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Review

A Solution of the Scalar Nonet Mass Puzzle

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Abstract

We present a short review dedicated to low-lying meson states. We present all meson nonets, which consist from up, down and strange light quarks. We consider the scalar nonet as a basic nonet. We work in the framework of the massless Nambu–Jona-Lasinio $U_R(3) \times U_L(3)$ quark model. The collective meson states are described through initially bare quark–antiquark pairs, whose condensates lead simultaneously to spontaneous breaking of the chiral and the flavour symmetry. After quantisation and the spontaneous breaking of the chiral symmetry, when quarks obtain constituent nonzero masses, they become dressed. We present an explanation of the inverse mass hierarchy of the low-lying nonet of the scalar mesons. The proposed explanation is based on symmetry principles. It is shown that, due to the flavour symmetry breaking, two isodoublets of $K_0^*(700)$ mesons play the role of Goldstone bosons. It is also proven that there exists a solution with almost degenerate masses of the $a_0(980)$ and $f_0(980)$ mesons and a zero mass of the $f_0(500)$ meson. Short description of the physical properties of other meson nonets is provided. In particular unique mass relations among the different nonets, which are experimentally confirmed, are presented.

Keywords: low-lying meson nonets; NJL model; scalar nonet mass puzzle

1. Introduction

In this review, we will provide a basic description of all low-lying meson resonances. Until now, there has not been an appropriate unified description of these states. During the last 50 years, numerous models [1] have been proposed for the description of these resonances including, for example, the bag model [2,3], exotic multi-quark states [4], hybrid gluon-quark states [5], molecular states [6], etc. Our approach is based on the assumption of the quark–antiquark structure of mesons. We will use the Nambu–Jona-Lasinio (NJL) model [7,8] for the description of these states. The NJL model was proposed analogously to Bogoljubov's mechanism for symmetry breaking and the quasi-particles method [9], which was developed earlier for superfluidity [10] and superconductivity [11,12]. Initially, this model was proposed for the description of the strong interacting hadrons: mesons and baryons. In our approach, we will use quarks instead of hadrons. Such an approach simultaneously allows one to follow the spontaneous breaking of physical symmetries and to describe all possible low-lying hadrons and, in particular, mesons.

As shown in the work of Nambu and Jona-Lasinio, the effective four fermion interaction can lead to spontaneous symmetry breaking and to excitation of new degrees of

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freedom—collective excited modes. The latter is the milestone of our approach. We have extended the NJL model on the basis of completeness and symmetry principles.

Quantum Chromodynamics (QCD) is the microscopic theory of strong interactions [13,14], which describes the interactions of quarks via the exchange of vector gauge particles—the gluons. This theory has an asymptotic freedom property [15,16], as a result of which, the interactions of quarks at high energies decrease. Hence, perturbation theory can be applied. However, at low energies, the coupling constant becomes unacceptably large, and perturbation theory cannot provide a quantitative description of the composite hadronic states—the resonances. Therefore, different phenomenological models are used, which include, to a certain extent, the basic properties of the QCD, such as chiral symmetry and its breaking, which leads to appearance of nonzero vacuum expectation values for different combinations of quark and gluon fields. One such model is the NJL model, which is successfully used for the description of hadron physics and for the mechanism of the spontaneous breaking of the chiral symmetry [17–21]. The NJL model has been used also in many other physics applications, for example, pion generalised parton distributions (GPD) at zero skewness [22], medium effects on the form factor of the ρ meson [23], the GPD nucleon in the NJL model [24,25], and so forth. A main feature of this model is that it is based on the chiral symmetry and explains its breaking in particle physics, analogously to superconductivity. The excitations in the superconductor can be described by the coherent mixture of electrons and holes. Similarly, the explanation of the meson states spectra can be provided in the framework of quark–antiquark degrees of freedom.

Another important characteristic of the NJL model is that it is based on special relativistic invariance and includes the Lorentz group as the symmetry group of hadronic excitations. Spin 1 excitations appear from the bilinear combinations of spinor fields with spin $1/2$ [26–28]. Besides these vector and axial–vector excitations, there appear pseudoscalar and scalar excitations, the latter ones needed for the chiral symmetry breaking.

In *relativistic* theory, the fundamental spinors with spin $1/2$, which are the building blocks of the representations with high spin, can be of two types: *right* and *left*. They, in addition to their *non-relativistic* quantum mechanical description, have a new quantum number *chirality* $\chi = \pm 1/2$, correspondingly. They are the fundamental spinors of two *different* compact groups of three-dimensional rotations $O(3)$. Their direct product is isomorphic to the non-compact Lorentz group $O(3,1)$. The representations of these groups are connected through the transformation of parity $\mathcal{P}$, which explains the notation of fundamental spinors: *right* and *left*.

The Weyl spinor ψ_α and its conjugated $\psi^*_\alpha \equiv \psi_{\dot{\alpha}}$ transform under the fundamental spinor representations of Lorentz groups $(1/2, 0)$ and $(0, 1/2)$, correspondingly. These first rank spinors describe particles with spin $1/2$. The spinor dotted $\dot{\alpha}$ and undotted α indices can take values 1 and 2. Weyl spinors with different types of indices are connected by $\mathcal{P}$-transformation of spatial reflection or $\mathcal{C}$-transformation of charge conjugation, and they are not separately invariant towards these transformations.

Spinors of higher ranks may be constructed using the fundamental Weyl spinors [29]. Any combination of Weyl spinors is reducible, because the spinor algebra contains invariant antisymmetric spinors $\epsilon^{\alpha\beta}$ and $\epsilon^{\dot{\alpha}\dot{\beta}}$ with undotted and dotted indices. Irreducible spinors of higher rank can be constructed by multiplying the symmetric combination of Weyl spinors with undotted indices and the symmetric combination of conjugated Weyl spinors with dotted indices. Different types of irreducible combinations of n-undotted and m-dotted fundamental Weyl spinors transform under the representations of Lorentz group $(n/2, m/2)$, and they describe particles with spin $j = (n+m)/2$ and chirality $\chi = (n-m)/2$.

The parity is conserved in strong interactions, and therefore, it is convenient to choose as spinor fields the Dirac spinors $\Psi\ (1/2, 0) \oplus (0, 1/2)$, which describe the quark states.

The conservation of the charge conjugation and the QCD colours in the strong interactions chooses among linear combinations only the quark–antiquark channels of excitation $\overline{\Psi}\mathcal{O}\Psi$, where $\mathcal{O}$ are the all possible combinations of the Dirac gamma matrices.

