

Article

Using Dual Double Fuzzy Semi-Metric to Study the Convergence

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Abstract: Convergence using dual double fuzzy semi-metric is studied in this paper. Two types of dual double fuzzy semi-metric are proposed in this paper, which are called the infimum type of dual double fuzzy semi-metric and the supremum type of dual double fuzzy semi-metric. Under these settings, we also propose different types of triangle inequalities that are used to investigate the convergence using dual double fuzzy semi-metric.

Keywords: dual double fuzzy semi-metric; double fuzzy semi-metric; fuzzy semi-metric space; triangle inequality; triangular norm

1. Introduction

The concept of fuzzy metric space proposed by Kramosil and Michalek [1] was inspired by the Menger space that is a special kind of probabilistic metric space by referring to Schweizer and Sklar [2–4], Hadžić and Pap [5], and Chang et al. [6]. Kaleva and Seikkala [7] proposed another concept of fuzzy metric space by considering the membership degree of the distance between any two different points. George and Veeramani [8,9] studied some properties of fuzzy metric spaces in the sense of Kramosil and Michalek [1]. Gregori and Romaguera [10–12] also extended the study of the properties of fuzzy metric spaces and fuzzy quasi-metric spaces in which the symmetric condition was not assumed.

The Hausdorff topology induced by the fuzzy metric space was studied in Wu [13]. In this paper, we shall propose the concept of double fuzzy-semi metric in fuzzy semi-metric space and study its convergent properties. The potential application for using the convergence of dual double fuzzy semi-metric is to study the new type of fixed point theorems in fuzzy semi-metric space by considering the Cauchy sequences, which will be the future research and may refer to the previous work of Wu [14] for studying the common coincidence points and common fixed points in fuzzy semi-metric spaces. Wu [15] studied the so-called fuzzy semi-metric space without assuming the symmetric condition. In the fuzzy semi-metric space (X, M) , the symmetric condition $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$ is not assumed to be true. Therefore, four kinds of triangle inequalities should be considered.

In order to obtain the new type of fixed point theorems in fuzzy semi-metric space, we need to study the convergence using dual double fuzzy semi-metric. Based on the concept of t-norm $*$, we shall firstly define the double fuzzy semi-metric by considering the mapping $\zeta : X^4 \times [0, +\infty) \rightarrow [0, 1]$ that is defined by:

$$\zeta(x, y; u, v, t) = M(x, y, t) * M(u, v, t),$$

where ζ is called a double fuzzy semi-metric.

The convergence using fuzzy semi-metric has been studied in Wu [16], where the infimum type of dual fuzzy semi-metric is the function $\Gamma^\downarrow(\lambda, \cdot, \cdot) : X \times X \rightarrow [0, +\infty)$ defined by:

$$\Gamma^\downarrow(\lambda, x, y) = \inf \{t > 0 : M(x, y, t) \geq 1 - \lambda\},$$

and the supremum type of dual fuzzy semi-metric is the function $\Gamma^\uparrow(\lambda, \cdot, \cdot) : X \times X \rightarrow [0, +\infty)$ defined by:

$$\Gamma^\uparrow(\lambda, x, y) = \sup \{t > 0 : M(x, y, t) \leq 1 - \lambda\}.$$

In this paper, we shall consider the double fuzzy semi-metric ζ to define the infimum and supremum types of dual double fuzzy semi-metric. The infimum type of dual double fuzzy semi-metric is the function $\Psi^\downarrow(\lambda, \cdot, \cdot; \cdot, \cdot) : X^4 \rightarrow [0, +\infty)$ defined by:

$$\Psi^\downarrow(\lambda, x, y; u, v) = \inf \{t > 0 : \zeta(x, y; u, v, t) \geq 1 - \lambda\},$$

and the supremum type of dual double fuzzy semi-metric is the function $\Psi^\uparrow : X^4 \rightarrow [0, +\infty)$ defined by:

$$\Psi^\uparrow(\lambda, x, y; u, v) = \sup \{t > 0 : \zeta(x, y; u, v, t) \leq 1 - \lambda\}.$$

Using the infimum and supremum types of dual fuzzy semi-metric $\Gamma^\downarrow(\lambda, x, y)$ and $\Gamma^\uparrow(\lambda, x, y)$, the convergence of sequences in (X, M) and the concept of Cauchy sequence in (X, M) have been studied in Wu [16]. In this paper, we study the extended convergence of sequences in (X, M) and the concept of joint Cauchy sequence in (X, M) using the infimum and supremum types of dual double fuzzy semi-metric $\Psi^\downarrow(\lambda, x, y; u, v)$ and $\Psi^\uparrow(\lambda, x, y; u, v)$. As we mentioned above, these convergences will be used in the near future to establish the new types of fixed point theorems in fuzzy semi-metric space (X, M) .

In Section 2, we review some basic properties of fuzzy semi-metric space that will be used for further discussion. In Section 3, we introduce the concept of double fuzzy semi-metric and derive the related triangle inequalities. In Sections 4 and 5, the concepts of infimum and supremum types of dual double fuzzy semi-metric are proposed, and their convergent properties and triangle inequalities are studied.

2. Fuzzy Semi-Metric Space

Let X be a nonempty universal set, and let M be a mapping defined on $X \times X \times [0, \infty)$ into $[0, 1]$. Then (X, M) is called a fuzzy semi-metric space if and only if the following conditions are satisfied:

- For any $x, y \in X$, $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
- $M(x, y, 0) = 0$ for all $x, y \in X$ with $x \neq y$.

We say that M satisfies the symmetric condition if and only if $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$. We say that M satisfies the strongly symmetric condition if and only if $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t \geq 0$. Since the symmetric condition is not assumed to be true in fuzzy semi-metric space, four kinds of triangle inequalities called \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft, \diamond\}$ were proposed by Wu [15].

Example 1. Let X be a universal set, and let $d : X \times X \rightarrow \mathbb{R}_+$ satisfy the following conditions:

- $d(x, y) \geq 0$ for any $x, y \in X$;
- $d(x, y) = 0$ if and only if $x = y$ for any $x, y \in X$;
- $d(x, y) + d(y, z) \geq d(x, z)$ for any $x, y, z \in X$.

Note that we do not assume $d(x, y) = d(y, x)$. For example, let $X = [0, 1]$. We define:

$$d(x, y) = \begin{cases} y - x & \text{if } y \geq x \\ 1 & \text{otherwise.} \end{cases}$$

Then $d(x, y) \neq d(y, x)$ and the above three conditions are satisfied. Now we take t-norm $*$ as $a * b = ab$ and define:

$$M(x, y, t) = \begin{cases} \frac{t}{t + d(x, y)} & \text{if } t > 0 \\ 1 & \text{if } t = 0 \text{ and } d(x, y) = 0 \\ 0 & \text{if } t = 0 \text{ and } d(x, y) > 0 \end{cases} = \begin{cases} \frac{t}{t + d(x, y)} & \text{if } t > 0 \\ 1 & \text{if } t = 0 \text{ and } x = y \\ 0 & \text{if } t = 0 \text{ and } x \neq y. \end{cases}$$

It is clear to see that $M(x, y, t) \neq M(y, x, t)$ for $t > 0$, since $d(x, y) \neq d(y, x)$. It is not hard to check that $(X, M, *)$ is a fuzzy semi-metric space satisfying the \bowtie -triangle inequality.

