 3. Out-of-control Average Run Length of the C_1^2-, C_1^3- and C_1^4- charts for a given design.

		Reference Sample Size m							
		50		**100**		**150**		**200**	
ARL_o	n	(a,b,j,r)	ARL_{out}	(a,b,j,r)	ARL_{out}	(a,b,j,r)	ARL_{in}	(a,b,j,r)	ARL_{out}
370	5	(6,45,2,2) (9,41,2,2) (11,38,2,2)	55.99 57.79 59.95	(10,91,2,2) (13,87,2,3) (22,98,2,3)	59.21 83.59 50.57	(12,122,2,2) (13,121,2,3) (20,119,2,3)	81.16 147.59 111.78	(18,187,2,2) (27,181,2,3) (34,173,2,3)	61.48 71.68 73.31
	11	(4,37,4,6) (5,33,5,6) (6,35,4,7)	237.35 312.24 133.71	(23,85,6,5) (33,87,6,5) (29,86,5,6)	74.64 49.88 41.85	(30,129,5,5) (32,127,5,6) (37,124,5,6)	50.99 67.52 60.43	(40,174,5,5) (42,170,5,6) (40,176,4,7)	48.76 68.88 46.98
	15	(18,46,8,7) (17,45,6,8) (12,33,6,8)	23.18 12.47 12.11	(14,74,8,7) (21,73,7,7) (33,87,6,7)	189.89 91.17 18.84	(26,118,7,7) (22,74,7,7) (34,90,6,7)	96.90 80.01 16.94	(28,152,8,7) (43,152,8,7) (54,167,7,8)	175.17 105.05 49.12
500	5	(6,48,2,2) (9,43,2,2) (12,43,2,2)	59.94 64.57 53.14	(8,82,2,2) (13,89,2,3) (21,97,2,3)	103.05 97.53 61.73	(11,122,2,2) (13,123,2,3) (21,123,2,3)	105.44 173.89 122.80	(16,175,2,2) (27,186,2,3) (35,180,2,3)	83.14 80.85 81.17
	11	(8,43,4,6) (15,41,5,5) (12,42,4,7)	57.85 38.62 29.45	(31,79,6,5) (33,88,6,5) (30,88,5,6)	75.43 57.37 41.85	(32,138,5,5) (30,126,5,6) (35,123,5,6)	48.01 89.98 80.30	(36,169,5,5) (39,167,6,6) (46,165,5,6)	71.79 117.15 83.11
	15	(16,46,7,7) (11,37,6,8) (9,31,5,8)	22.44 38.90 31.17	(23,93,6,7) (25,78,7,7) (34,92,6,7)	27.20 67.15 17.91	(26,119,7,7) (34,87,8,7) (22,70,6,7)	111.20 43.76 73.85	(37,163,7,7) (38,161,7,8) (47,161,7,8)	93.99 103.54 78.48

Each cell contains the out-of-control ARL-values attained for $\gamma = 0.8$ by C_1^{2-}(upper entry), C_1^3—(middle entry) and C_1^{4-}(lower entry) chart, respectively.

A similar numerical investigation has been carried out for the class of C_2^k-charts. More precisely, in Table 4, the out-of-control ARLs of the $C_2^2-, C_2^3 - C_2^4-$monitoring schemes are provided for the same designs displayed in Table 2.

Table 4. Out-of -control Average Run Length of the C_2^2-, C_2^3- and C_2^4- charts for a given design.

		Reference Sample Size m			
		100		**200**	
ARL_o	n	(a,b,c,d,i,j,r_1,r_2)	ARL_{out}	(a,b,c,d,i,j,r_1,r_2)	ARL_{out}
370	25	(6,43,55,92,5,21,1,1) (8,43,53,87,6,21,2,1) (13,45,56,85,5,20,2,1)	36.27 118.84 9.91	(9,83,103,183,5,21,1,1) (8,71,108,176,5,21,1,1) (9,91,118,170,5,21,2,1)	70.75 307.17 215.94
	30	(7,44,53,90,6,25,2,1) (9,43,52,86,6,25,2,1) (12,42,56,81,5,20,2,1)	32.89 32.80 4.98	(9,74,108,178,6,24,1,1) (13,72,117,175,6,25,2,1) (14,70,119,170,6,25,2,1)	67.75 72.17 91.94
500	25	(6,47,55,92,5,21,1,1) (7,43,53,87,6,21,2,1) (12,42,56,85,5,20,2,1)	36.69 203.41 13.67	(8,81,103,184,5,21,1,1) (13,75,117,179,5,21,1,1) (6,97,115,170,5,21,2,1)	112.97 51.25 487.09
	30	(7,44,47,90,6,25,2,1) (8,42,53,86,6,25,2,1) (12,48,56,81,5,20,2,1)	44.30 57.63 4.98	(9,76,105,178,6,24,1,1) (13,78,117,175,6,25,2,1) (14,75,114,167,5,20,2,1)	84.55 66.59 59.65

