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Generalized Beth–Uhlenbeck Approach to the (2+1)-Dimensional Gross–Neveu Model

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Abstract

We study the thermodynamics of the (2+1)-dimensional Gross–Neveu model inspired from graphene. We focus on the entropy density of the Gaussian fluctuation beyond the mean field. The full in-medium, momentum-dependent evaluation reveals that the fluctuations give a substantial contribution, even comparable to that of the mean field. We argue that the back-reaction from the fluctuations to the mean field should be included, which reduces the contribution mainly coming from the Landau-damping region. To treat this self-consistently, we use the generalized version of the Beth–Uhlenbeck approach for the entropy density. Compared with the standard Beth–Uhlenbeck formulation, the generalized version suppresses the low-energy contributions while preserving the bound-state effects. To illustrate this, we consider the respective contributions of the bound excitons and unbound fermions to the total entropy. This shows a sharper crossover between the degrees of freedom compared to the standard Beth–Uhlenbeck approach. This behavior is consistent with Mott-transition physics in two-dimensional materials.

Keywords: Beth–Uhlenbeck; thermodynamics; Gross–Neveu model

1. Introduction

The Gross–Neveu (GN) model is a quantum field theory of N Dirac fermion species characterized by a four-fermion contact interaction. Originally it was introduced by David Gross and André Neveu in 1974 [1] in (1+1) dimensions to demonstrate asymptotic freedom and spontaneous mass generation. Its (3+1)-dimensional counterpart was introduced thirteen years earlier by Yoichiro Nambu and Giovanni Jona-Lasinio [2] (NJL) as a model of locally interacting nucleons to describe the mass gap. Later, as quarks were being established as the fundamental degrees of freedom, the degrees of freedom for nucleons were replaced by the quark ones [3,4]. Since then, both GN and NJL models have been widely studied. The NJL model serves as a low-energy effective field theory of Quantum Chromodynamics (QCD) and is widely used to model the equation of state of matter under extreme conditions and to study the QCD phase diagram. In contrast, the renormalizable (1+1)-dimensional GN model is used as a theoretical playground for investigating various non-perturbative phenomena in interacting fermionic systems. The (2+1)-dimensional version of the Gross–Neveu model serves as a middle ground, as it shows rich phase structures [5] while being renormalizable [6] within the $1/N$ expansion [7]. The model attracted renewed interest after the discovery of graphene [8] and the subsequent rise in two-dimensional Dirac materials. The low-energy excitations near the high-symmetry points in

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the Brillouin zone (Dirac cone) of graphene behave as massless Dirac particles [9]. Due to the reduced Fermi velocity, the effective interaction strength can become sufficiently large to support non-perturbative phenomena [10], such as interaction-driven gap formation. The GN model provides a natural framework for describing the generation of an insulating gap in such systems [11]. In [12], the authors considered a specific version of the model respecting the symmetries of graphene. The paper presents the mean field approximation and the Gaussian fluctuations for the correlations. The Gaussian fluctuations and their effect on the thermodynamics were cast into the Beth–Uhlenbeck formalism following previous work on the NJL model, like in [13]. In our previous work [14], we numerically investigated the model. In particular, we showed the finite momentum effects on the fluctuations, namely, that bound states reappear when their momenta relative to the thermal system exceed the medium-dependent Mott momentum and that Landau-damping modes exist at lowest energies. The latter contribute substantially to the thermodynamics, sometimes being comparable to the mean fields. It is worth noting that the back-reaction, i.e., the effect of the fluctuations back to the mean field, is not considered in this framework. A self-consistent treatment of the thermodynamics is therefore required. There are a few examples [15–17] on how to include the effect of the correlations back to the mean field. Here, we will focus on the Φ-derivable approach to quantum field theory [18], which contains the generalized version of the Beth–Uhlenbeck one for a special choice of the Φ functional. We will demonstrate that the suppression of thermodynamic contributions from low-lying Landau-damping modes is related to the back-reaction effect. The back-reaction accounts for the partial inclusion of correlation effects in the mean-field description via medium-dependent self-energies, thus reducing the remaining explicit correlation contribution. In this article, we will compare the thermodynamical quantity entropy density for the standard (BU) and the generalized Beth–Uhlenbeck (gBU) approach. We also present the entropy fraction of bound excitons and constituent fermions as a measure for the medium dependence of the composition, comparable to the ionization degree discussed for nuclear [19,20] and electromagnetic [21,22] plasmas.