The following interesting observations will be used in further study.

Remark 1. Let (X, M) be a fuzzy semi-metric space.

- Suppose that M satisfies the \bowtie -triangle inequality. Then:

$$M(a, b, t_1) * M(b, c, t_2) * M(c, d, t_3) \leq M(a, c, t_1 + t_2) * M(c, d, t_3) \leq M(a, d, t_1 + t_2 + t_3).$$

In general, we have:

$$M(x_1, x_2, t_1) * M(x_2, x_3, t_2) * \cdots * M(x_p, x_{p+1}, t_p) \leq M(x_1, x_{p+1}, t_1 + t_2 + \cdots + t_p). \quad (1)$$

- Suppose that M satisfies the \triangleright -triangle inequality. Since:

$$M(a, b, t_1) * M(c, b, t_2) \leq \min \{M(a, c, t_1 + t_2), M(c, a, t_1 + t_2)\},$$

which implies:

$$M(a, b, t_1) * M(c, b, t_2) * M(d, c, t_3) \leq \min \{M(a, d, t_1 + t_2 + t_3), M(d, a, t_1 + t_2 + t_3)\}. \quad (2)$$

In general, we have:

$$\begin{aligned} & M(x_1, x_2, t_1) * M(x_3, x_2, t_2) * M(x_4, x_3, t_3) * \cdots * M(x_{p+1}, x_p, t_p) \\ & \leq \min \{M(x_1, x_{p+1}, t_1 + t_2 + \cdots + t_p), M(x_{p+1}, x_1, t_1 + t_2 + \cdots + t_p)\}. \end{aligned}$$

- Suppose that M satisfies the \triangleleft -triangle inequality. Since:

$$M(b, a, t_1) * M(b, c, t_2) \leq \min \{M(a, c, t_1 + t_2), M(c, a, t_1 + t_2)\},$$

which implies:

$$M(b, a, t_1) * M(b, c, t_2) * M(c, d, t_3) \leq \min \{M(a, d, t_1 + t_2 + t_3), M(d, a, t_1 + t_2 + t_3)\}. \quad (3)$$

In general, we have:

$$\begin{aligned} & M(x_2, x_1, t_1) * M(x_2, x_3, t_2) * M(x_3, x_4, t_3) * \cdots * M(x_p, x_{p+1}) \\ & \leq \min \{M(x_1, x_{p+1}, t_1 + t_2 + \cdots + t_p), M(x_{p+1}, x_1, t_1 + t_2 + \cdots + t_p)\}. \end{aligned}$$

- Suppose that M satisfies the \diamond -triangle inequality. Then:

$$\begin{aligned} M(a, b, t_1) * M(b, c, t_2) * M(d, c, t_3) &= M(b, c, t_1) * M(a, b, t_2) * M(d, c, t_3) \\ &\leq M(c, a, t_1 + t_2) * M(d, c, t_3) \leq M(a, d, t_1 + t_2 + t_3) \end{aligned} \quad (4)$$

and:

$$\begin{aligned} M(b, a, t_1) * M(c, b, t_2) * M(c, d, t_3) &\leq M(a, c, t_1 + t_2) * M(c, d, t_3) \\ &= M(c, d, t_3) * M(a, c, t_1 + t_2) \leq M(d, a, t_1 + t_2 + t_3). \end{aligned} \quad (5)$$

From Equation (4), we also have:

$$\begin{aligned} M(a, b, t_1) * M(c, b, t_2) * M(d, c, t_3) &= M(d, c, t_3) * M(c, b, t_2) * M(a, b, t_1) \\ &\leq M(d, a, t_1 + t_2 + t_3), \end{aligned} \quad (6)$$

which implies:

$$M(b, a, t_1) * M(b, c, t_2) * M(c, d, t_3) \geq M(a, d, t_1 + t_2 + t_3) \quad (7)$$

by referring to Equation (5). In general, we have the following cases:

- (a) If p is even, then:

$$\begin{aligned} M(x_1, x_2, t_1) * M(x_2, x_3, t_2) * M(x_4, x_3, t_3) * M(x_4, x_5, t_4) * M(x_6, x_5, t_5) \\ * M(x_6, x_7, t_6) * \cdots * M(x_p, x_{p+1}, t_p) \leq M(x_{p+1}, x_1, t_1 + t_2 + \cdots + t_p) \end{aligned}$$

and:

$$\begin{aligned} M(x_2, x_1, t_1) * M(x_3, x_2, t_2) * M(x_3, x_4, t_3) * M(x_5, x_4, t_4) * M(x_5, x_6, t_5) \\ * M(x_7, x_6, t_6) * \cdots * M(x_p, x_{p+1}, t_p) \leq M(x_1, x_{p+1}, t_1 + t_2 + \cdots + t_p). \end{aligned}$$

- (b) If p is odd, then:

$$\begin{aligned} M(x_1, x_2, t_1) * M(x_2, x_3, t_2) * M(x_4, x_3, t_3) * M(x_4, x_5, t_4) * M(x_6, x_5, t_5) \\ * M(x_6, x_7, t_6) * \cdots * M(x_p, x_{p+1}, t_p) \leq M(x_1, x_{p+1}, t_1 + t_2 + \cdots + t_p) \end{aligned}$$

and:

$$\begin{aligned} M(x_2, x_1, t_1) * M(x_3, x_2, t_2) * M(x_3, x_4, t_3) * M(x_5, x_4, t_4) * M(x_5, x_6, t_5) \\ * M(x_7, x_6, t_6) * \cdots * M(x_{p+1}, x_p, t_p) \leq M(x_{p+1}, x_1, t_1 + t_2 + \cdots + t_p). \end{aligned}$$

Let (X, M) be a fuzzy semi-metric space.

- We say that M is nondecreasing if and only if, given any fixed $x, y \in X$, $M(x, y, t_1) \geq M(x, y, t_2)$ for $t_1 > t_2 > 0$.
- We say that M is increasing if and only if, given any fixed $x, y \in X$, $M(x, y, t_1) > M(x, y, t_2)$ for $t_1 > t_2 > 0$.
- We say that M is symmetrically nondecreasing if and only if, given any fixed $x, y \in X$, $M(x, y, t_1) \geq M(y, x, t_2)$ for $t_1 > t_2 > 0$.
- We say that M is symmetrically increasing if and only if, given any fixed $x, y \in X$, $M(x, y, t_1) > M(y, x, t_2)$ for $t_1 > t_2 > 0$.

