

Article

Moving Mass Problem – Analytical Approximation of Transient Solution on Infinite Two-Layer Model

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Abstract

The problem of moving inertial objects traversing simplified models of railway tracks has an analytical solution in the transformed domain under the assumptions of linearity and longitudinal homogeneity of the model. However, the time-domain response usually requires numerical integration, which could only be avoided if the inverse Laplace transform could be evaluated by contour integration as a sum of residues. Such approach relies on the assumption that the function to be integrated is analytic except for isolated poles and vanishes asymptotically along the infinite contour. Unfortunately, this function exhibits step discontinuities that, in addition to violating the previous assumption, prevent, in several cases, the existence of poles, or there are poles that are not related to the dominant frequency, or the associated deflection is incorrect. This contribution presents a way of dealing with such situations. A new technique for the determination of dominant frequencies is proposed and validated. We show how to estimate deflections with acceptable accuracy not only at the contact point, but also along the structure without numerical integration. The technique is new and, to the author's knowledge, has not been previously published. Additionally, it is explained why the response in undamped cases resembles that of damped systems.

Keywords: rail track; integral transforms; analytical solution; equivalent flexibility; mass-induced frequencies; energy redistribution

1. Introduction

Combating climate change and reducing the carbon footprint are among the main challenges to be addressed by scientists in today's society. Promoting rail transport is one of the possible ways to contribute to this goal. As it is impossible to construct new railway lines all over the world, the question of how to adapt existing lines to increased demands on track capacity by increasing travel velocity, frequency of passages, and axle loads is pertinent. Such changes in railway operation and design must be supported by viable numerical models; however, detailed finite element analyses can still be time-consuming, and the resulting data may be difficult to interpret. For this reason, analytical and semi-analytical approaches remain relevant because they provide better physical insight into the problem and, due to appropriate simplifications, reduce the results to their essential features, which are easier to analyze. In addition, the use of dimensionless parameters provides an opportunity to investigate several design options within a common framework. This makes moving-load problems a very active research field. Besides their direct

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connection with transportation, these problems are also relevant to other industrial applications.

On the other hand, the increasing number of operational railway lines has raised significant environmental concerns, particularly regarding train-induced vibrations. Such vibrations may contribute to long-term structural deterioration, affect buildings containing sensitive equipment, and disturb urban populations through noise pollution. A recent state-of-the-art review [1] summarizes the modeling approaches used to describe these effects and illustrates the broad range of analytical and numerical models currently applied to train-induced environmental vibrations.

Properties such as critical velocity and the instability of moving inertial objects are among the leading characteristics to be investigated. The critical velocity is usually determined as the velocity at which a moving constant force enters into resonance with the supporting structure, leading to increased vibration. In this way, the lowest velocity of wave propagation in the supporting structure is identified. If the supporting structure is infinite and longitudinally homogeneous, this may theoretically induce infinite deflections. The steady-state solution removes time-dependent terms and, as such, all inertial effects. Experimental data indicate that this threshold has already been reached under real operating conditions [2], emphasizing the practical relevance of critical-velocity analysis for high-speed railway lines, particularly on soft ground.

Usually, the steady-state vibration pattern is rapidly achieved, and there is no need to involve transient vibrations. In addition, under the assumption of linearity, results can be superposed; therefore, there is no need to analyze a set of moving forces individually. This makes the moving-force problem relatively more accessible, and it has consequently been studied from different points of view for many years, starting with the simplest model of an Euler–Bernoulli beam traversed by a constant moving force. The problem was subsequently extended to harmonic and time-variable forces and to supporting structures resting on elastic foundations, including Winkler springs and more complicated foundation arrangements. These classical variants and their analytical solutions are comprehensively summarized in Frýba's monograph [3]. Foundation models going beyond the classical Winkler representation were discussed by Kerr [4], while Knothe and Grassie [5] addressed modeling aspects particularly relevant to railway track and vehicle-track interaction at high frequencies. A more recent review by Lamprea-Pineda et al. [6] provides an extended overview of railway applications, covering beam-on-foundation models from Winkler-type representations to elastic half-space formulations.

The problem of moving forces, either constant or harmonic, has therefore been studied in detail over the years and has progressed well beyond the cases discussed in [3]. More general supporting structures and analytical solution procedures have been introduced, including formulations for harmonic moving loads and their associated ground-vibration fields [7]. Such developments demonstrate that analytical approaches remain applicable even when the supporting medium becomes substantially more complex than the classical beam-on-Winkler-foundation model.

Other important groups of analyses add inertial properties to the moving object, considering a moving mass, moving oscillator, or simplified vehicle models. In such cases, it is worth considering transient vibrations because only in this way can the possibility of instability be revealed. When the supporting structure is finite, transient vibrations enter the full solution in an inherent way. However, there are only a few works dedicated to the full vibration pattern associated with moving inertial objects on infinite models. The distinction between moving-force and moving-inertial-object problems is important because the inertia of the moving object introduces additional coupling between its motion and the deformation of the supporting structure.

The moving-mass problem was first solved on a finite beam as early as 1929 by Jeffcott [8] using the method of successive approximations. A pioneering solution for an infinite railway track subjected to a moving, vibrating mass was subsequently presented by Duffy [9]. These early studies established the basis for treating the moving object and the supporting structure as a coupled dynamic system rather than considering the moving load as prescribed independently of the structural response.

Instability originates from anomalous Doppler waves, as exemplified in [10]. Most studies focus on identifying the conditions leading to instability, often using the D-decomposition method for a moving mass [11,12], a moving oscillator [13,14], or a moving two-mass oscillator [15]. The latter formulations account for additional internal degrees of freedom and therefore provide a closer representation of simplified vehicle dynamics. Other configurations, such as several moving oscillators on infinite beams [16], multi-span beams [17], or an infinite string [18], have also been examined. These developments extend the analysis from a single moving inertial object towards systems representing repeated or distributed vehicle components. Such approaches can then be extended to a moving bogie [19–21], providing a further step towards more realistic railway vehicle representations.

A different research direction focuses on sharp stiffness changes in the supporting structure and the associated transition radiation [22–24]. These studies investigate the additional wave phenomena generated when a moving system encounters a spatial change in structural properties and have also motivated analytical and energy-based approaches to the design of railway transition zones [25]. Thus, the treatment of spatial non-uniformity provides another example in which wave propagation and energy redistribution become important for understanding the response of railway structures.

Recently, other applications related to possible future transportation systems, such as hyperloop tubes, have also been investigated [26,27]. These studies consider very high velocities and associated instability mechanisms, further motivating the analysis of moving inertial objects at velocities substantially higher than those normally encountered in conventional railway applications.

Nevertheless, full deflection shapes are rarely calculated. A new approach based on dominant frequencies was introduced by the author of this paper. Its application to a one-layer model, including clarifications related to discontinuities in the equivalent flexibility of the model and an extension to the two-mass moving oscillator, is discussed in [28]. Owing to the step discontinuities in the equivalent flexibility, the sum of residues can only provide an approximation of the full solution. The full solution is then composed of the steady-state part, corresponding to the moving force, and the unsteady harmonic motion attributed to the moving mass, which forms the main part of the transient response. A detailed analysis of the discontinuities is presented for a more general model of a plane foundation in [29]. The approach was subsequently extended by introducing non-homogeneous initial conditions [30] and to the two-layer model [31]. A summary of some main features of layered models is presented in [32]. This approach relies heavily on the determination of the characteristic equation, with the resulting frequencies referred to as induced frequencies to distinguish them from natural frequencies. It was demonstrated that the agreement between numerical integration results and results obtained from these new formulas is very good. The transient vibration shapes were approximated by the so-called unsteady harmonic component guided by the dominant frequency (the mass-induced frequency). However, due to the single harmonic motion, the initial conditions may not be fully satisfied, leading to some initial discrepancy that is generally negligible. While the roots always exist in unstable cases, this is not the case for stable vibrations. Therefore, the main limitation was found to be associated with the step discontinuities in the equivalent flexibility of these models, which prevented a root of the characteristic equation from

being found in several cases. Additionally, a root can be located in a region where, due to surrounding discontinuities, the corresponding residuum does not correctly reflect the deflection shape.

In the absence of damping, one might expect the unsteady harmonic motion to exhibit a constant amplitude, as observed in a one-layer model [28]. However, this does not occur due to energy redistribution caused by the interlayer energy flux, and the resulting vibrations in two- and three-layer models resemble a damped response. This redistribution effectively acts as radiation damping; therefore, the observed decay in undamped layered models is not caused by physical damping, but by conservative energy transfer between layers. This effect, often interpreted as dispersive or radiation damping, plays a key role in the dynamic behavior of such systems. It is more pronounced in layered models with point masses than in plane models, where the mass is distributed and the term radiation damping is more appropriate.

This paper addresses a mathematical difficulty that does not seem to have been discussed extensively in the moving-load literature. By proposing the new tracing technique, it removes the necessity of numerical integration, which is prone to numerical errors. It extends the method from [28–33] and demonstrates how to overcome the difficulty raised by step discontinuities by using tracing or regularization techniques, so that the full vibration response can be approximated analytically without numerical integration for any velocity. Deflection shapes can then be determined from closed-form formulas at any time and any position without any previous analysis to reach the desired time, which is necessary in numerical approaches such as the finite element method. We also establish how and why some existing poles should be disregarded.

The new technique is basically analytical; the only numerical step consists of calculating the roots of a leading polynomial repeatedly. This new technique provides an approximation in the same sense as in the author's previous works and therefore has the same advantages and limitations. The new feature is that dominant frequencies can be found in cases where the previous approach fails and that the agreement of deflections along the spatial coordinate is improved.

It is worth mentioning that the analytical approach is also important for energy analysis, providing additional physical insight into wave propagation. Such information can be used for model tuning, for controlling or mitigating certain features, and generally for improving the design of railway tracks.

This paper focuses on the two-layer model, but the new technique presented is directly extendable to the three-layer model or to models assuming a plane foundation. This also makes it possible to illustrate the expected differences when the foundation mass is treated either as a set of lumped rigid masses or as a distributed parameter.

The paper is organized as follows. First, the model and its solution by closed-form formulas are given in Section 2. Section 3 is dedicated to the motivation for the new technique. Section 4 then discusses the tracing technique that is implemented to overcome the mathematical limitation in the characteristic equation. An alternative approach is included in Section 5. Section 6 exemplifies the selection of validation examples, which are separated into four cases, discussed in detail in dedicated subsections. The paper ends with a discussion of the results presented and conclusions in Sections 7 and 8, respectively.

2. Analysis Description, Assumptions and Solution

Layered models of railway tracks are widely used for the dynamic analysis of ballasted tracks and can be classified as one-, two-, or three-layer models, although additional layers may also be considered. The one-layer model corresponds to the beam on a Winkler–Pasternak foundation, which is commonly used in many applications. Additional layers in ballasted tracks separate the track components into a rail and sleepers' layer and

a ballast layer. For the ballast layer, it is important to correctly estimate the dynamically activated ballast mass, as well as the ballast cone vertical stiffness and the shear stiffness between adjacent cones. However, a separate ballast layer is present only in the three-layer model. In the two-layer model, there are two options: (i) the ballast layer replaces the foundation and the foundation is modeled by a rigid surface; or (ii) both layers, ballast and foundation, are replaced by an equivalent layer. The two-layer model can also be used in other applications with different interpretations of the layers, which leads to a broader range of admissible mechanical parameters.

The two-layer model traversed by a moving mass with an applied vertical force is illustrated in Figure 1. Given the explanation above, the second layer is designated in a general way as foundation. It is worth noting that the two-layer model was selected because it is the simplest model where the issue to be solved in this paper occurs.

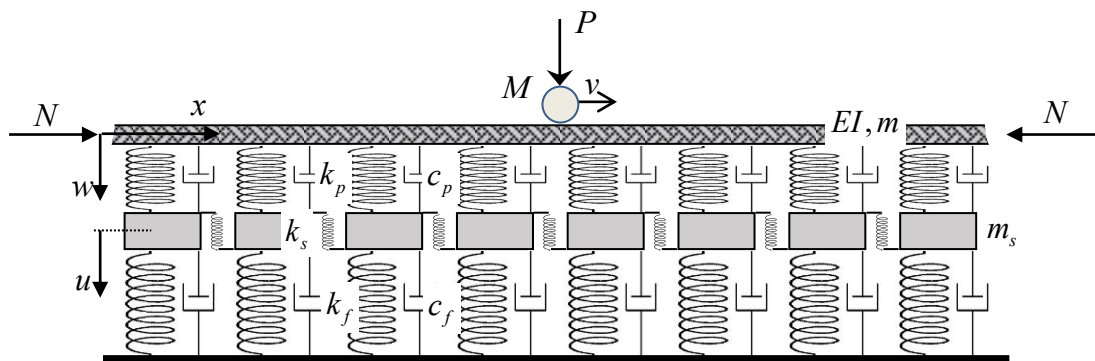


Figure 1. A two-layer model traversed by a single mass M , acted upon by a vertical force P and moving together at constant velocity v .

It is assumed that the moving mass remains in permanent contact with the beam and that no additional friction or other components are considered. The supporting structure is assumed to be infinitely long and uniform in the longitudinal direction, which also means that all parameters are considered to be distributed along the length, including those that appear as discrete components in Figure 1.

The beam is modeled according to Euler–Bernoulli beam theory, with bending stiffness EI and mass per unit length m , and may additionally be subjected to an axial force N . The stiffness parameters are represented by k , with subscripts p , b , s , and f denoting the rail pads, ballast, shear component, and foundation, respectively. Viscous damping coefficients are expressed by c , where the subscripts p , b , and f correspond to the rail pads, ballast, and foundation. The masses m are indexed by s and b for sleepers' layer and ballast. Displacements are represented by w for the beam and by u for the sleepers' layer.

The governing equations have been presented in previous author's studies, e.g., ref. [31]. The dynamic equilibrium equations of the beam and sleepers' layer in fixed coordinates read as:

$$\begin{aligned} EIw_{,xxxx} + Nw_{,xx} + mw_{,tt} + c_p(w_{,t} - u_{,t}) + k_p(w - u) &= p, \\ m_s u_{,tt} - k_s u_{,xx} - c_p(w_{,t} - u_{,t}) - k_p(w - u) + c_f u_{,t} + k_f u &= 0, \end{aligned} \quad (1)$$

where spatial variable x and time t have been omitted for simplicity.

The loading term is:

$$p(x, t) = (P - Mw_{0,tt}(t))\delta(x - vt). \quad (2)$$

All designations have been explained in the preceding text except for δ which denotes the Dirac delta function. Derivatives are designated by the respective variable in the

subscript position, preceded by a comma. Subscript 0 in Equation (2) indicates that the displacement is considered at the contact point (CP). After eliminating this additional unknown, transforming to moving coordinates, and introducing dimensionless parameters, Equation (1) becomes:

$$\ddot{w}_{,\xi\xi\xi\xi} + 4\eta_N \dot{w}_{,\xi\xi} + 4\alpha^2 \ddot{w}_{,\xi\xi} + 4\ddot{w}_{,\tau\tau} - 8\alpha \dot{w}_{,\xi\tau} + 8\eta_p (\dot{w}_{,\tau} - \alpha \dot{w}_{,\xi} - \ddot{u}_{,\tau} + \alpha \ddot{u}_{,\xi}) + 4\kappa_p (\dot{w} - \ddot{u}) = (8\eta_p - 4\eta_M \dot{w}_{,\tau\tau}) \delta(\xi), \quad (3)$$

$$\mu_s (\ddot{u}_{,\tau\tau} - 2\alpha \ddot{u}_{,\xi\tau} + \alpha^2 \ddot{u}_{,\xi\xi}) - 2\eta_p (\dot{w}_{,\tau} - \alpha \dot{w}_{,\xi} - \ddot{u}_{,\tau} + \alpha \ddot{u}_{,\xi}) - \kappa_p (\dot{w} - \ddot{u}) - \eta_s \ddot{u}_{,\xi\xi} + \ddot{u} + 2\eta_f (\ddot{u}_{,\tau} - \alpha \ddot{u}_{,\xi}) = 0. \quad (4)$$

Dimensionless parameters offer several advantages: (i) they allow a wide range of possible scenarios to be covered simultaneously, and (ii) they have numerically comparable values, which reduces numerical errors that commonly arise when real parameters differ by several orders of magnitude.

For the definition of dimensionless parameters, a reference Winkler beam characterized by EI , m , and k_f is used. The critical velocity of a constant moving force traversing the reference beam is given by:

$$v_{ref} = \sqrt[4]{\frac{4k_f EI}{m^2}} = \frac{1}{\chi} \sqrt{\frac{k_f}{m}} \quad \text{with} \quad \chi = \sqrt[4]{\frac{k_f}{4EI}}, \quad \alpha = \frac{v}{v_{ref}}. \quad (5)$$

and is used as the reference velocity for defining the velocity ratio. All other parameters are related to the reference beam. First, the mass, stiffness, and damping ratios are defined as:

$$\mu_s = \frac{m_s}{m}, \quad \mu_b = \frac{m_b}{m}, \quad \kappa_p = \frac{k_p}{k_f}, \quad \kappa_b = \frac{k_b}{k_f}, \quad \eta_p = \frac{c_p}{2\sqrt{mk_f}}, \quad \eta_b = \frac{c_b}{2\sqrt{mk_f}}, \quad \eta_f = \frac{c_f}{2\sqrt{mk_f}}. \quad (6)$$

Furthermore, the axial force and shear stiffness ratios are defined, together with the dimensionless spatial moving coordinate, time, and reference frequency:

$$\eta_N = \frac{N}{2\sqrt{k_f EI}}, \quad \eta_s = \frac{k_s}{2\sqrt{k_f EI}}, \quad \xi = r\chi, \quad \tau = \chi v_{ref} t = t\omega_{ref} \quad \text{with} \quad \omega_{ref} = \sqrt{\frac{k_f}{m}}. \quad (7)$$

Finally, all displacements are related to the static displacement produced by the force P applied to the reference beam, as given in Equation (8), where the moving mass ratio and the conversion of circular frequencies are also defined:

$$w_{st} = \frac{P\chi}{2k_f}, \quad \tilde{w} = \frac{w}{w_{st}}, \quad \tilde{u} = \frac{u}{w_{st}}, \quad \eta_M = \frac{M\chi}{m}, \quad \tilde{\omega} = \frac{\omega}{\chi v_{ref}} = \frac{\omega}{\omega_{ref}}. \quad (8)$$

The solution method follows the standard procedure. First, the Laplace transform is applied to account for the initial instant.

$$\tilde{F}(\xi, \bar{q}) = \int_0^\infty f(\xi, \tau) e^{-\bar{q}\tau} d\tau, \quad (9)$$

where $\bar{q} = iq$. Homogeneous initial conditions are assumed. The effect of non-homogeneous initial conditions was analyzed [30]. Nevertheless, the initial conditions do not affect the stability analysis.