The article is structured as follows. In the Section 2, we will introduce the model and very briefly present the method of mean field approximation along with Gaussian fluctuations. The Section 3 after that sketches the shortcomings of this method and makes a case for the generalized Beth–Uhlenbeck approach. The Section 4 provides some numerical results for the model introduced in the Section 2 to illustrate the point. Finally, we summarize and discuss the implications and future work.

2. Model and Methodology

The Lagrangian for the (2+1)-dimensional Gross–Neveu model consists of the free Dirac part with a four-fermion interaction of the form $(\bar{\psi}\Gamma_i\psi)^2$, where the vertices Γ_i reflect internal symmetries in different channels, typically a combination of Dirac gamma matrices. In the Euclidean signature, the Lagrangian,

$$\mathcal{L} = \bar{\psi}(\gamma_\nu\partial_\nu + m_0 - \gamma_3\mu)\psi - \sum_{i=1}^{4}\frac{G}{2N}(\bar{\psi}\Gamma_i\psi)^2, \quad (1)$$

where m_0 is the bare mass and μ is the chemical potential. The specific interaction channels that we consider are inspired by effective models for graphene. In particular, [23] discusses different symmetry channels available to graphene-like systems. In [12], the authors chose the interaction channels $\Gamma_i = \{I, \gamma_{45}, i\gamma_5, i\gamma_4\}$, which we used in our previous work [14] as well as in this article. The coupling constants are taken to be the same constant G. It is possible to consider different coupling constants for different channels, which results in the occurrence of non-zero condensates in different channels (different phases) in the vacuum;

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see [24] for a discussion. With the particular choice of high symmetry in the coupling constants in the present work, we have a non-zero condensate at the mean-field level only in the scalar channel.

To obtain the partition function, we first use a Hubbard–Stratonovich transformation by introducing auxiliary fields, Φ_i.

$$\mathcal{Z} = \int \prod_{i=1}^{4} \mathcal{D}\Phi_i \exp\left\{ -N \int d^3x \sum_{i=1}^{4} \frac{\Phi_i^2}{2G} + N \mathrm{Tr} \ln S^{-1} \right\}, \tag{2}$$

with $S^{-1} = \gamma_\nu \partial_\nu + m_0 - \mu\gamma_3 + \sum_{i=1}^{4} \Gamma_i \Phi_i$. Now, we split the auxiliary fields into a mean field and a fluctuation part, $\Phi_i \to \bar{\Phi}_i + \delta\Phi_i$. The mean field is assumed to be constant over space–time, while the fluctuations are considered small compared to the mean-field values. Then, the partition function can be written up to the second order in the expansion of $\delta\Phi_i$ as $\mathcal{Z} = \mathcal{Z}_{\mathrm{mf}} \mathcal{Z}_{\mathrm{fl}}$, where the mean-field part is

$$\mathcal{Z}_{\mathrm{mf}} = \exp\left\{ -N \int d^3x \sum_{i=1}^{4} \frac{\bar{\Phi}_i^2}{2G} - N \mathrm{Tr} \ln S_{\mathrm{mf}}^{-1} \right\}, \tag{3}$$

with $S_{\mathrm{mf}}^{-1} = S^{-1}(\Phi_i \to \bar{\Phi}_i)$. The fluctuation part is

$$\mathcal{Z}_{\mathrm{fl}} = \int \prod_{i=1}^{4} \mathcal{D}\delta\Phi_i \exp\left\{ -\sum_{i=1}^{4} \frac{\delta\Phi_i^2}{2} \int d^3x \left(\frac{N}{G} + 2N \mathrm{Tr} \ln(1 + S_{\mathrm{mf}}\Sigma) \right) \right\}, \tag{4}$$

where we introduced the notation $\Sigma = \sum_{i=1}^{4} \Gamma_i \delta\Phi_i$. In the mean field approximation, we first ignore the fluctuation term $\mathcal{Z}_{\mathrm{fl}}$ and find the mean-field value by extremizing the thermodynamic potential $\Omega_{\mathrm{mf}} = -(1/V) \ln \mathcal{Z}_{\mathrm{mf}}$, where V is the volume ($V = \beta\ell^2$, β is the inverse temperature and ℓ is the length scale). This results in the gap equation, $\delta\Omega_{\mathrm{mf}}/\delta\Phi_i = 0$. The fluctuations are calculated with the mean-field values obtained in the previous step. In [13], the authors suggest considering a generalized version of the gap equations where contributions from the fluctuations are also included in the determination of gaps

$$\frac{\partial(\Omega_{\mathrm{mf}} + \Omega_{\mathrm{fl}})}{\partial\Phi_i} = 0. \tag{5}$$