The following interesting results were modified from Wu [15] using the similar argument, which will be used in further discussion.

Proposition 1. (Wu [15]) *Let (X, M) be a fuzzy semi-metric space. Then we have the following properties:*

- (i) *If M satisfies the \bowtie -triangle inequality, then M is nondecreasing. If M satisfies the strict \bowtie -triangle inequality, then M is increasing.*
- (ii) *If M satisfies the \triangleright -triangle inequality or the \triangleleft -triangle inequality, then M is both nondecreasing and symmetrically nondecreasing. If M satisfies the strict \triangleright -triangle inequality or the strict \triangleleft -triangle inequality, then M is both increasing and symmetrically increasing.*
- (iii) *If M satisfies the \diamond -triangle inequality, then M is symmetrically nondecreasing. If M satisfies the strict \diamond -triangle inequality, then M is symmetrically increasing.*

3. Double Fuzzy Semi-Metric

Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$. Given any four elements $x, y, u, v \in X$, recall that the value $M(x, y, t)$ means the membership degree of the distance that is less than t between x and y , and the value $M(u, v, t)$ means the membership degree of the distance that is less than t between u and v . In this case, we can define a value:

$$\zeta(x, y; u, v, t) = \min \{M(x, y, t), M(u, v, t)\},$$

which means the membership degree of the distance that is simultaneously less than t between x and y and between u and v . In general, instead of considering the min function, we shall use the t-norm. The formal definition is given below.

Definition 1. Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$. We define the mapping $\zeta : X^4 \times [0, +\infty) \rightarrow [0, 1]$ by:

$$\zeta(x, y; u, v, t) = M(x, y, t) * M(u, v, t).$$

Then ζ is called a double fuzzy semi-metric.

Example 2. Continued from Example 1, we consider:

$$M(x, y, t) = \begin{cases} \frac{t}{t + d(x, y)} & \text{if } t > 0 \\ 1 & \text{if } t = 0 \text{ and } x = y \\ 0 & \text{if } t = 0 \text{ and } x \neq y. \end{cases}$$

If we take t-norm as $a * b = a \cdot b$, then the double fuzzy semi-metric can be obtained as:

$$\begin{aligned} \zeta(x, y; u, v, t) &= M(x, y, t) \cdot M(u, v, t) \\ &= \begin{cases} \frac{t}{t + d(x, y)} \cdot \frac{t}{t + d(u, v)} & \text{if } t > 0 \\ 1 & \text{if } t = 0 \text{ and } x = y \text{ and } u = v \\ 0 & \text{if } t = 0 \text{ and } x \neq y \text{ or } u \neq v. \end{cases} \end{aligned}$$

The potential application for considering the double fuzzy semi-metric is to study the new type of fixed point theorems in fuzzy semi-metric space.

Proposition 2. (Triangle Inequalities for Dual Fuzzy Semi-Metric) *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$. Given any $x, y, z, u, v, w \in X$, we have the following properties:*

(i) Suppose that M satisfies the \bowtie -triangle inequality. Then we have the inequality:

$$\zeta(x, z; u, w, t + s) \geq \zeta(x, y; u, v, t) * \zeta(y, z; v, w, s)$$

for $s, t > 0$.

(ii) Suppose that M satisfies the \triangleright -triangle inequality. Then we have the inequality:

$$\zeta(x, z; u, w, t + s) \geq \zeta(x, y; u, v, t) * \zeta(z, y; w, v, s)$$

for $s, t > 0$.

(iii) Suppose that M satisfies the \triangleleft -triangle inequality. Then we have the inequality:

$$\zeta(x, z; u, w, t + s) \geq \zeta(y, x; v, u, t) * \zeta(y, z; v, w, s)$$

for $s, t > 0$.

(iv) Suppose that M satisfies the \diamond -triangle inequality. Then we have the inequality:

$$\zeta(x, z; u, w, t + s) \geq \zeta(y, x; v, u, t) * \zeta(z, y; w, v, s)$$

for $s, t > 0$.

Proof. It suffices to prove part (i); we have:

$$\begin{aligned} \zeta(x, z; u, w, t + s) &= M(x, z, t + s) * M(u, w, t + s) \\ &\geq (M(x, y, t) * M(y, z, s)) * M(u, w, t + s) \\ &\quad \text{(using the } \bowtie\text{-triangle inequality and the increasing property of t-norm)} \\ &\geq (M(x, y, t) * M(y, z, s)) * (M(u, v, t) * M(v, w, s)) \\ &\quad \text{(using the } \bowtie\text{-triangle inequality and the increasing property of t-norm)} \\ &= (M(x, y, t) * M(u, v, t)) * (M(y, z, s) * M(v, w, s)) \\ &\quad \text{(using the associative and commutative properties of t-norm)} \\ &= \zeta(x, y; u, v, t) * \zeta(y, z; v, w, s). \end{aligned}$$

This completes the proof. \square

Definition 2. Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$, and let ζ be a double fuzzy semi-metric given by:

$$\zeta(x, y; u, v, t) = M(x, y, t) * M(u, v, t).$$

Given any fixed $x, y, u, v \in X$, we define the following concepts of monotonicity:

- The mapping $\zeta(x, y; u, v, \cdot)$ is said to be nondecreasing if and only if $\zeta(x, y; u, v, t_1) \geq \zeta(x, y; u, v, t_2)$ for $t_1 > t_2$. The mapping $\zeta(x, y; u, v, \cdot)$ is said to be increasing if and only if $\zeta(x, y; u, v, t_1) > \zeta(x, y; u, v, t_2)$ for $t_1 < t_2$.
- The mapping $\zeta(x, y; u, v, \cdot)$ is said to be symmetrically nondecreasing if and only if $\zeta(x, y; u, v, t_1) \geq \zeta(y, x; u, v, t_2)$ for $t_1 > t_2$. The mapping $\zeta(x, y; u, v, \cdot)$ is said to be symmetrically increasing if and only if $\zeta(x, y; u, v, t_1) > \zeta(y, x; u, v, t_2)$ for $t_1 < t_2$.
- The mapping $\zeta(x, y; u, v, \cdot)$ is said to be \triangleleft -semisymmetrically nondecreasing if and only if $\zeta(x, y; u, v, t_1) \geq \zeta(y, x; u, v, t_2)$ for $t_1 > t_2$. The mapping $\zeta(x, y; u, v, \cdot)$ is said to be \triangleleft -semisymmetrically increasing if and only if $\zeta(x, y; u, v, t_1) > \zeta(y, x; u, v, t_2)$ for $t_1 < t_2$.

- The mapping $\zeta(x, y; u, v, \cdot)$ is said to be \triangleright -semisymmetrically nondecreasing if and only if $\zeta(x, y; u, v, t_1) \geq \zeta(x, y; v, u, t_2)$ for $t_1 > t_2$. The mapping $\zeta(x, y; u, v, \cdot)$ is said to be \triangleright -semisymmetrically increasing if and only if $\zeta(x, y; u, v, t_1) > \zeta(x, y; v, u, t_2)$ for $t_1 < t_2$.