Today six types (flavours) of quarks are known. In 1995, the heaviest quark was discovered: the top quark [30,31]. It is the heaviest elementary particle, $m_t = 172.56 \pm 0.31$ GeV, found until now [32]. Its mean lifetime is too short to be able to produce composite hadronic states. Nevertheless, recently, the CMS [33] and ATLAS [34] collaborations have observed such hadronic states: top–antitop quark states. The mass term of the top quark appears from its interaction with the Higgs field, h, in particular, with its vacuum expectation value v:

$$\mathcal{L}_m = g_t\, \bar{t}\, \frac{v+h}{\sqrt{2}} t = m_t\, (\overline{t_R} t_L + \overline{t_L} t_R) + \frac{m_t}{v}\, h \cdot \bar{t}t. \tag{1}$$

In correspondence with the Standard model, the mass of the particles, which is obtained from interactions with Higgs $m_t = g_t v/\sqrt{2}$ is proportional to the Yukawa constant g_t, and it is called the *current* mass. Hence, the top quark has the strongest Yukawa interaction with Higgs. As is well known, the mass term (the term inside brackets in Equation (1)) breaks the right–left symmetry. Therefore, we are not going to consider in this review particles with heavy *current* masses.

The next massive quark is the bottom quark, discovered in 1977 [35]. The bottom and top quarks were predicted theoretically in the paper of Kobayashi and Maskawa [36], which explained CP-violation. CP-violation was discovered in 1964 [37]. The mass of the bottom quark is 4.183 ± 0.007 GeV [32].

The next lighter quark is the charm quark. It was theoretically predicted by Glashow, Iliopoulos and Maiani (GIM mechanism) [38]. Its mass was estimated in 1974 [39], several months before the discovery of the charm quark itself [40,41]. Its mass $m_c = 1.2730 \pm 0.0046$ GeV is also above the hadronic scale of order 1 GeV, and we do not consider this flavour in low-lying meson physics.

In cosmic rays, strange particles were detected, which were explained later as consisting of strange quarks. On the basis of all data, Gell-Mann [42] and Zweig [43,44] constructed a quark model including up, down and strange quarks. The masses of these particles are of the order of several MeV for up and down quarks and hundreds of MeV for the strange quark. Such small masses, as those of up and down quarks, cannot explain the masses of the nucleons, neutrons and protons, which consist of these quarks. Therefore, the Brout–Englert–Higgs mechanism [45,46], accepted in Particle Physics for the explanation of particle masses, is unable to explain the masses of the nuclei of the chemical elements of our Universe. Fortunately, Nambu has proposed a method of spontaneous dynamical symmetry breaking for the light quarks [47], which can explain the large (compared to the hadronic scale) *constituent* masses of the hadrons and, at the same time, can explain the smallness of the masses of the pseudoscalar mesons—the pions.

In the next section, we describe all low-lying mesons nonets. Section 3 introduces the NJL model for the scalar meson nonet and presents its effective potential. In Section 4, the spontaneous symmetry breaking mechanism of chiral and flavour symmetries is described, and the mass spectrum of the scalar mesons is obtained. A special solution, corresponding to the physical case of inverse hierarchy of scalar meson masses, is presented. In Section 5, we present a short description of the other meson nonets and provide the unique mass relations for the spin 1 mesons. The last section contains the conclusions of the review.

2. Meson Nonets

We use the three lightest quark flavours in the framework of the "Eightfold way" [48,49]. However, the fundamental spinor Ψ, aside from the flavour indices

$f = 1, \ldots, N_f$, $N_f = 3$, will have also colour indices [50,51]. The colour group $SU_C(N_C)$, where $N_C = 3$, presents the local symmetry group of the strong interactions of the QCD, and it is realised via the interactions of quarks with the gauge fields, the gluons. We will omit the colour index C in the bilinear combinations $\overline{\Psi^C}\mathcal{O}\Psi_C$, because the observed hadronic states are colourless as a result of the confinement of the colour states in QCD. The parameter of colour will explicitly appear only at the integration of the fundamental spinors over virtual loops. Hence, we will work with the $SU_f(3)$ flavour group. The antiquark–quark presentations there are defined in accordance with Equation (2)

$$\bar{\mathbf{3}} \times \mathbf{3} = \mathbf{8} + \mathbf{1}. \tag{2}$$

Then, the antiquark–quark states of the mesons are formed as an octet and a singlet of this flavour group; i.e., they form nonets, which are presented on the two-dimensional $I_3 - S$ plots in Figures 1–3, where the coordinates present the quantum numbers of the third isospin projection I_3 and the strangeness S.

The quark states are described by four component Dirac spinors. Therefore, their direct products contain 16 independent components in the antiquark–quark channel:

$$[(1/2,0) \oplus (0,1/2)] \otimes [(1/2,0) \oplus (0,1/2)] = (0,0) \oplus (0,0) \oplus (1/2,1/2) \oplus (1/2,1/2) \oplus (1,0) \oplus (0,1). \tag{3}$$

Although Nambu and Jona-Lasinio in their second paper [8] consider chiral invariant tensor–tensor interactions in the case of isotopical symmetry with two flavours, they could not include this interaction in their first work for one flavour [7]. The reason for this is the fact that, in the case of one flavour, it is impossible to construct a chiral invariant tensor–tensor interaction without derivatives. For the first time, the chiral invariant tensor–tensor interaction was introduced in the model NJL with one flavour in [52]. In the case of three flavours, the chiral invariant tensor–tensor interaction explains the presence of two additional nonets (Figure 3), which were considered in the literature as excited states. This very interaction is used in our model in contrast to the NJL model.

Under the Lorentz group, 16 independent components from the right-hand side of Equation (3) transform as a scalar (S) (the left panel in Figure 1), pseudoscalar (P) (the right panel in Figure 1), vector (V) (the left panel in Figure 2), axial–vector (A) (the right panel in Figure 2) and antisymmetric tensor of second rank $(R$ and $B)$ (the left and right panels in Figure 3), correspondingly. Their quantum numbers, according to the Lorentz transformations, are described in Table 1. Each column in the table corresponds to a singlet flavour state of a meson nonet, except for the two columns with the same quantum numbers 1^{--}.

Table 1. Quantum numbers of singlet flavour states of meson nonets.

$\overline{\Psi}\mathcal{O}\Psi$	$\overline{\Psi}\Psi$ (S)	$\overline{\Psi}\gamma^5\Psi$ (P)	$\overline{\Psi}\gamma_\mu\Psi$ (V)	$\overline{\Psi}\gamma_\mu\gamma^5\Psi$ (A)	$\overline{\Psi}\sigma_{0i}\Psi(R)$	$\overline{\Psi}\sigma_{ij}\Psi$ (B)
J^{PC}	0^{++}	0^{-+}	1^{--}	1^{++}	1^{--}	1^{+-}

Here Greek index $\mu = 0,1,2,3$ and Latin indexes $i,j = 1,2,3$.