However, in the present work, we restrict ourselves to the consideration of the former case only.

The fluctuation term is simplified further by expanding the log term in (4) as $\mathrm{Tr} \ln(1 + S_{\mathrm{mf}}\Sigma) \approx \mathrm{Tr}(S_{\mathrm{mf}}\Sigma) - \frac{1}{2}\mathrm{Tr}(S_{\mathrm{mf}}\Sigma S_{\mathrm{mf}}\Sigma)$. The linear term vanishes due to the gap equations, and we can evaluate the Gaussian integral to obtain,

$$\Omega_{\mathrm{fl}} = -\frac{1}{V} \ln \mathcal{Z}_{\mathrm{fl}} = \frac{1}{2} \sum_{i=1}^{4} \mathrm{Tr} \ln\left(\frac{N}{G} - \Pi_i \right), \tag{6}$$

where we have defined the polarization function $\Pi_i = -\frac{N}{V} \mathrm{Tr}(S_{\mathrm{mf}}\Gamma_i S_{\mathrm{mf}}\Gamma_i)$.

The expression can be cast into the Beth–Uhlenbeck form; see, for example, the case of the NJL model in [25],

$$\Omega_{\mathrm{fl}} = -\sum_i \int \frac{d^2q}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} g(\omega) \delta_i(\omega, q), \tag{7}$$

where the phase shift is $\delta_i(\omega, q) = \mathrm{Im} \ln\left(\frac{N}{G} - \Pi_i(\omega + i\eta, q) \right)$ and $g(\omega) = (e^{\beta\omega} - 1)^{-1}$ is the Bose distribution function.

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The entropy density, which is the temperature derivative of the pressure ($\mathcal{P} = -\Omega$), can be derived to have the form,

$$\mathcal{S}_{\mathrm{fl,\,BU}} = \sum_i \int \frac{d^2q}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\partial g(\omega)}{\partial T} \delta_i(\omega, q) \,. \tag{8}$$

3. Generalized Beth–Uhlenbeck Approach

The phase shift defined in the previous section satisfies Levinson's theorem [26]. It is zero in the absence of scattering states and exhibits a discontinuous jump by π when a bound state, the lowest excitation, appears. It remains constant up to the two-particle threshold ($2m$), beyond which scattering states contribute, and the phase shift decreases smoothly to zero as the correlations die down at larger frequencies. In the scalar channel, no bound state is present, so the phase shift remains zero below the threshold and becomes non-zero only in the continuum region above it. However, at a finite momentum, we see the appearance of a non-trivial bump in the space-like region ($\omega < q$) also known as the Landau-damping region; see figure 8(a) in [14] for an illustration. The small phase shifts in this region pick up disproportionately large contributions to the thermodynamical quantities due to the divergence of the bosonic distribution function at small energies (see Equation (7)). As noted in our previous work, as well as in the case of the Polyakov-loop-coupled NJL model in [27], these contributions can be comparable to the mean field values. It is worrying if the terms that we obtained by expanding around the mean-field compare or even surpass the mean-field contribution, as it questions the very validity of such an expansion scheme.

