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Theoretical Formulations of Integral-Type Frequency–Amplitude Relationships for Second-Order Nonlinear Oscillators

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Abstract

The development of simple and yet accurate formulations of frequency–amplitude relationships for non-conservative nonlinear oscillators is an important issue. The present paper is concerned with integral-type frequency–amplitude formulas in the dimensionless time domain and time domain to accurately determine vibrational frequencies of nonlinear oscillators. The novel formulation is a balance of kinetic energy and the work during motion of the nonlinear oscillator within one period; its generalized formulation permits a weight function to appear in the integral formula. The exact values of frequencies can be obtained when exact solutions are inserted into the formulas. In general, the exact solution is not available; hence, low-order periodic functions as trial solutions are inserted into the formulas to obtain approximate values of true frequencies. For conservative nonlinear oscillators, a powerful technique is developed in terms of a weighted integral formula in the spatial domain, which is directly derived from the governing ordinary differential equation (ODE) multiplied by a weight function, and integrating the resulting equation after inserting a general trial ODE to acquire accurate frequency. The free parameter is involved in the frequency–amplitude formula, whose optimal value is achieved by minimizing the absolute error to fulfill the periodicity conditions. Several examples involving two typical non-conservative nonlinear oscillators are explored to display the effectiveness and accuracy of the proposed integral-type formulations.

Keywords: non-conservative nonlinear oscillator; exact frequency–amplitude formula; periodicity conditions; trial ODE; integral-type frequency formulas; weight functions; parameter



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1. Introduction

To motivate the present study, we consider a conservative nonlinear oscillator:

$$\ddot{x}(t) + F(x) = 0, \quad (1)$$

where $F(x)$ is an odd function satisfying $xF(x) > 0$. We cite the following result [1–3]:

$$T = 2\sqrt{2} \int_0^A \frac{dx}{\sqrt{\int_x^A F(s)ds}}, \quad (2)$$

where T is the period of the oscillator to satisfy $x(T) = x(0) = A$ and $\dot{x}(T) = \dot{x}(0) = 0$.

Equation (2) is an integral formula in the spatial domain of x , which can be derived by multiplying Equation (1) by $\dot{x}$,

$$\frac{d}{dt}\dot{x}^2 = -2F(x)\dot{x}; \quad (3)$$

integrating it at the first quadrant of the periodic orbit yields

$$\dot{x} = \sqrt{2} \sqrt{\int_x^A F(s) ds}. \quad (4)$$

Then, Equation (2) follows readily.

Equation (2) is a well-known formula to exactly compute the period and frequency $\omega = 2\pi/T$, which is, however, limited to the conservative nonlinear oscillator. Equation (4) is the first integral of Equation (1). The integrand in Equation (2) is singular at $x = A$. To compute T , it needs to perform two integrals and is, in general, an implicit function of A .

We consider a general second-order nonlinear oscillator:

$$\begin{cases} \ddot{x}(t) + f(x, \dot{x}) = 0, \\ x(0) = A, \dot{x}(0) = 0, \end{cases} \quad (5)$$

where $A > 0$ is the amplitude of an oscillation of the oscillator.

Let $\dot{x} = y$; Equation (5) can be expressed via a system of two first-order ODEs:

$$\begin{cases} \dot{x}(t) = y, \\ \dot{y}(t) = -f(x, y), \\ x(0) = A, y(0) = 0. \end{cases} \quad (6)$$

According to the Bendixson–Dulac Theorem [3], a necessary condition for the existence of a periodic solution in a simply connected region $\Omega \subset \mathbb{R}^2$ is that $\partial f(x, y)/\partial y$ must change its sign inside Ω . If $\partial f(x, y)/\partial y \neq 0, \forall (x, y) \in \Omega$, there exists no periodic solution in Ω . He and Garcia [4] proved that Equation (5) possesses a periodic solution if and only if there exists a function $\dot{x} = \phi(x) \in C^1(\mathbb{R})$, such that

$$\frac{d\phi(x)}{dx} = -\frac{f(x, \phi(x))}{\phi(x)}, \phi(A) = 0. \quad (7)$$

Using the Lipschitz condition for the right-hand side, the existence of ϕ is guaranteed. As Equation (4) for Equation (1), $\dot{x} = \phi(x)$ is the first integral of Equation (5), if it allows a periodic solution.

Like $x F(x) > 0$ for Equation (1), $x f(x, \dot{x}) > 0$ is required for Equation (5). The main problem we are concerned with is the derivation of a simple integral form of vibrational frequency under the assumption of $x f(x, \dot{x}) > 0$, so that the periodic solution of Equation (5) is assumed to be existent.

Applying a similar operation in Equation (3) for Equation (5), we cannot simply obtain the relation between $\dot{x}$ and x as that in Equation (4) for the conservative oscillator. For a non-conservative nonlinear oscillator, if the first integral $\dot{x} = \phi(x)$ exists, we can derive a similar exact formula:

$$T = 4 \int_0^A \frac{1}{\phi(x)} dx, \quad (8)$$

which is like Equation (2) for the conservative nonlinear oscillator in Equation (1).

He and Garcia [4] made a breakthrough for the following separable non-conservative oscillator:

$$\begin{cases} \ddot{x}(t) = f_1(x)f_2(\dot{x}), \\ x(0) = A, \dot{x}(0) = 0. \end{cases} \quad (9)$$

In Corollary 1 of [4], He and Garcia obtained an exact integral formula:

$$\begin{cases} \int_0^{\phi(x)} \frac{\phi}{f_2(\phi)} d\phi = \int_A^x f_1(s) ds, \\ T = 4 \int_0^A \frac{1}{\phi(x)} dx. \end{cases} \quad (10)$$

For Equation (1), we have $f_1 = -F$ and $f_2 = 1$; inserting them into Equation (10) yields the same formula for T in Equation (2).

Rather than Equations (2) and (10), a direct relationship between the frequency and amplitude is more useful for a quick estimation of the frequency. For this purpose, a trial solution is often used to replace the exact solution by

$$x(t) = A \cos \omega t, \quad (11)$$

which satisfies the initial conditions in Equation (5), but with $\omega = \omega(A)$ an unknown function of A . The corresponding trial ODE for Equation (11) is

$$\ddot{x} + \omega^2 x = 0. \quad (12)$$

He [5] rewrote Equation (1) as

$$\ddot{x} + \frac{F(x)}{x} x = 0; \quad (13)$$

upon comparing Equations (12) and (13), He [5] suggested a very simple frequency–amplitude formula:

$$\omega^2 = \left. \frac{F(x)}{x} \right|_{x=NA}, \quad (14)$$

where $N = \sqrt{3}/2$ for non-singular oscillator [6], and $N = 0.8$ for singular oscillator [7].

In [8], Equation (13) was augmented to

$$\ddot{x} + \frac{x F(x)}{x^2} x = 0. \quad (15)$$

Integrating both the nominator and the denominator in the nonlinear stiffness term $x F / x^2$, a formula was suggested as follows [8,9]:

$$\omega^2 = \frac{\int_0^T x(t) F(x(t)) dt}{\int_0^T x^2(t) dt}, \quad (16)$$

which is an integral formula in the time domain of t . This formula was established by El-Dib [10] as an efficient formula. However, the theoretical foundation of Equation (16) needs a further certification, to which the present paper is concerned with.

Antola [11] introduced a weighted functional to determine the frequency:

$$\Phi(A) = \int_0^A W(x) [F(x) - \omega^2 x]^2 dx. \quad (17)$$

By minimizing Φ one can arrive to the following formula:

$$\omega^2 = \frac{\int_0^A xW(x)F(x)dx}{\int_0^A x^2W(x)dx}. \quad (18)$$

Equation (18) is a formal integral approach to the frequency formula in the spatial domain.

He's frequency–amplitude formula [12] to estimate the frequency of a nonlinear oscillator is simple and yet accurate. It, without needing a lot of computational costs, can provide a reasonable approximation that establishes a relationship between the vibration frequency and amplitude. He's formula has demonstrated its effectiveness across a broad spectrum of challenges in many fields, particularly valuable in obtaining the closed-form analytical solutions for oscillators of the Duffing type [4,13]. The relationship between the frequency and amplitude is a mainly concerned property of the nonlinear oscillators. For improving the precision of the relationship, there were many modifications of He's frequency formula [14–17]. Other modifications of He's frequency–amplitude formulation were summarized in [18,19]. Recently, many discussions of the frequency–amplitude formulas, including those of He's, were clarified in [20].

Equation (10) motivates us to develop the integral-type frequency formula for a general non-conservative nonlinear oscillator.

So far, different types of frequency formulas have been created one by one; there are still some challenging problems to set up a theoretical foundation of these formulas and derive a general type frequency–amplitude relationship in terms of the integrals of simple functions. Based on Equations (5) and (11), and other more complex trial solutions, we are going to theoretically derive some integral-type frequency–amplitude relationships for non-conservative nonlinear oscillators. Meanwhile, based on Equations (1) and (12), and other more complex trial ODEs, some general integral-type frequency–amplitude relationships for conservative nonlinear oscillators are to be derived. The idea of trial ODE is extended in the paper, and some weight functions are introduced to generalize the integral formulas. In many analytic methods with some trial functions as the bases to express the periodic solution of a nonlinear oscillator, there exist certain parameters in the approximation methods. A satisfactory choice of those parameters is of utmost important, because they intimately relate to the accuracy of the proposed analytic methods. By minimizing the error of periodicity conditions, the optimal value of the parameter in the proposed frequency–amplitude formula can be determined. The paper supplements the theoretical foundations for the existent or non-existent integral-type frequency–amplitude formulations in the literature, which are set up to lay out these methods. We are going to enlarge Equations (16) and (18) into a broad spectrum to involve the non-conservative nonlinear oscillator; meanwhile, we make a depth study of different integral-type frequency–amplitude formulations from the theoretical aspects.

The mentioned first integral for a non-conservative nonlinear oscillator can be treated as the energy integral. In addition, the so-called energy balance method appears in a vast literature. To name a few, Hill et al. [21] introduced an energy-based interpretation of backbone curves in a two-degree-of-freedom nonlinear oscillator, demonstrating how undamped, unforced modal backbones correspond to resonant peaks in the forced, damped system and thus providing critical insight into nonlinear resonance phenomena. Building on this framework, Hill et al. [22] developed an analytical method to detect isolated periodic solution branches (“isolas”) in weakly nonlinear structures, showing that the symmetry breaking gives rise to discrete backbone segments and simplifies numerical continuation strategies for uncovering these solution families. Sun et al. [23] extended the energy balance method to systems with non-conservative nonlinearities by incorporating damped nonlin-

ear normal modes (dNNMs), comparing complex nonlinear modes and extended periodic motion concepts to enhance the resonance prediction accuracy in practical engineering applications. Hill et al. [24] examined the role of phase-locking in internal resonance from an NNM perspective, proving that phase-locked coupling is a necessary condition for internal resonance and elucidating how backbone curves reflect the stiffness-modulation effects of non-phase-locked interactions. Together, these contributions deepen our theoretical understanding of backbone curves, isolas, dNNMs, and phase-locking effects, and they lay the groundwork for advanced analytical and numerical tools in nonlinear forced-response analysis. The energy-preserving principle was adopted by Liu et al. [25] for developing energy-preserving/group-preserving schemes to depict the nonlinear vibration behaviors of multi-coupled Duffing oscillators.

In the present paper, we do not attempt to develop the method for obtaining a highly accurate periodic solution in terms of harmonics for a non-conservative nonlinear oscillator. Instead of the present paper aims to the developments of highly accurate integral-type frequency–amplitude formulas to quickly estimate the frequency for a non-conservative nonlinear oscillator; unlike Equations (16) and (18), which are merely for conservative nonlinear oscillators.

Our results obtained are not applicable to the systems of Equation (5) of a general form, when the existence of a periodic solution is not guaranteed. For the general system (5), what conditions guarantee the existence of periodic solutions and how many periodic solutions exist is still an open problem. The present paper is strictly restricted to the systems with periodic solutions, which satisfy Equation (5) and $xf(x, \dot{x}) > 0$.

The present paper also excludes the non-conservative nonlinear oscillator which exhibits a limit cycle, because this sort nonlinear oscillator does not satisfy the condition $xf(x, \dot{x}) > 0$. For example, the van der Pol equation is given by

$$\ddot{x} + x + \mu(x^2 - 1)\dot{x} = 0, x(0) = A, \dot{x}(0) = 0, \quad (19)$$

where $\mu > 0$. Obviously $xf(x, \dot{x}) = x^2 + \mu(x^3 - x)\dot{x}$ not necessarily satisfies $xf(x, \dot{x}) > 0$. It is a self-sustained nonlinear oscillator with a unique limit cycle. However, it would be an interesting issue by extending Equation (7) to this sort of nonlinear oscillator in the near future.