Due to the fact that quantum numbers of the states $\overline{\Psi}\gamma_\mu\Psi$ and $\overline{\Psi}\sigma_{0i}\Psi$ coincide, their physical states $\omega(782)/\phi(1020)$ and $\omega' = \omega(1420)/\phi' = \phi(1680)$ are represented by almost maximal mixture between them [52]. Therefore, the left panels in Figures 2 and 3 present physical mixed states of nonets with singlet states $\overline{\Psi}\gamma_\mu\Psi$ and $\overline{\Psi}\sigma_{0i}\Psi$.

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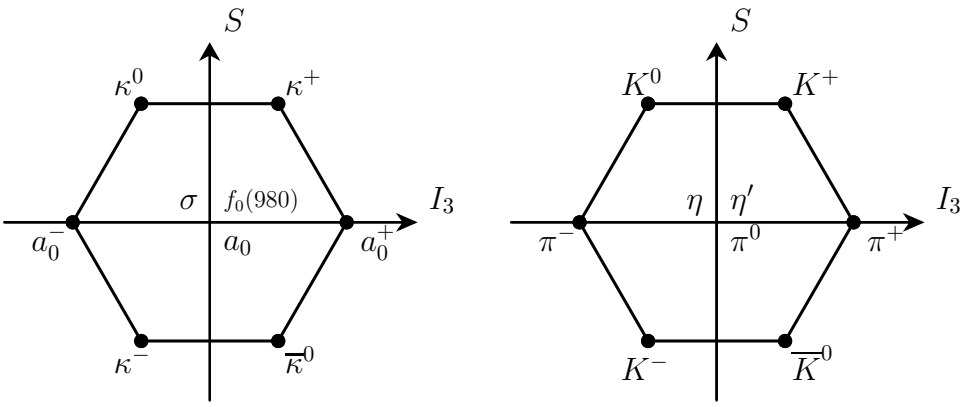


Figure 1. Left panel presents the low-lying scalar meson nonet. Right panel presents the low-lying pseudoscalar meson nonet.

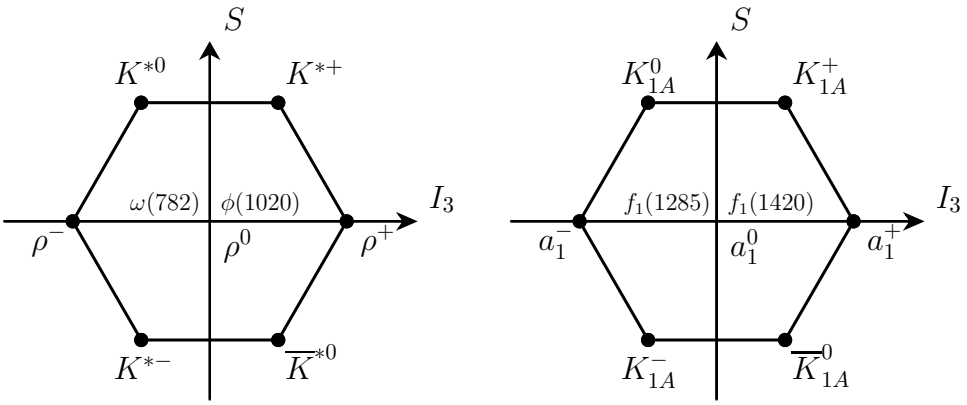


Figure 2. Left panel presents the low-lying vector meson nonet. Right panel presents the low-lying axial–vector meson nonet (A).

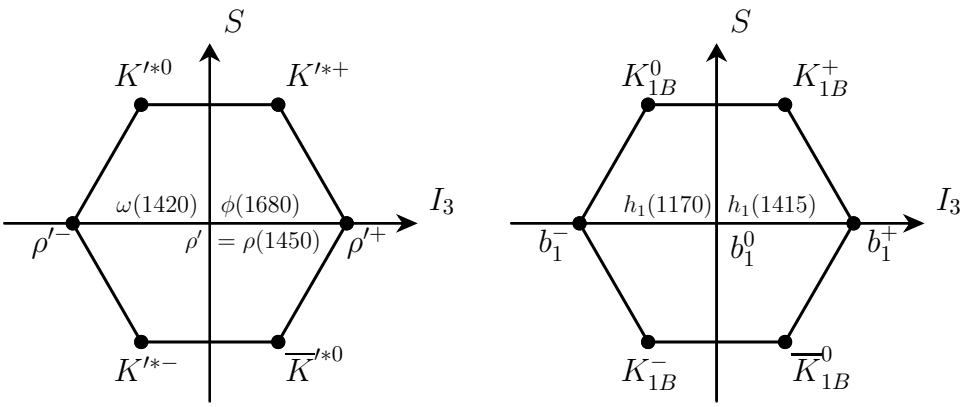


Figure 3. Left panel presents the low-lying vector prime meson nonet. Right panel presents the low-lying axial–vector meson nonet (B).

Only the scalar nonet (left panel in Figure 1) does not have mixing with any other nonets. Further on in this paper, we will consider the scalar nonet as the base nonet. However, there exists a problem connected with the explanation of the inverse hierarchy mass in the low-lying scalar nonet. The constituent quark models, which describe the mesons as quark–antiquark pairs, cannot explain this problem [53–59]. Among the first hypotheses explaining the inverse hierarchy is the work of Jaffe [4] proposing the diquark–antidiquark structure ($qq\bar{q}\bar{q}$) of the scalar mesons. Another explanation, preserving the quark-antiquark structure of scalar mesons, see [60], uses the $U_A(1)$ anomaly term in dynamical chiral symmetry breaking of chiral effective theories. While this mechanism

explains the difference between lowest-lying scalar meson $f_0(500)$ (or σ) and $a_0(980)$, it is not able to explain the mass of the $K_0^*(700)$ (or κ) meson [61].

Here, we discuss an alternative explanation of the smallness of the $K_0^*(700)$ meson mass proposed in our work [62]. It uses the well-known mechanism of spontaneous symmetry breaking of the $U_f(3)$ flavour symmetry to $SU_f(2)$ isotopic symmetry. Due to this symmetry breaking, $K_0^*(700)$ mesons with isospin $I = 1/2$ play the role of massless Bogoliubov–Nambu–Goldstone (BNG) bosons. This is a natural explanation of the small masses of the $K_0^*(700)$ mesons, similar to the explanation of the small pion masses.