The significant contributions to thermodynamics come from the lower frequencies of the correlated states. It should be noted that the composite states are constructed from fermionic degrees of freedom, and at such low energies, well below the threshold, treating the correlation as a strictly elementary boson might be inadequate and risks overcounting. Some parts of the correlations should be treated as corrections to the mean field. This effect of the correlation back to the mean field is also known as back-reaction. There are several attempts to quantify such effects for the case of the NJL model and its variants. For example, see [15,16,28,29], etc. However, in this article, we will focus on the correlations. To properly account for the thermodynamic contribution by the correlations, it is necessary to use a self-consistent treatment of the theory. One such treatment is the Φ-derivable approach [18]. Within this approach, the thermodynamic potential is constructed such that it remains stationary under variations in the propagators, ensuring conservation laws and consistency. The standard Beth–Uhlenbeck expression originates from the $\mathrm{Tr}\ln$ term (6), which encodes two-particle scattering states through the phase shift. In the Φ-derivable approach, additional contributions arise from the Φ functional through self-energy terms associated with two-particle correlations in the medium. These contributions are partially already included in the dressed propagators, and a self-consistent separation of mean-field and correlation effects modifies the phase-shift representation of the entropy density; see for a derivation [30],

$$\mathcal{S}_{\mathrm{fl,\,gBU}} = \sum_i \int \frac{d^2q}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\partial g(\omega)}{\partial T} \left(\delta_i(\omega, q) - \frac{\sin(2\delta_i(\omega, q))}{2} \right) . \tag{9}$$

When compared to (8), this expression has an additional term in the integrand. The additional term $-\sin(2\delta)/2$ accounts for the overlap between explicit correlation contributions and self-energy effects already encoded in the dressed propagators. For small phase shifts, i.e., the scattering states and the soft Landau-damping modes, this correction strongly suppresses their contribution. Near the bound state pole, the term vanishes, preserving all the effects of the bound state contributions. We illustrate this in Figure 1, by plotting the phase shift (δ) and the generalized phase shift ($\delta - \sin(2\delta)/2$).

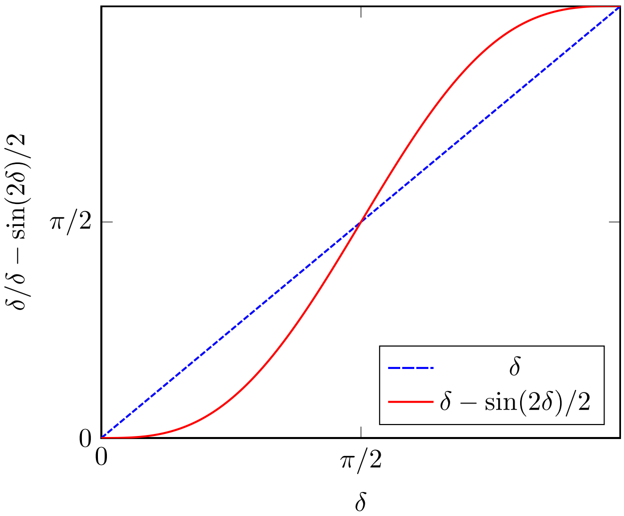

Figure 1. Comparison of the contribution coming from the phase shift in the Beth–Uhlenbeck approach and in the generalized Beth–Uhlenbeck approach. The generalized version suppresses the small phase shift and enhances the phases near π.

4. Results

For the numerical calculations, we use the model parameters from our previous work in [14]. The renormalized coupling constant (g) is defined as $\frac{1}{g} = \frac{1}{G} - \frac{\Lambda}{\pi}$. The mass scale defined by $M = \frac{\pi}{|g|}$ sets the scale of the theory. We use this scale for all the figures presented in this article. The cutoff used in this article is $\Lambda = 5M$. Finally, the free mass parameter m_0 is taken such that $\kappa = m_0/G = 0.046M^2$, which gives the vacuum value of pseudoscalar mass of $0.1M$. It is also possible to consider the chiral limit $m_0 = 0$. However, results for this case are presented here. The numerical code used in our work is publicly available [31] and can be used to obtain the results in the chiral limit. Note that in Equation (8) the momentum q is the momentum of the collective degrees of freedom and not of the constituent fermions. So the cutoff required for the integral to be regulated may differ from the cutoff Λ used as the regulator for the fermions. We follow in this article the convention from [27] and take the cutoff in the range $[\Lambda, 2\Lambda]$.

In Figure 2 we show the entropy density fluctuations in the Beth–Uhlenbeck (blue) (using Equation (8)) and the generalized Beth–Uhlenbeck (red) (using Equation (9)) approaches for scalar (b) and pseudoscalar channels (a). Both figures show a band for the entropy density, corresponding to the correlation cutoff between Λ and 2Λ. The lower line of the band represents the lower cutoff, and the upper one represents the larger cutoff. We note that the contribution to the fluctuation is significantly higher in the Beth–Uhlenbeck approach. In the pseudoscalar channel, bound states exist at low temperatures, which dissolve into the continuum at and above the Mott temperature. Since the generalized Beth–Uhlenbeck approach does not suppress genuine bound-state contributions, the fluctuation entropy obtained in both approaches agrees in the low-temperature regime.