Next, the Fourier transform is applied

$$F(p, \bar{q}) = \int_{-\infty}^\infty \tilde{F}(\xi, \bar{q}) e^{-ip\xi} d\xi \quad (10)$$

allowing Equations (3) and (4) to be written in a matrix form:

$$\begin{bmatrix} D_1 + 4D_2 & -4D_2 \\ -D_2 & D_2 + D_3 \end{bmatrix} \begin{Bmatrix} W \\ U \end{Bmatrix} = \begin{Bmatrix} \frac{8\eta_p}{iq} + 4\eta_M q^2 \tilde{W}(0, iq) \\ 0 \end{Bmatrix}, \quad (11)$$

where

$$\begin{aligned} D_1(p, q) &= p^4 - 4(q - \alpha p)^2 - 4\eta_N p^2, \\ D_2(p, q) &= 2i\eta_p(q - \alpha p) + \kappa_p, \\ D_3(p, q) &= -\mu_s(q - \alpha p)^2 + 2i\eta_f(q - \alpha p) + \eta_s p^2 + 1. \end{aligned} \quad (12)$$

From Equation (11), $\tilde{W}$ and U can be solved, but the term $\tilde{W}(0, iq)$ remains unknown. Therefore, $W(p, iq)$ is first expressed as:

$$W(p, iq) = \frac{1}{D(p, q)} \left(\frac{8\eta_p}{iq} + 4\eta_M q^2 \tilde{W}(0, iq) \right), \quad (13)$$

where

$$D(p, q) = D_1(p, q) + \frac{4D_2(p, q)D_3(p, q)}{D_2(p, q) + D_3(p, q)}. \quad (14)$$

The inverse Fourier transform is then applied

$$\tilde{W}(\xi, q) = \frac{1}{2\pi} \left[\frac{8\eta_p}{iq} + 4\eta_M q^2 \tilde{W}(0, q) \right] \int_{-\infty}^{\infty} \frac{1}{D(p, q)} e^{ip\xi} dp \quad (15)$$

and, by introduction of $\xi=0$ and using

$$K(\xi, q) = \int_{-\infty}^{\infty} \frac{1}{D(p, q)} e^{ip\xi} dp \quad (16)$$

it follows that

$$\tilde{W}(0, q) = \frac{4\eta_p}{iq(\pi - 2\eta_M q^2 K(0, q))} K(0, q). \quad (17)$$

After substitution:

$$\tilde{W}(\xi, q) = \frac{4\eta_p}{iq(\pi - 2\eta_M q^2 K(0, q))} K(\xi, q). \quad (18)$$

Similarly, for sleepers' layer:

$$\tilde{U}(\xi, q) = \frac{4\eta_p}{iq(\pi - 2\eta_M q^2 K(0, q))} S(\xi, q), \quad (19)$$

where

$$S(\xi, q) = \int_{-\infty}^{\infty} \frac{1}{D_1(p, q) + 4D_3(p, q) + \frac{D_1(p, q)D_3(p, q)}{D_2(p, q)}} e^{ip\xi} dp. \quad (20)$$

It is important to highlight that the wave numbers for the beam and sleepers' layer are the same, as they are given by the roots of the determinant of the system given in Equation (11).

Without loss of generality, $\tilde{W}(\xi, iq)$ and $\tilde{U}(\xi, iq)$ were switched to $\tilde{W}(\xi, q)$ and $\tilde{U}(\xi, q)$, respectively.

Finally, the inverse Laplace transform is applied:

$$f(t) = \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{a-iT}^{a+iT} e^{\bar{q}t} F(\bar{q}) d\bar{q}, \quad (21)$$

where a is a small number.

As in the previous works, first used in [33], it was found to be more convenient to switch the real and imaginary axes, so that the final beam and sleepers' layer deflection shapes can be calculated numerically as:

$$\tilde{w}(\xi, \tau) = \frac{1}{\pi} \int_{-ia-\infty}^{-ia+\infty} \frac{-2i\eta_p K(\xi, q)}{q(\pi - 2\eta_M q^2 K(0, q))} e^{iq\tau} dq, \quad (22)$$

$$\tilde{u}(\xi, \tau) = \frac{1}{\pi} \int_{-ia-\infty}^{-ia+\infty} \frac{-2i\eta_p S(\xi, q)}{q(\pi - 2\eta_M q^2 K(0, q))} e^{iq\tau} dq. \quad (23)$$

This change of variables is motivated by the fact that the original variable for the Laplace transform, as written after Equation (9), which is then used in the Bromwich integral in Equation (21) is $\bar{q} = iq$, therefore it is convenient to switch $\bar{q}$ with q which is used for the frequency variable. Then, the adaptation introduced is directly derived from the change of variables. Assuming that the method of contour integration can be used, one pole is readily obtained as $q = 0$, and its corresponding residue identifies the steady-state solution associated with the moving force. The other terms are residues corresponding to the poles identified as complex roots of the complex characteristic equation:

$$2\eta_M K(0, q)q^2 - \pi = 0. \quad (24)$$

It is worth noting that its form is the same for all models, because only the function $K(0, q)$, which is proportional to the equivalent flexibility of the model at the CP distinguishes different models. In what follows it will be simply referred as K -function.

From previous studies, it is clear that, in general, only one root is necessary to identify the dominant frequency. A reasonable approximation of the full vibration pattern is then given for the beam by

$$\tilde{w}(\xi, \tau) = \frac{4}{\pi} K(\xi, 0) - \frac{2K(\xi, q_{M_1})}{\pi + \eta_M q_{M_1}^3 K_{,q}(q_{M_1})} e^{iq_{M_1}\tau} - \frac{2K(\xi, q_{M_2})}{\pi + \eta_M q_{M_2}^3 K_{,q}(q_{M_2})} e^{iq_{M_2}\tau}, \quad q_{M_2} = \overline{q_{M_1}} \quad (25)$$

and for the sleepers' layer by

$$\tilde{u}(\xi, \tau) = \frac{4}{\pi} S(\xi, 0) - \frac{2S(\xi, q_{M_1})}{\pi + \eta_M q_{M_1}^3 K_{,q}(q_{M_1})} e^{iq_{M_1}\tau} - \frac{2S(\xi, q_{M_2})}{\pi + \eta_M q_{M_2}^3 K_{,q}(q_{M_2})} e^{iq_{M_2}\tau}, \quad q_{M_2} = \overline{q_{M_1}}. \quad (26)$$

First terms in Equations (25) and (26) identify the steady-state solution associated with the moving force, while the next two terms compose the unsteady harmonic part, i.e., the dominant part of the transient solution that specifies the contribution of the moving mass. These terms can be combined, after which the real part of q_M defines the argument of the trigonometric functions and the imaginary part defines the decay, if positive. Thus, unstable behavior is identified by a negative imaginary part of q_M . In what follows, no distinction is made between indices 1 and 2, and q_M with a positive real part is used to denote the mass-induced dominant frequency.

In Equations (25) and (26), the part resulting from the integration along the branch cuts is omitted. It was demonstrated in [28,29,31] that step discontinuities in the K -function occur only in the stable region and therefore never affect instability. In the case of true roots of Equation (24), in all cases tested in [28–30,32,33], this root indicated the dominant frequency and the only discrepancy between this analytical estimate and the numerical integration was identified in the initial times, given that a harmonic vibration will likely exhibit some differences in the initial conditions. It should be noted that the validation examples focused mainly on CP deflections and therefore differences in deflection shapes along the model, in cases where the true roots are located in a region separated from the unstable one by a discontinuity, were not identified. However, it was demonstrated that in cases of two true roots, only one represents the dominant frequency.

To analyze specific cases, ranges of allowable parameters must be identified. This was already presented in [31]:

$$\mu_s = 1 \div 6, \quad \kappa_p = 0.03 \div 50000. \quad (27)$$

However, the upper limit on κ_p can be even higher, theoretically tending to infinity, which would represent a rigid connection between the rail and the sleeper.

It has been shown in [28,29] that the step discontinuities in the K -function, which occur whenever one of the p -roots of $D(p,q)$ for a given smooth q -variation passes through a real value to the other half of the complex p -plane, may lead to situations where Equation (24) has no roots or has additional roots corresponding to non-dominant frequencies. It will additionally be demonstrated that some true roots may identify non-realistic beam and sleepers' layer deflection shapes. Since these situations only occur in stable cases, and therefore instability analyses are not affected, this is probably the main reason why this issue has been neglected by other researchers. Nevertheless, identifying the dominant unsteady harmonic vibrations remains important for further analyses and mitigation measures.

It is worth noting that an exact solution could be obtained by summing all residues, including those associated with non-dominant frequencies, and by adding a contribution obtained by numerical integration along the branch cuts, which would eliminate all discontinuities. However, the discontinuity locations have complicated shapes in the complex q -plane, and their positions change with velocity. Designing branch cuts to avoid them and performing numerical integration along them would be considerably more complicated than the numerical integration given by Equations (22) and (23), even for the relatively simple case of the two-layer model. Moreover, this is not the aim of the present paper. The aim is to identify the dominant frequency and obtain a reasonable analytical approximation of the transient part of the full solution.

3. Motivation

Dominant frequencies of the transient vibrations associated with a moving mass are important for several reasons, including, among others: (i) energy analysis; (ii) vibration mitigation; (iii) avoiding resonance with other external sources; (iv) approximation of full deflection shapes within a region of interest around the moving source. Additionally, (v) the imaginary part of the dominant frequency helps to estimate how rapidly transient vibrations, activated by any inhomogeneity, will cease. While dominant frequencies in the one-layer model can be mostly determined as roots of Equation (24), this is not the case for other models.

In the absence of damping, one might expect the unsteady harmonic motion to exhibit a constant amplitude, as observed in the one-layer model and in the two-layer model in the examples presented in [31]. However, when energy redistribution between layers is sufficiently strong, the resulting vibration patterns can resemble harmonically damped oscillations because this energy redistribution activates an apparent radiation damping. In such situations, the q_M -roots of the characteristic equation, Equation (24) do not exist because of the step discontinuity in the K -function within the region where they would otherwise appear.

Results published in [31] used parameters with relatively high κ_p , approaching an almost rigid connection between the rail and the sleepers. This produces a low phase shift between the beam and sleepers' layer vibrations and makes the model resemble a one-layer model. Consequently, the interlayer flux is low, and the K -function discontinuity can be disregarded, so that undamped cases can still exhibit real q_M in the subcritical velocity range.

In cases where the roots of Equation (24) do not exist, deflection shapes can be calculated by numerical integration. However, by observing the deflection at the CP, it can be observed that a dominant frequency guides this motion. When an estimate of this value is obtained, however, there is no indication of it in Equation (24). Additionally, the vibrations resemble those of damped cases, even when no damping is introduced.

Having obtained results from numerical integration, the dominant frequency is still difficult to estimate using standard procedures such as the Fast Fourier Transform (FFT), Power Spectral Density or Hilbert transforms. Regarding the real part, $q_M^r = \text{Re}(q_M)$, the FFT can provide a straightforward estimate. However, the frequency resolution depends on the length of the vibration signal, and numerical results at larger times are prone to numerical errors due to exponential functions, especially when calculated in MATLAB 26 using default double precision. Roughly speaking, obtaining the real part with a precision of 0.001 would be very difficult because this would require the vibration to be calculated up to $\tau = 1000$. Even this relatively limited precision is insufficient for analytical comparisons and validation. Using the Hilbert transform, the imaginary part $q_M^i = \text{Im}(q_M)$ can be estimated from the function's envelope, and the real part can also be recovered. However, here the situation is even worse due to the insufficient signal length. It is more efficient to estimate q_M^i manually from extreme values of the transient vibration of the CP.

As an example, the two-layer model characterized by mass and stiffness ratios $\mu_s = 5$ and $\kappa_p = 3$, with no damping, shear stiffness, or axial force, is considered. The model is subjected to a moving mass characterized by a moving mass ratio $\eta_M = 40$ and a velocity ratio $\alpha = 0.3$. The evolution of the CP displacement as a function of τ , obtained by numerical integration, is presented in Figure 2.

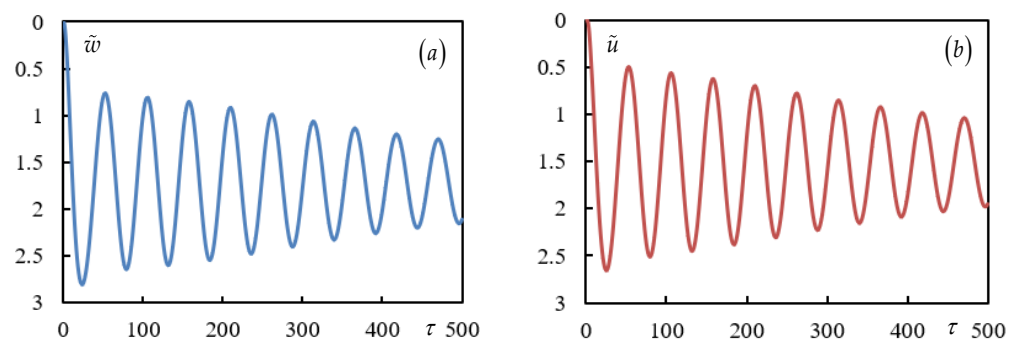


Figure 2. Deflection of the CP as a function of τ for $\mu_s = 5$, $\kappa_p = 3$, $\eta_M = 40$, $\alpha = 0.3$ and $\eta_N = \eta_s = \eta_p = \eta_f = 0$. (a) beam; (b) sleepers' layer.

While the transient part of the solution extracted at the CP as a function of τ in Figure 2 clearly exhibits a dominant frequency, examination of the characteristic Equation (24) provides no indication of the prospective q_M through any anomaly in the form of a local minimum.

As the q_M is impossible to determine as a root of Equation (24), a new technique is proposed. A rough estimate from numerical integration indicates $q_r = 0.1204$. The dominant frequency should have a positive imaginary part; however, there is a step discontinuity before reaching the expected value.

In Figure 3, the discontinuity in K -function along q_i within the interval -0.01 to 0.01 is plotted. It is demonstrated that discontinuities correspond to changes in the number of p -roots with positive imaginary part, n , which changes from 2 to 3 and then to 4. Discontinuity is more pronounced during the first transition than during the second one. It is also usually steeper in $\text{Im}(K)$ than in $\text{Re}(K)$. The second discontinuity is hardly visible unless the scale is enlarged; it then becomes clearly visible, at least in $\text{Im}(K)$, as demonstrated in part (d) of Figure 3.

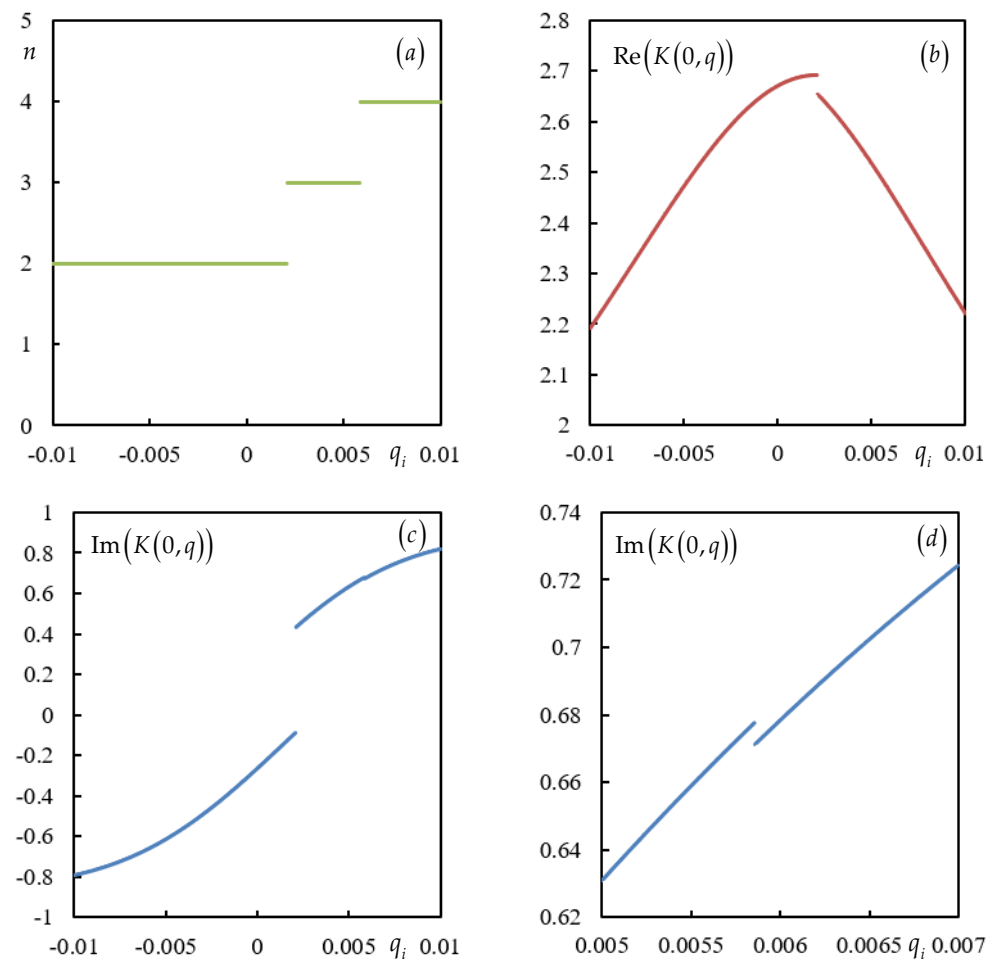


Figure 3. K-function distribution for prospective $q_r = 0.1204$ and $q_i = \langle -0.01; 0.01 \rangle$: (a) number of roots in the upper half-plane; (b) real part; (c) imaginary part; (d) rescaled imaginary part.

By adding rail pad damping $\eta_p = 0.036$, the discontinuity shifts and a root is found, as demonstrated in Figure 4. The exact value of this root is $q_M = 0.120514 + 0.002062i$.

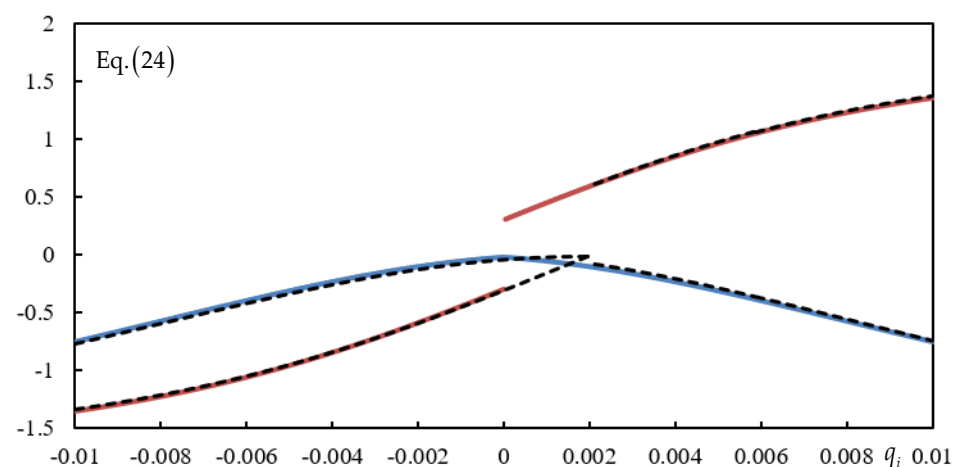


Figure 4. Value from Equation (24) for prospective $q_r = 0.1204$ and $q_i = \langle -0.01; 0.01 \rangle$. Red line: the real part; Blue line: the imaginary part; Black dashed line: values when $\eta_p = 0.036$ is added.

Additionally, upon closer examination of the discontinuity, it appears possible to regularize the K -function by smooth extension. The new regularized function F can then be used instead of K . Since continuity is preserved in unstable regions, i.e., when $\text{Im}(q) < 0$, it is possible to calculate the value of F for different q using the same p -roots but at their new positions. Therefore, the p -roots must be traced from the unstable region.