3. The Scalar Nonet Model

Let us consider the massless three flavour states $\Psi = (u\ d\ s)^T$ of the flavour group $U_f(3)$. Due to the fact that the states are massless, this group can be extended to $U_R(3) \times U_L(3)$. The Lagrangian of such interactions corresponding to the NJL model can be expressed as

$$\mathcal{L} = i\overline{\Psi}\partial\!\!\!/\,\Psi + \frac{G_0}{2}\left(\overline{\Psi}\,\Psi\right)^2 + \frac{\widetilde{G}_0}{2}\sum_{a=1}^{8}\left(\overline{\Psi}\,\lambda_a\Psi\right)^2, \tag{4}$$

where λ_a $(a = 1, \ldots, 8)$ are the Gell-Mann matrices. G_0 and $\widetilde{G}_0$ are *positive* constants of the self-interactions of the singlet state quark scalar current and the octet states of the quark scalar currents, correspondingly. They both have a dimension $[mass]^{-2}$. We discuss initially massless quarks, which obtain masses due to spontaneous breaking of the chiral symmetry.

The quantisation of this Lagrangian in perturbation theory on the dimensional constants G_0 and $\widetilde{G}_0$ leads to nonrenormalisable theory. To resolve this problem, we will use the linearisation method of this Lagrangian developed in [26,27]. As a result of the linearisation of the Lagrangian (Equation (4)) the following, equivalent on the classical level Lagrangian, is obtained:

$$\mathcal{L} = i\overline{\Psi}\partial\!\!\!/\,\Psi + g_0\overline{\Psi}\,\Psi\,S_0 - \frac{g_0^2}{2G_0}\,S_0^2 + \sum_{a=1}^{8}\left(g_a\overline{\Psi}\,\lambda_a\Psi\,S_a - \frac{g_a^2}{2\widetilde{G}_0}\,S_a^2\right), \tag{5}$$

where auxiliary fields are introduced:

$$S_0 = \frac{G_0}{g_0}\,\overline{\Psi}\Psi \quad \text{and} \quad S_a = \frac{\widetilde{G}_0}{g_a}\,\overline{\Psi}\,\lambda_a\Psi. \tag{6}$$

These fields play the role of collective excitation states of the corresponding currents. Here, g_0 and g_a are the dimensionless Yukawa coupling constants.

All the collective modes become dynamical due to the self-energy quantum corrections from fermion loops (Figure 4a). After proper normalisation of the kinetic terms of the collective modes, the relations

$$6g_0^2 N_C I_0 = 4g_a^2 N_C I_0 = 1 \tag{7}$$

are obtained. Here,

$$I_0 \equiv -i\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{\left(p^2 - m_0^2\right)^2} = \int\frac{\mathrm{d}^4p_E}{(2\pi)^4}\frac{1}{\left(p_E^2 + m_0^2\right)^2} > 0 \tag{8}$$

is a logarithmically divergent integral, which is positive in Euclidian momentum space.

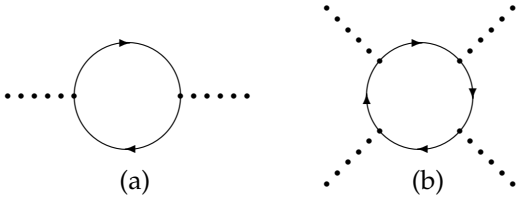


Figure 4. (**a**) Radiative correction to self-energy parts. (**b**) Radiative correction to self-interactions.

The small current mass m_0 for the quark, which is introduced here, allows avoiding the infrared divergences in the denominator, and we will neglect it in the numerator. Such quark mass introduction explicitly breaks the chiral symmetry, and it is called *soft* symmetry breaking. This symmetry breaking does not lead to extra ultraviolet divergences in the scalar particles masses.

Due to the *dynamical origin* of the kinetic terms, all the interactions in the NJL model are described by a single dimensionless coupling constant

$$g \equiv g_0 = \sqrt{2/3}\, g_a. \tag{9}$$

Another essential point of the NJL model is the generation of the self-interactions of the scalar fields (Figure 4b), which lead to a spontaneous dynamical breaking of the chiral and flavour symmetry.

Thus, the effective potential reads as follows:

$$\begin{aligned} V_{\text{eff}} = {} & \tfrac{\mu^2}{2} S_0^2 + \tfrac{\tilde{\mu}^2}{2} \sum_{a=1}^{8} S_a^2 + \tfrac{g^2}{2} S_0^4 + 3g^2 S_0^2 \sum_{a=1}^{8} S_a^2 + \tfrac{3g^2}{4} \left(\sum_{a=1}^{8} S_a^2 \right)^2 \\ & + \tfrac{3\sqrt{3}}{\sqrt{2}} g^2 S_0 S_3 \left(S_4^2 + S_5^2 - S_6^2 - S_7^2 \right) - \sqrt{2} g^2\, S_0 S_8^3 \\ & + 3\sqrt{2} g^2\, S_0 S_8 \left(S_1^2 + S_2^2 + S_3^2 - \tfrac{S_4^2 + S_5^2 + S_6^2 + S_7^2}{2} \right) \\ & + 3\sqrt{6} g^2 S_0 (S_1 S_4 S_6 + S_1 S_5 S_7 - S_2 S_4 S_7 + S_2 S_5 S_6). \end{aligned} \tag{10}$$

This potential has the following essential property: it depends only on one dimensionless constant g^2 and two dimensional mass parameters

$$\begin{aligned} \mu^2 &= g_0^2 / G_0 - 12 g_0^2 N_C I_2, \\ \tilde{\mu}^2 &= g_a^2 / \widetilde{G}_0 - 8 g_a^2 N_C I_2. \end{aligned} \tag{11}$$

Here,

$$I_2 \equiv i \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{p^2 - m_0^2} = \int \frac{\mathrm{d}^4 p_E}{(2\pi)^4} \frac{1}{p_E^2 + m_0^2} > 0 \tag{12}$$

is the positive quadratically divergent integral. The explicit form of the effective potential (10) depending on small number of parameters provides the possibility to find exact solutions of spontaneous symmetry breaking and the mass spectrum of the scalar mesons. We will discuss the solutions in the next section.