We highlight the innovation points of the present paper as follows:

1. A supplementary variable is introduced, which helps derive an integral-type frequency formula for a non-conservative nonlinear oscillator in the dimensionless time domain.
2. A novel integral-type frequency formula is derived for a non-conservative nonlinear oscillator in the time domain. Through a few lines of computations, the approximate value of the true frequency can be well estimated.
3. A novel integral-type frequency formula involving a weight function and parameter for a non-conservative nonlinear oscillator is derived. To meet the periodicity conditions by a parameter very accurate frequency can be obtained.
4. A generalized conservation law for a non-conservative nonlinear oscillator is derived in terms of a weight function. The resulting integral-type frequency formulas in the dimensionless time domain and time domain can be used to find accurate frequency.
5. For a conservative nonlinear oscillator, we derive three types of integral formulas in the dimensionless time domain, time domain, and spatial domain to quickly obtain the frequency with high accuracy.
6. For a non-conservative nonlinear oscillator, the integral formula in the spatial domain does not exist; unless the first integral exists.

2. Exact Formulations of Frequency–Amplitude Relationships for Nonlinear Oscillators

In this section, we derive the formulas depicting the frequency–amplitude relationship for Equation (5) in a dimensionless time domain and then in the time domain. The technique in the dimensionless time domain was first developed by Liu and Chang [26] as a numerical method to find the period and periodic solution of an n -dimensional nonlinear system of ordinary differential equations. It has not yet been applied to the second-order nonlinear oscillator in Equation (5) to compute the frequency using the derived formulations.

Letting

$$\tau = \frac{t}{T} \quad (20)$$

be a dimensionless time, considering

$$\dot{x}(t) = \frac{x'(\tau)}{T}, \dot{y}(t) = \frac{y'(\tau)}{T}, \quad (21)$$

where the prime denotes the differential with respect to τ , and by means of Equation (6), we come to

$$x'(\tau) = Ty(\tau), \quad (22)$$

$$y'(\tau) = -Tf(x(\tau), y(\tau)). \quad (23)$$

Theorem 1. We assume that the periodic solution of Equation (5) exists inside a simply connected region $\Omega \subset \mathbb{R}^2$ wherein $\partial f(x, y)/\partial y$ changes its sign. $f(x, y)$ satisfies

$$xf(x, y) > 0. \quad (24)$$

The frequency for Equation (5) reads as follows:

$$\omega = \frac{2\pi \int_0^1 z(\tau) f(x(\tau), y(\tau)) d\tau}{\int_0^1 y^2(\tau) d\tau}, \quad (25)$$

where

$$z(\tau) = z_0 + \int_0^\tau y(\xi) d\xi, 0 < \tau \leq 1 \quad (26)$$

is a supplementary variable satisfying $zf > 0$. Furthermore, by means of Equations (22), (25) and (26),

$$\omega^2 = \frac{4\pi^2 \int_0^1 x(\tau) f(x(\tau), y(\tau)) d\tau}{\int_0^1 x'(\tau)^2 d\tau} \quad (27)$$

can be derived.

Proof. We introduce a supplementary variable $z(\tau)$ given by Equation (26), which satisfies

$$z'(\tau) = y(\tau), z(0) = z_0, \quad (28)$$

where z_0 is an initial value of $z(\tau)$.

The following identity is easily verified:

$$z(\tau)y'(\tau) = [z(\tau)y(\tau)]' - z'(\tau)y(\tau). \quad (29)$$

Multiplying Equation (23) by $z(\tau)$ generates

$$z(\tau)y'(\tau) = -Tz(\tau)f(x(\tau), y(\tau)), \quad (30)$$

which using Equations (28) and (29) yields

$$[z(\tau)y(\tau)]' - y^2(\tau) = -Tz(\tau)f(x(\tau), y(\tau)). \quad (31)$$

The periodicity conditions for a periodic motion in the τ -domain are

$$x(1) = x(0) = A, y(1) = y(0) = 0. \quad (32)$$

The integration of Equation (31) from $\tau = 0$ to $\tau = 1$ is

$$z(1)y(1) - z(0)y(0) - \int_0^1 y^2(\tau)d\tau = -T \int_0^1 z(\tau)f(x(\tau), y(\tau))d\tau. \quad (33)$$

By means of $y(1) = y(0) = 0$ in Equation (32), one has $z(1)y(1) - z(0)y(0) = 0$; hence, the period T can be derived from Equation (33) as follows:

$$T = \frac{\int_0^1 y^2(\tau)d\tau}{\int_0^1 z(\tau)f(x(\tau), y(\tau))d\tau}. \quad (34)$$

In terms of $\omega = 2\pi/T$, we can prove Equation (25).

It can be seen that because the initial condition for y is $y(0) = 0$, $z(0)$ is immaterial; hence, in Equation (28), the initial value of $z(\tau)$ can be an arbitrary constant. Later, we will take $z_0 = A/T$. If $y(0) \neq 0$, we can prove $[z(1) - z(0)]y(0) = 0$ to simplify the formula; it is implied by the periodicity of y , such that $z(\tau) = z_0 + \int_0^\tau y(s)ds$ is also a periodic function of τ with $z(1) = z(0) = z_0$. As shown by Equation (34), $z(\tau)$ only influences the period but not the system of ODEs in Equation (6).

Comparing Equation (28) to Equation (22), one has

$$y(\tau) = \frac{x'(\tau)}{T}, z'(\tau) = \frac{x'(\tau)}{T} \Rightarrow z(\tau) = \frac{x(\tau)}{T}, \quad (35)$$

if $z(0) = z_0 = A/T$ is taken as the initial condition of the ODE in Equation (28). Inserting Equation (35) for $y(\tau)$ and $z(\tau)$ into Equation (34) yields

$$T^2 = \frac{\int_0^1 x'(\tau)^2 d\tau}{\int_0^1 x(\tau)f(x(\tau), y(\tau))d\tau}, \quad (36)$$

which can be arranged to Equation (27) using $T^2 = 4\pi^2/\omega^2$. According to the assumption (24), $\int_0^1 x(\tau)f(x(\tau), y(\tau))d\tau > 0$, T^2 in Equation (36) is well-defined. The proof is complete. $\square$

The assumption $xf(x, y) > 0$ in Equation (24) is similar to $xF(x) > 0$ for Equation (1).

Now we go back to the time domain formulation of the frequency with the help of Equations (21) and (36), which can generate a simple integral formula to compute the frequency in the t -domain. Correspondingly, Equation (27) is a frequency formula in the τ -domain.

Theorem 2. Under the same assumptions in Theorem 1, the frequency for Equation (5) can be determined by

$$\int_0^{2\pi/\omega} \dot{x}^2(t)dt = \int_0^{2\pi/\omega} x(t)f(x(t), y(t))dt, \quad (37)$$

where $y(t) = \dot{x}(t)$, which signifies that the kinetic energy and work are balanced within one period of the motion of the oscillator.

Proof. Using $d\tau = dt/T$ derived from Equation (20) and inserting Equation (21) for $x'(\tau) = T\dot{x}(t)$ into Equation (36), we have

$$T^2 = \frac{T^2 \int_0^T \dot{x}^2(t) dt}{\int_0^T x(t)f(x(t), y(t)) dt}. \quad (38)$$

Dividing both sides by T^2 produces

$$\int_0^T \dot{x}^2(t) dt = \int_0^T x(t)f(x(t), y(t)) dt. \quad (39)$$

Then, using $T = 2\pi/\omega$, Equation (37) follows readily. $\square$

For an extension of Theorem 1, we consider a generalization of $z'(\tau) = y(\tau)$ given by Equation (28) to the following result.

Theorem 3. A general frequency formula for Equation (5) is given by

$$\omega = \frac{2\pi \int_0^1 z(\tau)f(x(\tau), y(\tau)) d\tau}{\int_0^1 y(\tau)g(y(\tau)) d\tau}, \quad (40)$$

where $z(\tau)$ satisfies

$$z'(\tau) = g(y(\tau)), z(0) = z_0, \quad (41)$$

$zf > 0$, and $g(y)$ is an integrable function of y satisfying $yg > 0$.

Proof. Inserting Equation (41) for $z'(\tau)$ into the identity (29), we have

$$z(\tau)y'(\tau) = [z(\tau)y(\tau)]' - y(\tau)g(y(\tau)), \quad (42)$$

which being combined into Equation (30) generates

$$[z(\tau)y(\tau)]' - y(\tau)g(y(\tau)) = -Tz(\tau)f(x(\tau), y(\tau)). \quad (43)$$

Integrating Equation (43) from $\tau = 0$ to $\tau = 1$ and using $y(1) = y(0) = 0$ renders

$$T = \frac{\int_0^1 y(\tau)g(y(\tau)) d\tau}{\int_0^1 z(\tau)f(x(\tau), y(\tau)) d\tau}. \quad (44)$$

In terms of $\omega = 2\pi/T$, we can prove Equation (40). $\square$

When the double-integral Formula (2) is applicable to the conservative system (1) to compute the frequency, after rigorous proofs carried out for Theorems 1–3, we have the single-integral Formulas (27), (37) and (40) to compute the frequency for a non-conservative system (5), of which suitable conditions for the existence of these integral formulas were given. These results are undoubtedly correct.

Remark 1. In this section, we have derived three integral-type Formulas (27), (37), and (40) to compute the frequency. The key points are the introduction of a supplementary variable z in Equations (28) and (41) and the identity in Equation (29). These integral equations are all exact formulas, providing the exact value of the frequency if an exact solution is inserted into these formulas. As explained in Theorem 2, Equation (37) is a conservation law in the time domain for the balance of kinetic energy and work. Correspondingly, Equation (27) is a conservation law in the

dimensionless time domain, and Equation (40) can be viewed as a generalized conservation law in the dimensionless time domain.

Equation (16) is a special case of Equation (37) in Theorem 2. For the trial ODE (12), kinetic energy is equal to potential energy, which leads to

$$\int_0^T \dot{x}^2(t) dt = \int_0^T \omega^2 x^2(t) dt; \quad (45)$$

inserting it into the left-hand side of Equation (37) generates Equation (16) with $f(x, y) = F(x)$ for conservative nonlinear oscillator.

Remark 2. To demonstrate the exactness of Equation (37) in Theorem 2, we derive an exact solution of a nonlinear ODE in Appendix A. We prove that the following ODE:

$$\ddot{x} + 9x - 6A^{2/3}x^{1/3} = 0, x(0) = A, \dot{x}(0) = 0 \quad (46)$$

has an exact solution $x(t) = A \cos^3 t$. By applying Equation (37) to determine the frequency of Equation (46), we suppose that the exact solution is

$$x(t) = A \cos^3 \omega t = \frac{3A}{4} \cos \omega t + \frac{A}{4} \cos 3\omega t, \quad (47)$$

and we can prove that $\omega = 1$ is the frequency of Equation (46). Inserting Equation (47) into the left-hand side of Equation (37) yields

$$\int_0^{2\pi/\omega} \dot{x}^2(t) dt = \frac{9A^2\pi\omega}{8}. \quad (48)$$

Inserting Equations (46) and (47) with $f = 9x - 6A^{2/3}x^{1/3}$ into the right-hand side of Equation (37) yields

$$\int_0^{2\pi/\omega} x(t)f(x(t), y(t)) dt = \frac{9A^2\pi}{8}. \quad (49)$$

Comparing Equations (48) and (49) it is obvious that when $\omega = 1$, Equation (37) is satisfied. Equation (37) holds for the exact solution. Indeed, Equation (37) is an exact integral formula to determine the exact value of frequency, if the exact solution $x(t)$ is given and inserted into the formula. However, in general, we do not have an exact solution for a given nonlinear oscillator; in practice, Equation (37) is in turn used as a convenient tool to find an approximate frequency by inserting a trial solution.

3. Applications of the Integral-Type Formulas to Compute Frequency

Notice that when x and y can be obtained exactly, Equations (27) and (37) provide the exact value of ω ; however, the latter one is an implicit form of the frequency equation. In general, the exact solution of a nonlinear oscillator is not available. To be an approximation of ω , we assume that Equation (11) is a suitable approximation of the real solution x for the oscillatory system satisfying $x(0) = A$ and $\dot{x}(0) = 0$.