In Figure 3, we compare the total entropy fluctuation density for both approaches. Once again, we note that the difference is minimal at small temperatures and also in the high temperature limit, where all the fluctuations die out, leaving only the mean-field part. Only in the intermediate region of temperatures ($T \sim M$), these two methods produce dissimilar results. It becomes clearer if one considers the compositions (Figure 4), i.e., the fraction of entropy (S/S_{tot}) carried by each individual degree of freedom. The pseudoscalar collective particles dominate at low temperature, while the constituent fermion species have the dominant contribution to the entropy as the collective modes melt at increasing temperatures. The scalar modes only have a significant contribution slightly below the Mott temperature. It is instructive to compare this result with the ionization degree of strongly correlated fermionic systems undergoing the Mott transition. For example, see [32–34] for two-dimensional materials, and [35] for the case of nuclear matter. The papers show the ionization degree, i.e., the fraction of free electrons in the mixture of free electrons and bound excitons, as a function of density. Below and above the Mott transition, the excitons and the free electrons dominate, respectively. A sharp transition between the two occurs at the Mott transition density. The generalized Beth–Uhlenbeck approach shows a sharper transition of degrees of freedom. It is worth noting that in realistic excitonic systems, the ionization degree remains finite even at very low densities and approaches zero only at a finite density before the Mott transition sets in. In contrast, GN- and NJL-type models effectively incorporate a confinement-like mechanism. As a result, at very low temperatures, only the bound excitons exist as relevant degrees of freedom, which translates to zero ionization degree. This qualitative difference reflects the limitations of contact-interaction models in capturing the physics of dilute exciton gases at extremely low densities.

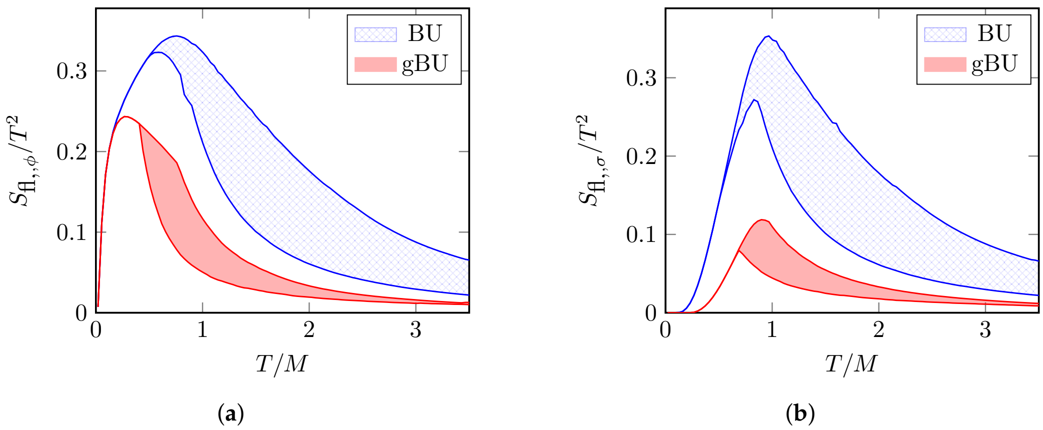

Figure 2. Entropy density fluctuations for pseudoscalar (**a**) and scalar (**b**) channels in the Beth–Uhlenbeck (blue) and the generalized Beth–Uhlenbeck (red) approaches.