In conclusion, there are two options: (i) some parameters with negligible influence on the problem at hand can be changed so that a true root q_M is found; or (ii) the K -function can be regularized by smooth extension in an unambiguous way to obtain a new function F , and a root q_M can then be determined from a modified equation:

$$2\eta_M F(0, q)q^2 - \pi = 0. \quad (28)$$

Option (ii) constitutes the main new contribution of this paper; it is more general than option (i) and will therefore be discussed and validated extensively in the following sections. Option (i) has limited applicability and will consequently be illustrated only briefly. It will be demonstrated that the final vibration can be identified through regularization of the model by adjusting parameters that have a relatively small influence on the total energy rate.

4. Roots Tracing

To solve the characteristic Equation (24) for the complex roots q_M , several iterative techniques have already been suggested in previous author's works [28]. To accomplish this, the K -function has to be evaluated several times. Exact values can be obtained by contour integration. Nevertheless, as already mentioned, once the number of n related to p -roots changes, a step discontinuity arises.

This section explains one of the possible root-tracing procedures, embedded directly in the iterative technique used to determine the roots of Equation (24), which is detailed in previous author's works, e.g., ref. [28].

The initial estimate must be located in the unstable region, as this is the region where the K -function is continuous. The iterative procedure proceeds as follows. Let q_0 be an initial estimate, which must have a negative imaginary part. Let $p_{0j}, j=1\dots 6$ denote the roots of $D(p, q_0)=0$, and let $K(0, q_0)$ be calculated directly by contour integration. The roots in the upper-half of the complex p -plane are separated as the tracing roots, $p'_{0j}, j=1, 2$. They are always two such roots. The next estimate is then obtained as:

$$q_1 = \beta(2\eta_M K(0, q_0)q_0^2 - \pi) + (1 - \beta)q_0, \quad (29)$$

where β is a convenient parameter, usually between 0.01 and 0.5. After that, an estimate of the new tracing p -roots is calculated using:

$$p'_{1j} = p'_{0j} - \frac{D(p'_{0j}, q_1)}{D_{,p}(p'_{0j}, q_1)}, j=1, 2. \quad (30)$$

The p -roots are then calculated from $D(p, q_1)=0$, and the new tracing roots are identified as those closest to p'_{1j} . These roots are designated as $\hat{p}'_{1j}$. The next frequency is then obtained from:

$$q_2 = \beta(2\eta_M F(0, q_1)q_1^2 - \pi) + (1 - \beta)q_1, \quad (31)$$

where $F(0, q_1)$ is not obtained by contour integration, but is instead calculated using the traced roots, $\hat{p}'_{1j}, j=1, 2$ in a similar manner. The process is repeated from Equation (30) until the required accuracy is achieved. The convergence criteria are the deviation from

zero in Equation (28) and the deviation between two consecutive frequencies q_i and q_{i+1} . Both criteria are checked separately for the real and imaginary parts. This procedure works quite well; nevertheless, it is necessary to ensure that q does not change abruptly. This can be achieved by adapting the controlling parameter β . For low velocities, the required accuracy is usually achieved with fewer than 50 iterations using $\beta=0.5$. When approaching PCV or CV, where the gradients of functions involved are higher, it is necessary to decrease the values of β to maintain the convergence of the iterations. In this case, the number of iterations can reach higher values, around 200 or even more. However, the calculations are very fast, so the increase in the number of iterations is not very significant. It is useful to record the criteria throughout the iterations to interrupt or adapt the process in cases of non-convergence. The final q_M is not affected by β ; only an unnecessarily low β is responsible for a higher number of iterations.

In regions with high gradients of the F -function, convergence of the iterative procedure is generally not achieved, even for a very low value of β . Therefore, it was necessary to identify the roots using the Complex Bisection Method. This method was implemented as follows: first, a rectangular region where the prospective q_M should be located was identified; then, a grid of points was introduced and Equation (28) was evaluated at each of these grid points. Next, zero-level intersections were found for the real and imaginary parts. The rectangular region was then reduced so that the zero-level intersections could be approximated by straight lines. Finally, the intersection of these lines was calculated, and the rectangular domain was reduced around this new position, and the process was repeated until the required precision. This iterative process is commonly applied in solving complex equations. These two iterative techniques proved to be more efficient than the Argument Principle.

Continuity of F -value, as well as of the real and imaginary parts of tracing roots $\hat{p}'_{ij}, j=1,2$ must be preserved when approaching the final q_M from any direction originating in the unstable region. Usually, two or three options are sufficient. The first elimination of an unsuitable q_M should be performed along a line with fixed q_M^r .

However, it can happen that tracing is mathematically correct, but the final F -value is different. This is illustrated in the example that follows. In this example, one of the final traced roots loses its positive imaginary part, even though at the final stage $n=4$. Such a F -value is usually inappropriate, which can be recognized from the resulting unusual deflection shape. As will be seen in the validation examples, however, such values can exceptionally still identify a valid traced q_M . The data selected corresponds to one of the cases discussed later in this paper, namely: $\mu_s=4$, $\kappa_p=1.1$. There is also a small amount of damping and almost negligible shear stiffness, as exemplified later; these parameters are not important for the explanation that follows.

Let $\alpha=1.053$ be assumed. The target $q=1.2233\cdot 10^{-2}+i8.8731\cdot 10^{-3}$, at which the F -function is to be evaluated, was selected during a detailed analysis of this case. Tracing started at $q_i=-1\cdot 10^{-5}$ and used 2000 steps with geometric spacing in q_i with a ratio of 0.5, ensuring smaller steps farther from the unstable region. Tracing with constant q_r terminated with $F(0,q)=(3.8-i9.49)\cdot 10^{-13}$, while uniform spacing in q_r starting with $2.2328\cdot 10^{-3}$ ended with $F(0,q)=-4.658+i3.431$. Figures 5 and 6 demonstrate that tracing was correctly accomplished in both cases.

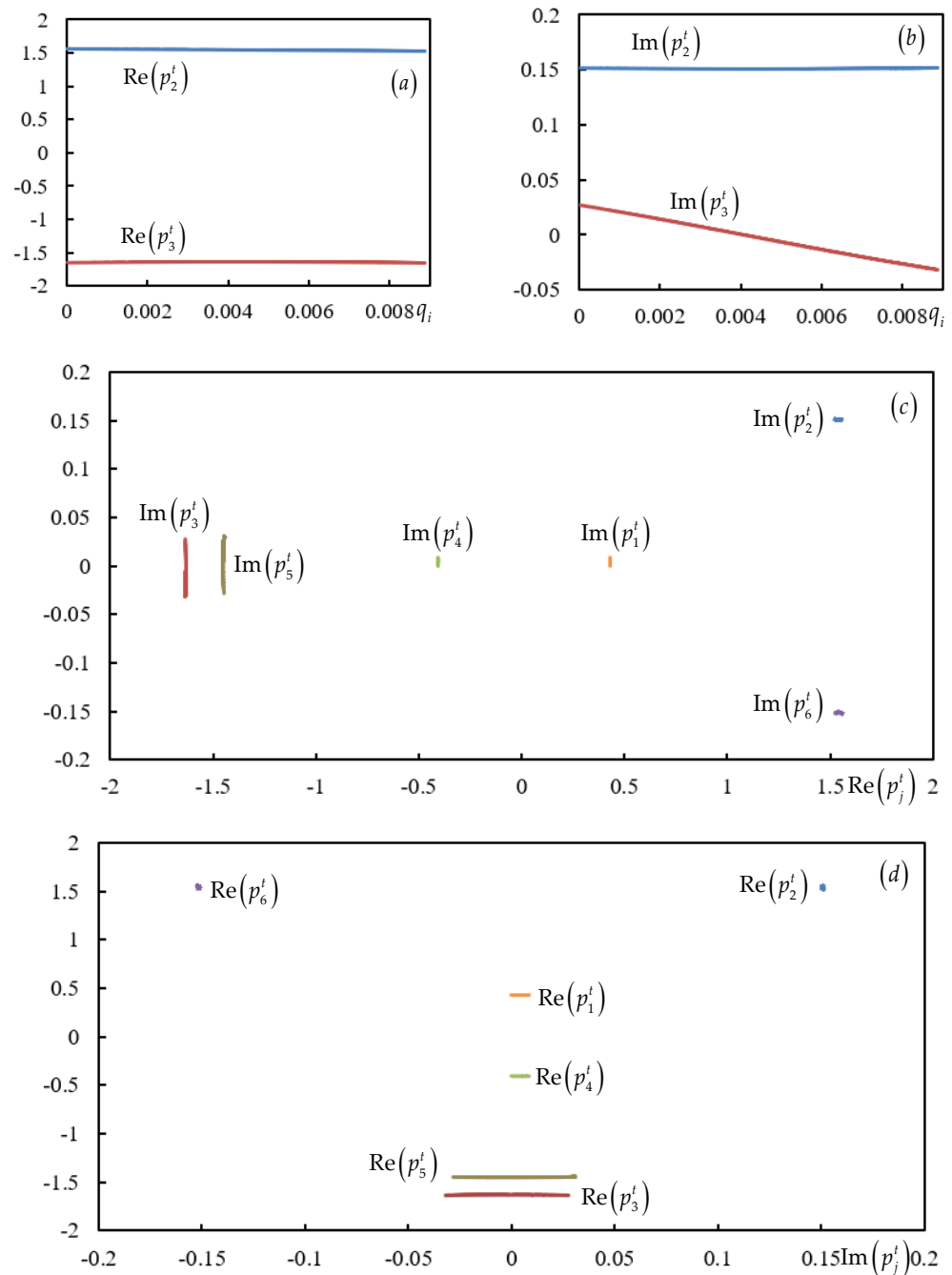
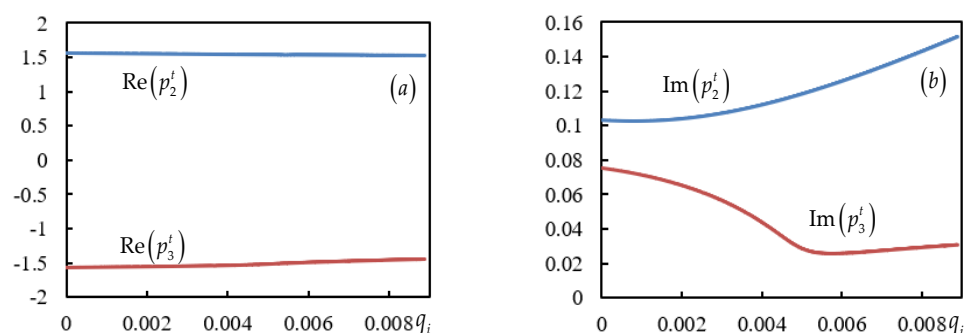


Figure 5. Tracing with constant q_r : (a) real part of selected roots; (b) imaginary part of selected roots; (c) imaginary part of all roots as a function of their real part; (d) real part of all roots as a function of their imaginary part (orange: 1; blue: 2; red: 3; green: 4; khaki: 5; violet: 6).



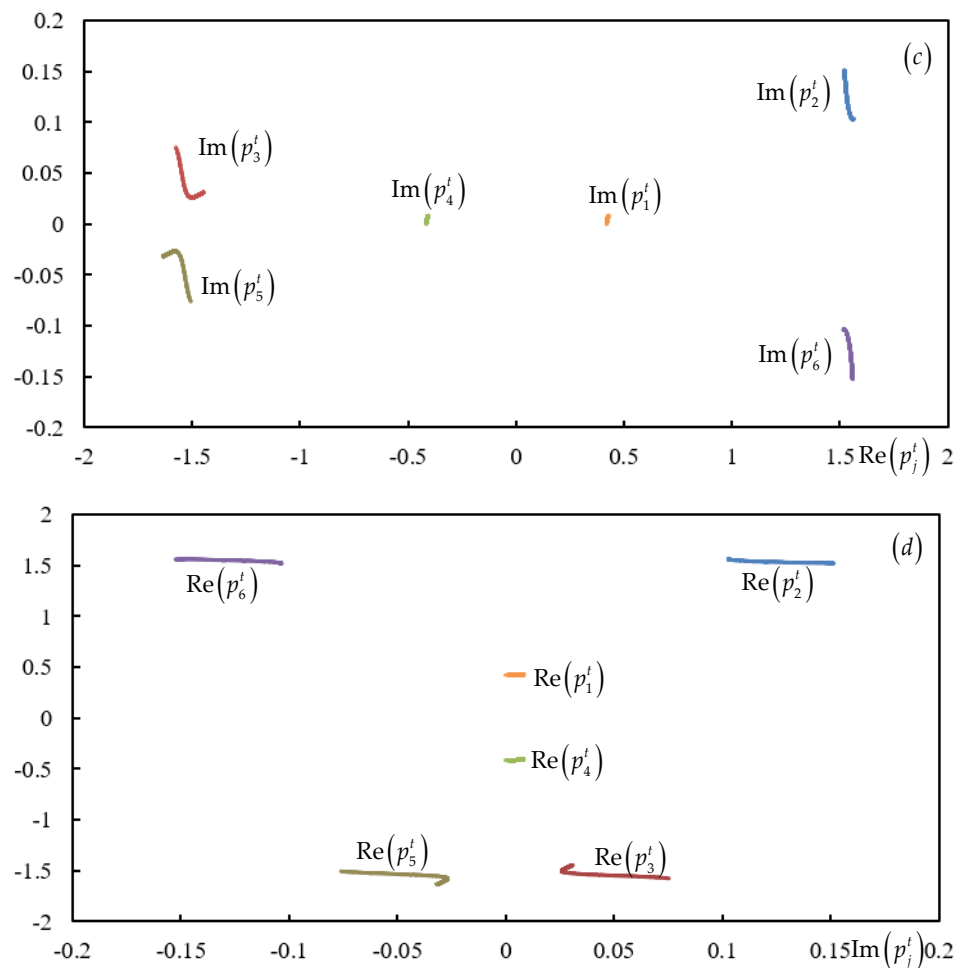


Figure 6. Tracing along inclined path in q_r : (a) real part of selected roots; (b) imaginary part of selected roots; (c) imaginary part of all roots as a function of their real part; (d) real part of all roots as a function of their imaginary part (orange: 1; blue: 2; red: 3; green: 4; khaki: 5; violet: 6).

Figures 5 and 6 show the real and imaginary parts of all six p -roots along the tracing paths from different perspectives. Parts (c) and (d) of these figures are not very clear, but this is intentional, as the purpose is to show that all roots remain sufficiently separated and therefore that no incorrect switching occurred during tracing. As seen in part (b) of Figure 5, root number 3 changes the sign of its imaginary part, whereas in Figure 6b it remains on the same side of the real axis.

In general, it is preferable to identify complete frequency lines, as they are generally smooth except in small regions around the critical velocity (CV) or pseudo-critical velocity (PCV). It should be noted that, in the unstable region, when $q_i < 0$, then $n=2$, that is, these two roots have a positive imaginary part, and a rule of selection the roots with the largest imaginary part can be established. Hereafter, this rule will be referred to as the fundamental rule. It will be demonstrated in the examples section that the traced roots can be determined, in most cases, by this rule, even if afterwards $n=3$ or $n=4$, the latter being much more common than the former. This can significantly speed up root identification, especially when the entire frequency line is followed. In such cases, the new q_M can be estimated by linear or quadratic extension of the frequency line identified so far, the tracing can be omitted, and the roots at these specific positions can be used to calculate the F -function. Then, the number of iterations can be reduced by approximately a factor of 5. More details about the p -roots positions will be given in the examples section, because, unfortunately, this rule fails in some cases. When the fundamental rule fails, strict

tracing according to Equations (29)–(31) must be used. Proximity between estimated positions and new positions of the traced roots is sufficient to follow their path.

5. Alternative Approach

The alternative approach, exemplified in Section 3 on Figure 4, is based on the idea of modifying some parameters of the problem at hand that have a negligible influence on the behavior, in order to allow for the existence of the true q_M .

Without having correct results, it is difficult to decide which parameters are significant and why. To help make informed decisions, the energy contribution is analyzed. Energy conservation is fulfilled for any deflection shape, regardless of whether it corresponds to the correct solution or not. Nevertheless, it is possible to quantify how important each part is for global conservation. Since the input is localized and associated with the Dirac delta function, the mechanical energy is differentiated with respect to time and integrated along the model with respect to the spatial coordinate. Therefore, the energy balance is applied to the energy rate.

The moving force responsible for the steady-state deflection can be omitted, and attention can be focused solely on the moving mass.

Assuming that the dominant frequency q_M is known, deflection shapes can be calculated for each wave number, transformed into real-valued functions, and expressed either in moving coordinates or, with the help of a kinematic invariant, in fixed coordinates. The entire wave ahead of the CP and behind the CP can then be used separately in the following formulas for moving coordinates:

$$U_{b,\tau} = \tilde{w}_{,\xi\xi\xi} (\tilde{w}_{,\xi\xi\xi} - \alpha \tilde{w}_{,\xi\xi\xi}) , \quad U_{p,\tau} = 4\kappa_p (\tilde{w} - \tilde{u}) (\tilde{w}_{,\tau} - \alpha \tilde{w}_{,\xi} - \tilde{u}_{,\tau} + \alpha \tilde{u}_{,\xi}) \quad (32)$$

$$U_{f,\tau} = 4\tilde{u} (\tilde{u}_{,\tau} - \alpha \tilde{u}_{,\xi}) , \quad U_{s,\tau} = 4\eta_s \tilde{u}_{,\xi} (\tilde{u}_{,\xi} - \alpha \tilde{u}_{,\xi\xi}) \quad (33)$$

$$T_{b,\tau} = 4 (\tilde{w}_{,\tau\tau} - 2\alpha \tilde{w}_{,\xi\tau} + \alpha^2 \tilde{w}_{,\xi\xi}) (\tilde{w}_{,\tau} - \alpha \tilde{w}_{,\xi}) , \quad T_{s,\tau} = 4\mu_s (\tilde{u}_{,\tau\tau} - 2\alpha \tilde{u}_{,\xi\tau} + \alpha^2 \tilde{u}_{,\xi\xi}) (\tilde{u}_{,\tau} - \alpha \tilde{u}_{,\xi}) \quad (34)$$

$$Q_p = 8\eta_p (\tilde{w}_{,\tau} - \alpha \tilde{w}_{,\xi} - \tilde{u}_{,\tau} + \alpha \tilde{u}_{,\xi})^2 , \quad Q_f = 8\eta_f (\tilde{u}_{,\tau} - \alpha \tilde{u}_{,\xi})^2 \quad (35)$$

where U denotes potential energy, T kinetic energy, and Q rate of energy loss. The subscripts have the same meaning as those used for the basic parameters (as specified in Section 2). The contribution of the axial force is omitted for simplicity, as it is not used in the following examples; however, it can be readily adapted from the shear contribution. The designations for displacements are not modified, even though, for the purposes of this analysis, only the transient (unsteady harmonic) part of the full solution is used.