Proposition 3. Let (X, M) be a fuzzy semi-metric space along with a t -norm $*$. Given any fixed $x, y, u, v \in X$, the double fuzzy semi-metric ζ satisfies the following properties:

- Suppose that M satisfies the \bowtie -triangle inequality. Then the mapping $\zeta(x, y; u, v, \cdot)$ from $[0, \infty)$ into $[0, 1]$ is nondecreasing.
- Suppose that M satisfies the \triangleright -triangle inequality or the \triangleleft -triangle inequality. Then the mapping $\zeta(x, y; u, v, \cdot)$ from $[0, \infty)$ into $[0, 1]$ is simultaneously nondecreasing, symmetrically nondecreasing, \triangleleft -semisymmetrically nondecreasing, and \triangleright -semisymmetrically nondecreasing.
- Suppose that M satisfies the \diamond -triangle inequality. Then the mapping $\zeta(x, y; u, v, \cdot)$ from $[0, \infty)$ into $[0, 1]$ is symmetrically nondecreasing.

Proof. Part (i) of Proposition 1 says that the mappings $M(x, y, \cdot)$ and $M(u, v, \cdot)$ from $[0, \infty)$ into $[0, 1]$ are nondecreasing. According to the increasing property of t -norm, we conclude that the mapping $\zeta(x, y; u, v, \cdot)$ from $[0, \infty)$ into $[0, 1]$ is nondecreasing, which proves part (i).

Part (ii) can be obtained from part (ii) of Proposition 1, and part (iii) can be obtained from part (iii) of Proposition 1. This completes the proof. \square

By using the strictly increasing property of t -norm, the proof of Proposition 3 is still valid for obtaining the following results.

Proposition 4. Let (X, M) be a fuzzy semi-metric space along with a t -norm $*$. Suppose that the t -norm satisfies the strictly increasing property. Given any fixed $x, y, u, v \in X$, the double fuzzy semi-metric ζ satisfies the following properties:

- Suppose that M satisfies the strict \bowtie -triangle inequality. Then the mapping $\zeta(x, y; u, v, \cdot)$ from $[0, \infty)$ into $[0, 1]$ is increasing.
- Suppose that M satisfies the strict \triangleright -triangle inequality or the strict \triangleleft -triangle inequality. Then the mapping $\zeta(x, y; u, v, \cdot)$ from $[0, \infty)$ into $[0, 1]$ is simultaneously increasing, symmetrically increasing, \triangleleft -semisymmetrically increasing, and \triangleright -semisymmetrically increasing.
- Suppose that M satisfies the strict \diamond -triangle inequality. Then the mapping $\zeta(x, y; u, v, \cdot)$ from $[0, \infty)$ into $[0, 1]$ is symmetrically increasing.

Let (X, M) be a fuzzy semi-metric space. The motivation for considering the following two concepts can refer to Wu [16].

- M is said to satisfy the canonical condition if and only if:

$$\lim_{t \rightarrow +\infty} M(x, y, t) = 1 \text{ for any fixed } x, y \in X.$$

- M is said to satisfy the rational condition if and only if:

$$\lim_{t \rightarrow 0+} M(x, y, t) = 0 \text{ for any fixed } x, y \in X.$$

Proposition 5. Let (X, M) be a fuzzy semi-metric space along with a t -norm $*$.

- Suppose that M satisfies the canonical condition. If the t -norm $*$ is left-continuous at 1 with respect to the first or second argument, then we have:

$$\lim_{t \rightarrow +\infty} \zeta(x, y; u, v, t) = 1. \quad (8)$$

- (ii) Suppose that M satisfies the rational condition. If the t -norm $*$ is right-continuous at 0 with respect to the first or second argument, then we have:

$$\lim_{t \rightarrow 0+} \zeta(x, y; u, v, t) = 0. \quad (9)$$

Proof. To prove part (i), the canonical condition says that:

$$\lim_{t \rightarrow +\infty} M(x, y, t) = 1 = \lim_{t \rightarrow +\infty} M(u, v, t).$$

The left-continuity of t -norm $*$ at 1 also says that:

$$\lim_{t \rightarrow +\infty} \zeta(x, y; u, v, t) = \left(\lim_{t \rightarrow +\infty} M(x, y, t) \right) * \left(\lim_{t \rightarrow +\infty} M(u, v, t) \right) = 1 * 1 = 1.$$

To prove part (ii), the rational condition says that:

$$\lim_{t \rightarrow 0+} M(x, y, t) = 0 = \lim_{t \rightarrow 0+} M(u, v, t).$$

The right-continuity of t -norm $*$ at 0 also says that:

$$\lim_{t \rightarrow 0+} \zeta(x, y; u, v, t) = \left(\lim_{t \rightarrow 0+} M(x, y, t) \right) * \left(\lim_{t \rightarrow 0+} M(u, v, t) \right) = 0 * 0 = 0.$$

This completes the proof. \square

Example 3. Continued from Example 1, it is not hard to check that M satisfies both the canonical and rational conditions. Suppose that we take t -norm as $a * b = a \cdot b$. Then Proposition 5 says that:

$$\lim_{t \rightarrow +\infty} \zeta(x, y; u, v, t) = 1 \text{ and } \lim_{t \rightarrow 0+} \zeta(x, y; u, v, t) = 0.$$

4. Convergence Based on the Infimum

From Definition 1, we see that the double fuzzy semi-metric ζ is a mapping from $X^4 \times [0, \infty)$ into $[0, 1]$. Here, we shall consider its dual sense by considering the mapping from $(0, 1] \times X^4$ into $[0, \infty)$. The formal definition is given below.

Definition 3. Let (X, M) be a fuzzy semi-metric space along with a t -norm $*$. We also assume that M satisfies the canonical condition, and that the t -norm $*$ is left-continuous at 1 with respect to the first or second argument. Given any fixed $x, y, u, v \in X$ and any fixed $\lambda \in (0, 1]$, we consider the following set:

$$\Pi^\downarrow(\lambda, x, y; u, v) = \{t > 0 : \zeta(x, y; u, v, t) \geq 1 - \lambda\},$$

which is used to define a mapping $\Psi^\downarrow(\lambda, \cdot, \cdot; \cdot, \cdot) : X^4 \rightarrow [0, +\infty)$ by:

$$\Psi^\downarrow(\lambda, x, y; u, v) = \inf \Pi^\downarrow(\lambda, x, y; u, v) = \inf \{t > 0 : \zeta(x, y; u, v, t) \geq 1 - \lambda\}.$$

In this case, the mapping Ψ^\downarrow from $(0, 1] \times X^4$ into $[0, \infty)$ is called the infimum type of dual double fuzzy semi-metric.