Each cell contains the out-of-control ARL-values attained for $\gamma = 0.7$ by C_2^{2-}(upper entry), C_2^{3-}(middle entry) and C_2^{4-}(lower entry) chart respectively.

If we consider the same case study mentioned earlier, namely let us assume that the practitioner works with a reference sample of size $m = 100$ in order to reach an in-control ARL equal to 370, then, under the Lehmann alternatives with parameter γ equal to 0.7, the

$C_2^2-, C_2^3 - C_2^4-$monitoring schemes appearing in Tables 1 and 3 achieve an out-of-control *ARL* equal to 36.69, 57.63 and 13.67, respectively.

4. Numerical Comparisons

In this section, we carry out extensive numerical experimentation to shed light on the efficacy of the new control charts and their robustness features under both in-control and out-of-control situations. The computations are accomplished with the aid of theoretical results discussed previously. It is evident that, traditionally, whenever a new control chart is established as a generalization of existing ones, direct comparison is highly recommended. Therefore, we next investigate the performance of the proposed control charts against those established [13,14].

A customary approach weighing two different control charts calls for determining a common in-control *ARL* and then examine the corresponding out-control ARLs. Throughout the next lines, we compare the behavior of the C_1^k-, C_2^k-monitoring schemes to those established by [13,14], respectively.

Table 5 offers several numerical comparisons between the C_1^2-control scheme and the nonparametric chart introduced by [13]. We assume that a reference sample of size $m = 100$ is available, while test samples of size $n = 5$ are then drawn from the process in order to decide whether it is in- or out-of-control. Both competing schemes are designed such that an in-control *ARL* near to 500 is achieved. The design parameters of the chart of [11] are determined as $a = 5, b = 95, j = 3, r = 2$ (Competitor 1, hereafter) and its exact in-control ARL equals 458.07.

Throughout Table 5, two different scenarios are considered for the distribution of the underlying process. We first consider the case of a process with in-control distribution as the normal distribution with parameters 0 and 1. The out-of-control distribution is normal distribution with possible shifts in mean and/or standard deviation (equal to θ units and δ units, respectively).

The first part of Table 5 clearly reveals that, under the assumption that the process is normally distributed, the C_1^2-control chart performs better than Competitor 1, in terms of out-of-control ARL values, in all cases considered. For example, let us consider the case where the process mean has shifted $\theta = 0.5$ units and the corresponding standard deviation has also shifted $\delta = 0.05$ units. As is easily observed by the aid of Table 5, the C_1^2-monitoring scheme achieves an out-of-control ARL equal to 37.91, while the corresponding ARL–value for Competitor 1 is 59.08.

On the one hand, it is quite useful to investigate the out-of-control performance of the proposed monitoring schemes for different underlying distributions. Therefore, the case of a process with in-control distribution, the Laplace distribution (0,1), is also considered. The second part of Table 5 clearly reveals that, under the assumption that the process follows the Laplace distribution, the C_1^2-control chart performs once again better than Competitor 1, in terms of out-of-control ARL values, in all cases considered. For example, let us recall the same out-of-control situation as before, namely, let us assume that the process mean has shifted $\theta = 0.5$ units and the corresponding standard deviation has also shifted $\delta = 0.05$ units. In that case, the superiority of the C_1^2-monitoring scheme against Competitor 1 is even more evident. Indeed, the C_1^2-chart achieves an out-of-control ARL equal to 84.65, while the corresponding ARL–value for the Competitor 1 is 187.85.

Table 5. ARL values of the C_1^2- control charts against competitive schemes under different process distributions and several shifts θ, δ (m = 100, n = 5).