To complete the energy balance, Equations (32)–(35) are applied separately to the waves ahead of and behind the CP. A sufficient integration domain is then selected and, after integration, all contributions are summed. This time-dependent total energy rate must match the input energy rate supplied by the moving mass, given by

$$P_M = -4\eta_M \tilde{w}_{,\tau\tau} (\tilde{w}_{,\tau} - \alpha \tilde{w}_{,\xi}) \Big|_{\xi=0} \quad (36)$$

where both types of waves can be introduced, ahead of or behind the CP, because only the value at the CP matters. Alternatively, the energy flux across the contact region:

$$S_+ = \tilde{w}_{,\xi\xi\xi} (\tilde{w}_{,\tau} - \alpha \tilde{w}_{,\xi}) - \tilde{w}_{,\xi\xi} (\tilde{w}_{,\xi\tau} - \alpha \tilde{w}_{,\xi\xi}) - 4\eta_s \tilde{u}_{,\xi} (\tilde{u}_{,\tau} - \alpha \tilde{u}_{,\xi}) \Big|_{\xi=0+} \quad (37)$$

can be used. Here, for the correct value at the CP, the separation of waves is important. Equation (37) is presented for waves ahead of the CP. Similarly, for the waves behind the CP, the difference must match the energy rate input. Several terms cancel each other due to the continuity of the deflection shape and its first and second spatial derivatives; therefore, only the term containing the third derivative of the beam deflection remains. As

expected, this balance is equivalent to the shear-force equilibrium at the contact point; therefore, Equation (37) can be simplified to

$$S_+^{red} = \tilde{w}_{,\xi\xi\xi} \left(\tilde{w}_{,\tau} - \alpha \tilde{w}_{,\xi} \right) \Big|_{\xi=0+} \quad (38)$$

reducing the energy balance to the well-known relation

$$-4\eta_M \tilde{w}_{,\tau\tau} \Big|_{\xi=0} - \tilde{w}_{,\xi\xi\xi} \Big|_{\xi=0+} + \tilde{w}_{,\xi\xi\xi} \Big|_{\xi=0-} = 0 \quad (39)$$

Wave separation is not only convenient for obtaining the correct values at the CP, but also for determining the integration length, because the waves behind the CP typically extend over a larger distance from the CP.

6. Justification of Selected Cases and Results

6.1. Cases Selected for Validation

Dominant frequencies determined using the methods described in the previous section are presented for several selected cases to provide validation. It is impossible to run examples over the full range of admissible parameters; nevertheless, each selected case spans a broad range of physical parameters. Therefore, the selected cases are considered sufficient to demonstrate the robustness of the proposed technique by covering qualitatively different regimes.

The same limitations as those observed for cases with true roots q_M , discussed in previous publications, are also encountered here. First, the unsteady harmonic component is unable to reproduce the initial conditions accurately; therefore, there is always a small, generally acceptable, difference at the initial times. This difference is negligible when the initial transient amplitude matches the steady-state deflection and increases with the difference between these two quantities. As it affects only a very narrow time interval, it can be disregarded.

Second, withing a narrow α -interval around the CV or PCV, the vibrations are over damped and dominated by the steady-state component. Consequently, the transient amplitude is relatively low and may contain a strong initial component that is difficult to approximate by harmonic motion. As the transient part ceases rapidly, leaving only the steady-state vibration, these cases are not particularly relevant to the objectives of the new analytical technique. Both limitations are considered acceptable given the advantages of the analytical approach compared with numerical integration.

Regarding the assessment of the accuracy of the q_M -estimates obtained, visual verification is, in fact, the most appropriate measure. Validation is therefore provided by comparing deflection shapes as a function of τ or ξ , obtained by numerical integration and by the proposed analytical approach. This provides the most direct and effective means of assessing the quality of the approximation.

Four cases are selected, and variants of these cases are also considered.

Only the first one has three resonances, because q_M is particularly important in cases with only one resonance. In such cases, the onset of instability can be shifted far beyond the PCV, creating a large α -interval in which the q_M and corresponding vibration patterns are important to determine.

The leading parameters of the selected cases are:

- Case 1: $\mu_s = 3$ and $\kappa_p = 5$;
- Case 2: $\mu_s = 5$ and $\kappa_p = 3$;
- Case 3: $\mu_s = 4$ and $\kappa_p = 4$;
- Case 4: $\mu_s = 4$ and $\kappa_p = 1.1$.

The first two cases are accompanied by a lower mass ratio $\eta_M = 40$, last two cases serve as limiting cases, in which the moving mass is related to the full typical axle load.

Thus, Cases 1 and 2 were selected to demonstrate the difference between one and three resonances. Case 3 was selected as a limiting case, and Case 4 as a specific situation related to Case 3, in which three asymptotes to the instability lines are recovered after a region with only one asymptote. Namely, considering κ_p with one decimal digit of precision, three asymptotes exist for $\kappa_p = 0.1$. The interval $\kappa_p \in \langle 0.2; 1.0 \rangle$ contains only one asymptote, while three asymptotes are recovered starting from $\kappa_p = 1.1$.

Undamped cases with no shear stiffness must use tracing from the very beginning, i.e., from the lowest α . In some cases, the step discontinuity can be disregarded, but generally, tracing provides better results because q_M should have a positive imaginary part, even if its value is negligible. Regarding instability, the absence of damping does not lead to behavior resembling the undamped one-layer model. Instability does not start at the CV but is shifted to higher velocities due to the activated radiation damping. Within the unstable region, true q_M exist.

Damped cases still have true q_M at some lower velocities, but tracing becomes necessary beyond a certain point. Difficulties are encountered around the higher CV, where stability is recovered. These difficulties are mainly numerical because the F -function exhibits high gradients in this region, and therefore increased numerical precision is required for its determination.

If a true q_M exists but is located within the region where $n=4$, its traced version also exists. These values are usually very close to each other, and therefore the deflection at the CP is essentially the same. Nevertheless, the deflection along ξ is always better approximated using the traced version. The differences are usually more pronounced in the sleepers' layer.

6.2. Case 1

Case 1 was selected because it has two CVs, corresponding to three resonances. However, the middle resonance is the false critical velocity (FCV), which has no influence on instability or on the other features of the cases analyzed [23].

No variants of this case are analyzed; only the basic option defined by $\mu_s = 3$, $\kappa_p = 5$ and $\eta_M = 40$ is considered. All other parameters are zero. This is therefore a fully undamped case in which all q_M must be traced, except for those in the unstable ones. No true q_M exists in the stable region.

Some basic characteristics are listed in Table 1.

Table 1. Basic characteristics of Case 1.

CV1	FCV	CV2	AS1	AS2	AS3
0.490606	0.491454	1.553524	0.490606	1.553524	1.553524
SW1	SW2	SW3	SW4	SW5	
0.490639	0.492201	0.585806	1.553206	1.553551	

Besides the abbreviations already defined, SW denotes a switch between stable and unstable regions, while AS denotes a vertical asymptote to the instability lines.

There is a recovery of stability in the first region (between AS1 and AS2) due to a particular evolution of the first instability-line branch, as shown in Figure 7. In part (a) of this figure, only the scaled part of the first instability-line branch is shown including the region where it is crossed by $\eta_M = 40$; in part (b) the corresponding q_r is given for both regions.

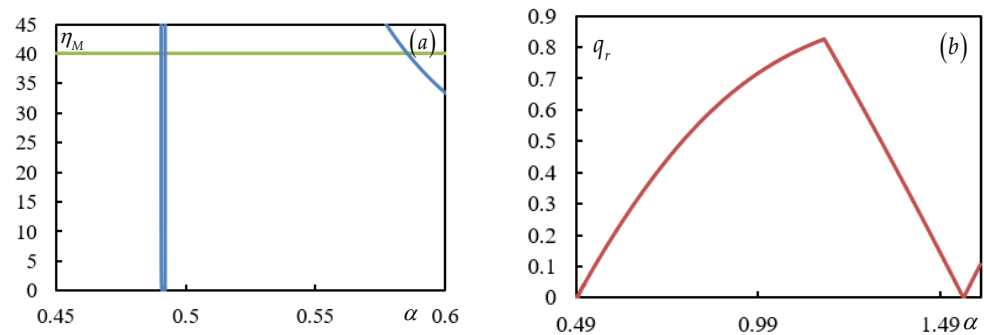


Figure 7. Case 1: (a) First branch of instability lines; (b) corresponding q_r for both branches, (horizontal green line marks the value of η_M).

One would expect another recovery around CV2. However, CV2, AS2, and AS3 are numerically practically indistinguishable, and SW4 and SW5 exhibit negligible differences with respect to them. It is necessary to highlight that the instability lines and their asymptotes have to be calculated with negligible damping for numerical stability. $\eta_f = 1 \cdot 10^{-20}$ was selected for this purpose. By lowering this value even further, CV2, AS2, and AS3 converge. It can thus be concluded that the interlayer flux is not strong enough at higher velocities, where this case resembles the fully undamped situation with respect to the loss and recovery of instability, although not with respect to the vibration pattern.

Therefore, in Figure 8, the frequency line is shown only up to $\alpha = 0.7$, because there is no point in showing frequency lines for higher velocities where true q_M exist.

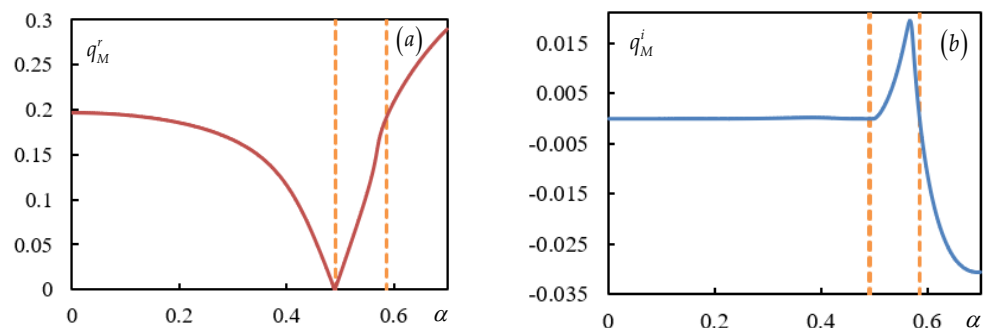


Figure 8. Case 1: (a) Real part of q_M ; (b) Imaginary part of q_M . (SW1, SW2 and SW3 are represented by orange dotted lines in increasing order).

In Figure 8, only SW1, SW2, and SW3 are added. Nevertheless, SW1 and SW2 appear almost coincident. From Table 1, it can be seen that instability is first lost slightly above CV1, indicating a small amount of damping in the form of activated radiation damping.

Figure 8 also shows a decrease in q_M^r and an increase in q_M^i around CV1. The increase in q_M^i is not very clear because of the scale, as there is another, much larger increase at higher α .

It is noteworthy that all traced q_M are obtained from the fundamental rule of p -roots selection.

Deflections at the CP from several traced q_M and one exact q_M from the unstable region are validated in Figure 9.

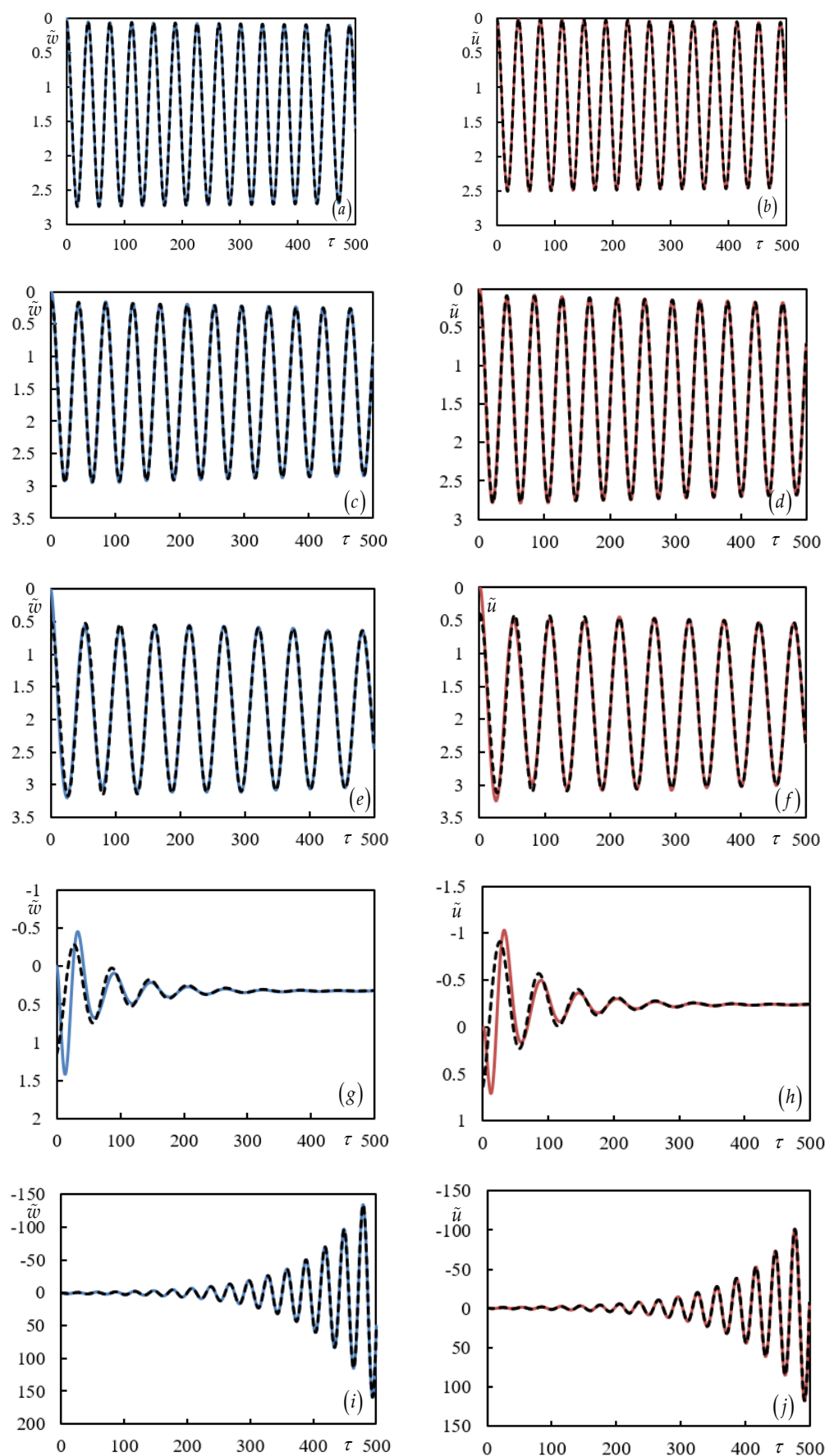
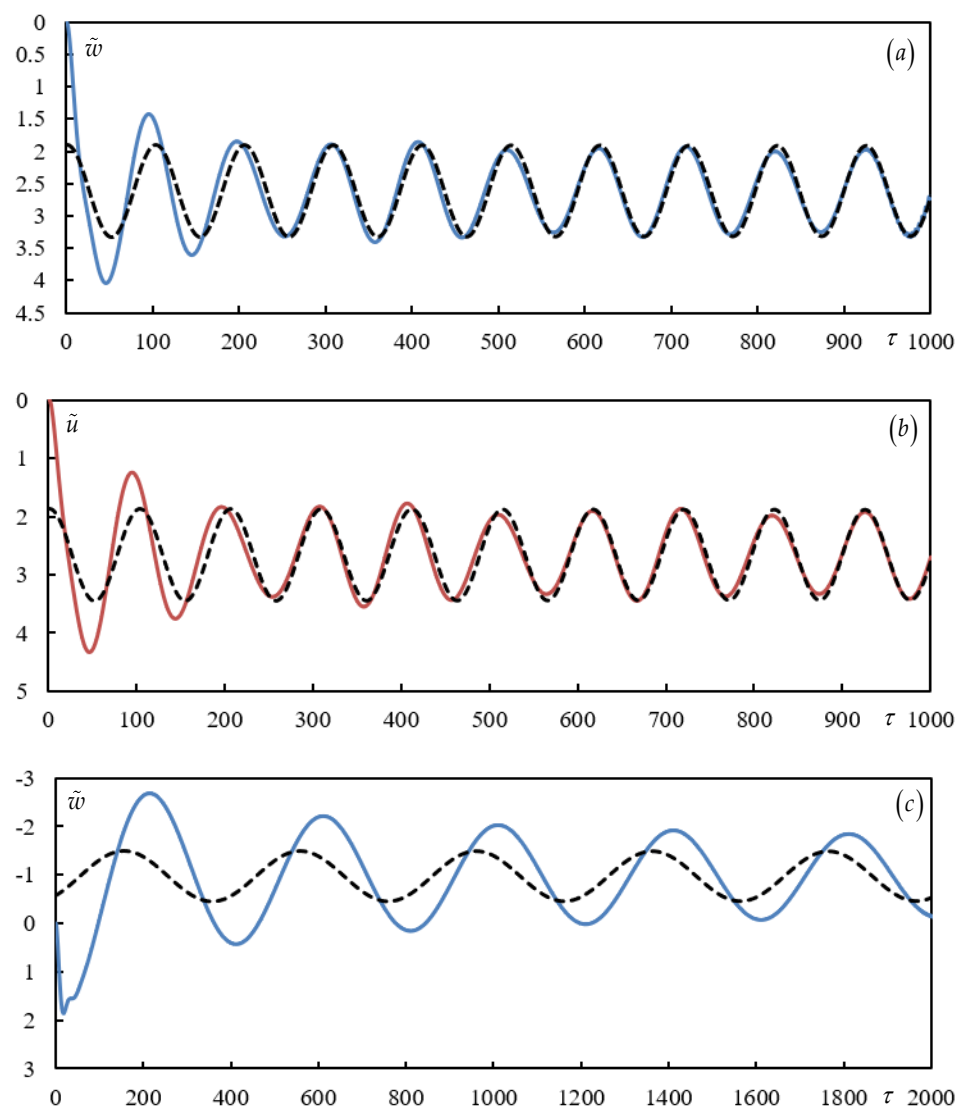


Figure 9. Case 1: Deflection of the CP as a function of τ : (a) $\alpha=0.3$ beam; (b) $\alpha=0.3$ sleepers' layer; (c) $\alpha=0.35$ beam; (d) $\alpha=0.35$ sleepers' layer; (e) $\alpha=0.4$ beam; (f) $\alpha=0.4$ sleepers' layer; (g) $\alpha=0.45$ beam; (h) $\alpha=0.45$ sleepers' layer; (i) $\alpha=0.5$ beam; (j) $\alpha=0.5$ sleepers' layer.

layer; (g) $\alpha = 0.55$ beam; (h) $\alpha = 0.55$ sleepers' layer; (i) $\alpha = 0.6$ beam; (j) $\alpha = 0.6$ sleepers' layer (blue solid line: numerical integration—beam; red solid line: numerical integration—sleepers' layer; black dotted line: analytical approximation).

Figure 9a,b show that agreement would be excellent for even lower velocities where the effect of radiation damping would be less visible. Very good agreement was obtained for the range of velocities in Figure 9.

As already mentioned, around CV1 the agreement is worse, and a longer time interval must be analyzed to allow the unsteady harmonic vibration to approach the numerical integration results, as demonstrated in Figure 10. The same feature occurs for true q_M , as reported in previous author's publications.



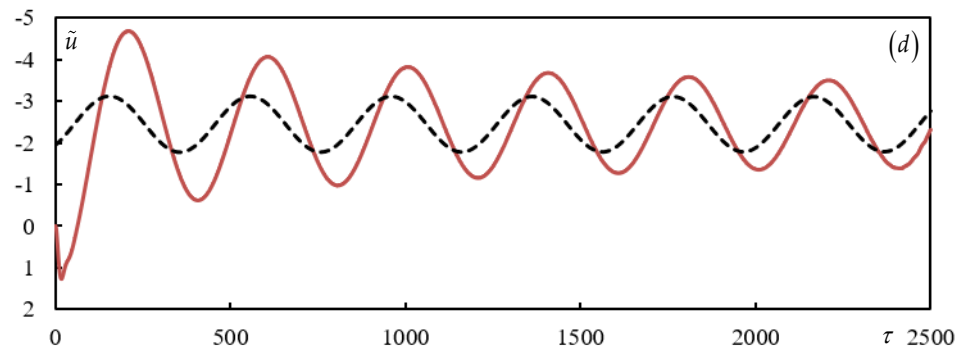


Figure 10. Case 1: Deflection of the CP as a function of τ : (a) $\alpha = 0.45$ beam; (b) $\alpha = 0.45$ sleepers' layer; (c) $\alpha = 0.5$ beam; (d) $\alpha = 0.5$ sleepers' layer (blue solid line: numerical integration—beam; red solid line: numerical integration—sleepers' layer; black dotted line: analytical approximation).