In terms of $\tau = t/T$, using $\omega T = 2\pi$ and by Equation (11), we have

$$x(\tau) = A \cos \omega T \tau = A \cos 2\pi \tau, x'(\tau) = -2\pi A \sin 2\pi \tau. \quad (50)$$

Inserting them into Equation (27) in Theorem 1 yields

$$\omega^2 = \frac{2}{A} \int_0^1 \cos 2\pi \tau f(A \cos 2\pi \tau, -\omega A \sin 2\pi \tau) d\tau, \quad (51)$$

which is a neater formula to compute ω for a given function $f(x, \dot{x})$ to depict the nonlinear oscillator.

Directly inserting Equation (11) into Equation (37) in Theorem 2, we have

$$\omega = \frac{1}{\pi A} \int_0^{2\pi/\omega} \cos \omega t f(A \cos \omega t, -\omega A \sin \omega t) dt. \quad (52)$$

It is the counterpart of Equation (51) in the time domain. Equation (52) is not an exact formula, because it is obtained from Equation (37) by inserting a trial solution in Equation (11); however, it is a good approximation of the true frequency.

When we apply the above formulas to compute the frequency for a conservative nonlinear oscillator, we replace $f(x, \dot{x})$ by $F(x)$.

We must emphasize that the accuracy of Equations (51) and (52) is merely of the first order, owing to the use of a first-order trial solution in Equation (11). When the degree of nonlinearity of a nonlinear oscillator is increased, or the oscillation amplitude is increased, higher-order trial solutions must be considered to enhance the accuracy of ω . For most examples to be shown below, we will adopt second-order and third-order harmonic functions as the trial solutions.

3.1. Duffing Oscillator

3.1.1. Initial Conditions $x(0) = A$ and $\dot{x}(0) = 0$

We assess the integral formula in Equation (52) derived from Theorem 2 for the Duffing oscillator:

$$\ddot{x}(t) + x(t) + \varepsilon x^3(t) = 0, x(0) = A, \dot{x}(0) = 0. \quad (53)$$

Here, the restoring force is a cubic odd function:

$$F(x) = x + \varepsilon x^3, \quad (54)$$

which after inserting Equation (11) for $x = A \cos \omega t$ leads to

$$F(A \cos \omega t) = A \cos \omega t + \varepsilon A^3 \cos^3 \omega t = \left[A + \frac{3\varepsilon A^3}{4} \right] \cos \omega t + \frac{\varepsilon A^3}{4} \cos 3\omega t. \quad (55)$$

Inserting Equation (55) into Equation (52) yields

$$\begin{aligned} \omega &= \frac{1}{\pi A} \left[A + \frac{3\varepsilon A^3}{4} \right] \int_0^{2\pi/\omega} \cos^2 \omega t dt + \frac{1}{\pi A} \frac{\varepsilon A^3}{4} \int_0^{2\pi/\omega} \cos \omega t \cos 3\omega t dt \\ &= \frac{1}{\pi} \left[1 + \frac{3\varepsilon A^2}{4} \right] \frac{1}{2} \int_0^{2\pi/\omega} [1 + \cos 2\omega t] dt = \frac{1}{\pi} \left[1 + \frac{3\varepsilon A^2}{4} \right] \frac{\pi}{\omega}, \end{aligned} \quad (56)$$

where the last term in the first row gives no contribution to ω , owing to the orthogonality between $\cos \omega t$ and $\cos 3\omega t$:

$$\int_0^{2\pi/\omega} \cos \omega t \cos 3\omega t dt = \frac{1}{2} \int_0^{2\pi/\omega} [\cos 2\omega t + \cos 4\omega t] dt = 0.$$

It follows from Equation (56) that

$$\omega = \sqrt{1 + \frac{3\varepsilon A^2}{4}}. \quad (57)$$

This is a well-known formula for the Duffing oscillator (53).

In Table 1 the exact value of

$$\omega_e = \pi \left(2\sqrt{2} \int_0^{\pi/2} \frac{d\theta}{\sqrt{2 + \varepsilon A^2(1 + \sin^2 \theta)}} \right)^{-1} \quad (58)$$

is compared to that computed by Equation (57), whose accuracy is dropped down to one order for large amplitude A .

Table 1. Comparing the frequencies with different values of A for Equation (53) with $\varepsilon = 2$.

A	0.5	1	3	5	10
Equation (57)	1.1726039	1.5811388	3.8078866	6.2048368	12.288206
Exact ω	1.1707815	1.5691058	3.7365995	6.0772487	12.024950
Equation (65)	1.1707813	1.5691057	3.7365996	6.0772493	12.024951
α	0.12345	2.01725×10^{-2}	4.771×10^{-4}	6.4985×10^{-5}	4.09413×10^{-6}

3.1.2. Improving the Accuracy of ω

As shown in Table 1 for the Duffing oscillator with the trial solution $x = A \cos \omega t$ to derive the frequency formula in Equation (57), its accuracy is only of the first order. For a large amplitude of oscillation, we can consider a more complex elliptic function, as shown below for the trial solution. A more complicated trial solution can obtain a better value of frequency; however, the computational cost to derive the frequency formula is also increased. To overcome this drawback, we will introduce the parameter in the weight function to derive the frequency formula, and then seek the accurate value of frequency by adjusting the parameter to meet the periodicity conditions.

Below, we improve the accuracy of ω by adopting the integral-type frequency Formula (40) in Theorem 3. To enhance the accuracy of ω , we consider the following $g(y)$ in Theorem 3:

$$g(y) = y + \alpha y^3, \quad (59)$$

where $\alpha > 0$ is a parameter to be given, and $y g(y) = y^2 + \alpha y^4 > 0$ is indeed a requirement in Theorem 3. When $\alpha = 0$, the integral-type frequency formula was already derived in Equation (51), whose accuracy of ω is of first-order.

Using $x(\tau) = A \cos \theta$ and $y(\tau) = -\omega A \sin \theta$, where $\theta = 2\pi\tau$, and Equations (41), (55) and (59), we can derive

$$g(y(\tau)) = -\left[\omega A + \frac{3\alpha\omega^3 A^3}{4} \right] \sin \theta + \frac{\alpha\omega^3 A^3}{4} \sin 3\theta, \quad (60)$$

$$z(\tau) = \left[\frac{\omega A}{2\pi} + \frac{3\alpha\omega^3 A^3}{8\pi} \right] \cos \theta - \frac{\alpha\omega^3 A^3}{24\pi} \cos 3\theta, \quad (61)$$

$$F(x(\tau)) = \left[A + \frac{3\varepsilon A^3}{4} \right] \cos \theta + \frac{\varepsilon A^3}{4} \cos 3\theta. \quad (62)$$

Inserting them into Equation (40) yields

$$36\alpha A^2 \omega^4 + b_0 \omega^2 - c_0 = 0, \quad (63)$$

where

$$b_0 = 48 - 36\alpha A^2 - 9\alpha\varepsilon A^3 + \alpha\varepsilon A^4, c_0 = 48 + 36\varepsilon A^2. \quad (64)$$

If $\alpha = 0$, Equation (63) recovers to Equation (57).

If $\alpha \neq 0$, the frequency is given by

$$\omega = \sqrt{\frac{1}{72\alpha A^2} \left[\sqrt{b_0^2 + 144\alpha A^2 c_0} - b_0 \right]}, \quad (65)$$

where the optimal value of α is determined by

$$\min_{\alpha > 0} |x(T) - A|; \quad (66)$$

$|x(T) - A|$ is the absolute error of the periodicity condition $x(T) = x(0) = A$. $x(T)$ is computed by applying the fourth-order Runge–Kutta method to integrate Equation (53) with a finer time increment $\Delta t = T/N$, where $N = 500$ is taken. The accuracy of Equation (65) is still with five orders even with large amplitudes of $A = 5$ and $A = 10$. The improvement of the frequency obtained by Equation (65) over Equation (57) is about three to five orders as tabulated in Table 1.

Although a simple trial solution $x = A \cos \omega t$ in Equation (11) was inserted into Equation (40) in Theorem 3, the resulting formula (65) is still very accurate. The key point lies in a free parameter α being appeared in Equation (65), which is determined by matching the periodicity condition, such that highly accurate frequency can be achieved. The term αy^3 in Equation (59) plays a role to correct Equation (57), such that the resultant frequency formula in Equation (65) is more accurate than Equation (57) for large amplitude vibration.

3.1.3. Check the Accuracy of Equation (37) in Theorem 2

Remark 2 has explained the exactness of Equation (37) in Theorem 2 by examining an exact solution in Equation (46). Continuing this line, we can check the accuracy of Equation (37) for the Duffing oscillator in Equation (53). According to [27], the exact solution of the Duffing oscillator is given by

$$x(t) = A \operatorname{cn}[t\sqrt{1 + \varepsilon A^2}; k], \quad (67)$$

where cn is the Jacobian elliptic function. Up to the third-order approximation of Equation (67), we consider

$$x(t) = C \cos \omega t + D \cos 3\omega t + E \cos 5\omega t, \quad (68)$$

where

$$k = \sqrt{\frac{\varepsilon A^2}{2(1 + \varepsilon A^2)}}, \quad (69)$$

$$K(k) = \int_0^{\pi/2} \frac{ds}{\sqrt{1 - k^2 \sin^2 s}}, \quad (70)$$

$$q = \exp\left[\frac{-\pi K(k')}{K(k)}\right], k' = \sqrt{1 - k^2}, \quad (71)$$

$$C = \frac{2\pi A}{kK(k)} \frac{\sqrt{q}}{1 + q}, \quad (72)$$

$$D = \frac{2\pi A}{kK(k)} \frac{\sqrt{q^3}}{1 + q^3}, \quad (73)$$

$$E = \frac{2\pi A}{kK(k)} \frac{\sqrt{q^5}}{1 + q^5}, \quad (74)$$

and $K(k)$ is the first kind complete elliptic integral.

Inserting Equation (68) into Equation (37), and after some operations we can derive

$$\omega^2 = \frac{4(C^2 + D^2 + E^2) + e_0 \varepsilon}{4C^2 + 36D^2 + 100E^2}, \quad (75)$$

where

$$e_0 = 3(C^4 + D^4 + E^4) + 4C^3D + 12(C^2D^2 + C^2E^2 + D^2E^2 + C^2DE + CD^2E). \quad (76)$$

In Table 2, the exact value obtained from Equation (58) is compared to that computed by Equation (75). It proves that the accuracy of Equation (37) for evaluating the frequency is very good.

As shown in Table 2 for the Duffing oscillator with the third-order trial solution in Equation (68) to derive the frequency formula in Equation (75), the accuracy of the frequency is enhanced upon comparing to that in Table 1. The computational cost to derive the frequency formula (75) is increased upon comparing to that in the derivation of Equation (57).

Table 2. Comparing the frequencies obtained from Equation (75) to the exact ones with different values of A for Equation (53) with $\varepsilon = 1$.

A	0.5	0.8	1	1.3	1.5
Exact ω	1.089158178779	1.213875303604	1.317776064966	1.495894806808	1.625676614802
Equation (75)	1.089158178773	1.213875303048	1.317776061670	1.495894787721	1.625676572213

3.1.4. Initial Conditions $x(0) = A$ and $\dot{x}(0) = B_0$

Next, we derive a simple formula for the Duffing oscillator by giving the general initial conditions:

$$x(0) = A, \dot{x}(0) = B_0. \quad (77)$$

Notice that $y(\tau)$ cannot have a constant term, say C_0 ; otherwise, $x(\tau)$ with a term $C_0 T \tau$, which is not a periodic function of τ . Because $y(\tau)$ is a periodic function of τ and does not have a constant term, $z(\tau)$ defined in Equation (26) is a periodic function. Even for $y(0) \neq 0$, the results in Theorems 1-3 are still applicable, since in Equation (33) the constant term is zero, i.e.,

$$z(1)y(1) - z(0)y(0) = [z(1) - z(0)]y(0) = 0,$$

because $z(\tau)$ defined in Equation (26) is indeed a periodic function satisfying $z(1) = z(0)$.