Example 4. Continued from Example 2, we have:

$$\Pi^\downarrow(\lambda, x, y; u, v) = \left\{ t > 0 : \frac{t}{t + d(x, y)} \cdot \frac{t}{t + d(u, v)} \geq 1 - \lambda \right\} = \left\{ t > 0 : t \geq \frac{C + \sqrt{C^2 + D}}{2} \right\},$$

where:

$$C = \frac{(d(x, y) + d(u, v))(1 - \lambda)}{\lambda} \text{ and } D = \frac{d(x, y) \cdot d(u, v) \cdot (1 - \lambda)}{\lambda}.$$

We also have:

$$\Psi^\downarrow(\lambda, x, y; u, v) = \inf \Pi^\downarrow(\lambda, x, y; u, v) = \left\{ t > 0 : t \geq \frac{C + \sqrt{C^2 + D}}{2} \right\} = \frac{C + \sqrt{C^2 + D}}{2}.$$

The potential application of dual double fuzzy semi-metric will be used to study the fixed point theorems in fuzzy semi-metric space. However, we first need to claim that the set $\Pi^\downarrow(\lambda, x, y; u, v)$ is nonempty. Suppose that $\Pi^\downarrow(\lambda, x, y; u, v) = \emptyset$. The definition says that $\zeta(x, y; u, v, t) < 1 - \lambda$ for all $t > 0$; that is:

$$\lim_{t \rightarrow +\infty} \zeta(x, y; u, v, t) \leq 1 - \lambda < 1,$$

which contradicts Equation (8). Therefore, Definition 3 is well-defined and $\Pi^\downarrow(\lambda, x, y; u, v) \neq \emptyset$.

Remark 2. The following observations will be useful for further discussion.

- For any $\lambda \in (0, 1]$, we have:

$$\Psi^\downarrow(1, x, y; u, v) = \inf \{t > 0 : \zeta(x, y; u, v, t) \geq 0\} = \inf \{t > 0\} = 0,$$

and:

$$\begin{aligned} \Psi^\downarrow(\lambda, x, x; u, u) &= \inf \{t > 0 : \zeta(x, x; u, u, t) \geq 1 - \lambda\} \\ &= \inf \{t > 0 : 1 \geq 1 - \lambda\} = \inf \{t > 0\} = 0. \end{aligned} \quad (10)$$

- Given any fixed $x, y, u, v \in X$, if $\lambda_1 > \lambda_2$, then:

$$\Pi^\downarrow(\lambda_2, x, y; u, v) \subseteq \Pi^\downarrow(\lambda_1, x, y; u, v) \text{ and } \Psi^\downarrow(\lambda_1, x, y; u, v) \leq \Psi^\downarrow(\lambda_2, x, y; u, v). \quad (11)$$

Proposition 6. Let (X, M) be a fuzzy semi-metric space along with a t -norm $*$. We also assume that M satisfies the canonical condition, and that the t -norm $*$ is left-continuous at 1 with respect to the first or second argument. Given any fixed $x, y, u, v \in X$, suppose that the following conditions are satisfied:

$$\Pi^\downarrow(0+, x, y; u, v) \equiv \bigcap_{0 < \lambda \leq 1} \Pi^\downarrow(\lambda, x, y; u, v) \neq \emptyset,$$

and:

$$\{t > 0 : \zeta(x, y; u, v, t) = 1\} \neq \emptyset.$$

Then we have:

$$\Pi^\downarrow(0+, x, y; u, v) = \{t > 0 : \zeta(x, y; u, v, t) = 1\}. \quad (12)$$

Moreover, the following limit exists:

$$\lim_{\lambda \rightarrow 0+} \Psi^\downarrow(\lambda, x, y; u, v) = \sup_{0 < \lambda \leq 1} \Psi^\downarrow(\lambda, x, y; u, v). \quad (13)$$

Proof. The assumption $\Pi^\downarrow(0+, x, y; u, v) \neq \emptyset$ says that we can consider $t \in \Pi^\downarrow(0+, x, y; u, v)$, i.e., $\zeta(x, y; u, v, t) \geq 1 - \lambda$ for all $\lambda \in (0, 1]$. Then we obtain $\zeta(x, y; u, v, t) \geq 1$ by taking $\lambda \rightarrow 0+$, which shows that $\zeta(x, y; u, v, t) = 1$, i.e.:

$$\Pi^\downarrow(0+, x, y; u, v) = \bigcap_{0 < \lambda \leq 1} \Pi^\downarrow(\lambda, x, y; u, v) \subseteq \{t > 0 : \zeta(x, y; u, v, t) = 1\}.$$

On the other hand, suppose that $\zeta(x, y; u, v, t) = 1$. Then $\zeta(x, y; u, v, t) = 1 \geq 1 - \lambda$ for all $\lambda \in (0, 1]$. Therefore, we obtain $t \in \Pi^\downarrow(0+, x, y; u, v)$, which implies the desired equality (Equation (12)). Further, the inequality (Equation (11)) says that the limit (Equation (13)) exists. This completes the proof. \square

Proposition 7. Suppose that (X, M) is a fuzzy semi-metric space along with a t -norm $*$. We also assume that M satisfies the canonical and rational conditions, and that the t -norm $*$ is left-continuous at 1 and right-continuous at 0 with respect to the first or second argument. If M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft, \diamond\}$, then, for any fixed $x, y, u, v \in X$ with $x \neq y$ or $u \neq v$, we have $\Psi^\downarrow(\lambda, x, y; u, v) > 0$ for $\lambda \in (0, 1)$.

Proof. We first consider the case of M , satisfying the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. From Equation (10), we need to consider $x \neq y$ or $u \neq v$. We want to assume $\Psi^\downarrow(\lambda, x, y; u, v) = 0$ for $\lambda \in (0, 1)$ to obtain a contradiction. Using the concept of infimum from $\Psi^\downarrow(\lambda, x, y; u, v)$, given any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that $\zeta(x, y; u, v, t_\epsilon) \geq 1 - \lambda$ and:

$$t_\epsilon < \Psi^\downarrow(\lambda, x, y; u, v) + \epsilon = \epsilon.$$

Parts (i) and (ii) of Proposition 3 say that the mapping $\zeta(x, y; u, v, \cdot)$ from $[0, \infty)$ into $[0, 1]$ is nondecreasing. Therefore, we obtain:

$$\zeta(x, y; u, v, \epsilon) \geq \zeta(x, y; u, v, t_\epsilon) \geq 1 - \lambda.$$

Since ϵ can be any positive real number, using Equation (9), we must have:

$$0 = \lim_{\epsilon \rightarrow 0+} \zeta(x, y; u, v, \epsilon) \geq \zeta(x, y; u, v, t_\epsilon) \geq 1 - \lambda,$$

which contradicts $0 < \lambda < 1$.