For the velocities selected for demonstration, $\alpha = 0.5$ in Figure 10c,d exhibit the poorest behavior because this value is practically coincident with the CV. The fit would be better at larger times; however, due to the numerical precision of the numerical integration, this could not be demonstrated. The poorer fit is not caused by the fact that q_M was traced, but by the ability of the unsteady harmonic component associated with the dominant frequency to represent the transient solution. Thus, this is an inherent feature of the present type of analysis and is not related to the new approach presented in this paper. In any case, these cases are of limited practical interest because the steady-state deflection at the contact point is negative and the vibration pattern is strongly dominated by the steady-state component.

6.3. Case 2

By changing the leading values to $\mu_s = 5$ and $\kappa_p = 3$, a case with only one resonance is obtained. $\eta_M = 40$ is kept. As this case is more relevant to the objectives of this paper, several variants are considered, as listed below:

- Variant 1: $\eta_f = \eta_s = \eta_p = 0$;
- Variant 2: $\eta_f = 0.01$, $\eta_s = \eta_p = 0$;
- Variant 3: $\eta_f = 0.01$, $\eta_s = 0.1$, $\eta_p = 0$;
- Variant 4: $\eta_f = 0.01$, $\eta_s = 0.1$, $\eta_p = 0.05$.

The resonance value is calculated from analytical expressions. Damping is omitted because, when damping is included, true resonance never occurs and the CV is therefore indistinguishable from the PCV. Consequently, CV is calculated for Variant 1 and for Variant 3 with damping removed, which also serves for Variant 4.

First, basic characteristics for Case 2, Variant 1 are listed in Table 2.

Table 2. Basic characteristics of Case 2, Variant 1.

CV	AS1	AS2	AS3	SW1	SW2	SW3
1.347984	0.399933	1.347984	1.347984	0.582612	1.347673	1.348005

Results of parametric analysis confirming the existence of the PCV and the exact value of CV are shown in Figure 11.

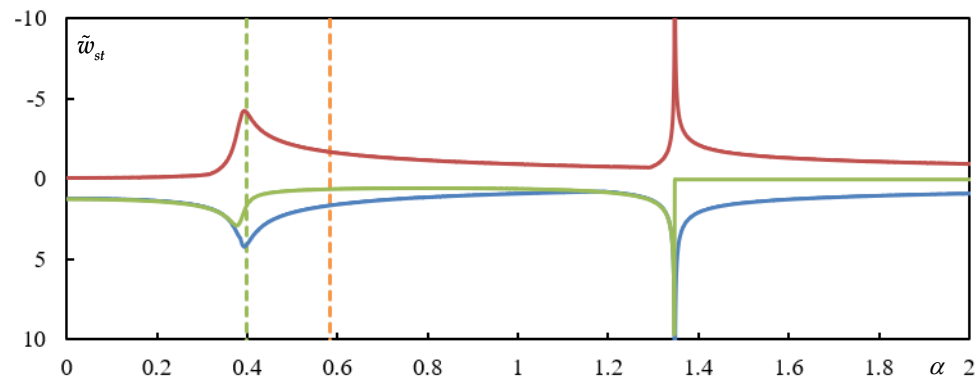


Figure 11. Case 2, Variant 1: Parametric analysis with respect to α of the beam steady-state deflection. (maximum and minimum deflections over the entire beam: blue and red solid line; deflection at the CP: green solid line; AS1: green dotted line; SW1: orange dotted line).

The PCV value obtained from the parametric analysis is ambiguous because the extreme values at the CP and the maximum and minimum values over the entire beam are obtained for different α . For this reason, PCV is marked by AS1. Vertical lines marking CV, AS2, and AS3 are not included in Figure 11 because they are located at the resonance, which would make the graph unclear. Moreover, their values are practically indistinguishable.

As in Case 1, frequency line is smooth around CV and stability does not recover. Although Table 2 indicates SW2 and SW3, they were calculated using negligible damping $\eta_f = 1 \cdot 10^{-20}$. As this value is reduced, SW2 and SW3 approach each other and eventually coincide. The instability lines are calculated by finding a real-valued K -function. This search should also be adapted to the tracing procedure because, when the discontinuity is located at $q_i = 0$, identification of q_r for which $\text{Im}(K)$ switches sign is compromised. As this is not the objective of this paper, it is sufficient to conclude that undamped cases do not recover stability around the higher CV, as demonstrated by the frequency lines. At such high velocities, the radiation damping is not strong enough to change the typical distribution of instability lines for undamped cases, which is characterized by a very steep increase/decrease in η_M in the vicinity of AS followed by negligible η_M over the remainder of the region.

In Figure 12a, part of the first branch is shown to delimit SW1, and the corresponding q_r is plotted over the region encompassing CV in Figure 12b.

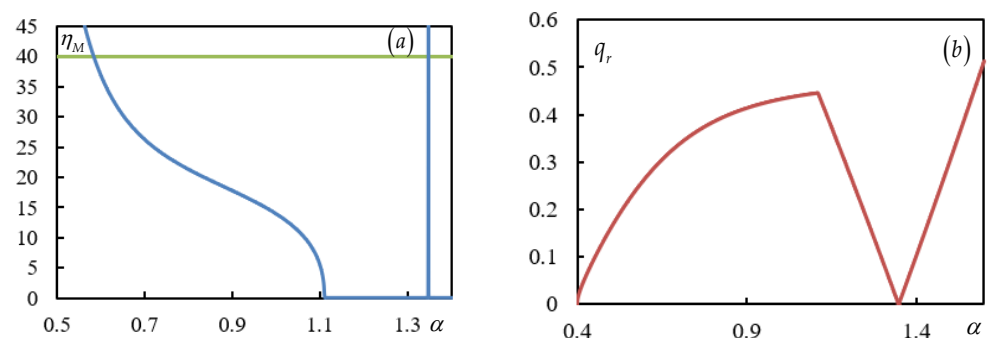


Figure 12. Case 2, Variant 1: (a) First branch of the instability lines; (b) corresponding q_r for both branches, (horizontal green line marks the value of η_M).

Unlike Case 1, there is no unexpected stability recovery in the first region, between AS1 and AS2, and SW1 is located well ahead of AS1 (see Figure 11), demonstrating that

the interlayer flux around the PCV is relatively strong and that the activated radiation damping shifts the onset of instability to higher velocities.

For completeness, frequency lines are shown over a larger range in Figure 13.

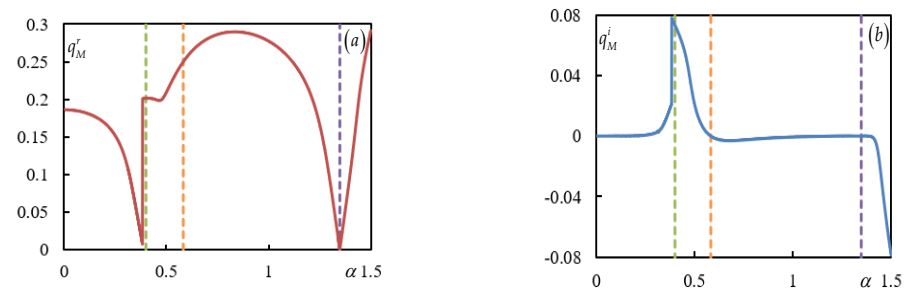


Figure 13. Case 2, Variant 1: (a) Real part of q_M ; (b) Imaginary part of q_M . (AS1: green dotted line; SW1: orange dotted line; CV: violet dotted line).

Except in the unstable region, which starts at SW1, all q_M -values are obtained by tracing with the help of the fundamental rule. There are no true roots in the stable region. Although the true q_M covers a relatively large interval here, it is included to demonstrate that there is no stability recovery.

q_M^i have extremely small negative values around CV. The lowest value in the graph in Figure 12 is obtained for $\alpha = 1.3479$ and equals $q_M^i = -9.31 \cdot 10^{-27}$. Its determination required careful application of the Complex Bisection Method and a large number of digits in the calculation.

A new feature is detected around the PCV. Around the PCV, there is a sudden jump in q_M -values. In fact, the branch for higher values of α could be extended to overlap with lower α -range; however, the additional q_M do not contribute significantly to the deflection at the contact point because of the high value of q_M^i . In any case, this is the region where differences are expected because the steady-state vibration is dominant and is therefore not particularly relevant to the objectives of this paper. It would be difficult to obtain agreement between the CP deflection obtained by numerical integration and the analytical estimate using either of these two branches, since the CP deflection is represented by a typical overdamped evolution that rapidly approaches the steady-state value. This also makes the exact location of this jump uncertain.

Very good agreement is obtained for all α except in the region approximately from 0.36 to 0.37, where there is practically no transient vibration. As these graphs are similar to those included in Figures 9 and 10, they are not included. Only the case with $\alpha = 0.3$, which was used in Section 3, is shown in Figure 14.

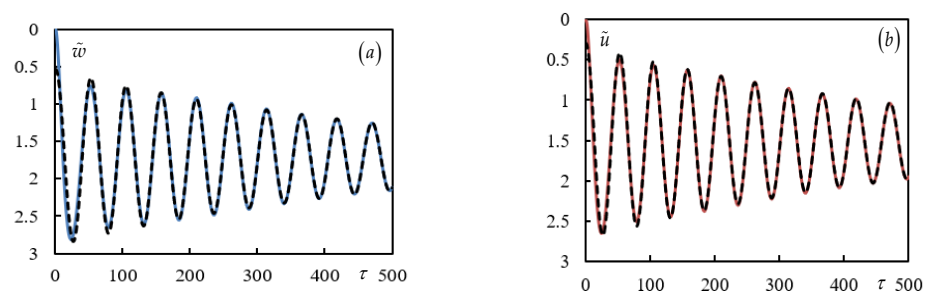


Figure 14. Case 2, Variant 1: Deflection of the CP as a function of τ for $\alpha = 0.3$: (a) beam; (b) sleepers' layer (blue solid line: numerical integration—beam; red solid line: numerical integration—sleepers' layer; black dotted line: analytical approximation).

Related deflections along the beam and sleepers' layer are also included for $\tau = 409$ and $\tau = 444$ in Figure 15. The large time value was selected to ensure an exact match at the CP; the two values then correspond to an intermediate value and a local maximum.

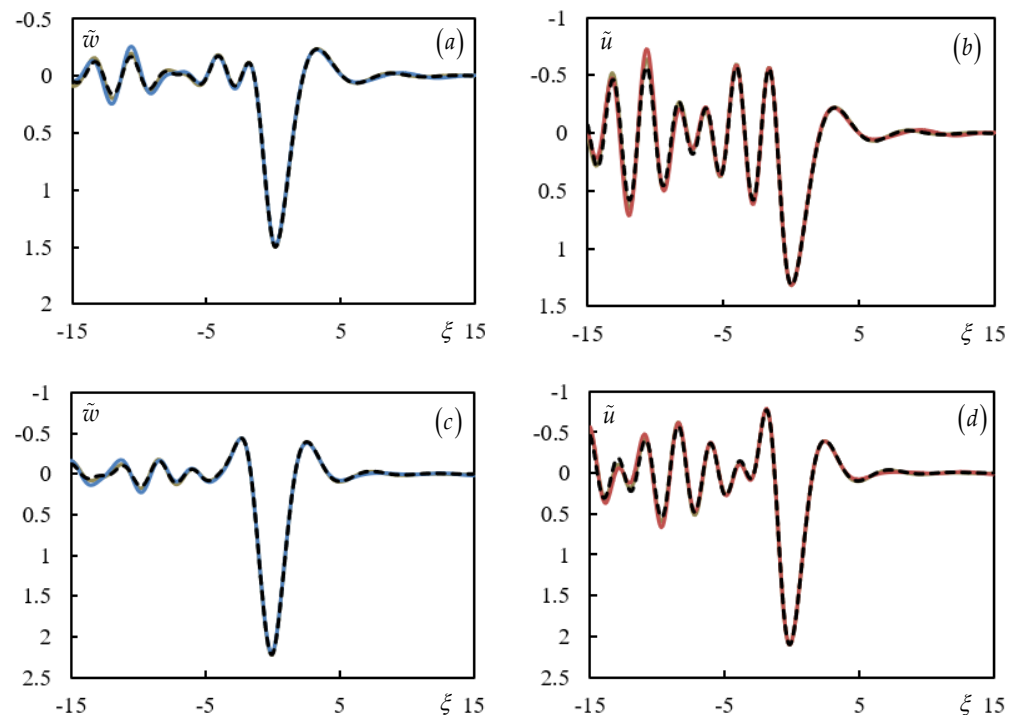


Figure 15. Case 2, Variant 1: Deflections as a function of ξ for $\alpha=0.3$: (a) $\tau=409$, beam; (b) $\tau=409$, sleepers' layer (c) $\tau=444$, beam; (d) $\tau=444$, sleepers' layer (blue solid line: numerical integration—beam; red solid line: numerical integration—sleepers' layer; black dotted line: analytical approximation with adaptation; khaki solid line: analytical approximation without adaptation).

It can be concluded from Figure 15 that the agreement is very good. It is necessary to point out that the F -value identifies the CP deflection well. However, when the corresponding p -roots are used consistently to determine the deflection along ξ , i.e., when the traced p -roots define the wave ahead of the CP and the remaining p -roots define the wave behind it, some waves may not fulfill the boundary conditions at infinity.

For the case shown in Figure 15, $n=4$. Two of such p -roots are the traced roots and are therefore used to form the wave ahead of the CP. The other two are used for the wave behind the CP and are therefore not fulfilling the boundary conditions at $\xi=-\infty$. Then also the two reminding roots are equally implemented to form the wave behind the CP. Figure 15 demonstrates that this fact is negligible within a reasonable range of interest around the CP. Alternatively, the problematic roots can be used for shape determination without their imaginary parts, which does not violate continuity at the CP. Both versions are plotted in Figure 15, and it can be seen that the non-adapted option, i.e., the one that retains the imaginary parts of problematic roots, provides an even better approximation in this particular case. Such a conclusion cannot be made generally, as there are cases in which omitting the imaginary parts provides better results. Since the adapted option is more general, it is considered preferable. These differences are better visualized in a rescaled version in Figure 16; nevertheless, one cannot disregard the fact that the numerical integration results may be affected by numerical errors.

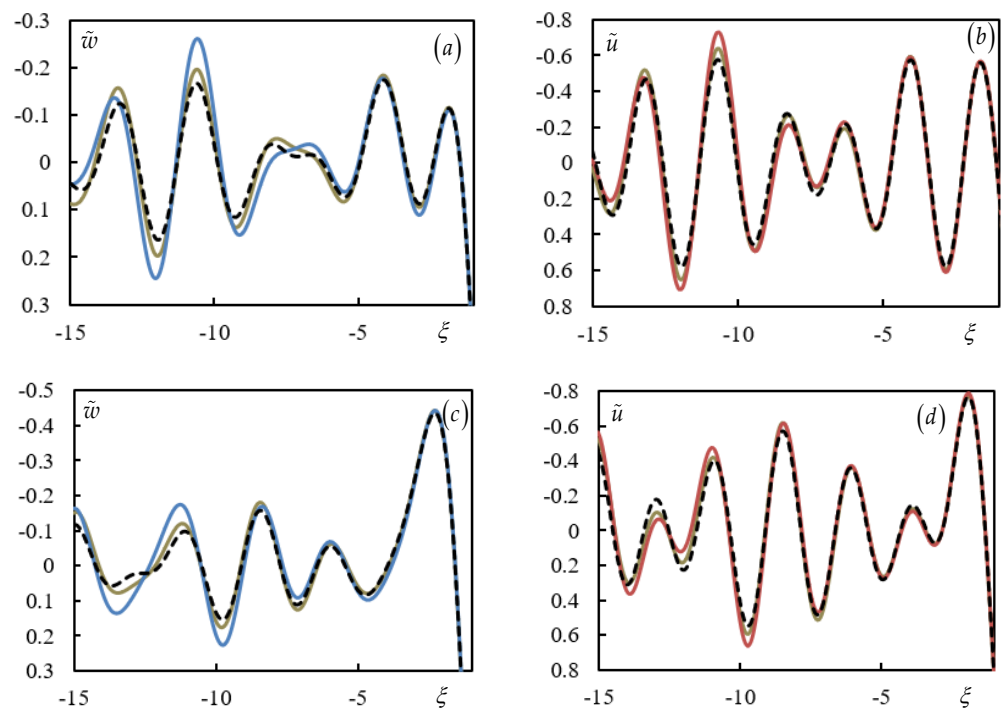


Figure 16. Case 2: Variant 1: Rescaled deflections as a function of ξ for $\alpha=0.3$: (a) $\tau=409$, beam; (b) $\tau=409$, sleepers' layer (c) $\tau=444$, beam; (d) $\tau=444$, sleepers' layer (blue solid line: numerical integration—beam; red solid line: numerical integration—sleepers' layer; black dotted line: analytical approximation with adaptation; khaki solid line: analytical approximation without adaptation).

This case with $\alpha=0.3$ was also implemented in the alternative approach. Either η_f or η_p can be gradually increased until the discontinuity shifts in such a way that a true q_M is found. This happens for $\eta_f = 0.027$ ($\eta_p = 0$) and for $\eta_p = 0.036$ ($\eta_f = 0$).

To obtain the energy rate contributions, it is necessary to integrate along the model length, separately for the waves ahead of the CP and behind it. The functions used are analytical and therefore the integration does not cause complications. It would be possible to derive an analytical form for the values resulting from the integration; however, as this approach was not explored further, the necessary values were calculated in symbolic software. $\xi=100$ and $\xi=200$ were implemented and no differences were found. At first glance, $\eta_f = 0.027$ might appear to introduce a smaller deviation from the undamped case due to its lower value, but this is not the case. The smaller deviation from the original case should be estimated from energy loss at the CP as a function of τ ; namely, the case with less dissipated energy should be chosen. This comparison is presented in Figure 17 for waves ahead of the CP, but the same conclusion is valid for waves behind the CP.

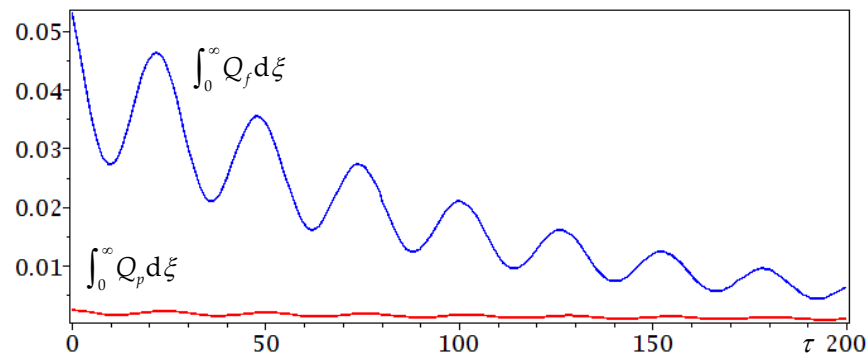


Figure 17. Case 2: Variant 3: Total energy-rate loss due to waves ahead of the CP for $\alpha=0.3$ and modified damping: blue— $\eta_p=0.036$, $\eta_f=0$; red— $\eta_p=0$, $\eta_f=0.027$.