4. Symmetry Breaking and Mass Spectrum of Scalar Mesons

As a result of the spontaneous symmetry breaking, the scalar fields obtain nonzero vacuum expectation values. The physical vacuum conserves the electric charge and the quark flavour. Hence, only the scalar fields S_0, S_3 and S_8 can have nonzero vacuum expectation values, because these fields interact with the diagonal combinations of quark–antiquark flavours. We find the minimum of the effective potential (10) differentiating the

potential only on these degrees of freedom. The following system of equations are obtained as a result of the minimisation:

$$\begin{cases} \left.\dfrac{\partial V_{\text{eff}}}{\partial S_0}\right|_{\substack{S_0=\langle S_0\rangle \\ S_3=\langle S_3\rangle \\ S_8=\langle S_8\rangle}} = \mu^2\langle S_0\rangle + 2g^2\langle S_0\rangle^3 + 6g^2\langle S_0\rangle\left(\langle S_3\rangle^2 + \langle S_8\rangle^2\right) \\ \qquad\qquad -\sqrt{2}g^2\langle S_8\rangle^3 + 3\sqrt{2}g^2\langle S_3\rangle^2\langle S_8\rangle = 0, \\ \left.\dfrac{\partial V_{\text{eff}}}{\partial S_3}\right|_{\substack{S_0=\langle S_0\rangle \\ S_3=\langle S_3\rangle \\ S_8=\langle S_8\rangle}} = \left[\tilde{\mu}^2 + 3g^2\left(2\langle S_0\rangle^2 + \langle S_3\rangle^2 + \langle S_8\rangle^2 + 2\sqrt{2}\langle S_0\rangle\langle S_8\rangle\right)\right]\langle S_3\rangle = 0, \\ \left.\dfrac{\partial V_{\text{eff}}}{\partial S_8}\right|_{\substack{S_0=\langle S_0\rangle \\ S_3=\langle S_3\rangle \\ S_8=\langle S_8\rangle}} = \tilde{\mu}^2\langle S_8\rangle + 3g^2\left(2\langle S_0\rangle^2 + \langle S_3\rangle^2 + \langle S_8\rangle^2 - \sqrt{2}\langle S_0\rangle\langle S_8\rangle\right)\langle S_8\rangle \\ \qquad\qquad +3\sqrt{2}g^2\langle S_0\rangle\langle S_3\rangle^2 = 0. \end{cases} \tag{13}$$

Note that, due to the neutrality of the vacuum expectation values, $\langle S_a\rangle = 0$ for $a = 1,2,4,5,6,7$. It is also known that $SU_f(2)$ group cannot be spontaneously broken [63], and $\langle S_3\rangle = 0$ is the right solution of the system (13), while $\langle S_0\rangle \neq 0$ and $\langle S_8\rangle \neq 0$ acquire nonzero vacuum expectation values.

The spontaneous symmetry breaking takes place only at strong coupling constants G_0 and $\widetilde{G}_0$ when the mass parameters are negative, i.e., $\mu^2 < 0$ and $\tilde{\mu}^2 < 0$. We introduce the dimensionless variables $x = 3\sqrt{2}g\langle S_0\rangle/\sqrt{-\tilde{\mu}^2}$, $z = 3g\langle S_8\rangle/\sqrt{-\tilde{\mu}^2}$ and $r^2 = \mu^2/\tilde{\mu}^2$ in order to avoid the irrational coefficients and the dimensional parameters in (13).

Using the linearised Lagrangian (5), the constituent quark masses can be derived as

$$m_u = m_d = -\sqrt{\frac{-\tilde{\mu}^2}{18}}(x+z), \tag{14}$$

$$m_s = -\sqrt{\frac{-\tilde{\mu}^2}{18}}(x-2z). \tag{15}$$

The first and the third equations of the system (13) in the new dimensionless variables read as follows:

$$\begin{cases} x^2 + 6z^2 - 2\dfrac{z^3}{x} = 9r^2, \\ x^2 - xz + z^2 = 3. \end{cases} \tag{16}$$

The square of the masses of the scalar meson isotriplet $a_0^{\pm}(980) = (S_1 \mp iS_2)/\sqrt{2}$, $a_0^0(980) = S_3$ can be found as

$$\mathcal{M}_i^2 = \left.\frac{\partial^2 V_{\text{eff}}}{\partial S_i^2}\right|_{\substack{S_0=\langle S_0\rangle \\ S_8=\langle S_8\rangle}} = 9\sqrt{2}g^2\langle S_0\rangle\langle S_8\rangle = (-\tilde{\mu}^2)\,xz, \text{ where } i = 1,2,3. \tag{17}$$

This result shows that, for non-negative square masses, the vacuum expectation values of S_0 and S_8 should have the same signs. Note that the solutions of system (16) are invariant with respect to the simultaneous sign changing of the vacuum expectation values: $x \to -x$ and $z \to -z$. Therefore, for definiteness, we will search for solutions with positive vacuum expectation values.

The squares of the masses of the scalar mesons κ with isospin $I = 1/2$: $\kappa^{\pm} = (S_4 \mp iS_5)/\sqrt{2}$, $\kappa^0 = (S_6 - iS_7)/\sqrt{2}$, $\bar{\kappa}^0 = (S_6 + iS_7)/\sqrt{2}$,

$$\mathcal{M}_s^2 = \left.\frac{\partial^2 V_{\text{eff}}}{\partial S_s^2}\right|_{\substack{S_0=\langle S_0\rangle \\ S_8=\langle S_8\rangle}} = \tilde{\mu}^2 + 3g^2\left(2\langle S_0\rangle^2 - \sqrt{2}\langle S_0\rangle\langle S_8\rangle + \langle S_8\rangle^2\right) = 0, \tag{18}$$

are equal to zero, where $s = 4, 5, 6, 7$, and the last equality follows from the third equation of the system (13) with $\langle S_3 \rangle = 0$. The fact that the squares of the masses are equal to zero for these states is a direct consequence of Goldstone's theorem [64,65].

Now, let us calculate the squares of the masses of the isosinglet states with isospin $I = 0$, S_0 and S_8 and their mixing:

$$\mathcal{M}_0^2 = \frac{\partial^2 V_{\text{eff}}}{\partial S_0^2}\bigg|_{\substack{S_0=\langle S_0\rangle \\ S_8=\langle S_8\rangle}} = \mu^2 + 6g^2\Big(\langle S_0\rangle^2 + \langle S_8\rangle^2\Big) = (-\tilde{\mu}^2)\,\frac{2}{9}\left(x^2 + \frac{z^3}{x}\right), \tag{19}$$

$$\begin{aligned}\mathcal{M}_8^2 = \frac{\partial^2 V_{\text{eff}}}{\partial S_8^2}\bigg|_{\substack{S_0=\langle S_0\rangle \\ S_8=\langle S_8\rangle}} &= \tilde{\mu}^2 + 3g^2\Big(2\langle S_0\rangle^2 - 2\sqrt{2}\langle S_0\rangle\langle S_8\rangle + 3\langle S_8\rangle^2\Big) \\ &= (-\tilde{\mu}^2)\,\tfrac{z}{3}(2z - x),\end{aligned} \tag{20}$$

$$\mathcal{M}_{08}^2 = \frac{\partial^2 V_{\text{eff}}}{\partial S_0 \partial S_8}\bigg|_{\substack{S_0=\langle S_0\rangle \\ S_8=\langle S_8\rangle}} = 3g^2\Big(4\langle S_0\rangle - 2\sqrt{2}\langle S_8\rangle\Big)\langle S_8\rangle = (-\tilde{\mu}^2)\,\frac{\sqrt{2}\,z}{3}(2x - z) \neq 0. \tag{21}$$