We begin with

$$x(t) = A \cos \omega t + B \sin \omega t, \quad (78)$$

where for satisfying the initial condition $\dot{x}(0) = B_0$ in Equation (77),

$$B = \frac{B_0}{\omega}. \quad (79)$$

We insert the following terms into Equation (37) in Theorem 2:

$$\begin{aligned} \dot{x}^2(t) &= (\omega B \cos \omega t - \omega A \sin \omega t)^2, \\ x(t)F(x(t)) &= x^2(t) + \varepsilon x^4(t) = (A \cos \omega t + B \sin \omega t)^2 + \varepsilon (A \cos \omega t + B \sin \omega t)^4, \end{aligned} \quad (80)$$

which leads to the following frequency equation:

$$4\omega^4 - (3\varepsilon A^2 + 4)^2 \omega^2 - 3\varepsilon B_0^2 = 0, \quad (81)$$

and recovers to Equation (57) if $B_0 = 0$. Then, a simple formula is given as follows:

$$\omega = \sqrt{\frac{1}{8} \left[3\epsilon A^2 + 4 + \sqrt{(3\epsilon A^2 + 4)^2 + 48\epsilon B_0^2} \right]}. \quad (82)$$

The exact period is given by [26]:

$$T = 4\sqrt{2} \int_0^{\pi/2} \frac{d\theta}{\sqrt{2 + \epsilon a_0^2 (1 + \sin^2 \theta)}}, \omega = \frac{2\pi}{T}, \quad (83)$$

$$a_0 = \sqrt{\sqrt{\frac{1}{\epsilon^2} + \frac{1}{\epsilon} (2B_0^2 + 2A^2 + \epsilon A^4)} - \frac{1}{\epsilon}}, \quad (84)$$

where a_0 is the maximal displacement.

Table 3 compares ω computed by Equations (82) and (83). For Equation (82), the accuracy of frequency is good when a weak nonlinearity of the oscillator with the values of $A = B_0 = 0.1$ is considered.

Table 3. Comparing the frequencies with different values of ϵ for Equation (53) with the initial conditions $x(0) = 0.1$ and $\dot{x}(0) = 0.1$.

ϵ	0.1	0.2	0.3	0.4	0.5	1
Exact ω	1.0007491	1.0014965	1.0022420	1.0029859	1.0037280	1.0074128
Equation (83)	1.0014989	1.0029955	1.0044899	1.0059822	1.0074722	1.0148900

3.2. A Singular Oscillator

We consider a singular oscillator [28,29]:

$$\ddot{x}(t) + \frac{1}{x(t)} = 0, x(0) = A, \dot{x}(0) = 0, \quad (85)$$

where the restoring force $F(x) = 1/x$ is singular at $x = 0$. Recently, Kontomaris and Malamou [30] explained the singular oscillator from physical considerations and mechanical analogies.

Inspired by the work in [28], we take the following second-order trial solution:

$$x(t) = (1 + \beta)A \cos \omega t - \beta A \cos 3\omega t. \quad (86)$$

Inserting it and $F(x) = 1/x$ into Equation (37) in Theorem 2 yields the formula for ω :

$$\omega = \frac{\sqrt{2}}{A\sqrt{10\beta^2 + 2\beta + 1}}, \quad (87)$$

which is a new formula for the singular oscillator.

Like α determined by Equation (66), a suitable criterion to determine the optimal value of β is

$$\min_{\beta \geq 0} \left| x\left(\frac{T}{4}\right) \right|. \quad (88)$$

The new method offers an easy way to raise the precision of ω by inserting the above optimal value of β into the formula (87) derived from Theorem 2 for the singular oscillator.

For the singular oscillator, the exact value is [29]

$$\omega_e = \frac{\sqrt{2\pi}}{2A}. \quad (89)$$

According to Equation (88), we take $\beta = 0.09319$, which is inserted into Equation (87), and the resultant value is quite close to the exact one in Equation (89); the relative error is very small:

$$\left| \frac{\omega_e - \omega}{\omega_e} \right| = 6.198 \times 10^{-6}. \quad (90)$$

Given $A = 1$, in Figure 1 the periodic solution (86) is compared to the following second-order one [29]:

$$x(t) = A \frac{1.183131284 \cos(1.2482546A^{-1}t)}{1 + 0.183131284 \cos(2.4965092A^{-1}t)}. \quad (91)$$

They are quite close with the maximal difference being 3.79×10^{-2} .

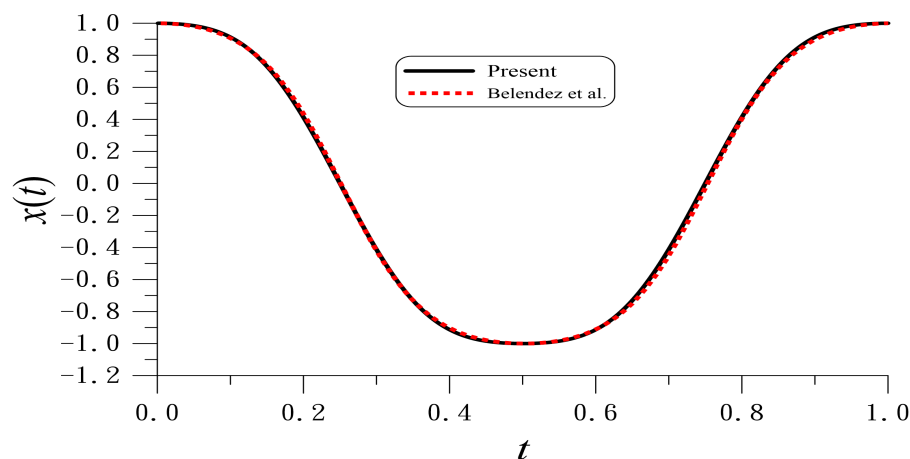


Figure 1. For the nonlinear singular oscillator up to the second-order solution, comparing the present periodic solution and the periodic solution obtained by Belendez et al. [29].

3.3. Mickens' Oscillator

We consider the Mickens' oscillator [31]:

$$\ddot{x}(t) + x(t)[1 + \dot{x}^2(t)] = 0, x(0) = A, \dot{x}(0) = 0. \quad (92)$$

The assumption $xf(x, y) > 0$ in Equation (24) is satisfied by Equation (92), due to $xf(x, y) = x^2(1 + y^2) > 0$.

Suppose that a second-order trial periodic solution is given by

$$x(t) = (1 + \beta)A \cos \omega t - \beta A \cos N\omega t, \quad (93)$$

where $N \geq 2$ is an integer, and like α, β is determined by Equation (66).

Inserting Equation (93) and $f(x, \dot{x}) = x + x\dot{x}^2$ into Equation (37) in Theorem 2 yields the following formula for ω :

$$\omega = \sqrt{\frac{2\beta^2 + 2\beta + 1}{2A_0}}, \quad (94)$$

where

$$A_0 = \frac{(1 + \beta)^2}{2} + \frac{N^2\beta^2}{2} - \frac{(1 + \beta)^4 A^2}{8} - \frac{N^2\beta^2(1 + \beta)^2 A^2}{4} - \frac{N^2\beta^4 A^2}{8} \quad (N = 2), \quad (95)$$

$$A_0 = \frac{(1 + \beta)^2}{2} + \frac{N^2\beta^2}{2} - \frac{(1 + \beta)^4 A^2}{8} - \frac{N^2\beta^2(1 + \beta)^2 A^2}{4} - \frac{\beta(1 + \beta)^3 A^2}{4} - \frac{N^2\beta^4 A^2}{8} \quad (N > 2). \quad (96)$$

Given $A = 1$, in Figure 2 the periodic solutions (93) with $N = 2$ and $N = 3$ are compared to the exact one. For $N = 2$, the optimal value is $\beta = -0.0429825$, which leads to $\omega = 1.136775916$. For $N = 3$, the optimal value is $\beta = -0.0221884$, which leads to $\omega = 1.136775502$. $N = 3$ is better than $N = 2$.

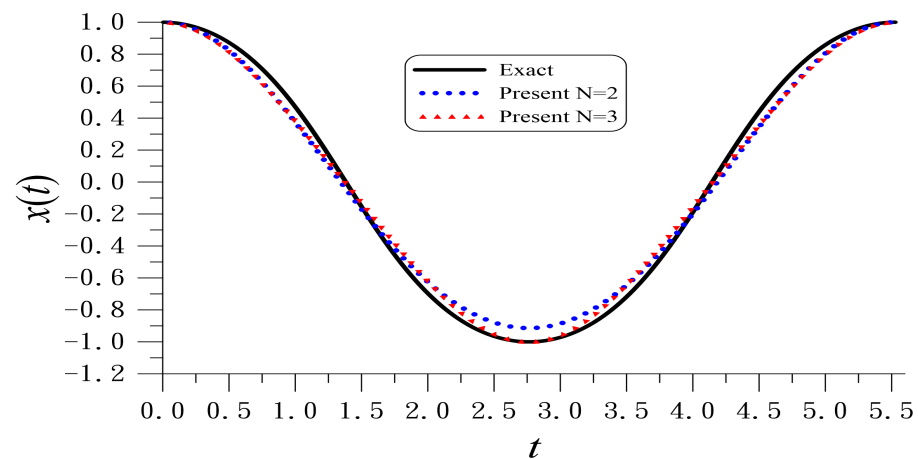


Figure 2. For the Mickens' nonlinear oscillator comparing present periodic solutions with $N = 2$ and $N = 3$ to the exact one.

The major difference between the Mickens curves obtained by $N = 2$ and $N = 3$ is the portion between $t = 2.25$ and $t = 3.5$, where the curve for $N = 3$ is closer to the exact solution than the curve for $N = 2$.

Table 4 compares ω computed by different values of N . It can be seen that N has little influence on the value of ω .

Table 4. Comparing the frequencies with different values of N for Equation (93) with the initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$.

N	2	3	4	5	6
ω	1.136775916	1.136775502	1.136775453	1.136775488	1.136776357

In [31], the following integral formula was derived:

$$T = 4A \int_0^1 \frac{dx}{\sqrt{\exp[A^2(1-x^2)] - 1}}. \quad (97)$$

When $A = 1$, $\omega = 1.137424014$ is obtained, which is close to the above $\omega = 1.136775916$ with $N = 2$, and $\omega = 1.136775502$ with $N = 3$.

Mickens' oscillator possesses the following first integral:

$$\dot{x} = \sqrt{\exp(A^2 - x^2) - 1} =: \phi(x). \quad (98)$$

It is easy to check the following equation:

$$\ddot{x} = \frac{-x \exp(A^2 - x^2) \dot{x}}{\sqrt{\exp(A^2 - x^2) - 1}} = \frac{-x(\dot{x}^2 + 1) \dot{x}}{\dot{x}} = -x(\dot{x}^2 + 1), \quad (99)$$

which is just Equation (92). From Equation (98) the period given by Equation (97) follows immediately from Equation (10) by inserting $\phi(x)$ in Equation (98).

For comparison we derive an approximation of $x(t)$ using the first-order harmonic balance method [31,32]:

$$x(t) = A \cos \frac{2t}{\sqrt{4 - A^2}}, \quad (100)$$

whose frequency is

$$\omega = \frac{2}{\sqrt{4 - A^2}}. \quad (101)$$

When $A = 1$, we obtain $\omega = 1.1547005$. Upon comparing to the exact value $\omega_e = 1.136775916$, the accuracy of Equation (101) is not good.

To raise the accuracy of ω using the harmonic balance method, we can consider a second-order trial solution:

$$x(t) = (A - B) \cos \omega t + B \cos 3\omega t, \quad (102)$$

where B and ω are unknown constants. After inserting Equation (102), and

$$\begin{aligned} \dot{x} &= -(A - B)\omega \sin \omega t - 3B\omega \sin 3\omega t, \\ \ddot{x} &= -(A - B)\omega^2 \cos \omega t - 9B\omega^2 \cos 3\omega t, \\ \dot{x}^2 &= \frac{1}{2}(A - B)\omega^2 + \frac{9B^2\omega^2}{2} + \left(\frac{7B}{2} - \frac{A}{2}\right)(A - B)\omega^2 \cos 2\omega t \\ &\quad - 3B(A - B)\omega^2 \cos 4\omega t - \frac{9B^2}{2} \cos 6\omega t \end{aligned}$$

into Equation (92), balancing the terms $\cos \omega t$ and $\cos 3\omega t$, and through a lengthy manipulation, we can derive two coupled nonlinear algebraic equations:

$$(A^2 - 4B^2 + 3AB - 4)\omega^2 + 4 = 0, \quad (103)$$

$$[(A - B)^2(A - 3B) + 18B(2 - B^2)]\omega^2 - 4B = 0. \quad (104)$$

However they are fifth-order nonlinear equations to hinder a closed-form formula for ω , unlike the formula (94) obtained from Theorem 2.