Finally, this conclusion is supported by an analysis of the deflections at the CP. Figure 18 clearly demonstrates that the modification with $\eta_p=0.036$ has a negligible influence, while the one with $\eta_f=0.027$ introduces more damping.

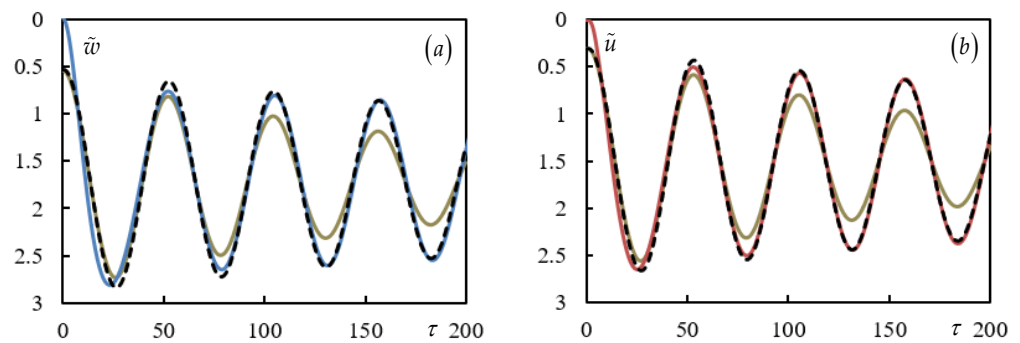


Figure 18. Deflection of the CP as a function of τ for $\alpha=0.3$: (a) beam; (b) sleepers' layer (blue solid line: numerical integration—beam; red solid line: numerical integration—sleepers' layer; black dotted line: adapted by $\eta_p=0.036$ —analytical approximation; khaki solid line: adapted by $\eta_f=0.027$ —analytical approximation).

Also, deflection shapes are practically equal for the original undamped case and the one with $\eta_p=0.036$, as it is demonstrated in Figure 19 for $\tau=80$.

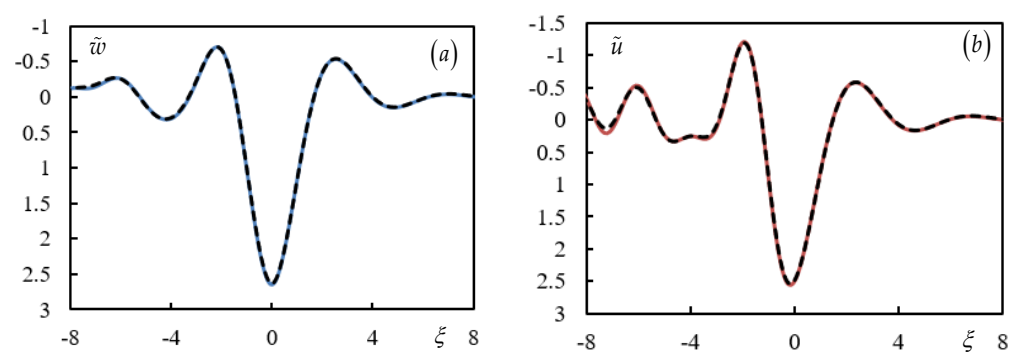


Figure 19. Deflections as a function of ξ for $\alpha=0.3$ and $\tau=80$: (a) beam; (b) sleepers' layer (blue solid line: numerical integration—beam; red solid line: numerical integration—sleepers' layer; black dotted line: adapted by $\eta_p=0.036$ —numerical integration).

Unfortunately, as already mentioned in Section 3, this approach is not universal and can be successfully applied only within a limited interval of velocities.

Case 2, Variant 2 is obtained from Variant 1 by adding low foundation damping, $\eta_f = 0.01$, slight modifications with respect to the previous variant are observed.

First, the basic characteristics are listed in Table 3. CV is retained for the reasons mentioned earlier. AS1 can now be determined exactly for the given damping and is therefore slightly higher than before, as is SW1.

Table 3. Basic characteristics of Case 2, Variant 2.

CV	AS1	SW1	True q_M
1.347984	0.401168	0.586695	0.001–0.276; 0.571–

Table 3 has another entry. As damping shifts the discontinuity in the K -function, the true q_M can be calculated up to a certain α , which is specified there. Additionally, the true q_M are recovered for α slightly lower than SW1. For such true q_M , $n=2$.

Due to damping, stability is expected to recover around CV. However, this region is of limited significance and therefore α is analyzed only up to $\alpha=0.7$.

Frequency lines are shown in Figure 20. It can be observed that there is no change when the true q_M transitions to the traced q_M .

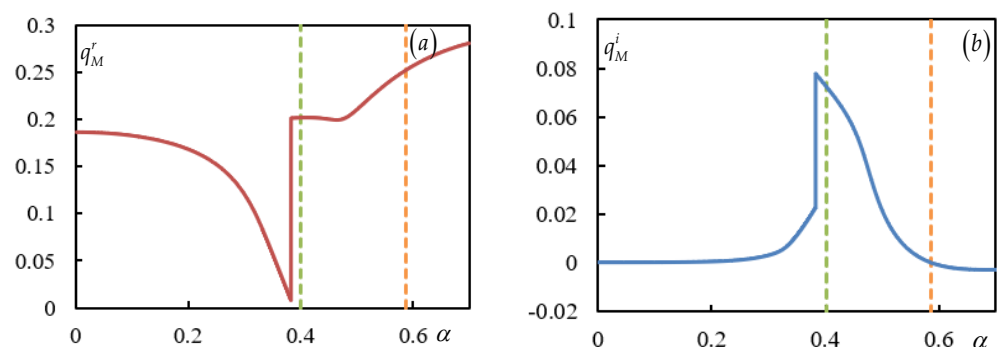


Figure 20. Case 2, Variant 2: (a) Real part of q_M ; (b) Imaginary part of q_M . (AS1: green dotted line; SW1: orange dotted line).

Around the PCV, marked in Figure 20 by AS1, there is again a sudden jump in q_M -values. The evolution of q_M is very similar to that of the previous variant; the only difference is that some q_M are true.

Regarding the deflection shapes at the CP or along ξ , the conclusions are the same as for the previous variants.

Case 2, Variant 3 is obtained from the previous case by adding low shear stiffness. The basic characteristics are summarized in Table 4.

Table 4. Basic characteristics of Case 2, Variant 3.

CV	AS1	SW1	True q_M
1.348327	0.420377	0.599720	0.001–0.308; 0.587–

It should be noted that the true q_M in the initial part of the α -region can be calculated using three p -roots with positive imaginary part, up to $\alpha=0.141$, because in this region $n=3$. However, the deflection shape along the model is better estimated from traced roots whenever $n>2$, as demonstrated further in Variant 4. These roots should

therefore be replaced by the traced ones. Here, the fundamental rule fails. Either the tracing must be applied, or, before selecting the two roots with the largest imaginary part, a root that lies far from the others, as an outsider, should be disregarded. As this decision can be ambiguous, it is better to apply strict tracing as defined by Equations (29)–(31).

The subsequent true q_M are from the region with $n=2$. The region with tracing, $\alpha \in (0.309; 0.587)$, uses the fundamental rule and the true q_M from the region with $n=2$ are recovered again within the stable region, slightly before SW1.

The fundamental rule is also violated around AS1 on the branch with high q_M . As before, rigorous tracking is recommended.

The frequency lines are plotted in Figure 21, demonstrating again that traced q_M smoothly extend the parts with true q_M . For the same reasons as in the previous subsection, the analysis is performed up to $\alpha = 0.7$.

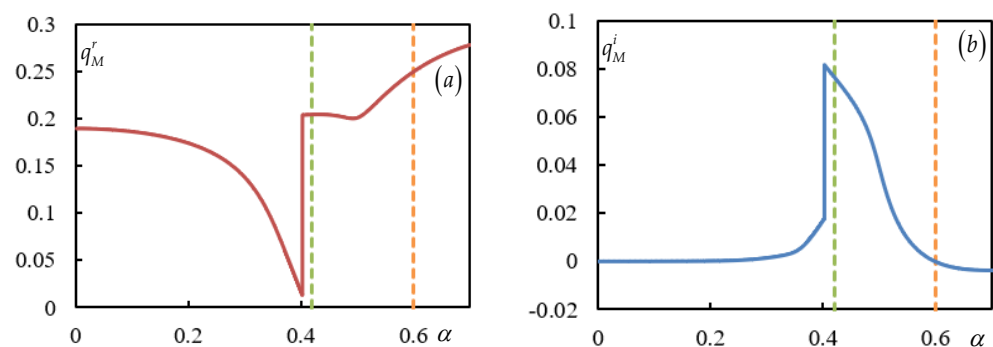


Figure 21. Case 2, Variant 3: (a) Real part of q_M ; (b) Imaginary part of q_M . (AS1: green dotted line; SW1: orange dotted line).

An interesting feature is revealed by the deflection shapes. Assuming $\alpha = 0.14$, the true q_M exists, and the deflections at the CP are given in Figure 22.

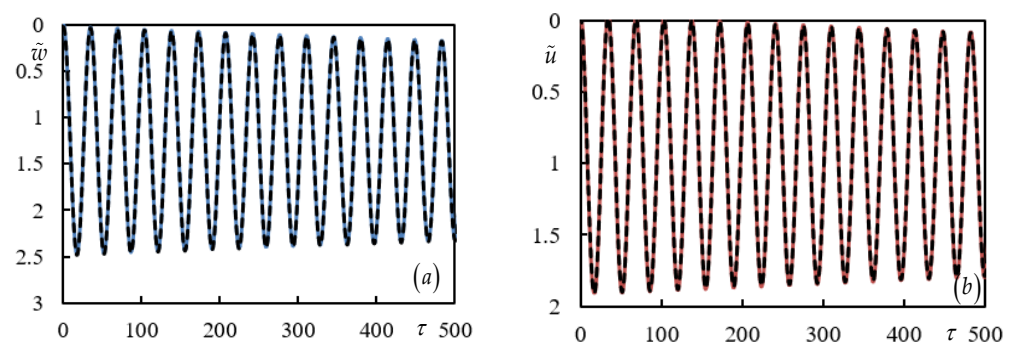


Figure 22. Case 2, Variant 3: Deflection of the CP as a function of τ for $\alpha = 0.14$: (a) beam; (b) sleepers' layer (blue solid line: numerical integration—beam; red solid line: numerical integration—sleepers' layer; black dotted line: analytical approximation).

Calculating the root using the fundamental rule yields a value that is practically indistinguishable from the previous one. However, the deflection shape obtained with the traced root and adaptation fits the numerical integration results better, as demonstrated in Figure 23 for $\tau = 100$. Therefore, the traced q_M can have better properties than the true one, as will also be demonstrated in further sections.

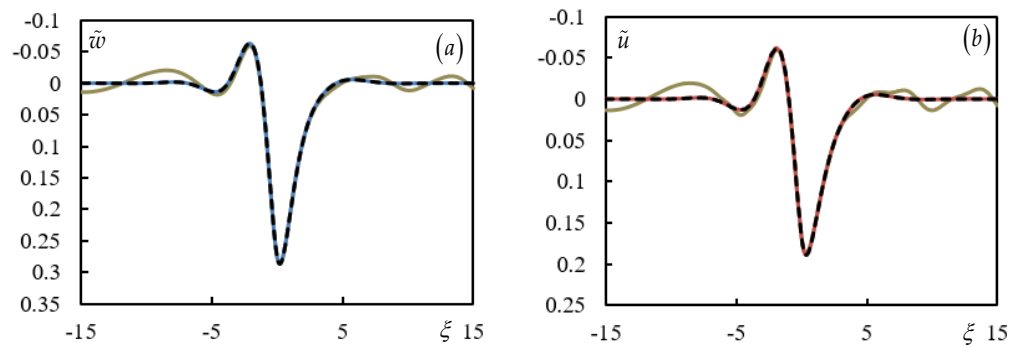


Figure 23. Case 2, Variant 3: Deflections as a function of ξ for $\alpha=0.14$ and $\tau=100$: (a) beam; (b) sleepers' layer (blue solid line: numerical integration—beam; red solid line: numerical integration—sleepers' layer; black dotted line: analytical approximation with adaptation; khaki solid line: analytical approximation for true q_M).

For $\alpha=0.33$, only traced root q_M exists. The deflections at the CP are given in Figure 24.

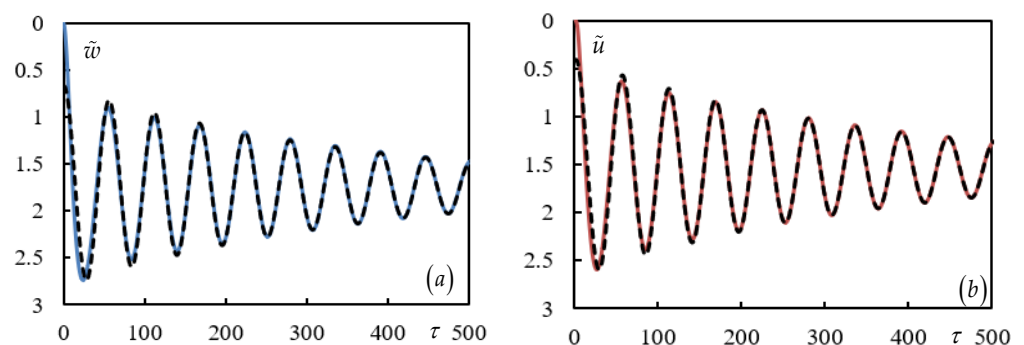


Figure 24. Case 2, Variant 3: Deflection of the CP as a function of τ for $\alpha=0.33$: (a) beam; (b) sleepers' layer (blue solid line: numerical integration—beam; red solid line: numerical integration—sleepers' layer; black dotted line: analytical approximation).

In Figure 25 it is demonstrated that for instance for $\tau=200$, analytical approximation with adaptation is practically identical to that without adaptation, and agreement with numerical integration results is very good.

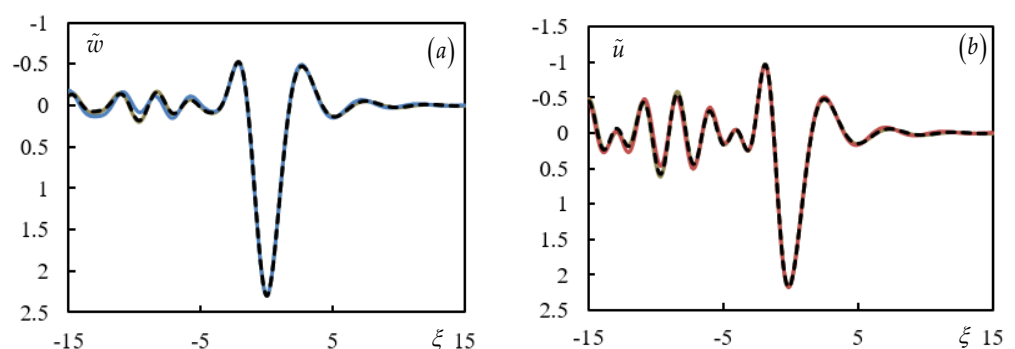


Figure 25. Case 2, Variant 3: Deflections as a function of ξ for $\alpha=0.33$ and $\tau=200$: (a) beam; (b) sleepers' layer (blue solid line: numerical integration—beam; red solid line: numerical integration—sleepers' layer; black dotted line: analytical approximation with adaptation; khaki solid line: analytical approximation without adaptation).

In Case 2, Variant 4, damping is further increased by adding the other component. The basic characteristics are summarized in Table 5.

Table 5. Basic characteristics of Case 2, Variant 4.

CV	AS1	SW1	True q_M
1.348327	0.420969	0.601043	0.001–0.341; 0.587–

This case represents the variant with the largest region of true q_M , but the initial part with three p -roots with positive imaginary part is again retained only up to $\alpha = 0.141$, as in the previous variant. Also, recovery of the true q_M starts at the same α as in the previous variant.

The frequency lines are plotted in Figure 26 demonstrating once more that traced q_M are smoothly extending the parts with true q_M and that the step around PCV (AS1) is retained. All other features detailed in previous variants are maintained.

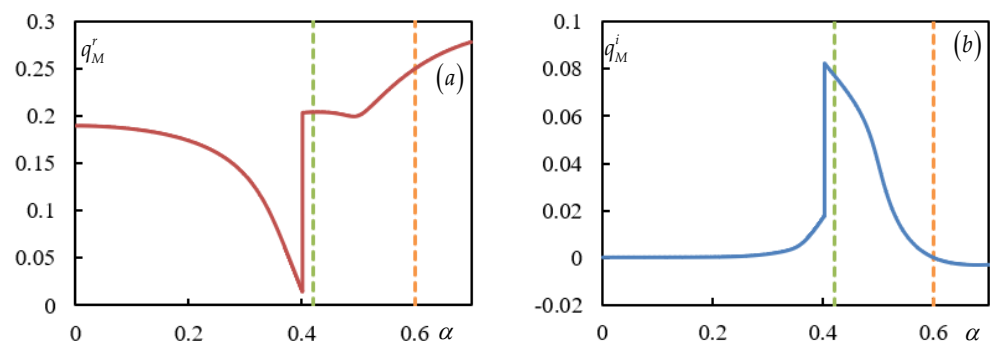


Figure 26. Case 2, Variant 4: (a) Real part of q_M ; (b) Imaginary part of q_M . (AS1: green dotted line; SW1: orange dotted line).

6.4. Case 3

Case 3 was selected due to a different combination of parameters, $\mu_s = \kappa_p = 4$, and the use of a limit value for η_M . It has only one resonance, like Case 2. The model is lightly damped and uses negligible shear stiffness. Damping is defined as a multiplication by $1 \cdot 10^{-5}$ applied to the real parameters k_f and k_p , yielding $\eta_f = 8.63 \cdot 10^{-4}$ and $\eta_p = 3.45 \cdot 10^{-3}$. $k_s = 0.01N$, which, together with the other parameters gives $\eta_s = 6.98 \cdot 10^{-10}$. The limit value for η_M is based on a mass corresponding to a typical axle load yielding $\eta_M = 2227.38$. An undamped variant will also be analyzed. Both options were studied up to $\alpha = 1.7$.

First, Figure 27 demonstrates how CV and AS are influenced by the value of κ_p .

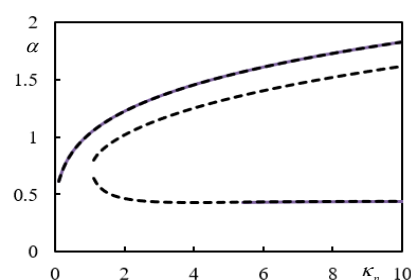


Figure 27. CV and AS as a function of κ_p (violet solid line: CV; black dotted line: AS).

The dotted line in Figure 27 is not closed due to the spacing between consecutive κ_p -values selected as 0.01. Figure 28 further exemplifies how AS is related to the steady-state solution. The parametric analysis is performed without damping to make a clear separation between CV and PCV. It can be seen that PCV is now well formed: extreme values along the entire beam are achieved at the same positions, for $\alpha = 0.431$, with a spacing between α -values of 0.001. However, the maximum value at the CP is achieved for $\alpha = 0.426$.