		***Normal Distribution* (0 + θ, 1 + δ)**		***Laplace Distribution* (0 + θ, 1 + δ)**	
θ	δ	C_1^2-*Chart* $a = 12, b = 84, j = 3, r = 2$	*Competitor* **1**	C_1^2-*Chart* $a = 12, b = 84, j = 3, r = 2$	*Competitor* **1**
0	0	475.84	458.07	475.84	458.07
0.25	0	176.43	248.92	263.59	374.63
0.5	0	45.77	81.88	108.07	257.35
1	0	6.30	10.00	13.82	84.08
1.5	0	2.60	2.63	3.39	22.29
0.25	0.05	124.01	160.15	192.43	268.20
0.5	0.05	37.91	59.08	84.65	187.85
1	0.05	6.23	8.81	12.68	65.12
1.5	0.05	2.67	2.75	3.42	18.61
0.25	0.10	91.21	109.40	145.26	198.46
0.5	0.10	32.11	44.65	68.14	141.62
1	0.10	6.17	7.90	11.76	51.92
1.5	0.10	2.51	2.52	3.44	15.89
0.25	0.15	69.60	78.44	112.82	151.09
0.5	0.15	27.67	35.01	56.10	109.75
1	0.15	6.10	7.19	10.99	42.42
1.5	0.15	2.68	2.84	3.46	13.84
0.25	0.20	54.74	58.51	89.79	117.88
0.5	0.20	24.19	28.27	47.09	87.09
1	0.20	6.04	6.61	10.36	35.39
1.5	0.20	2.29	2.45	3.48	12.24

In what follows, we investigate the out-of-control performance of the class of C_2^k-control charts and provide several numerical comparisons versus the monitoring scheme introduced by [14]. We assume that a reference sample of size $m = 100$ is available, while the in-control process distribution is supposed to be the Exponential with parameter λ = 1 and the shifts we are wishing to capture are created by a change in the parameter λ. Test samples of size $n = 25$ are then drawn from the process in order to decide whether it is in- or out-of-control. Both competing schemes are designed such that an in-control ARL near to 500 is achieved. The design parameters of the chart of [14] are determined as $a = 2,\ b = 48,\ c = 49,\ d = 99,\ i = 4,\ j = 21,\ r_1 = 1,\ r_2 = 1$ (Competitor 2, hereafter) and its exact in-control ARL equals to 497.21.

As is easily observed, the C_2^4-control chart is, under Exponential distribution, superior to Competitor 2 in all cases examined. For instance, if the process mean of the underlying in-control distribution has shifted $0.05(0.10)$ units, C_2^4-monitoring scheme achieves (see Table 6) an out-of-control ARL equal to 319.50 (184.13), while the corresponding ARL-value for the Competitor 2 is 464.96 (396.84).

Table 6. ARL values of the C_2^4- control charts against competitive schemes under Exponential distribution and several shifts (m = 100, n = 25).

	Exponential Distribution (λ)	
Shift	C_2^4-*Chart* a=12, b=42, c=56, d=85, i=5, j=20,r_1=2, r_2=1	*Competitor* 2
0.000	492.12	497.21
0.025	402.57	485.98
0.050	319.50	464.96
0.075	246.08	434.80
0.100	184.13	396.84
0.125	134.15	353.09
0.150	95.51	306.01
0.175	66.80	258.20
0.200	46.23	212.12
0.225	31.95	169.80
0.250	22.27	132.63
0.275	15.84	101.32
0.300	11.60	75.90
0.325	8.81	55.97
0.350	6.96	40.77
0375	5.72	29.45
0.400	4.87	21.18

5. Discussion and Some Conclusions

In the present work, a new class of nonparametric Shewhart-type control chart based on order statistics with signaling runs-type rules is set up and studied in some detail. The monitoring statistics correspond to suitably chosen order statistics from sequential test samples which are drawn from the underlying process. The proffered schemes utilize well-known runs-type rules. The run length of the new distribution-free control charts is investigated for both in- and out-of-control situations. Several numerical results disclose the capability of the proposed schemes for detecting a possible shift of the underlying distribution process. More precisely, the proposed control charts seem to perform competitively under different distribution models, such as the Normal distribution, the Laplace or the Exponential distribution. Moreover, the new monitoring schemes are proved to be able to detect small shifts in either mean or dispersion of the underlying distribution. Due to the existence of several design parameters, the proposed family of control charts offers to the practitioner the flexibility to achieve a pre-determined level of performance with great precision. It is also concluded that the proposed charts perform well even if the test sample size is quite small. The latter remark seems practically useful since, in real life applications, quite often the underlying process is monitored by the aid of small samples. Finally, it is of some future research interest to implement runs-type rules to additional nonparametric control charts for enhancement of their performance.

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