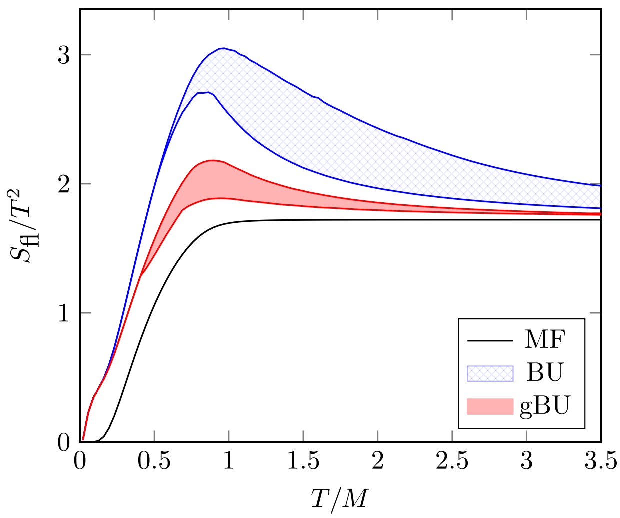

Figure 3. Total normalized (dimensionless) entropy density for the case of Beth–Uhlenbeck (blue) and generalized Beth–Uhlenbeck formalism (red).

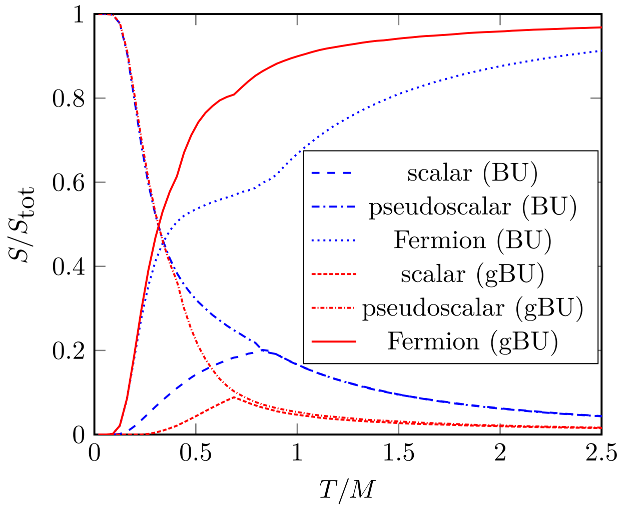

Figure 4. Fraction of the total entropy carried in BU (blue) and gBU (red) by the scalar (loosely dashed and tightly dashed lines), pseudoscalar channels (loosely dash-dotted and tightly dash-dotted lines), and the constituent fermions (solid and dotted lines). The figure is shown for the collective mode cutoff of Λ. For the other cutoffs, the qualitative feature of the figure remains unchanged.

5. Discussion

We have computed and compared the entropy density of the fluctuations in the Beth–Uhlenbeck approach and its generalized version. The contribution from the fluctuations (in BU) is significant, which warranted us to include back-reaction and to look for a self-consistent treatment. The generalized Beth–Uhlenbeck formula, derived from the Φ-derivable approach, includes the back-reaction of the two-particle correlations on the mean-field. By subtracting the term $\frac{1}{2}\sin(2\delta_i)$, it strongly suppresses the contribution from the continuum states, while preserving the thermodynamics of the bound excitons. The result is a significant reduction in the entropy fluctuations at intermediate temperatures $T \sim M$. Low and high temperature regimes are largely unaffected. Consequently, it shows a sharper crossover from bound excitons to free fermions, which is a signature of Mott-like dissociation of the bound state to the plasma of constituent fermions. This behavior mirrors the ionization degree observed in theoretical and experimental studies of excitons in two-dimensional materials.

The article demonstrates that the generalized Beth–Uhlenbeck formulation offers a thermodynamically more consistent description of strongly correlated fermionic systems. Nevertheless, there are a few limitations that should be addressed. The effect of back-reaction on correcting the mean-field needs to be addressed. The cutoff dependence for the collective mode integral persists, although it is noticeably weaker in the gBU case. Usage of a cutoff scheme other than the 3D sharp cutoff used here, or usage of non-local coupling, can remove such dependence. Finally, the analysis was carried out at zero chemical potential. The inclusion of finite chemical potentials should shed light on density-driven Mott-dissociation physics due to phase space occupation (Pauli blocking) as well.

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Data Availability Statement: The data used to plot the figures in the paper is produced using the code authored by B.M., which is publicly available at https://github.com/biplab37/PhaseGN (accessed on 3 April 2026).

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Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

GN	Gross–Neveu
NJL	Nambu–Jona-Lasinio
QCD	Quantum Chromodynamics
BU	Beth–Uhlenbeck
gBU	generalized Beth–Uhlenbeck
NCN	Narodowe Centrum Nauki

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