For the classical methods like the harmonic balance method and the perturbation method, the higher-order value of the frequency is hard to achieve, which requires performing a lot of algebraic operations. In general, the closed-form formula for ω is not available. In contrast, the advantage of the proposed integral formula is easy to derive the formula in closed form, and we can quickly compute quite accurate frequencies merely through a few lines of computations.

In Equation (93) if we replace the second term by $\sin N\omega t$,

$$x(t) = (1 + \beta)A \cos \omega t - \beta A \sin N\omega t, \quad (105)$$

it would be a failure to satisfy the initial conditions, and Equation (92) cannot be satisfied by balancing the harmonic terms.

3.4. Quintic Duffing Oscillator

We consider a quintic Duffing oscillator [33]:

$$\ddot{x}(t) + x(t) + \varepsilon x^5(t) = 0, x(0) = A, \dot{x}(0) = 0. \quad (106)$$

Beléndez et al. [33] using the technique of cubication through the Chebyshev polynomials to derive an approximation of Equation (106):

$$\ddot{y}(t) + \alpha y(t) + \beta y^3(t) = 0, y(0) = 1, \dot{y}(0) = 0, \quad (107)$$

where $y = x/A$ and

$$\alpha = 1 + \frac{15\varepsilon A^4}{16}, \beta = \frac{5\varepsilon A^4}{4}. \quad (108)$$

For Equation (107) the exact solution is given by

$$y(t) = \text{cn}[t\sqrt{\alpha + \beta}; k], k = \frac{\beta}{2(\alpha + \beta)}. \quad (109)$$

Up to third-order approximation, we consider

$$y(t) = C_0 \cos \omega t + D_0 \cos 3\omega t + E_0 \cos 5\omega t, \quad (110)$$

where

$$K(k) = \int_0^{\pi/2} \frac{ds}{\sqrt{1 - k^2 \sin^2 s}}, \quad (111)$$

$$q = \exp \left[\frac{-\pi K(k')}{K(k)} \right], k' = \sqrt{1 - k^2}, \quad (112)$$

$$C_0 = \frac{2\pi}{kK(k)} \frac{\sqrt{q}}{1+q}, \quad (113)$$

$$D_0 = \frac{2\pi}{kK(k)} \frac{\sqrt{q^3}}{1+q^3}, \quad (114)$$

$$E_0 = \frac{2\pi}{kK(k)} \frac{\sqrt{q^5}}{1+q^5}. \quad (115)$$

Inserting Equation (110) into Equation (37) in Theorem 2 with $F = \alpha y + \beta y^3$, and through some operations we can derive

$$\omega^2 = \frac{\alpha(C_0^2 + D_0^2 + E_0^2) + \beta e_0}{4C_0^2 + 36D_0^2 + 100E_0^2}, \quad (116)$$

where

$$e_0 = 3(C_0^4 + D_0^4 + E_0^4) + 4C_0^3 D_0 + 12(C_0^2 D_0^2 + C_0^2 E_0^2 + D_0^2 E_0^2 + C_0^2 D_0 E_0 + C_0 D_0^2 E_0). \quad (117)$$

In Table 5, the exact value of the frequency obtained from

$$\omega_e = \frac{\pi}{2} \int_0^1 \frac{dx}{\sqrt{1 - x^2 + \frac{1}{3}(1 - x^6)}} \quad (118)$$

is compared to that computed by Equation (116).

Table 5. Comparing the frequencies obtained from Equation (116) to the exact ones with different values of A for Equation (106) with $\varepsilon = 1$.

A	0.1	0.3	0.5	0.8
Exact ω	1.0077211	1.0102241	1.0270160	1.1263184
Equation (116)	1.0077192	1.0026500	1.0193425	1.1204448

4. A Generalized Conservation Law

Instead of Theorem 3 by specifying $z'(\tau) = g(y(\tau))$ in Equation (41), we directly specify $z(\tau)$ as a function of x , given by

$$z(\tau) = G(x(\tau)). \quad (119)$$

Then we can prove the following result.

Theorem 4. A general weighted integral formula for the frequency of Equation (5) is given by

$$\omega^2 = \frac{4\pi^2 \int_0^1 G(x(\tau)) f(x(\tau), y(\tau)) d\tau}{\int_0^1 G'(x(\tau)) x'(\tau)^2 d\tau}, \quad (120)$$

where $G(x)$ is a differentiable weight function of x satisfying $G' > 0$ and $Gf > 0$.

Proof. By means of Equation (119), we can derive

$$z'(\tau) = G'(x(\tau)) x'(\tau), \quad (121)$$

which is used to replace g in Equation (40); inserting $y = x'/T$ and Equation (119) into Equation (40) in Theorem 3, we can obtain

$$\omega = \frac{2\pi \int_0^1 G(x(\tau))f(x(\tau), y(\tau))d\tau}{\frac{1}{T} \int_0^1 G'(x(\tau))x'(\tau)^2 d\tau} = \frac{4\pi^2 \int_0^1 G(x(\tau))f(x(\tau), y(\tau))d\tau}{\omega \int_0^1 G'(x(\tau))x'(\tau)^2 d\tau}, \quad (122)$$

which is Equation (120). $\square$

To present Theorem 4 in the time domain, we can derive a very simple integral formulation of the frequency–amplitude relationship as follows.

Theorem 5. *If $G(x)$ is a differentiable weight function of x satisfying*

$$f(x(t), y(t))G(x(t)) > 0, G'(x(t)) > 0, \quad (123)$$

then the following identity holds:

$$\int_0^T G'(x(t))\dot{x}^2(t)dt = \int_0^T f(x(t), y(t))G(x(t))dt, \quad (124)$$

which is a generalization of Equation (37) in Theorem 2, and can be used to exactly determine the frequency.

Proof. Inserting

$$\tau = \frac{t}{T} = \frac{\omega t}{2\pi}, x'(\tau) = T\dot{x}(t) \quad (125)$$

into Equation (120) generates

$$\omega^2 = \frac{4\pi^2 \frac{\omega}{2\pi} \int_0^T G(x(t))f(x(t), y(t))dt}{T^2 \frac{\omega}{2\pi} \int_0^T G'(x(t))\dot{x}^2(t)dt} = \frac{\omega^2 \int_0^T G(x(t))f(x(t), y(t))dt}{\int_0^T G'(x(t))\dot{x}^2(t)dt}. \quad (126)$$

Upon canceling ω^2 on both sides we can derive Equation (124). $\square$

As an extension of Equation (37) in Theorem 2, Equation (124) expresses a generalized balance equation for kinetic energy and work within one period, which is called a generalized conservation law for the motion of an oscillator.

Corollary 1. *If Equation (124) is inserted by the trial solution $x(t) = A \cos \omega t$, the frequency–amplitude relationship can be simplified to the following approximate formula:*

$$\frac{\omega^2 A^2}{2} \int_0^T G'(A \cos \omega t)[1 - \cos 2\omega t]dt = \int_0^T f(A \cos \omega t, -\omega A \sin \omega t)G(A \cos \omega t)dt, \quad (127)$$

where $Gf > 0$ and $G' > 0$ are required.

Proof. Equation (127) can be obtained simply by inserting $x(t) = A \cos \omega t$ into Equation (124). $\square$

Corollary 2. *If $G(x)$ is a differentiable weight function of x satisfying*

$$f(x(t), \dot{x}(t))G(x(t)) > 0, x(t)G(x(t)) > 0, \quad (128)$$

then the following frequency formula in the time domain is available for Equation (5):

$$\omega^2 = \frac{\int_0^T f(x(t), \dot{x}(t))G(x(t))dt}{\int_0^T x(t)G(x(t))dt}. \quad (129)$$

Proof. Upon using $G'(x(t))\dot{x}(t) = \dot{G}(x(t))$, the left-hand side of Equation (124) can be written as

$$\int_0^T G'(x(t))\dot{x}^2(t)dt = \int_0^T \dot{G}(x(t))\dot{x}(t)dt = G(x(t))\dot{x}(t)|_0^T - \int_0^T G(x(t))\ddot{x}(t)dt. \quad (130)$$

Utilizing the periodicity condition $G(x(T))\dot{x}(T) = G(x(0))\dot{x}(0)$ and inserting the trial ODE (12) for $\ddot{x}$ generates

$$\int_0^T G'(x(t))\dot{x}^2(t)dt = \omega^2 \int_0^T x(t)G(x(t))dt. \quad (131)$$

By means of Equations (131) and (124), one can derive Equation (129). $\square$

Notice that Equation (16) is recovered from Equation (129) with $G(x) = x$. We generalize Equation (16) to the integral formula for a non-conservative nonlinear oscillator, which is equipped with a weight function. However Equations (127) and (129) are not the exact integral formulas, because the trial solution $x(t) = A \cos \omega t$ in Equation (11) is used in Corollary 1, while the trial ODE (12) is inserted into Equation (130) to replace $\ddot{x}$ in Corollary 2.

Remark 3. As a continuation of Remarks 1 and 2, we must emphasize that all the formulas in Theorems 1–5 are exact, like Equation (2) for a conservative nonlinear oscillator. The frequency formulas in Theorems 1–5 are simpler than Equation (2) even for a conservative nonlinear oscillator. However, when the exact solutions are not available for the nonlinear oscillators, we must insert some simple trial solutions in the formulas to obtain the approximate values of the true frequency.

5. Applications of Time-Domain Frequency Formulas

5.1. Tapered Beam's Oscillator

The free vibration of a tapered beam is governed by [13]:

$$\ddot{x}(t) + \frac{x(t) + ax(t)\dot{x}^2(t) + bx^3(t)}{1 + ax^2(t)} = 0, x(0) = A, \dot{x}(0) = 0, \quad (132)$$

where $a > 0$ and $b > 0$ are constants.

The assumption $xf(x, y) > 0$ in Equation (24) is satisfied by Equation (132) due to $xf(x, y) = (x^2 + ax^2y^2 + bx^4)/(1 + ax^2) > 0$.

Using He's frequency–amplitude formula [13], one has

$$\omega = \sqrt{\frac{4 + 3bA^2}{4 + 2aA^2}}. \quad (133)$$

Inserting Equation (11) and

$$f(x, \dot{x}) = \frac{x + ax\dot{x}^2 + bx^3}{1 + ax^2}, G(x) = x + ax^3, G'(x) = 1 + 3ax^2, \quad (134)$$

with $Gf = x^2 + ax^2\dot{x}^2 + bx^4 > 0$ and $G' = 1 + 3ax^2 > 0$ required in Theorem 5, into Equation (127) renders

$$\frac{\omega^2 A^2}{2} \int_0^T [1 + 2aA^2 \cos^2 \omega t][1 - \cos 2\omega t]dt = \int_0^T [A^2 \cos^2 \omega t + bA^4 \cos^4 \omega t]dt. \quad (135)$$

It can be arranged to

$$\frac{\omega^2 A^2}{2} \left(1 + \frac{aA^2}{2}\right) = \frac{A^2}{2} + \frac{3bA^4}{8} \Rightarrow (4 + 2aA^2)\omega^2 = 4 + 3bA^2, \quad (136)$$

which is the same as Equation (133).

The tapered beam's oscillator possesses the following first integral:

$$\dot{x} = \phi(x) = \sqrt{\frac{2(A^2 - x^2) + b(A^4 - x^4)}{2(1 + ax^2)}}. \quad (137)$$

From Equation (137), the period can be computed by means of Equation (10). The proof of Equation (137) is relegated to Appendix B.

5.2. Generalized Singular Oscillator

We are interested in the following generalized singular oscillator [10,34]:

$$\ddot{x}(t) + \frac{1}{x^q(t)} = 0, x(0) = A, \dot{x}(0) = 0, \quad (138)$$

where $q > 0$ is an odd integer.

By adopting Equation (127) to compute the frequency for Equation (138) it is better to choose $G(x) = x^p$ with $p - q \geq 0$, and $p - q$ is an even integer to satisfy Equation (123); hence, p is an odd integer, and $GF = x^{p-q} > 0$ and $G' = px^{p-1} > 0$ as required in Corollary 1.

Before the work to derive the formula for ω , we need the following result.