Now we assume that M satisfies the \diamond -triangle inequality. Suppose that $\Psi^\downarrow(\lambda, y, x; v, u) = 0$ for $\lambda \in (0, 1)$. Part (iii) of Proposition 3 says that the mapping $\zeta(x, y; u, v, \cdot)$ is symmetrically nondecreasing. Therefore, we can similarly obtain:

$$\zeta(x, y; u, v, \epsilon) \geq \zeta(y, x; v, u, t_\epsilon) \geq 1 - \lambda.$$

This completes the proof. \square

Proposition 8. Let (X, M) be a fuzzy semi-metric space along with a t -norm $*$. We also assume that M satisfies the canonical condition, and that the t -norm $*$ is left-continuous at 1 with respect to the first or second argument. Given any fixed $x, y, u, v \in X$ and $\lambda \in (0, 1)$, we have the following properties:

(i) If $\epsilon > 0$ is sufficiently small satisfying $\Psi^\downarrow(\lambda, x, y; u, v) > \epsilon$, then we have:

$$\zeta(x, y; u, v, \Psi^\downarrow(\lambda, x, y; u, v) - \epsilon) < 1 - \lambda. \quad (14)$$

(ii) Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. For any $\epsilon > 0$, we have:

$$\zeta(x, y; u, v, \Psi^\downarrow(\lambda, x, y; u, v) + \epsilon) \geq 1 - \lambda \quad (15)$$

(iii) Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft\}$. For any $\epsilon > 0$, we have:

$$\zeta(x, y; u, v, \Psi^\downarrow(\lambda, y, x; u, v) + \epsilon) \geq 1 - \lambda \quad (16)$$

and:

$$\zeta(x, y; u, v, \Psi^\downarrow(\lambda, x, y; v, u) + \epsilon) \geq 1 - \lambda \quad (17)$$

(iv) Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. For any $\epsilon > 0$, we have:

$$\zeta(x, y; u, v, \Psi^\downarrow(\lambda, y, x; v, u) + \epsilon) \geq 1 - \lambda \quad (18)$$

Proof. To prove part (i), we assume that:

$$\zeta(x, y; u, v, \Psi^\downarrow(\lambda, x, y; u, v) - \epsilon) \geq 1 - \lambda.$$

The definition of Ψ^\downarrow says that $\Psi^\downarrow(\lambda, x, y; u, v) \leq \Psi^\downarrow(\lambda, x, y; u, v) - \epsilon$. This contradiction implies $\zeta(x, y; u, v, \Psi^\downarrow(\lambda, x, y; u, v) - \epsilon) \leq 1 - \lambda$.

To prove part (ii), using the concept of infimum from $\Psi^\downarrow(\lambda, x, y; u, v)$, given any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that $\zeta(x, y; u, v, t_\epsilon) \geq 1 - \lambda$ and $t_\epsilon < \Psi^\downarrow(\lambda, x, y; u, v) + \epsilon$. Parts (i) and (ii) of Proposition 3 says that the mapping $\zeta(x, y; u, v, \cdot)$ is nondecreasing. Therefore, we obtain:

$$\zeta(x, y; u, v, \Psi^\downarrow(\lambda, x, y; u, v) + \epsilon) \geq \zeta(x, y; u, v, t_\epsilon) \geq 1 - \lambda.$$

To prove part (iii), using the concept of infimum from $\Psi^\downarrow(\lambda, y, x; u, v)$, given any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that $\zeta(y, x; u, v, t_\epsilon) \geq 1 - \lambda$ and $t_\epsilon < \Psi^\downarrow(\lambda, y, x; u, v) + \epsilon$. Part (ii) of Proposition 3 says that the mapping $\zeta(y, x; u, v, \cdot)$ is \triangleleft -semisymmetrically nondecreasing. Therefore, we obtain:

$$\zeta(x, y; u, v, \Psi^\downarrow(\lambda, y, x; u, v) + \epsilon) \geq \zeta(y, x; u, v, t_\epsilon) \geq 1 - \lambda.$$

Since the mapping $\zeta(x, y; u, v, \cdot)$ is also \triangleright -semisymmetrically nondecreasing, we can similarly obtain another inequality.

Since the mapping $\zeta(x, y; u, v, \cdot)$ is semisymmetrically nondecreasing, using parts (ii) and (iii) of Proposition 3, we can similarly obtain part (iv). This completes the proof. \square

Proposition 9. Let (X, M) be a fuzzy semi-metric space along with a t -norm $*$. We also assume that M satisfies the canonical condition, and that the t -norm $*$ is left-continuous at 1 with respect to the first or second argument. Given any fixed $x, y, u, v \in X$ and $\lambda \in (0, 1)$, the following statements hold true:

- (i) Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft, \diamond\}$. If $t > \Psi^\downarrow(\lambda, x, y; u, v)$, then we have $\zeta(x, y; u, v, t) \geq 1 - \lambda$.
- (ii) If $0 < t < \Psi^\downarrow(\lambda, x, y; u, v)$, then we have the following properties:
 - Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Then we have $\zeta(x, y; u, v, t) < 1 - \lambda$.
 - Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft\}$. Then we have $\zeta(y, x; u, v, t) < 1 - \lambda$ and $\zeta(x, y; v, u, t) < 1 - \lambda$.
 - Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. Then we have $\zeta(y, x; v, u, t) < 1 - \lambda$.

Proof. To prove part (i), the inequality $t > \Psi^\downarrow(\lambda, x, y; u, v)$ says that there exists $\epsilon > 0$, satisfying $t \geq \Psi^\downarrow(\lambda, x, y; u, v) + \epsilon$. Therefore, we consider the following cases:

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Parts (i) and (ii) of Proposition 3 say that the mapping $\zeta(x, y; u, v, \cdot)$ is nondecreasing. Therefore, using Equation (15), we obtain:

$$\zeta(x, y; u, v, t) \geq \zeta(x, y; u, v, \Psi^\downarrow(\lambda, x, y; u, v) + \epsilon) \geq 1 - \lambda.$$

- Suppose that M satisfies the \diamond -triangle inequality. Part (iii) of Proposition 3 says that the mapping $\zeta(x, y; u, v, \cdot)$ is symmetrically nondecreasing. Therefore, using Equation (18), we obtain:

$$\zeta(x, y; u, v, t) \geq \zeta(y, x; v, u, \Psi^\downarrow(\lambda, x, y; u, v) + \epsilon) \geq 1 - \lambda.$$

To prove part (ii), the inequality $0 < t < \Psi^\downarrow(\lambda, x, y; u, v)$ says that there exists $\epsilon > 0$, satisfying $t \leq \Psi^\downarrow(\lambda, x, y; u, v) - \epsilon$. Therefore, we consider the following cases:

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Parts (i) and (ii) of Proposition 3 say that the mapping $\zeta(x, y; u, v, \cdot)$ is nondecreasing. Therefore, using Equation (14), we obtain:

$$\zeta(x, y; u, v, t) \leq \zeta(x, y; u, v, \Psi^\downarrow(\lambda, x, y; u, v) - \epsilon) < 1 - \lambda.$$

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft\}$. Part (ii) of Proposition 3 says that the mapping $\zeta(x, y; u, v, \cdot)$ is \triangleleft -semisymmetrically nondecreasing. Therefore, using Equation (14), we obtain:

$$\zeta(y, x; u, v, t) \leq \zeta(x, y; u, v, \Psi^\downarrow(\lambda, x, y; u, v) - \epsilon) < 1 - \lambda.$$

We can similarly obtain another inequality using the fact that the mapping $\zeta(x, y; u, v, \cdot)$ is also \triangleright -semisymmetrically nondecreasing.