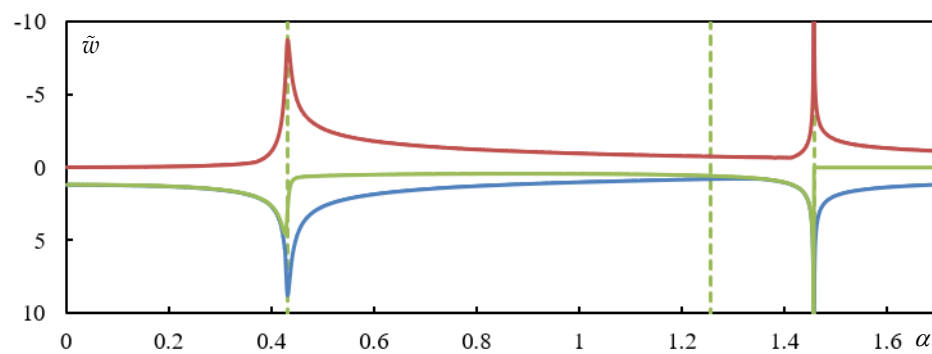


Figure 28. Case 3: Parametric analysis with respect to α of the beam steady-state deflection. (maximum and minimum deflections over the entire beam: blue and red solid line; deflection at the CP: green solid line; AS1, AS2 and AS3 in increasing order: green dotted lines).

Here, the combination of parameters in the damped version allows the determination of true q_M within quite large α -regions, which also causes the jump at PCV to be less pronounced than in Case 2. Nevertheless, there are true q_M calculated in regions with $n=4$, which will be replaced by their traced versions because the traced q_M identify the deflections along the beam and the sleepers' layer better, as already demonstrated for Case 2, Variant 3. However, for the CP deflections, both q_M -versions provide a good approximation.

Some basic characteristics are listed in Table 6.

Table 6. Basic characteristics of Case 3.

CV	AS1	AS2	AS3	SW1	SW2	SW3
1.456381	0.431898	1.256581	1.457498	0.438047	1.255548	1.464113
1.456381	0.431795	1.456381	1.456381	0.438969	-	-
True q_M		Fundamental Rule		Strictly Traced	Adapted q_M	
0.001–0.402		0.403–0.437		1.454–1.462	1.417–1.454	
0.438–1.417		1.418–1.453			1.463–1.464	
1.465–		1.463–1.464				
0.438–		0.001–0.437				

The last row of each part of Table 6 refers to the undamped variant of this case; however, it should be recalled that AS were calculated with negligible damping of $\eta_f = 1 \cdot 10^{-20}$. In the undamped version, the fundamental rule can be applied from the lowest α until $\alpha = 0.437$, which is the last stable α . Then, the system is unstable and will never recover.

The last column in Table 6 indicates the range of α for which true q_M exists, although in the region with $n=4$. As already mentioned, these roots will be switched to their traced version. Frequency lines are plotted in Figures 29 and 30 for the original and undamped variants, respectively.

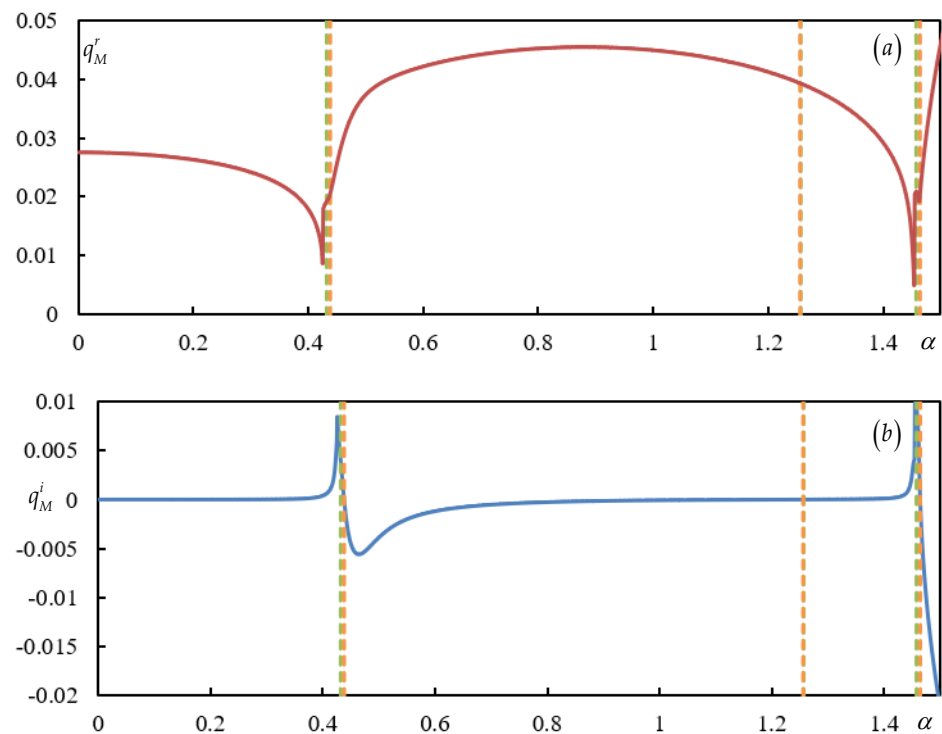


Figure 29. Case 3, Variant 4: (a) Real part of q_M ; (b) Imaginary part of q_M . (AS1, AS2 and AS3 in increasing order: green dotted lines; SW1, SW2 and SW3 in increasing order: orange dotted lines).

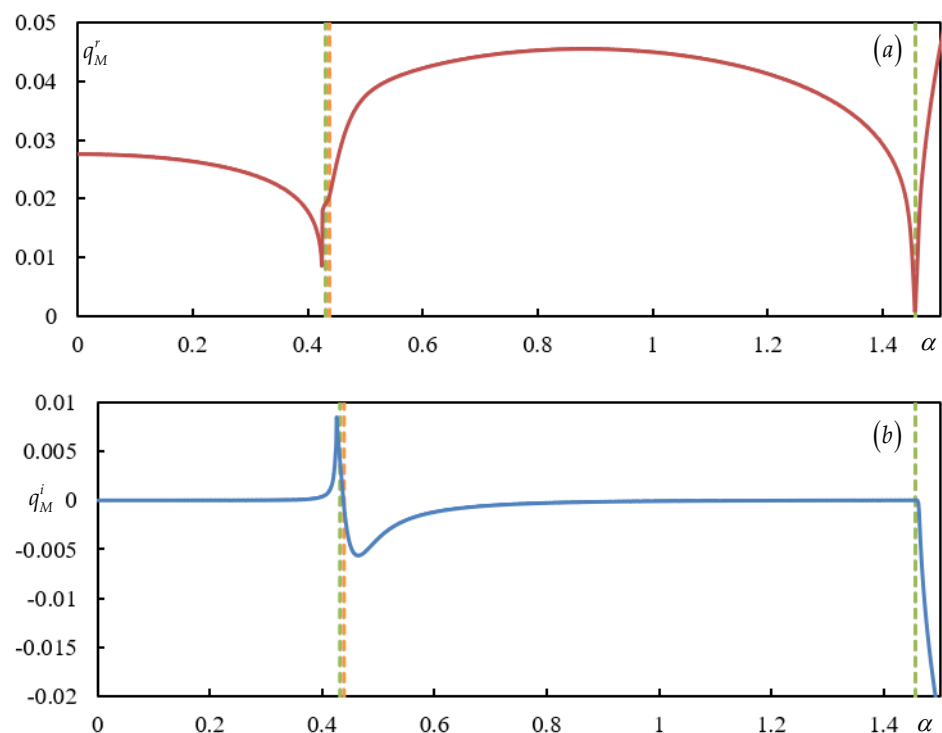


Figure 30. Case 3, undamped variant: (a) Real part of q_M ; (b) Imaginary part of q_M . (AS1, AS2 and AS3 in increasing order: green dotted lines; SW1: orange dotted lines).

As an example of excellent agreement between numerical integration and analytical approximation, $\alpha=1.43$ is selected, as there exist true q_M from the region with $n=4$ and traced q_M obtained by the fundamental rule. In Figure 31, the deflection of the CP

along τ is plotted and in Figure 32, the deflections with respect to ξ for $\tau=130$ are shown.

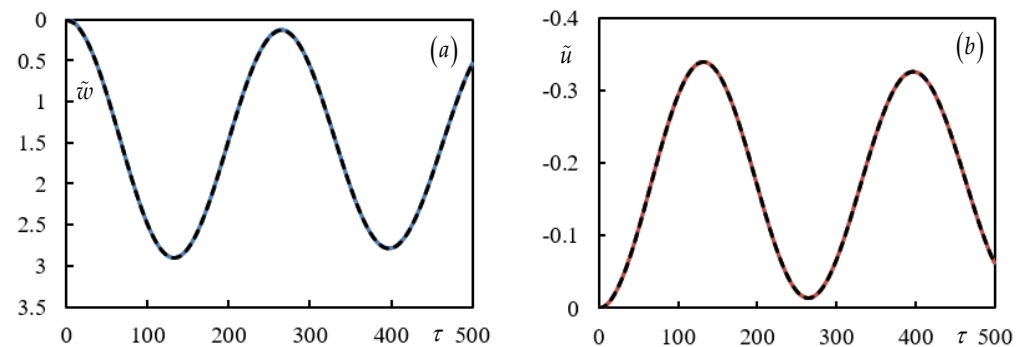


Figure 31. Case 3: Deflection of the CP as a function of τ for $\alpha=1.43$: (a) beam; (b) sleepers' layer (blue solid line: numerical integration—beam; red solid line: numerical integration—sleepers' layer; black dotted line: traced q_M ; khaki solid line: true q_M , this line is not visible as it is overlaid by other lines).

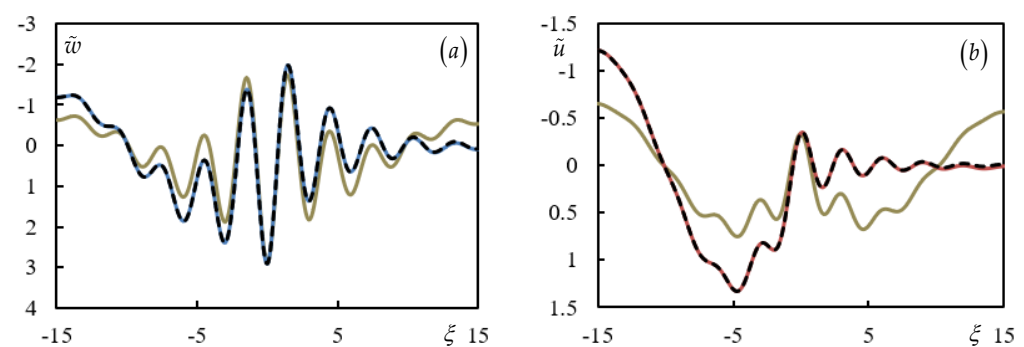


Figure 32. Case 3: Deflections as a function of ξ for $\alpha=1.43$ and $\tau=130$: (a) beam; (b) sleepers' layer (blue solid line: numerical integration—beam; red solid line: numerical integration—sleepers' layer; black dotted line: traced q_M with no adaptation; khaki solid line: true q_M).

Figure 32 clearly demonstrates that the true roots q_M from regions with $n=4$ must be adapted by tracing.

Finally, it is worth noting that it can happen, as in Case 2, that there are two valid traced roots for the same α . This usually happens within the region around CV or PCV, where the agreement between the numerical integration and analytical approximation is not very good and the steady-state solution is dominant. In this case, this happens for $\alpha \in (0.425; 0.428)$. Usually, the step should occur around α , where the maximum value of the steady-state solution at the CP is achieved, but, as in Case 2, the exact position is difficult to confirm.

6.5. Case 4

Case 4 is, in fact, a variant of the Case 3, just $\kappa_p=1.1$ instead of previous $\kappa_p=4$ is used. It was selected to analyze the effect on the behavior related to the fact that, after a region with only one AS, three AS exist again, as justified in Figure 27. However, AS1 and AS2 are strangely shifted with respect to the expected positions. An undamped variant will also be considered. It will be demonstrated that this unusual AS behavior will not influence the new technique proposed in this paper and that all features and conclusions are maintained.

In Figure 33, the positions AS1 and AS2 with respect to PCV and CV are shown. A significant shift ahead of AS1 from the position of PCV is noticed. PCV is ambiguous because extreme values are achieved for different α .

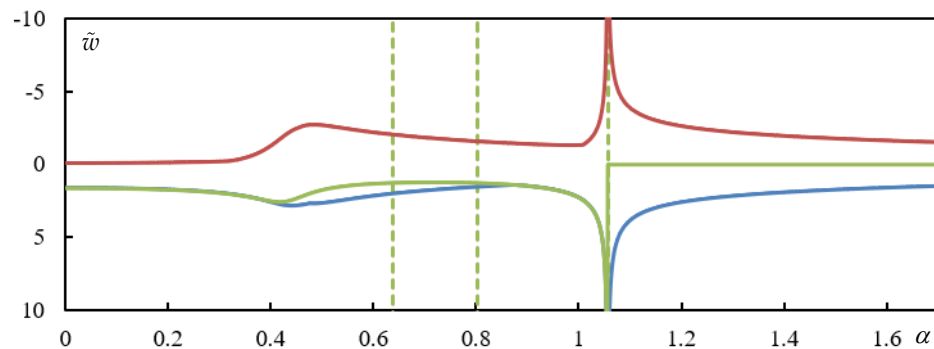


Figure 33. Case 4: Parametric analysis with respect to α of the beam steady-state deflection. (maximum and minimum deflections over the entire beam: blue and red solid line; deflection at the CP: green solid line; AS1, AS2 and AS3 in increasing order: green dotted line).

The AS positions are also checked using the instability lines, plotted in Figure 34.

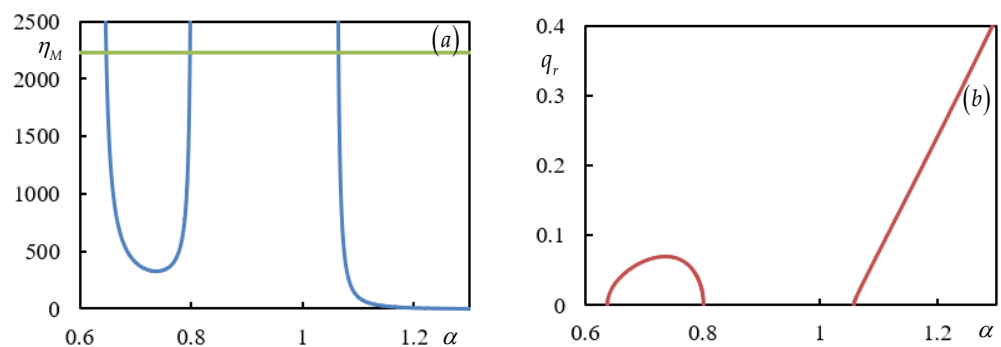


Figure 34. Case 4: (a) Both branches of instability lines; (b) corresponding q_r , (horizontal green line marks the value of η_M).

It can be concluded from Figure 34 that there is no strange formation in the instability lines, and SW1, SW2 and SW3 are predicted regularly.

Some basic characteristics of this case are listed in Table 7.

Table 7. Basic characteristics of Case 4.

CV	AS1	AS2	AS3	SW1	SW2	SW3
1.056776	0.638349	0.802500	1.058316	0.648102	0.798114	1.064207
1.056776	0.632485	1.056776	1.056776	0.640684	-	-
True q_M		Fundamental Rule	Strictly Traced		Adapted q_M	
0.001–0.382		0.383–0.533	1.054–1.061		1.023–1.053	
0.534–1.022		1.023–1.052			1.061–1.064	
1.065–		1.062–1.064				
0.438–		0.001–0.437				

For the undamped variant, the fundamental rule can be applied from the lowest α until $\alpha=0.437$, which is the last stable α . Then, the system is unstable and will never recover stability. The last row of each part of Table 7 refers to this case. However, it should

again be recalled that AS1, AS2 and AS3 were calculated with negligible damping, $\eta_f = 1 \cdot 10^{-20}$.

Also here, the last column in Table 7 indicates the range of α for which true q_M exist, although in the region with $n=4$. As already demonstrated, it is preferable to change these roots to their traced versions, because they better estimate the deflection shapes along ξ . Differences in q_M -values are again negligible and therefore, as for the CP, both versions provide for a good approximation.

The frequency lines are plotted in Figures 35 and 36 for the original and undamped variant, respectively.

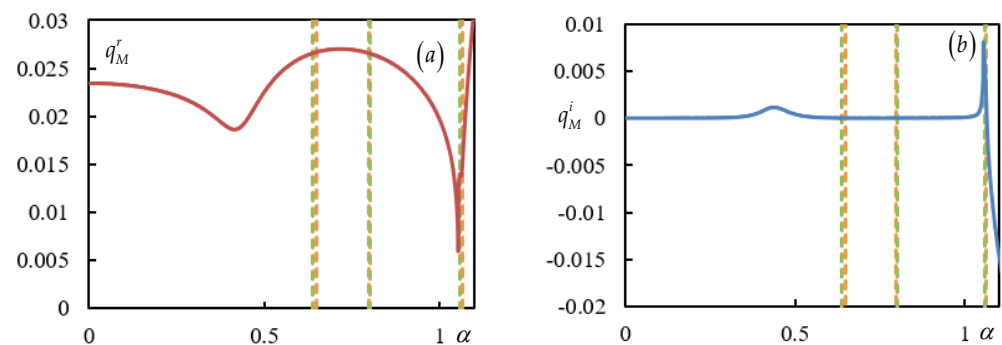


Figure 35. Case 4: (a) Real part of q_M ; (b) Imaginary part of q_M . (AS1, AS2 and AS3 in increasing order: green dotted line; SW1, SW2 and SW3 in increasing order: orange dotted line).

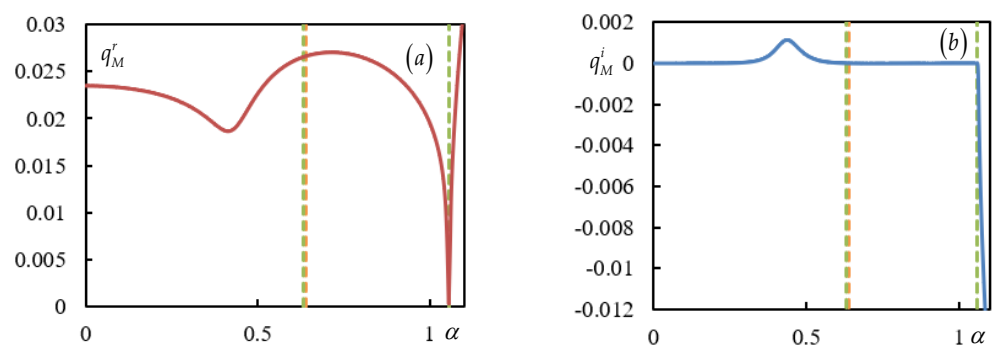


Figure 36. Case 4: undamped variant: (a) Real part of q_M ; (b) Imaginary part of q_M . (AS1, AS2 and AS3 in increasing order: green dotted line; SW1: orange dotted line).

It can be concluded that the decreased q_M^r (minimum at $\alpha=0.415$ for both, damped and undamped) and the increased q_M^i (maximum at $\alpha=0.437$ for both, damped and undamped) are located, as before, around PCV (maximum value of the steady-state deflection at the CP at $\alpha=0.418$, maximum and minimum steady-state deflection over the entire beam at $\alpha=0.442$ and $\alpha=0.484$). However, the evolution is smooth, and a step is noticed only around CV. This indicates that the more pronounced PCV, the steeper the change in q_M , but there is no exact α to match all these extremes. Nevertheless, it is demonstrated that this feature, identified in other cases, is preserved and not affected by the position of AS1.