From the last equation, it follows that there exists a non-trivial mixing between these states. This can be described by the non-diagonal matrix:

$$\mathcal{M}_{I=0}^2 = (-\tilde{\mu}^2)\begin{pmatrix} A & B \\ B & C \end{pmatrix}, \tag{22}$$

where $A = \frac{2}{9}\left(x^2 + \frac{z^3}{x}\right)$, $B = \frac{\sqrt{2}}{3}z(2x - z)$, and $C = \frac{1}{3}z(2z - x)$ The diagonalisation of this matrix leads to two eigenvalues:

$$\mathcal{M}_\sigma^2 = (-\tilde{\mu}^2)\,\frac{A + C - \sqrt{(A - C)^2 + 4B^2}}{2}, \tag{23}$$

$$\mathcal{M}_{f_0(980)}^2 = (-\tilde{\mu}^2)\,\frac{A + C + \sqrt{(A - C)^2 + 4B^2}}{2}. \tag{24}$$

They correspond to the squares of the masses of the physical states σ and $f_0(980)$.

Among different solutions of system (16), there exists only one solution that corresponds to the physical reality with proper mass spectrum for the scalar nonet. We introduce a dimensionless parameter $\delta = r^2 - 1$, which shows the deviation of ratio of the mass parameters μ^2 and $\tilde{\mu}^2$ from unity. Then, the solution for the vacuum expectation value $\langle S_0 \rangle$ can be expressed as

$$x^2 = 2 + 2\sqrt{1 - \delta^2}\cos\left(\frac{1}{3}\arccos\frac{1 - 3\delta^2}{\sqrt{(1 - \delta^2)^3}} - \frac{2\pi}{3}\right), \tag{25}$$

for $0 \leq \delta^2 \leq \delta_0^2 = 2\sqrt{3} - 3$. The positive vacuum expectation value $\langle S_8 \rangle$ can be found from the second equation of the system (16) as

$$z = \frac{x + \sqrt{3(4 - x^2)}}{2}. \tag{26}$$

So, the zero masses (18) of the κ mesons result from the flavour violation of the $U_f(3)$ group. The reduced squares of the masses of the meson isotriplet $a_0(980)$ and the two isosinglets σ and $f_0(980)$ are presented in Figure 5.

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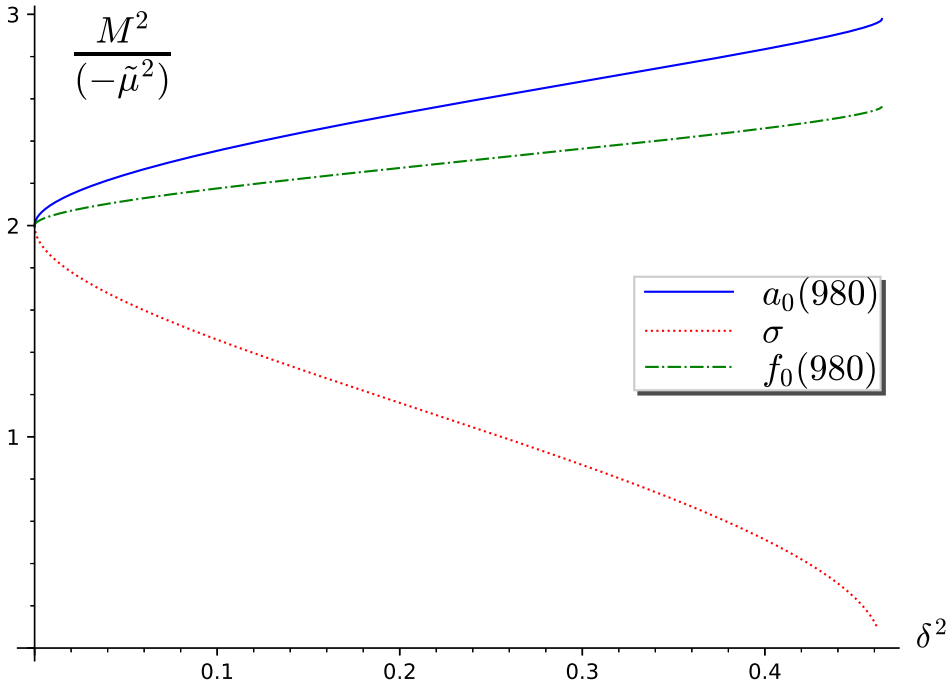


Figure 5. The figure presents the dependence of the reduced squares of masses of σ, $f_0(980)$ and $a_0(980)$ mesons on δ^2 (from [62]).

The solution ((25) and (26)) is more attractive from a physical point of view. Namely, for $\delta^2 \to \delta_0^2$, the masses of the isotriplet $a_0(980)$ and isosinglet $f_0(980)$ are almost degenerate and heavy, while the σ meson mass tends to zero, and it equals zero for $\delta^2 = \delta_0^2$. From the equations of system (16), it follows that the parameter r^2 is larger than one: $r^2 > 1$. So, the mass ratio $\mu^2/\tilde{\mu}^2 = r^2 > 1$. This mass ratio depends on the initial coupling constants of the Lagrangian (4) and means that G_0 is larger than $\frac{2}{3}\widetilde{G}_0$. The expected physical ranges of the free parameters μ^2 and $\tilde{\mu}^2$ can be determined from Figure 5 at the physical point $\delta^2 = \delta_0^2$:

$$\begin{aligned} \sqrt{-\mu^2} &\sim 765 \pm 30 \text{ MeV}, \\ \sqrt{-\tilde{\mu}^2} &\sim 590 \pm 20 \text{ MeV}. \end{aligned} \tag{27}$$

At $\delta_0^2 = 0$, the symmetry $U_f(3)$ is restored, but the chiral symmetry is broken, because at $x = 1, z = 2$: $m_u = m_d = -m_s \neq 0$. This explains the degeneracy of the masses of the three mesons at that point in Figure 5: $\mathcal{M}_{a_0(980)} = \mathcal{M}_{f_0(980)} = \mathcal{M}_\sigma$.

Let us analyse the mixing at $\delta = \delta_0 = \sqrt{2\sqrt{3} - 3}$. The physical states σ and $f_0(980)$ are defined by the states S_0 and S_8 from the following relation:

$$\begin{pmatrix} \sigma \\ f_0 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} S_0 \\ S_8 \end{pmatrix}, \tag{28}$$

where the mixing angle φ is given by

$$\varphi = \frac{1}{2} \arccos \frac{C - A}{\sqrt{(C - A)^2 + 4B^2}}. \tag{29}$$

An illustration of this mixing angle is presented in Figure 6.