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. Parts (ii) and (iii) of Proposition 3 say that the mapping $\zeta(x, y; u, v, \cdot)$ is symmetrically nondecreasing. Therefore, using Equation (14), we obtain:

$$\zeta(y, x; v, u, t) \leq \zeta(x, y; u, v, \Psi^\downarrow(\lambda, x, y; u, v) - \epsilon) < 1 - \lambda.$$

This completes the proof. \square

Proposition 10. Let (X, M) be a fuzzy semi-metric space along with a t -norm $*$. We also assume that M satisfies the canonical condition, and that the t -norm $*$ is left-continuous at 1 with respect to the first or second argument. Given any fixed $x, y, u, v \in X$ and $\lambda \in (0, 1)$, the following statements hold true:

- If $\zeta(x, y; u, v, t) < 1 - \lambda$, then we have the following properties:
 - Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft, \diamond\}$. Then we have $t \leq \Psi^\downarrow(\lambda, x, y; u, v)$.
 - Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft\}$. Then we have $t \leq \Psi^\downarrow(\lambda, y, x; u, v)$ and $t \leq \Psi^\downarrow(\lambda, x, y; v, u)$.
 - Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. Then we have $t \leq \Psi^\downarrow(\lambda, y, x; v, u)$.
- Suppose that the t -norm $*$ satisfies the strictly increasing property. If $\zeta(x, y; u, v, t) = 1 - \lambda$ for $t > 0$, then we have the following properties.
 - Suppose that M satisfies the strict \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. If $\Psi^\downarrow(\lambda, x, y; u, v) > 0$, then we have $t = \Psi^\downarrow(\lambda, x, y; u, v)$.
 - Suppose that M satisfies the strict \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft\}$. If $\Psi^\downarrow(\lambda, y, x; u, v) > 0$, then we have $t = \Psi^\downarrow(\lambda, y, x; u, v)$, and if $\Psi^\downarrow(\lambda, x, y; v, u) > 0$, then we have $t = \Psi^\downarrow(\lambda, x, y; v, u)$.
 - Suppose that M satisfies the strict \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. If $\Psi^\downarrow(\lambda, y, x; v, u) > 0$, then we have $t = \Psi^\downarrow(\lambda, y, x; v, u)$.
- If $\zeta(x, y; u, v, t) \geq 1 - \lambda$, then we have the following properties:

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Then we have $t \geq \Psi^\downarrow(\lambda, x, y; u, v)$.
- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft\}$. Then we have $t \geq \Psi^\downarrow(\lambda, y, x; u, v)$ and $t \geq \Psi^\downarrow(\lambda, x, y; v, u)$.
- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. Then we have $t \geq \Psi^\downarrow(\lambda, y, x; v, u)$.

Proof. To prove part (i), three cases are separately considered below:

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Using the contrapositive statement of part (i) of Proposition 9, we can obtain the desired result.
- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft\}$. According to the concept of infimum, given any $\epsilon > 0$, there exists $t_\epsilon > 0$, satisfying $\zeta(y, x; u, v, t_\epsilon) \geq 1 - \lambda$ and $t_\epsilon < \Psi^\downarrow(\lambda, y, x; u, v) + \epsilon$. Part (ii) of Proposition 3 says that the mapping $\zeta(x, y; u, v, \cdot)$ is \triangleleft -semisymmetrically nondecreasing. Therefore, if $t > t_\epsilon$ then $\zeta(x, y; u, v, t) \geq \zeta(y, x; u, v, t_\epsilon)$, which contradicts $\zeta(x, y; u, v, t) < 1 - \lambda$. It says that:

$$t \leq t_\epsilon < \Psi^\downarrow(\lambda, y, x; u, v) + \epsilon.$$

Since ϵ can be any positive real number, we must have $t \leq \Psi^\downarrow(\lambda, y, x; u, v)$. We can similarly obtain another inequality using the fact of the mapping $\zeta(x, y; u, v, \cdot)$ being \triangleright -semisymmetrically nondecreasing.

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. According to the concept of infimum, given any $\epsilon > 0$, there exists $t_\epsilon > 0$, satisfying $\zeta(y, x; v, u, t_\epsilon) \geq 1 - \lambda$ and $t_\epsilon < \Psi^\downarrow(\lambda, y, x; v, u) + \epsilon$. Parts (ii) and (iii) of Proposition 3 say that if $t > t_\epsilon$ then $\zeta(x, y; u, v, t) \geq \zeta(y, x; v, u, t_\epsilon)$, which contradicts $\zeta(x, y; u, v, t) < 1 - \lambda$. It says that:

$$t \leq t_\epsilon < \Psi^\downarrow(\lambda, y, x; v, u) + \epsilon.$$

Since ϵ can be any positive real number, we must have $t \leq \Psi^\downarrow(\lambda, y, x; v, u)$.

To prove part (ii), three cases are separately considered below:

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. According to the concept of infimum, given any $\epsilon > 0$, there exists $t_\epsilon > 0$, satisfying $\zeta(x, y; u, v, t_\epsilon) \geq 1 - \lambda$ and $t_\epsilon < \Psi^\downarrow(\lambda, x, y; u, v) + \epsilon$. Regarding the strict property, parts (i) and (ii) of Proposition 4 say that if $t > t_\epsilon$ then $\zeta(x, y; u, v, t) > \zeta(x, y; u, v, t_\epsilon)$, which contradicts $\zeta(x, y; u, v, t) = 1 - \lambda$. It says that:

$$t \leq t_\epsilon < \Psi^\downarrow(\lambda, x, y; u, v) + \epsilon.$$

Since ϵ can be any positive real number, we must have $t \leq \Psi^\downarrow(\lambda, x, y; u, v)$. Now we assume that $t < \Psi^\downarrow(\lambda, x, y; u, v)$. The first case of part (ii) of Proposition 9 says that $\zeta(x, y; u, v, t) < 1 - \lambda$, which also contradicts $\zeta(x, y; u, v, t) = 1 - \lambda$. Therefore, we must have $t = \Psi^\downarrow(\lambda, x, y; u, v)$.