To validate the proposed technique, $\alpha=0.4$ and $\alpha=1.05$ are selected. There is only traced q_M for $\alpha=0.4$. In Figure 37 deflection at the CP is plotted and in Figure 38 the deflection along ξ is plotted for $\tau=167$.

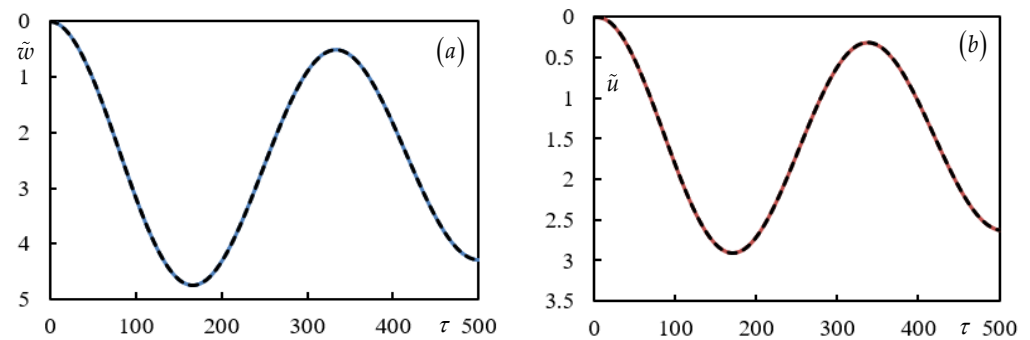


Figure 37. Case 4: Deflection of the CP as a function of τ for $\alpha=0.4$: (a) beam; (b) sleepers' layer (blue solid line: numerical integration—beam; red solid line: numerical integration—sleepers' layer; black dotted line: traced q_M).

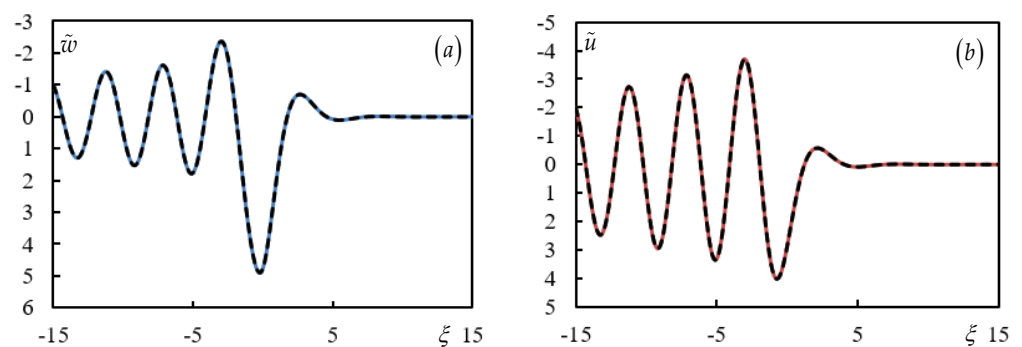


Figure 38. Case 4: Deflections as a function of ξ for $\alpha=0.4$ and $\tau=167$: (a) beam; (b) sleepers' layer (blue solid line: numerical integration—beam; red solid line: numerical integration—sleepers' layer; black dotted line: traced q_M with no adaptation).

It can be concluded from Figures 37 and 38 that the agreement is excellent.

There are both q_M for $\alpha=1.05$, true q_M from the region with $n=4$ and traced q_M . In Figure 39, the deflection at the CP is plotted, and in Figures 40 and 41, the deflection along ξ is plotted for $\tau=275$.

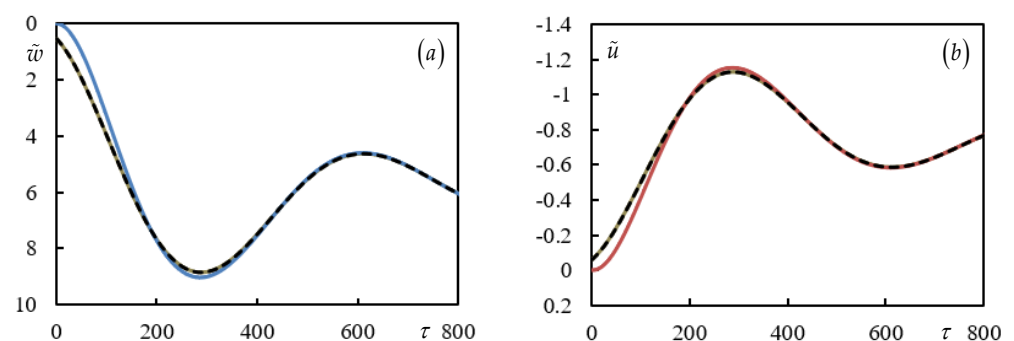


Figure 39. Case 4: Deflection of the CP as a function of τ for $\alpha=1.05$: (a) beam; (b) sleepers' layer (blue solid line: numerical integration—beam; red solid line: numerical integration—sleepers' layer; black dotted line: traced q_M ; khaki solid line: true q_M).

It can be concluded from Figure 39 that the difference between results obtained using the true and the traced q_M is negligible. Agreement between the numerical integration

and analytical approximation is slightly worse at low times due to the proximity to CV, but still very good.

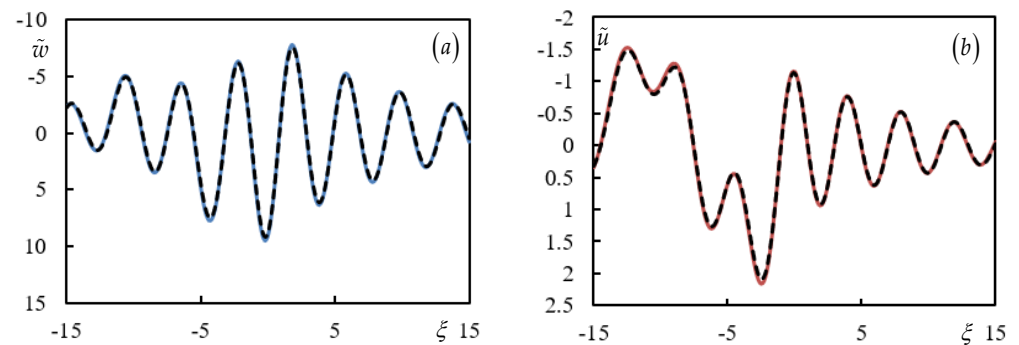


Figure 40. Case 4: Deflections as a function of ξ for $\alpha=1.05$ and $\tau=275$: (a) beam; (b) sleepers' layer (blue solid line: numerical integration—beam; red solid line: numerical integration—sleepers' layer; black dotted line: traced q_M with no adaptation).

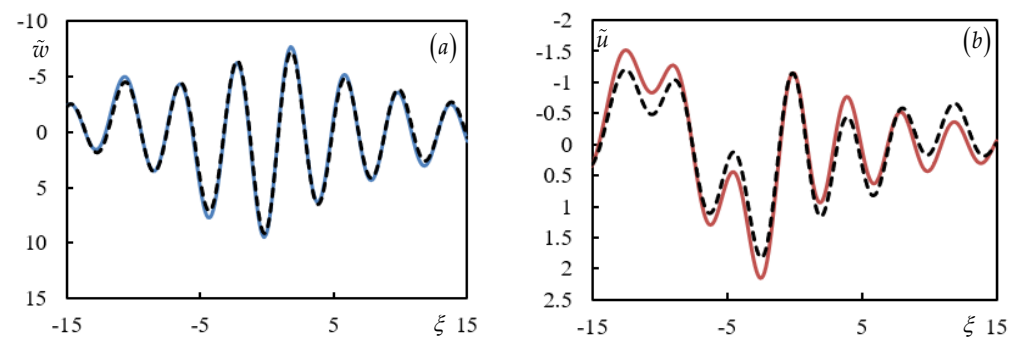


Figure 41. Case 4: Deflections as a function of ξ for $\alpha=1.05$ and $\tau=275$: (a) beam; (b) sleepers' layer (blue solid line: numerical integration—beam; red solid line: numerical integration—sleepers' layer; black dotted line: true q_M).

Figures 40 and 41 prove once again that the true q_M should be replaced by its traced version to obtain better agreement along the model and not only at the CP, because the agreement in Figure 40 is excellent, much better than in Figure 41.

To conclude, it is demonstrated that it can happen that the traced q_M is obtained from traced roots, however contrary to the fundamental rule, one of the roots has negative imaginary part. This is somewhat unusual because then only one p -root has a positive imaginary part for the wave ahead of the CP. Here, it happens, for instance, for $\alpha=1.06$, as indicated in Table 7. In Figure 42, the roots changes along q_i are shown using separately the real and imaginary parts of Equation (28). It is demonstrated that both parts achieved zero at the end.

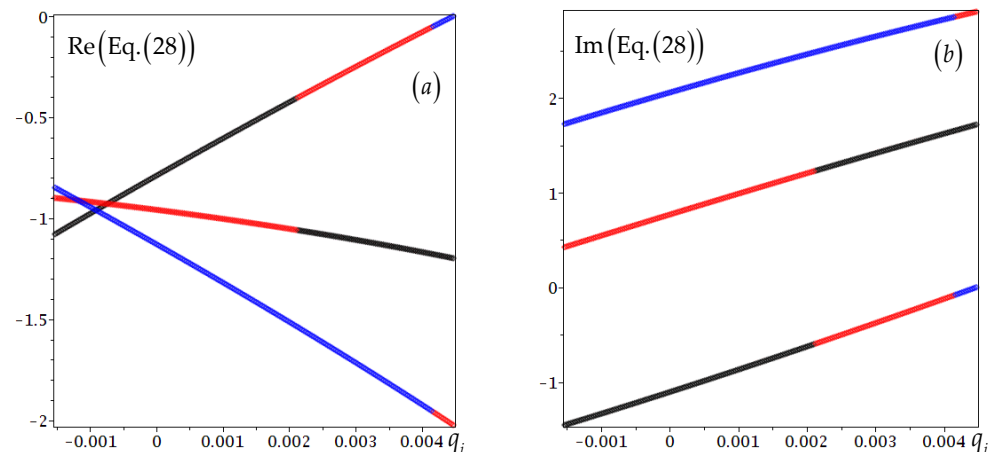


Figure 42. Case 4: Tracing along q_i for $\alpha = 1.06$: (a) Real part of Equation (28); (b) Imaginary part of Equation (28) (different colors designate different order, which is specific to the symbolic software used).

Figure 43 demonstrates that this is a valid traced root. The agreement between the numerical integration results and the analytical estimate is slightly compromised due to the α proximity to CV.

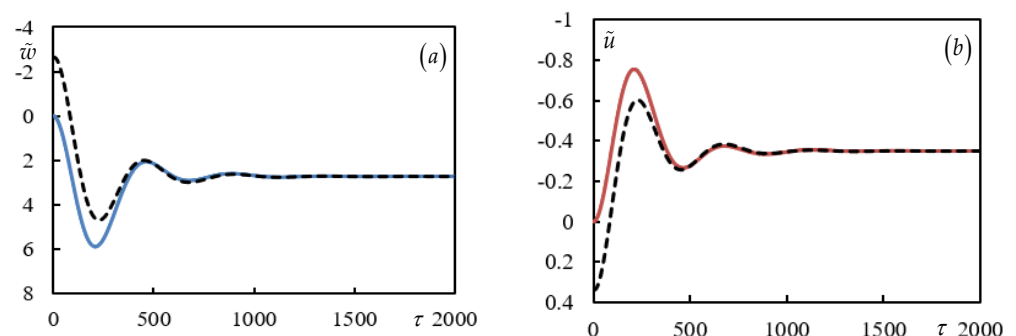


Figure 43. Case 4: Deflection of the CP as a function of τ for $\alpha = 1.06$: (a) beam; (b) sleepers' layer (blue solid line: numerical integration—beam; red solid line: numerical integration—sleepers' layer; black dotted: traced q_M).

To confirm the agreement along ξ , higher-precision numerical integration must be used. This case is dominated by the steady-state deflection; therefore, the transient part is low and more sensitive to errors. However, no other case of this specific case has been found to be more suitable; these cases appear only in Cases 3 and 4 and only around CV. Figure 44 demonstrates that, also here, the agreement is quite good for the shape without adaptation. A high time $\tau = 500$ had to be selected because, for such a comparison, agreement at the CP must be fulfilled. The full vibration is plotted together with the steady-state part as in other figures. It was decided not to plot separately only the transient vibration because, due to the high $\tau = 500$, the transient part is relatively low and the comparison could be misleading because the numerical integration requires some approximations. The steady-state part is included in Figure 44 for comparison, to demonstrate that there is still some transient vibration to be compared with, which approves this selection as a valid option.

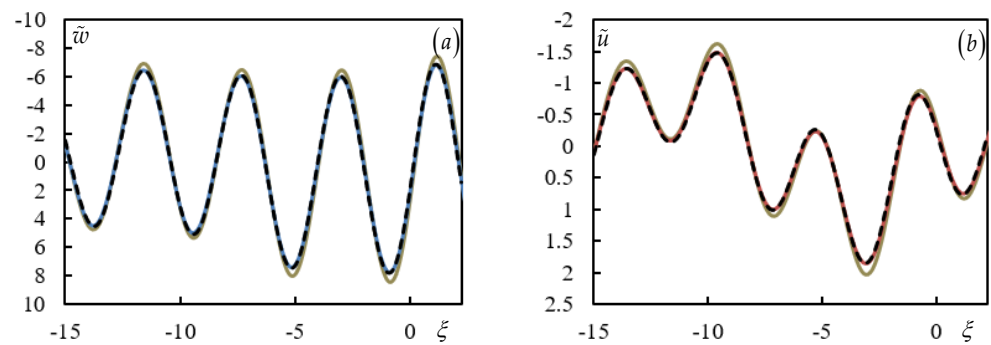


Figure 44. Case 4: Deflections as a function of ξ for $\alpha=1.06$ and $\tau=500$: (a) beam; (b) sleepers' layer (blue solid line: numerical integration—beam; red solid line: numerical integration—sleepers' layer; black dotted line: traced q_M with no adaptation, khaki solid line: steady-state solution).

7. Discussion

The new technique proposed in this paper to identify the dominant frequency and approximately reconstruct the unsteady harmonic component of the vibration patterns associated with a moving mass traversing the two-layer model is validated on several test cases. As there is no rigorous mathematical proof of this technique, many cases had to be selected to demonstrate its robustness. The agreement with the numerical integration results is presented visually using overlaid graphs, since, as explained in Section 3, with respect to the deflection of the contact point, the Fourier and Hilbert transforms do not provide sufficient accuracy at the dominant frequency due to the insufficiently long time intervals in which transient vibrations occur. Furthermore, any norm or numerical measure applied to the difference between shapes can distort the conclusions about the frequency value and can hide good agreement within the region of interest at the expense of a region that is less important. This applies especially to the deflection along the model as a function of the spatial coordinate. This approach to validation has also been implemented in the author's previous works.

In the cases analyzed, it was demonstrated that there is no region of α where it would not be possible to find a dominant frequency q_M by tracing. It was also shown that the true q_M from a region with $n=4$ or $n=3$ should be replaced by its traced version to better approximate vibration shapes along the model. Finally, it was demonstrated that there is generally only one dominant q_M , and thus all other q_M , either traced or true, should be disregarded because they do not contribute well to the vibration pattern.

The new technique does not contradict previous results presented in [31], because in those publications only cases with true q_M were presented, and not much importance was placed on the deflection shapes along the model. As already mentioned, the undamped versions of these cases had negligible step discontinuity in the K -function, which could therefore be disregarded.

The new technique has three main advantages with respect to the previous one, which used only true q_M :

- It allows a traced dominant q_M to be determined for any α ;
- It provides a substantially better approximation of the beam and sleepers' layer deflections in the region relevant to the moving interaction without p -roots adaptation, or along the full model if p -roots adaption is implemented.
- It demonstrates that the mathematically admissible continuation beyond the K -function discontinuity provides a better fit to the dominant frequency than the true q_M .

Nevertheless, the new technique maintains the same limitations as the previous one. These are inherent to the possibility of representing the transient vibration by a single harmonic motion:

- There is a small difference between the numerical integration results at the CP and the analytical approximation at low times. This deviation is generally negligible when the highest vibration amplitude approaches the steady-state deflection.
- Around CV and PCV, where the steady-state vibrations are dominant, agreement of the analytical prediction of the transient part with the numerical integration results is worse. This is, however, not important because, as mentioned above, the steady-state part is dominant, and the transient part decays rapidly even in undamped cases.

It was also demonstrated that, with the exception of the one-layer model, layered models incorporate inter-layer energy flux, which activates apparent radiation damping. Therefore, an undamped response may appear to be highly damped, which can be utilized in design guidelines aimed at vibration mitigation. This feature, besides causing decay in vibration amplitudes, shifts the onset of instability above the lower CV or PCV, unlike in an undamped one-layer model, where instability always starts at the CV. This shift is not further verified at higher velocities; therefore, around the higher CV there is no recovery of stability, in the same way as in the undamped one-layer model.

In addition to root tracing, an alternative approach was proposed that focused on adjusting the smallest possible damping that still provides reasonable results. Such an analysis should be complemented by an energy evaluation to select the type of damping that dissipates the smallest possible amount of energy.

8. Conclusions

In this paper, a new technique allowing quick analytical prediction of the dominant frequency and deflection shapes associated with the transient vibration of a constant mass traversing a two-layer model of the railway track at constant velocity is proposed and validated.

Dominant frequencies are important not only for vibration analysis and mitigation measures, but also for further analysis of normal and anomalous Doppler waves. Although this is primarily a theoretical study, the presented results also have practical relevance. From an engineering standpoint, the real part of the dominant frequency characterizes the oscillatory component of the transient response and can be related to the frequency of excitation generated by a moving wheel. Consequently, it provides information that can be used to identify and avoid conditions that may lead to resonance in moving vehicles. The imaginary part characterizes the rate at which the transient vibrations decay and therefore provides information about the duration of the transient response.

It is important to note that any inhomogeneity in the track or structural system can generate transient vibrations. The proposed approach provides a means of assessing these vibrations and of investigating how their characteristics change with the input parameters. Thus, although the model involves several simplifications, the results can be used as a theoretical and computational basis for railway engineering applications, particularly for assessing the influence of inhomogeneities, identifying potentially critical excitation conditions, and tailoring the relevant input parameters to control the transient response.

This new technique is readily extendable to proximate moving objects and to the three-layer model, which indicate directions for future studies. Additionally, models with a plane foundation will also be analyzed, and the level of radiation damping will be compared with that of models implementing discrete point masses in the supporting structure.

The analyses presented in this paper have several limitations, primarily the assumption that the supporting structure is longitudinally homogeneous and that the moving mass is idealized as a point mass in permanent fixed contact with the guiding beam. These limitations suggest possibilities for future extensions to models with discrete supports, contact springs, and the possibility of loss of contact.

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Abbreviations

The following abbreviations are used in this manuscript:

AS	Vertical asymptote to instability line
CP	Contact point
CV	Critical velocity
FCV	False critical velocity
PCV	Pseudo critical velocity
FFT	Fast Fourier Transform
SW	Switch between stable and unstable region

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