Lemma 1. For any positive integer $n \geq 2$, we have

$$\int_0^T \cos^n \omega t dt = \frac{n-1}{n} \int_0^T \cos^{n-2} \omega t dt. \quad (139)$$

Proof. Utilizing the technique of integration by parts, we have

$$\int \cos^n \omega t dt = \frac{1}{n\omega} \cos^{n-1} \omega t \sin \omega t + \frac{n-1}{n} \int \cos^{n-2} \omega t dt. \quad (140)$$

Then it follows that

$$\int_0^T \cos^n \omega t dt = \frac{1}{n\omega} [\cos^{n-1} \omega T \sin \omega T - \cos^{n-1} 0 \sin 0] + \frac{n-1}{n} \int_0^T \cos^{n-2} \omega t dt. \quad (141)$$

Because of $\omega T = 2\pi$ and $\sin 2\pi = 0$, Equation (141) reduces to Equation (139). $\square$

Now inserting $F = x^{-q}$, $G(x) = x^p$, $G'(x) = px^{p-1}$ into Equation (127) yields

$$A^{p+1} \omega^2 p \int_0^T [\cos^{p-1} \omega t - \cos^{p+1} \omega t] dt = A^{p-q} \int_0^T \cos^{p-q} \omega t dt, \quad (142)$$

where $[1 - \cos 2\omega t]/2$ is replaced by $\sin^2 \omega t = 1 - \cos^2 \omega t$.

Repeatedly using Lemma 1, the right-hand side of Equation (142) is

$$\alpha_1 := \int_0^T \cos^{p-q} \omega t dt = \frac{p-q-1}{p-q} \times \frac{p-q-3}{p-q-2} \times \cdots \times \frac{p-q-(2k_1-1)}{p-q-(2k_1-1)+1}, \quad (143)$$

of which the product is up to $p-q-(2k_1-1) = 1$, where k_1 is the number of the fractional terms in the product. Correspondingly, we have

$$\alpha_2 := \int_0^T \cos^{p-1} \omega t dt = \frac{p-2}{p-1} \times \frac{p-4}{p-3} \times \cdots \times \frac{p-2k_2}{p-2k_2+1}, \quad (144)$$

$$\alpha_3 := \int_0^T \cos^{p+1} \omega t dt = \frac{p}{p+1} \times \frac{p-2}{p-1} \times \cdots \times \frac{p-2(k_3-1)}{p-2(k_3-1)+1}, \quad (145)$$

where k_2 is the number of the fractional terms in the product of the first equation, and k_3 is the number of the fractional terms in the product of the second equation. It is obvious that $\alpha_2 > \alpha_3$.

As a consequence, the following result holds for Equation (138).

Theorem 6. For the generalized singular oscillator in Equation (138), the frequency–amplitude formula is given by

$$\omega = \sqrt{\frac{\alpha_1}{p(\alpha_2 - \alpha_3)A^{q+1}}}, \quad (146)$$

where α_1 , α_2 and α_3 are constants given in Equations (143)–(145).

Proof. Inserting Equations (143)–(145) into Equation (142) yields Equation (146). $\square$

Notice that for the generalized singular oscillator in Equation (138), the following formula for ω was proposed by Garcia [35]:

$$\omega = \sqrt{\frac{(N+2) \int_0^A x^N F(x) dx}{A^{N+2}}}, \quad (147)$$

where $N = 1, 2, \dots$

Theorem 7. For the conservative nonlinear oscillator in Equation (1), a powerful formula for ω is given by Equation (147).

Proof. Taking the product of x^N to Equation (1) and integrating it by considering its amplitude $x(0) = A$, we have

$$\int_{\frac{T}{4}}^0 x^N(t) \ddot{x}(t) \dot{x}(t) dt + \int_0^A x^N F(x) dx = 0. \quad (148)$$

Inserting

$$x^N = A^N \cos^N \omega t, \dot{x}(t) = -\omega A \sin \omega t, \ddot{x} = -\omega^2 A \cos \omega t \quad (149)$$

into the first term in Equation (148), we have

$$\int_{\frac{T}{4}}^0 x^N(t) \ddot{x}(t) \dot{x}(t) dt = -A^{N+2} \omega^3 \int_{\frac{T}{4}}^0 \cos^{N+1} \omega t (-\sin \omega t dt) = -\frac{\omega^2 A^{N+2}}{N+2}. \quad (150)$$

It follows from Equations (148) and (150) that

$$-\frac{\omega^2 A^{N+2}}{N+2} + \int_0^A x^N F(x) dx = 0, \quad (151)$$

which can be arranged to Equation (147). $\square$

The presented derivation is suitable for the general conservative nonlinear oscillator in Equation (1), not limited to the generalized singular oscillator in Equation (138). Our approach to Equation (147) is simpler than that given in [35].

Inserting $F = x^{-q}$ into Equation (147) for the generalized singular oscillator yields

$$\omega = \sqrt{\frac{N+2}{(N-q+1)A^{q+1}}}, \quad (152)$$

which bears certain similarity to Equation (146). However, these two formulas are not the same because the coefficient is not equal, i.e., $(N+2)/(N-q+1) \neq \alpha_1/[p(\alpha_2 - \alpha_3)]$. Even we give the same value of $N = p$, they are unequal; for example, for $N = p = 5$ and $q = 3$, $(N+2)/(N-q+1) = 7/3$ is not equal to $\alpha_1/[p(\alpha_2 - \alpha_3)] = 8/5$.

As an application of Equations (146) and (152), we consider Equation (138) with $q = 3$. The exact value of ω is

$$\omega = \frac{\pi}{2A^2} \approx \frac{1.571}{A^2}. \quad (153)$$

By means of Equation (146) with $p = q = 3$, we have

$$\omega = \frac{2\sqrt{2}}{\sqrt{3}A^2} \approx \frac{1.632}{A^2}. \quad (154)$$

On the other hand, inserting $N = q = 3$ into Equation (152) yields

$$\omega = \frac{\sqrt{5}}{A^2} \approx \frac{2.236}{A^2}. \quad (155)$$

We improve it by inserting $N = 5$ into Equation (152),

$$\omega = \frac{\sqrt{\frac{7}{3}}}{A^2} \approx \frac{1.528}{A^2}, \quad (156)$$

which is more accurate than Equation (155).

6. A Weighted Integral Formula for Conservative Nonlinear Oscillators

The formulations of frequency in the τ -domain and t -domain are also applicable to the conservative nonlinear oscillator, if $f(x, \dot{x})$ is replaced by $F(x)$. In this section, and the next two sections, we are going to derive simpler formulas in the spatial domain, namely the x -domain, to compute the frequency for a conservative nonlinear oscillator. The new formulations are based on the idea of a trial ODE to replace the acceleration term $\ddot{x}$ in the formulation. A more general result than Theorem 7 can be deduced by introducing a weight function.

Theorem 8. For the conservative nonlinear oscillator in Equation (1), an integral-type frequency–amplitude formula is given by

$$\omega^2 = \frac{\int_0^A \rho(x)F(x)dx}{\int_0^A x\rho(x)dx}, \quad (157)$$

where $\rho(x) > 0$ is a weight function satisfying $\rho(x)F(x) > 0$ and $x\rho(x) > 0$ in the range of $x \in [0, A]$.

Proof. Taking the product of $\rho(x) > 0$ to Equation (1) yields

$$\rho(x)\ddot{x} + \rho(x)F(x) = 0, \quad (158)$$

which by inserting the trial ODE $\ddot{x} = -\omega^2 x$ for $\ddot{x}$, becomes

$$\omega^2 x\rho(x) = \rho(x)F(x). \quad (159)$$

Integrating both sides from $x = 0$ to $x = A$, and considering its amplitude $x(0) = A$, we can derive Equation (157). $\square$

It is not an easy issue to determine the optimal weight function $\rho(x)$ in its general form used in Equation (157) for a general conservative nonlinear oscillator. In general, we take $\rho(x) = x^p$ to be a power function for an easier manner to compute the integrals $\int_0^A \rho(x)F(x)dx$ and $\int_0^A x\rho(x)dx$ in Equation (157).

If we take $\rho(x) = x^p, p \geq 0$, a suitable criterion to determine p is

$$\min_{p \geq 0} \left| x \left(\frac{T}{4} \right) \right| \quad (160)$$

for the singular oscillator, or the periodicity condition

$$\min_{p \geq 0} |x(T) - A| \quad (161)$$

for the non-singular oscillator. Equation (160) is an original criterion for the singular oscillator.

As an application of Equation (157), we take $\rho = x^4$:

$$\omega^2 = \frac{\int_0^A x^4 F(x)dx}{\int_0^A x^5 dx}, \quad (162)$$

which was introduced by He and Liu [36] for a modified frequency–amplitude formulation of the fractal vibration systems.

We take $\rho = x^4$ in Equation (157) derived from Theorem 8, such that Equation (162) is the same as that proposed by He and Liu [36]. For other system we take $\rho = x^p$ and determine p by Equation (161).

In addition the conditions $\rho(x) > 0$, $\rho(x)F(x) > 0$ and $x\rho(x) > 0$, the principle for choosing $\rho(x)$ is that the integral terms $\int_0^A x\rho(x)dx$ and $\int_0^A \rho(x)F(x)dx$ can be integrated explicitly, without resorting on the numerical integrations. In this manner, we can derive the explicit formula for ω by means of Equation (157).

6.1. Quintic Duffing Oscillator

We apply Equation (157) in Theorem 8 to the quintic Duffing oscillator (106). Inserting $\rho = x^p$ and $F = x + \varepsilon x^5$ into Equation (157) generates

$$\omega^2 = 1 + \frac{(p+2)\varepsilon A^4}{p+6}. \quad (163)$$

This formula is much simpler than that given in Equation (116).

The frequency is a quantity to measure the periodicity of the solution for a periodic system. Therefore $T = 2\pi/\omega$ must abide the rule of $x(T) = x(0) = A$, that is $x(2\pi/\omega) - A = 0$. When Equation (163) is used to compute ω , the best value of p is selected such that the condition $x(T) - A = 0$ can be satisfied. Usually, the condition $x(T) - A = 0$ cannot be solved exactly, because $x(T)$ is a highly nonlinear and implicit function of T . Hence, we turn it to be a minimization problem in Equation (161), which is easily implemented into a computer program to determine the optimal value of p .

In view of Table 6, we can observe that Equation (163) with $p = 5.5$ is more accurate than Equation (116) when $A \geq 0.5$.

Table 6. Comparing the frequencies obtained from Equation (163) with $p = 5.5$ to the exact ones with different values of A for Equation (106) with $\varepsilon = 1$.

A	0.1	0.3	0.5	0.8
Equation (163)	1.0000328	1.0026499	1.0202687	1.1262143
Exact ω	1.0077211	1.0102241	1.0270160	1.1263184
Equation (116)	1.0077192	1.0026500	1.0193425	1.1204448

6.2. Singular Oscillators

We consider two singular oscillators with $q = 1$ and $q = 3$ in Equation (138). Inserting $\rho(x) = x^p$ into Equation (157), it immediately leads to

$$\omega = \sqrt{\frac{p+2}{pA^2}}, \omega_e = \sqrt{\frac{\pi}{2A^2}} (q = 1), \quad (164)$$

$$\omega = \sqrt{\frac{p+2}{(p-2)A^4}}, \omega_e = \frac{\pi}{2A^2} (q = 3). \quad (165)$$

According to Equation (160), we plot the values of $|x(T/4)|$ in a finer interval $p \in [3.5, 3.51]$ for $q = 1$, and $p \in [4.72, 4.732]$ for $q = 3$ in Figure 3. Because the curves are obtained by the integration of the singular ODE (138), which is infinite at $x = 0$, they are irregular in some range of p . In Figure 3, one global minimum and two local minima are remarked. To satisfy the requirement in Equation (160), we take the optimal value of p at the global minimum point.

For $q = 1$, the minimal point is $p = 3.503965$, rendering $\omega = 1.2533084045$, which is close to the exact one with the relative error 4.57×10^{-6} . This result is slightly better than that obtained by Equation (87). For $q = 3$, the minimal point is $p = 4.7265$, rendering $\omega = 1.57069485897$, which is close to the exact one with the relative error 6.46×10^{-5} .