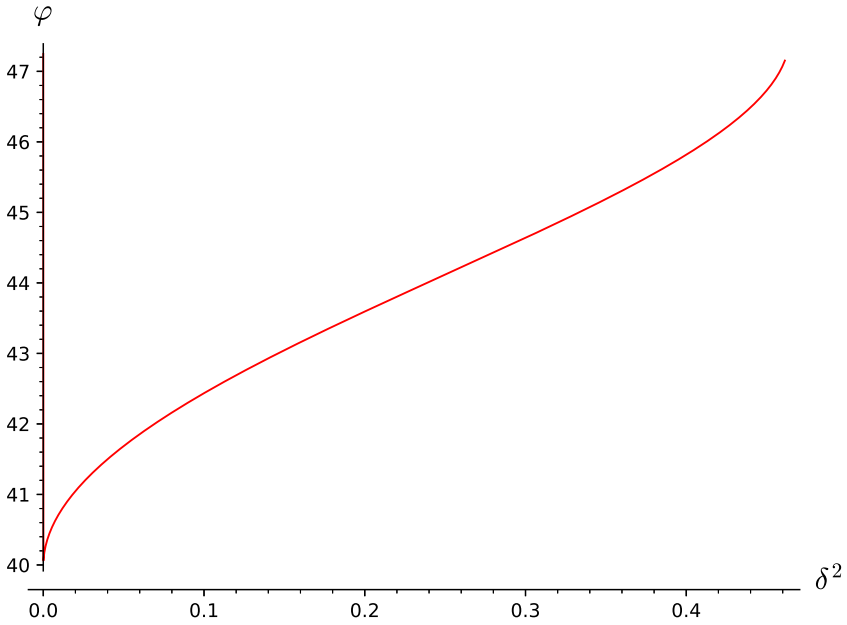


Figure 6. The figure presents the mixing angle φ corresponding to the solution for the physical case (from [62]).

The formula connecting the pure states $(u\bar{u} + d\bar{d})/\sqrt{2}$ and $s\bar{s}$ to the physical states σ and $f_0(980)$ reads as follows:

$$\begin{pmatrix} \sigma \\ f_0 \end{pmatrix} = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} s\bar{s} \\ \frac{u\bar{u}+d\bar{d}}{\sqrt{2}} \end{pmatrix}, \tag{30}$$

where the mixing angle ω is related to the mixing angle φ by the relation

$$\omega = \arctan\sqrt{2} - \varphi. \tag{31}$$

It is interesting to note that $\omega = \omega_0$ at point $\delta = \delta_0$, where $\omega_0 \approx 54.7^\circ - 47.5^\circ = 7.2^\circ$. This very small angle corresponds to a nearly ideal mixing. Therefore, contrary to the common assumption that $f_0(980)$ meson is almost a pure $s\bar{s}$ state, we obtain that it is actually the σ meson that has an almost pure $s\bar{s}$ state, while the $f_0(980)$ meson is close to the $(u\bar{u} + d\bar{d})/\sqrt{2}$ state. This finding has been confirmed by the recent experimental results of the CMS [66] and ALICE [67] Collaborations.

5. Description of the Other Nonets

The flavour group $SU_f(N_f)$ is the group of global symmetry of QCD in case of equal masses of quarks with different flavours. Hence, in contrast to the gauge colour group $SU_C(3)$, the flavour group represents an approximate symmetry. First, a relatively good approximation to the real case of hadron resonances is provided by the group of isotopical symmetry $SU_f(2)$ [68]. Moreover, as far as the two quarks u and d from the fundamental spinor

$$\Psi = \begin{pmatrix} u \\ d \end{pmatrix} \tag{32}$$

can be assumed massless in comparison with the hadrons' masses, it can be supposed that the QCD symmetry group is a wider group, namely $SU_R(2) \times SU_L(2)$. The spontaneous breaking of the chiral symmetry removes the degeneracy between the masses of the boson resonances multiplets with opposite parity. Corresponding to the Goldstone theorem [64,65], the representatives of the pseudoscalar multiplet with quantum num-

bers 0^{-+} are massless, while for the scalar multiplet, with quantum numbers 0^{++}, they are massive.

Hence, in the antiquark-quark channel

$$\bar{\mathbf{2}} \times \mathbf{2} = \mathbf{3} + \mathbf{1}, \tag{33}$$

a triplet of almost massless mesons with quantum numbers $1^-(0^-)$ and a triplet of massive scalar mesons with quantum numbers $1^-(0^+)$ should exist. Here, by $I^G(J^P)$, we denote the quantum numbers of the isotopic multiplet, where I is the eigenvalue of the isotopic spin, while G is the eigenvalue of the $\mathcal{G}$-parity operator. The latter presents the product of the charge conjugation C and the rotation at 180 degrees around the second axis y in the isotopical space. The introduction of the operator of $\mathcal{G}$-parity is necessary, because the electrically charged components of the multiplet are not the eigenvalues of the operator of the charge conjugation C. $\mathcal{G}$-parity of the isotopical multiplet is defined according to the formula $G = C(-1)^I$, where C is the charge conjugation parity of the truly neutral component of the multiplet.

From the mesons table [32], it is seen that the pion triplet (π^+, π^0, π^-) with masses $m_{\pi^\pm} \approx 139.6$ MeV and $m_{\pi^0} \approx 135.0$ MeV and with quantum numbers $1^-(0^-)$, as well as the triplet $a_0(980)$ of mesons (a_0^+, a_0^0, a_0^-) with masses $m_{a_0} \approx 980$ MeV and quantum numbers $1^-(0^+)$, really exist. (The difference between masses of the charged and neutral pions is due to the difference of the masses of u and d quarks and the electromagnetic interactions.) The small deviation from zero of the π meson mass is defined by the Gell-Mann–Oakes–Renner relation [69]

$$m_\pi^2 = -\frac{\langle \bar{u}u + \bar{d}d \rangle}{F_\pi^2} \, (m_u + m_d), \tag{34}$$

and it is due to the small but nonzero current masses of quarks

$$m_u + m_d \approx 10 \text{ MeV}. \tag{35}$$

(As far as the quark masses depend on the scale of their determination, here, we present their values at 1 GeV.)

The quarks' *current* masses, which enter the Lagrangian of the perturbation theory in QCD at small distances, are different from the *constituent* masses of the quarks, which appear due to spontaneous breaking of the chiral symmetry at large distances, as a result of nonperturbative effects. The constituent masses constitute the dominating part of nucleons' masses. The spontaneous chiral symmetry breaking leads also to nonzero vacuum expectations values

$$\langle \bar{u}u \rangle = \langle \bar{d}d \rangle \approx -(250 \text{ MeV})^3, \tag{36}$$

and to the constant of pion decay

$$F_\pi \approx 130.7 \text{ MeV}. \tag{37}$$

The interpretation of the singlet in (33) as $I = 0$ requires some additional clarification. The singlet state **1** at the classical level is invariant under the phase transformations of the group $U(1)$. However, due to anomalies, the group of chiral transformations $U_A(1)$ [70,71] is no longer a symmetry group at the quantum level. Therefore, an existence of massless or light particles with these quantum numbers is not to be expected. Another reason that leads to difficulties of the interpretation of the isoscalar states is the existence of the quite light strange quark with a mass of a hundred MeV. The states $\bar{s}\mathcal{O}s$ have the same quantum

numbers as the singlet states $(\bar{u}\mathcal{O}u + \bar{d}\mathcal{O}d)\sqrt{2}$. Hence, the physically observed meson states should be a mixture of the latter. Such a situation is realised exactly for pseudoscalar isosinglets. On the basis of the classification of the flavour group $SU_f(3)$, the physical states of the η meson with mass $m_\eta \approx 547.9$ MeV and η' meson with mass $m_{\eta'} \approx 957.8$ MeV can be presented as almost pure octet states

$$\eta \approx \eta_8 = \frac{i}{\sqrt{6}}\Big(\bar{u}\gamma^5 u + \bar{d}\gamma^5 d - 2\,\bar{s}\gamma^5 s\Big), \tag{38}$$

and singlet states

$$\eta' \approx \eta_0 = \frac{i}{\sqrt{3}}\Big(\bar{u}\gamma^5 u + \bar{d}\gamma^5 d + \bar{s}\gamma^5 s\Big). \tag{39}$$

We have discussed up to now the well studied sector of scalar and pseudoscalar meson resonances. Let us discuss now the meson states with spin 1. As far as a complete theory $SU_f(3)$ does not exist yet, we will describe only the classification of meson hadronic resonances in the channels with isovector $I = 1$ and isosinglet $I = 0$. Due to the lightness and almost equality of u and d-quark masses and the diagonal structure of the isosinglet states, this model provides a good quantitative description of the mass spectrum (Figure 7a) and of the dynamical properties of the mesons in these channels [52].

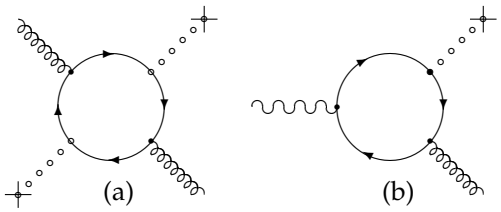


Figure 7. Contributions of S_0 vacuum expectation values to the mass terms (**a**), mixing between vector mesons and between pseudoscalar and axial–vector mesons (**b**).

This model provides only a qualitative description of the states with $I = 1/2$ with strange quarks, because of the considerable difference between the masses of the light quarks and the strange quark. For example, the properties of the low-lying vector state $K^*(892)$ should be connected with the properties of its partner $K'^* = K^*(1410)$, as in the case of light nonstrange mesons. However, quantitatively, the mixing will be different and will include the strange quark mass m_s.

For strange quarks, even the neutral states are no longer the eigenvalues states of the charge conjugation operator C; hence, the axial–vector meson A and the axial–vector meson B are described by the same quantum numbers $1/2(1^+)$, and they can mix. This way the physical states $K_1(1270)$ and $K_1(1400)$ also are a mixture of pure states K_{1A} and K_{1B}.

There also exists a mixing between the pions of the pseudoscalar meson nonet and a_1-mesons of the axial–vector nonets (Figure 7b). This mixing may be removed by a renormalisation of the pion field. b_1 mesons of the axial–vector nonet have unique quantum numbers 1^{+-} and do not mix with the other meson nonets. Maximal mixings existing between vector fields V and R (Figure 7b) form the physical vector nonets in the left panels of Figures 2 and 3. This leads to the following ratios among masses of the three different nonets, which should be equal to one. Using the experimental data [32] for isotriplets

$$R_{I=1} = \frac{2\Big(m_{\rho'} - m_\rho\Big)^2 + 3m_{\rho'}m_\rho}{3\big(m_{b_1}\big)^2} = 0.96 \pm 0.03, \tag{40}$$

and isosinglets

$$R_{I=0} = \frac{2(m_{\omega'} - m_{\omega})^2 + 3m_{\omega'}m_{\omega}}{3(m_{h_1})^2} = 1.01 \pm 0.07, \tag{41}$$

we get good agreements.

The mass of the $h_1(1415)$ meson with hidden strange flavour, $s\bar{s}$, is predicted from the relation [52]

$$m^2_{h_1(1415)} = \frac{2\left(m_{\phi'} - m_{\phi}\right)^2 + 3m_{\phi'}m_{\phi}}{3} = 1415 \pm 13 \text{ MeV}. \tag{42}$$

6. Conclusions

In this short review, we discussed all nonets of the low-lying meson states. We provided their description in the framework of Nambu–Jona-Lasinio massless quark model. The scalar mesons nonet is studied in detail. Its collective meson states are described through quark–antiquark pairs, whose condensates lead simultaneously to spontaneous breaking of the chiral $U_R(3) \times U_L(3)$ symmetry and the flavour $SU_f(3)$ symmetry. Due to these symmetries breaking, there appear *constituents* masses of quarks. We have calculated the mass spectrum of the scalar mesons and their mixing. This mass spectrum provides an explanation of the inverse mass hierarchy of the scalar mesons nonet, where κ and σ mesons are light states. We find that $f_0(980)$ meson is close to $(u\bar{u} + d\bar{d})/\sqrt{2}$ state. This finding is confirmed by the recent experimental results of the CMS [66] and ALICE [67] collaborations.

A short description of the physical properties of other meson nonets is provided. In the framework of the isotopical $SU_f(2)$ group, unique mass relations among the different nonets are presented, which are in accordance with the experimental data. Using these mass relations, the $h_1(1415)$ mass is predicted, and such a particle with the predicted mass value was experimentally discovered [72].

It is impressive that, using just one Dirac spinor with three flavours, it is possible to describe simultaneously the six nonets of the low-lying mesons, namely 54 states. The properties of the two vector nonets, which we have presented, do not correspond to the wrong assumptions that the primed vector nonet is in the $2\ ^3S_1$ state. As shown in our works and also by lattice calculations [73], these states correspond to the $^3D_1$ state.

The presented model will be completed after the explicit introduction of the *current* quark masses, using the flavour $SU_f(3)$ group. Such an approach, namely the introduction of nonzero *current* quark masses, was proposed in [74].

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Abbreviations

The following abbreviations are used in this manuscript:

BNG	Bogoliubov–Nambu–Goldstone
NJL	Nambu–Jona-Lasinio
QCD	Quantum Chromodynamics
GIM	Glashow–Iliopoulous–Maiani
GPD	generalised parton distributions

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