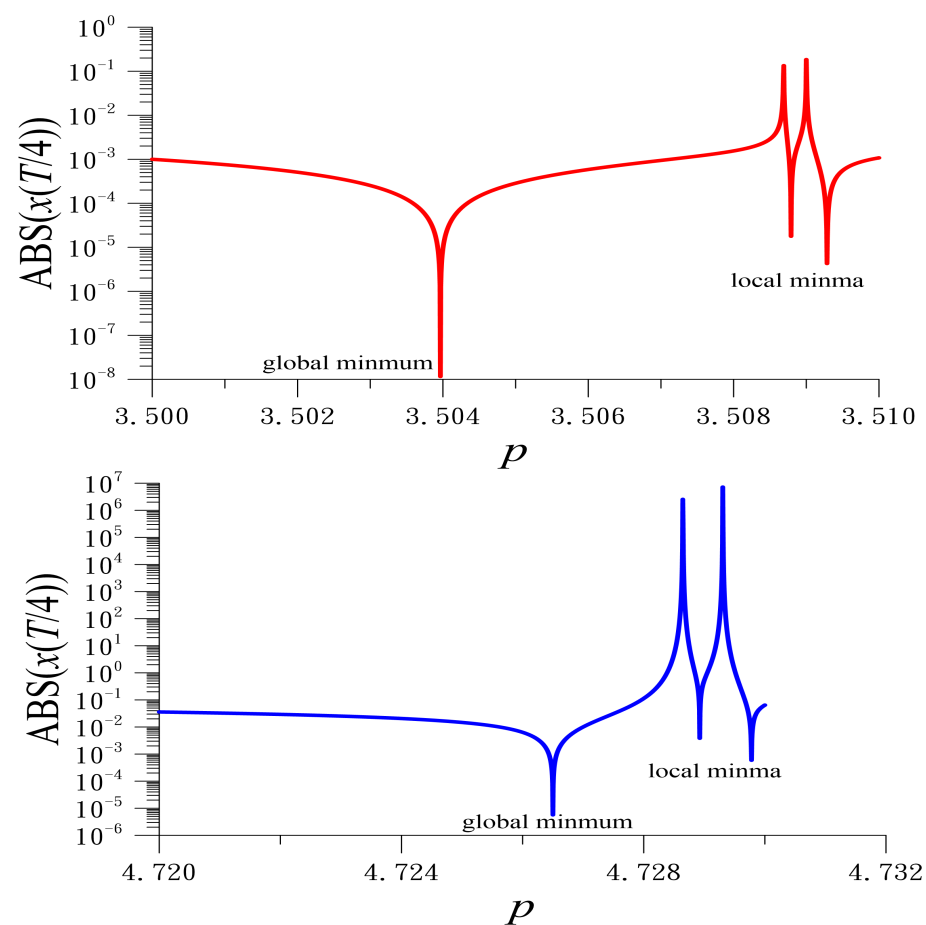


Figure 3. For two singular oscillators with $q = 1$ and $q = 3$, plotting the curves to find the minimal points in finer intervals. Because the curves are obtained by the integration of the singular ODE, they are irregular in some ranges.

For the singular oscillator with $q = 3$, upon setting $W = x^{p-1}$ into Equation (18), the same formula can be derived in Equation (165).

Now we check the theoretical foundation for Equation (18) derived by Antola [11]. Letting

$$\Phi(x) = \int_0^x W(s)[F(s) - \omega^2 s]^2 ds, \quad (166)$$

and from Equation (1), we have the following system of first-order ODEs:

$$\begin{cases} \dot{x}(t) = y, x(0) = A, \\ \dot{y}(t) = -F(x), y(0) = 0, \\ \dot{\Phi}(t) = W(x)[F(x) - \omega^2 x]^2 y, \Phi(0) = 0, \end{cases} \quad (167)$$

where ω is computed from Equation (18) by inserting $F = x^{-q}$, $q = 3$, and $W = x^p$. We can integrate this system using the fourth-order Runge–Kutta method.

We plot the values of $\Phi(A)$ vs. p in an interval $p \in [3, 4.4]$ in Figure 4. The minimal point is $p = 3.7805$, and the corresponding ω is $\omega = 1.6329931618$, which compared to the exact value $\omega_e = 1.5707963267$ is with the relative error 3.96×10^{-2} . We can conclude that the weighted functional method, as advocated by Antola [11], is less accurate than the method derived in Theorem 8.

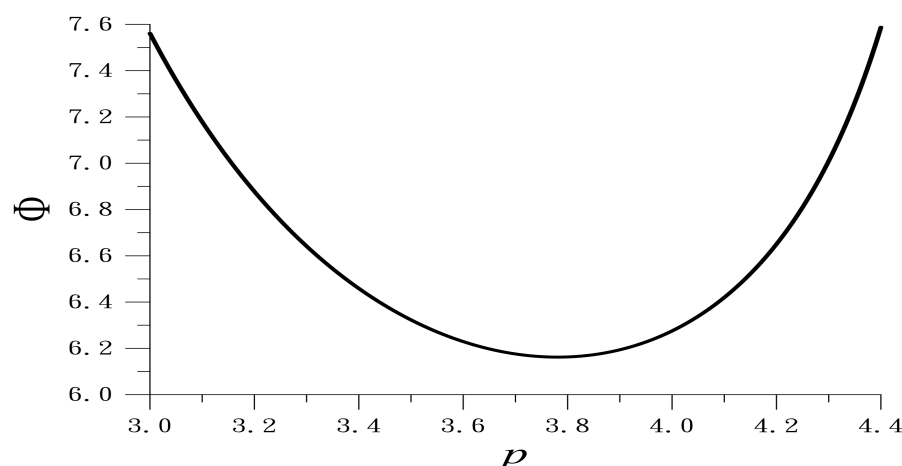


Figure 4. For the singular oscillator with $q = 3$, plotting the curve to find the minimal point of the weight functional in Equation (166).

7. A More Complex Trial ODE

Rather than $\ddot{x} = -\omega^2 x$ (or the trial solution $x = A \cos \omega t$) used in the derivation of Theorem 8, we consider a more complex trial solution given in Equation (86). If $\beta = -1/4$ for Equation (86), we can prove that the corresponding ODE is

$$\ddot{x} + 9\omega^2 x - 6\omega^2 A^{2/3} x^{1/3} = 0, \quad (168)$$

whose proof is relegated to Appendix A.

It is interesting that

$$x(t) = \frac{3A}{4} \cos \omega t + \frac{A}{4} \cos 3\omega t = A \cos^3 \omega t \quad (169)$$

is the unique solution of Equation (168) by subjecting to initial conditions $x(0) = A$ and $\dot{x}(0) = 0$.

It follows from Equation (169) that

$$\dot{x}(t) = -\frac{3A\omega}{4} \sin \omega t - \frac{3A\omega}{4} \sin 3\omega t, \quad (170)$$

$$\ddot{x}(t) = -\frac{3A\omega^2}{4} \cos \omega t - \frac{9A\omega^2}{4} \cos 3\omega t. \quad (171)$$

By means of Equations (169) and (170) the initial conditions $x(0) = A$ and $\dot{x}(0) = 0$ are satisfied obviously. Using Equations (169) and (171), we can derive

$$\begin{aligned} \ddot{x} + 9\omega^2 x - 6\omega^2 A^{2/3} x^{1/3} &= -\frac{3A\omega^2}{4} \cos \omega t - \frac{9A\omega^2}{4} \cos 3\omega t \\ &+ \frac{27A\omega^2}{4} \cos \omega t + \frac{9A\omega^2}{4} \cos 3\omega t - 6\omega^2 A^{2/3} A^{1/3} \cos \omega t \\ &= \left[\frac{27A\omega^2}{4} - \frac{3A\omega^2}{4} - \frac{24A\omega^2}{4} \right] \cos \omega t = 0. \end{aligned} \quad (172)$$

It is Equation (168). We proved that Equation (169) is a unique solution of Equation (168), subjecting to initial conditions $x(0) = A$ and $\dot{x}(0) = 0$.

Theorem 9. For the conservative nonlinear oscillator in Equation (1), if the trial ODE is Equation (168), then an integral-type formula for ω is given by

$$\omega^2 = \frac{\int_0^A \rho(x) F(x) dx}{\int_0^A \rho(x) (9x - 6A^{2/3} x^{1/3}) dx}, \quad (173)$$

where $\rho(x) > 0$ is a weight function satisfying $\rho(x)F(x) > 0$ and $\rho(x)(9x - 6A^{2/3} x^{1/3}) > 0$ in the range of $x \in [0, A]$.

Proof. Inserting Equation (168) for $\ddot{x}$ into Equation (158), we have

$$\omega^2 \rho(x) (9x - 6A^{2/3} x^{1/3}) = \rho(x) F(x). \quad (174)$$

Integrating both sides from $x = 0$ to $x = A$, and considering its amplitude $x(0) = A$, we can derive Equation (173). $\square$

Now, we apply Theorem 9 to the soft oscillator [35,37]:

$$\begin{cases} \ddot{x}(t) + x^{1/3}(t) = 0, \\ x(0) = A, \dot{x}(0) = 0, \end{cases} \quad (175)$$

whose exact frequency is given by

$$\omega_e = \frac{2\pi}{T_e} \approx \frac{1.072271055}{A^{1/3}}, T_e = 4\sqrt{\frac{2}{3}} \int_0^1 \frac{ds}{\sqrt{1-s^{4/3}}}. \quad (176)$$

Integrating from $x(0) = A$ to $x(T/4) = 0$ for the following first-order ODE derived from Equation (175):

$$\dot{x} = \sqrt{\frac{3}{2}} \sqrt{A^{4/3} - x^{4/3}}, \quad (177)$$

the error is $|x(T/4)| = 3.1 \times 10^{-6}$. Equation (177) is the first integral of Equation (175).

Inserting $F(x) = x^{1/3}$ and $\rho(x) = x^p$ into Equation (173) generates a simple formula:

$$\omega = \sqrt{\frac{p+2}{3pA^{2/3}}}. \quad (178)$$

To meet the requirement in Equation (160) the optimal value $p = 0.82032788$ is obtained, of which $\omega = 1.070520625$ is obtained, and

$$\left| \frac{\omega - \omega_e}{\omega_e} \right| = 1.63 \times 10^{-3}. \quad (179)$$

The error $|x(T/4)| = 2.79 \times 10^{-10}$ is smaller than $|x(T_e/4)| = 3.1 \times 10^{-6}$ obtained by the exact value of T_e . It is interesting that ω in Equation (178) with the optimal value $p = 0.82032788$ for $A = 1$ is more accurate than the exact one ω_e in Equation (176). Using Equation (176) to compute ω_e , a singular integral $\int_0^1 ds / \sqrt{1-s^{4/3}}$ is encountered when s nears 1, which may cause a less accurate value of ω_e .

The nonlinear oscillator (168) and its solution (169) has infinite frequency. To define, we insert $\alpha = 0.5$ into

$$T = \frac{4}{3\sqrt{\alpha}} \int_0^A \frac{dx}{\sqrt{A^{3/2} x^{4/3} - x^2}} = \frac{4}{3\sqrt{\alpha}} \int_0^1 \frac{dy}{\sqrt{y^{4/3} - y^2}}, \omega = \frac{2\pi}{T} \approx 1.3614134762. \quad (180)$$

Inserting ω into Equations (168) and (169), we compare the exact solution to that obtained by the fourth-order Runge–Kutta method in Figure 5; they are almost coincident. The closed orbit is not smooth near the cusp point $(x, \dot{x}) = (0, 0)$. The trajectory does not cross the cusp point, which immediately returns before reaching the cusp points on $t = T/4$ and $t = 3T/4$.

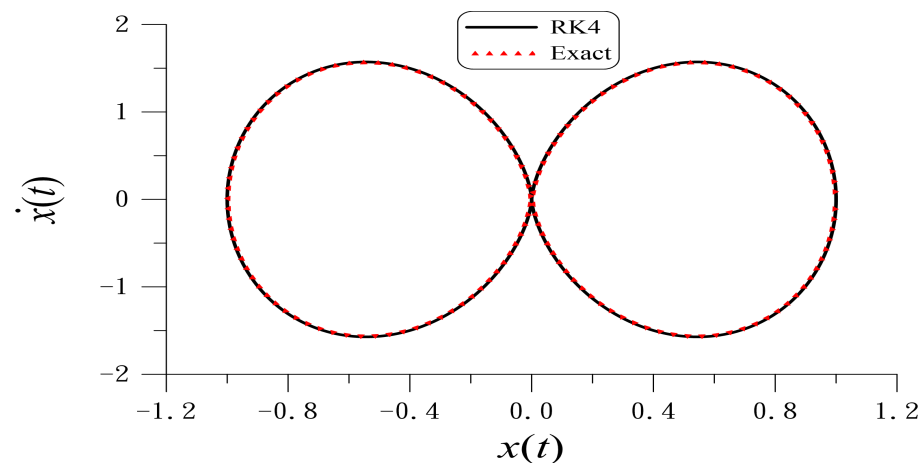


Figure 5. For a linear-soft oscillator comparing the exact solution and that computed by RK4. They coincide.

Remark 4. We mention a counterexample for $N = 3$ of Garcia's conjecture. For Equation (147), the conjecture of Garcia [35] was proposed that an optimal integer value N to approximate the true amplitude-period relationship is $N = 3$. We consider the nonlinear oscillator (168) with a parameter $\omega^2 = \alpha > 0$:

$$\ddot{x} + 9\alpha x - 6\alpha A^{2/3} x^{1/3} = 0, \quad (181)$$

where 9α signifies the linear spring constant, while $6\alpha A^{2/3}$ is a soft spring constant.

Inserting $F = 9\alpha x - 6\alpha A^{2/3} x^{1/3}$ with $\alpha = 0.5$ and $N = 3$ into Equation (147) as derived in [35] yields

$$\omega = \sqrt{\frac{9N\alpha}{3N+4}} \approx 1.0190493307, \quad (182)$$

which is quite different from the value 1.3614134762 in Equation (180). Hence, the optimal integer is not $N = 3$ as conjectured by Garcia [35].

8. A Generalized Trial ODE for Conservative Nonlinear Oscillators

For a given, more complex trial solution, the corresponding trial ODE may not be obtained easily. To generalize the frequency formula, we can relax this constraint by introducing a trial function in the trial ODE.

Theorem 10. Let

$$\ddot{x} = -\omega^2 H(x) \quad (183)$$

be a trial ODE, where $H(x) \neq F(x)$. For the conservative nonlinear oscillator in Equation (1) an integral formula for ω is given by

$$\omega^2 = \frac{\int_0^A \rho(x) F(x) dx}{\int_0^A \rho(x) H(x) dx}, \quad (184)$$

where $\rho(x) > 0$ is a weight function satisfying $\rho(x)F(x) > 0$ and $\rho(x)H(x) > 0$ in the range of $x \in [0, A]$.

Proof. Taking the product of $\rho(x) > 0$ to Equation (1) yields

$$\rho(x)\ddot{x} + \rho(x)F(x) = 0, \quad (185)$$

which by inserting Equation (183) for $\ddot{x}$ becomes

$$\omega^2 \rho(x) H(x) = \rho(x) F(x). \quad (186)$$

Integrating both sides from $x = 0$ to $x = A$, and considering its amplitude $x(0) = A$, we can derive Equation (184). $\square$

For the Duffing oscillator in Section 3.1, we take $\rho(x) = x^4$ and

$$H(x) = x + \beta A^{-1} x^2, \quad (187)$$

where the factor A^{-1} is employed to balance the power of A when $x = A$ is taken. Then, by inserting $F(x) = x + \varepsilon x^3$, $\rho(x) = x^4$ and Equation (187) into Equation (184), we can derive

$$\omega^2 = \frac{28 + 21\varepsilon A^2}{28 + 24\beta}. \quad (188)$$

If $\beta = 0$, we recover to the frequency formula given in Equation (57). However, $\beta = 0$ is not the optimal value. As in Equation (66) for α , we can obtain the optimal value of β . Formula (188) is simpler than the one in Equation (65) derived from Theorem 3 for the Duffing oscillator.

In Table 7 the exact value of ω_e given in Equation (58) is compared to that computed by Equation (188). Upon comparing to Table 1, the improvement of accuracy of ω is apparent.

Table 7. Comparing the frequencies with different values of A for the Duffing oscillator (53) with $\varepsilon = 2$.

A	0.5	1	3	5	10
Exact ω	1.1707815	1.5691058	3.7365995	6.0772487	12.024950
Equation (188)	1.1707815	1.5691057	3.7365995	6.0772478	12.024950
β	0.00363495	0.0179625	0.044940075	0.049501492	0.05164157

When we consider the following initial conditions:

$$x(0) = A_0, \dot{x}(0) = B_0, \quad (189)$$

Equation (188) is still applicable but with the following maximal displacement A :

$$A = \sqrt{\sqrt{\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}(2B_0^2 + 2A_0^2 + \varepsilon A_0^4)} - \frac{1}{\varepsilon}}. \quad (190)$$

The formula (188) for this case is much simpler than Equation (82) derived from Theorem 2 for the Duffing oscillator. Table 8 compares ω to the exact ones with the values of $A_0 = B_0 = 0.1$. Comparing Table 8 to Table 3, the improvement of the accuracy of ω is significant.

Table 8. Comparing the frequencies with different values of ε for Equation (190) with the initial conditions $x(0) = 0.1$ and $\dot{x}(0) = 0.1$; present values are computed by Equations (188) and (190).

ε	0.1	0.2	0.3	0.4	0.5	1
Exact	1.0007491	1.0014965	1.0022420	1.0029859	1.0037280	1.0074128
Present	1.0007492	1.0014965	1.0022420	1.0029859	1.0037280	1.0074128
β	0	4.3408×10^{-7}	9.7173×10^{-7}	1.71946×10^{-6}	2.67450×10^{-6}	1.046398×10^{-5}

Then, we raise the values of A_0 and B_0 to $A_0 = 1$ and $B_0 = 0.5$. Table 9 compares ω to the exact ones with different values of ε for Equation (190). The accuracy of the frequency can be appreciated when large values of ε are considered.

Table 9. Comparing the frequencies with different values of ε for Equation (190) with the initial conditions $x(0) = 1$ and $\dot{x}(0) = 0.5$; present values are computed by Equations (188) and (190).

ε	0.5	0.6	0.8	1.0	1.5
Exact	1.1959625395	1.2292135193	1.2920371106	1.3508823476	1.4851401782
Present	1.1959625391	1.2292135177	1.2920371102	1.3508823469	1.4851401772
β	4.508772×10^{-3}	5.71318×10^{-3}	8.071193×10^{-3}	$1.0297463013 \times 10^{-2}$	$1.516603181 \times 10^{-2}$

9. Conclusions

The theoretical foundations of integral-type formulations were incorporated into the frequency–amplitude formulas to estimate the frequency for both conservative and non-conservative nonlinear oscillators. For a newly received model of a nonlinear oscillator, if there exists no simple method to accurately estimate the vibration frequency, the proposed integral-type methods can quickly produce a good approximation of the vibration frequency with a few lines of computation. The frequency–amplitude formulas were developed as the integral-type equations in the τ -domain and t -domain for the non-conservative nonlinear oscillators. Because they are derived mathematically from an identity and the governing equations without adopting any approximation, the frequency obtained is exact if one inserts the exact solution into the formulas. Suitable conditions for these formulas were given, and the results are correct because they are derived through a rigorous mathematical procedure.

In general, the exact solution is not available; hence, we inserted a simple trial solution into the formulas to obtain the approximate value of the true frequency of the nonlinear oscillator. The basic requirement of the trial solution is that it must satisfy the given initial conditions.

For two examples in Sections 3.1.1 and 5.1, we took the simplest trial solution $x = A \cos \omega t$ to derive the frequencies in order to compare them to the results obtained by other methods. A more complicated trial solution or trial ODE can obtain a better value of frequency; however, the computational cost to derive the frequency formula is also increased. To overcome this drawback, we have introduced a parameter in the weight function to derive the frequency formula, and then sought an accurate value of the frequency by adjusting the parameter to meet the periodicity conditions. For most examples, we have employed more complex trial solutions to obtain more accurate frequencies.

The frequency–amplitude formulas were also developed as the integral type in the x -domain for conservative nonlinear oscillators. The idea of trial ODE is introduced, which, being inserted into the integration of the governing equation, can generate very simple yet effective formulas to compute the frequency. After introducing the weight functions, these formulas are generalized to more powerful ones, which involve some parameters. When the parameter in the formulas is obtained by minimizing the absolute error of the periodicity condition, the accuracy of the frequency obtained from the new frequency–amplitude formulas can be significantly increased by several orders. A more complex trial ODE is introduced. It enlarged the application range of the integral-type frequency–amplitude formulas. Many examples of both conservative and non-conservative nonlinear oscillators were examined to confirm the efficiency and accuracy of the proposed theoretical integral methods.

Consequently, for the conservative nonlinear oscillators, we have three integral-type formulations of the frequency in the τ -domain, t -domain, and x -domain. For the first two types, the integral formulations are exact. When some simple trial solutions are inserted into the formulas, we can obtain the approximate values of the frequency. The x -domain formulations are not the exact ones, because the acceleration term $\ddot{x}$ was replaced by the trial ODE. However, its advantage is that we do not need to use the trial solution to compute the frequency, and its form is simpler than that in the τ -domain and t -domain. For the non-conservative nonlinear oscillator, the x -domain formulation does not exist, because the integral term $\int_0^A \rho(x) f(x, \dot{x}) dx$ cannot be expressed as a function of A .

The method based on the balance of kinetic energy and work during the oscillation period is conceptual simple. It allowed us to quickly obtain the integral-type formulas for determining the oscillator oscillation frequency. Even with complex trial functions up to three orders, the algebraic manipulations to derive the explicit formula of frequency are quite saving.

Main points for the novel contributions of the present paper are summarized as follows.

- The theoretical contributions were substantial, with a comprehensive study on deriving and validating the integral-type formulas for estimating the vibrational frequencies of both conservative and non-conservative nonlinear oscillators.
- Significant contributions were made to the field of nonlinear oscillations by providing theoretically sound and practically useful integral-type frequency–amplitude formulas.
- Three integral-type integral formulas for conservative nonlinear oscillator were constructed, respectively, in the dimensionless time domain, time domain, and spatial domain.

- Novel integral-type frequency formulas were derived for a non-conservative nonlinear oscillator in the dimensionless time domain and time domain. Through a few lines of computations, the approximate value of frequency can be well estimated by these formulas.

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Appendix A

In this appendix we derive Equation (168). The term $\cos 3\omega t$ in Equation (86) can be expressed as

$$\cos 3\omega t = 4 \cos^3 \omega t - 3 \cos \omega t. \quad (\text{A1})$$

Hence, Equation (86) becomes

$$4\beta A \cos^3 \omega t - (1 + 4\beta) A \cos \omega t + x = 0. \quad (\text{A2})$$

If $\beta = -1/4$, it reduces to

$$A \cos^3 \omega t = x, \quad (\text{A3})$$

and then

$$\cos \omega t = \frac{x^{1/3}}{A^{1/3}}. \quad (\text{A4})$$

Inserting Equations (A3) and (A4) into Equation (A1) yields

$$\cos 3\omega t = \frac{4x}{A} - \frac{3x^{1/3}}{A^{1/3}}. \quad (\text{A5})$$

Taking twice differentials of Equation (86) yields

$$\ddot{x} = -\omega^2(1 + \beta)A \cos \omega t + 9\omega^2\beta A \cos 3\omega t. \quad (\text{A6})$$

Inserting Equations (A4) and (A5) and taking $\beta = -1/4$, we can obtain

$$\ddot{x} = -\frac{3}{4}\omega^2 A \frac{x^{1/3}}{A^{1/3}} - \frac{9}{4}\omega^2 A \left[\frac{4x}{A} - \frac{3x^{1/3}}{A^{1/3}} \right]. \quad (\text{A7})$$

It can be arranged to Equation (168).

Appendix B

In this appendix we derive Equation (137). It follows from Equations (7) and (132) that

$$\frac{d\phi}{dx} = -\frac{1}{\phi} \frac{x + bx^3 + ax\phi^2}{1 + ax^2}, \quad (\text{A8})$$

where $\dot{x} = \phi$. Letting

$$\varphi = \phi^2, \quad (\text{A9})$$

Equation (A8) becomes

$$\frac{d\varphi}{dx} = -\frac{2x + 2bx^3 + 2ax\varphi}{1 + ax^2}, \quad (\text{A10})$$

which is a linear ODE for φ . Considering the integrating factor

$$I(x) = \exp \left[\int \frac{2ax}{1+ax^2} dx \right] = 1 + ax^2, \quad (\text{A11})$$

and the condition $\varphi(A) = \varphi^2(A) = 0$, the solution of $\varphi(x)$ is

$$(1 + ax^2)\varphi(x) = - \int_A^x (2s + 2bs^3)ds = A^2 - x^2 + \frac{b}{2}(A^4 - x^4). \quad (\text{A12})$$

It can be arranged to

$$\varphi(x) = \frac{2(A^2 - x^2) + b(A^4 - x^4)}{2(1 + ax^2)}. \quad (\text{A13})$$

Then, Equation (137) was derived using Equation (A9).

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