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft\}$. We can similarly obtain $t \leq \Psi^\downarrow(\lambda, y, x; u, v)$. Now we assume that $t < \Psi^\downarrow(\lambda, y, x; u, v)$. The second case of part (ii) of Proposition 9 says that $\zeta(x, y; u, v, t) < 1 - \lambda$, which also contradicts $\zeta(x, y; u, v, t) = 1 - \lambda$. Therefore, we must have $t = \Psi^\downarrow(\lambda, y, x; u, v)$. Another result can be similarly obtained.
- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. We can similarly obtain $t \leq \Psi^\downarrow(\lambda, y, x; v, u)$. Now we assume that $t < \Psi^\downarrow(\lambda, y, x; v, u)$. The third case of part (ii) of Proposition 9 says that $\zeta(x, y; u, v, t) < 1 - \lambda$, which also contradicts $\zeta(x, y; u, v, t) = 1 - \lambda$. Therefore, we must have $t = \Psi^\downarrow(\lambda, y, x; v, u)$.

Part (iii) can be obtained from the contrapositive statement of part (ii) of Proposition 9. This completes the proof. \square

Proposition 11. Let (X, M) be a fuzzy semi-metric space along with a t -norm $*$. We also assume that M satisfies the canonical condition, and that the t -norm $*$ is left-continuous at 1 with respect to the first or second argument. Given any fixed $x, y, u, v \in X$ and $\lambda \in (0, 1)$, the following statements hold true:

- (i) Suppose that the mapping $\zeta(x, y; u, v, \cdot) : (0, \infty) \rightarrow [0, 1]$ is left-continuous on $(0, \infty)$. If $\Psi^\downarrow(\lambda, x, y; u, v) > 0$, then we have:

$$\zeta(x, y; u, v, \Psi^\downarrow(\lambda, x, y; u, v)) \leq 1 - \lambda. \quad (19)$$

- (ii) Suppose that the mapping $\zeta(x, y; u, v, \cdot) : (0, \infty) \rightarrow [0, 1]$ is right-continuous on $(0, \infty)$. Then the following statements hold true:

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. If $\Psi^\downarrow(\lambda, x, y; u, v) > 0$, then we have:

$$\zeta(x, y; u, v, \Psi^\downarrow(\lambda, x, y; u, v)) \geq 1 - \lambda. \quad (20)$$

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft\}$. If $\Psi^\downarrow(\lambda, y, x; u, v) > 0$, then we have:

$$\zeta(x, y; u, v, \Psi^\downarrow(\lambda, y, x; u, v)) \geq 1 - \lambda, \quad (21)$$

and if $\Psi^\downarrow(\lambda, x, y; v, u) > 0$, then we have:

$$\zeta(x, y; u, v, \Psi^\downarrow(\lambda, x, y; v, u)) \geq 1 - \lambda. \quad (22)$$

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. If $\Psi^\downarrow(\lambda, y, x; v, u) > 0$, then we have:

$$\zeta(x, y; u, v, \Psi^\downarrow(\lambda, y, x; v, u)) \geq 1 - \lambda. \quad (23)$$

Proof. By applying $\epsilon \rightarrow 0+$ to the inequality Equation (14), we obtain Equation (19), which proves part (i). By applying $\epsilon \rightarrow 0+$ to the inequality (Equation (15)), we obtain Equation (20), which proves part (ii). The other inequalities can be similarly obtained by parts (iii) and (iv) of Proposition 8. This completes the proof. \square

In order to establish the triangle inequalities for the infimum type of dual double fuzzy semi-metric, we provide a useful lemma.

Lemma 1. (Wu [16]) Suppose that the t -norm $*$ is left-continuous at 1 with respect to the first or second argument. For any $a \in (0, 1)$ and any $p \in \mathbb{N}$, there exists $r \in (0, 1)$ such that:

$$\overbrace{r * r * \dots * r}^{p \text{ times}} > a.$$

Theorem 1. (Triangle Inequalities for Dual Double Fuzzy Semi-Metric) Let (X, M) be a fuzzy semi-metric space along with a t -norm $*$. We also assume that M satisfies the canonical condition, and that the t -norm $*$ is left-continuous at 1 with respect to the first or second argument. Given any fixed $\mu \in (0, 1]$ and any fixed and distinct $x_1, x_2, \dots, x_p, y_1, y_2, \dots, y_p \in X$, we have the following inequalities:

(i) Suppose that M satisfies the \bowtie -triangle inequality. Then, there exists $\lambda \in (0, 1)$, satisfying:

$$\begin{aligned} \Psi^\downarrow(\mu, x_1, x_p; y_1, y_p) &\leq \Psi^\downarrow(\lambda, x_1, x_2; y_1, y_2) + \Psi^\downarrow(\lambda, x_2, x_3; y_2, y_3) + \cdots \\ &\quad + \Psi^\downarrow(\lambda, x_{p-2}, x_{p-1}; y_{p-2}, y_{p-1}) + \Psi^\downarrow(\lambda, x_{p-1}, x_p; y_{p-1}, y_p) \end{aligned} \quad (24)$$

$$\begin{aligned} \Psi^\downarrow(\mu, x_1, x_p; y_p, y_1) &\leq \Psi^\downarrow(\lambda, x_1, x_2; y_2, y_1) + \Psi^\downarrow(\lambda, x_2, x_3; y_3, y_2) + \cdots \\ &\quad + \Psi^\downarrow(\lambda, x_{p-2}, x_{p-1}; y_{p-1}, y_{p-2}) + \Psi^\downarrow(\lambda, x_{p-1}, x_p; y_p, y_{p-1}) \end{aligned} \quad (25)$$

$$\begin{aligned} \Psi^\downarrow(\mu, x_p, x_1; y_1, y_p) &\leq \Psi^\downarrow(\lambda, x_p, x_{p-1}; y_{p-1}, y_p) + \Psi^\downarrow(\lambda, x_{p-1}, x_{p-2}; y_{p-2}, y_{p-1}) \\ &\quad + \cdots + \Psi^\downarrow(\lambda, x_3, x_2; y_2, y_3) + \Psi^\downarrow(\lambda, x_2, x_1; y_1, y_2) \end{aligned}$$

$$\begin{aligned} \Psi^\downarrow(\mu, x_p, x_1; y_p, y_1) &\leq \Psi^\downarrow(\lambda, x_p, x_{p-1}; y_p, y_{p-1}) + \Psi^\downarrow(\lambda, x_{p-1}, x_{p-2}; y_{p-1}, y_{p-2}) \\ &\quad + \cdots + \Psi^\downarrow(\lambda, x_3, x_2; y_3, y_2) + \Psi^\downarrow(\lambda, x_2, x_1; y_2, y_1). \end{aligned}$$

(ii) Suppose that M satisfies the \triangleright -triangle inequality. Then, there exists $\lambda \in (0, 1)$, satisfying:

$$\begin{aligned} &\max \left\{ \Psi^\downarrow(\mu, x_1, x_p; y_1, y_p), \Psi^\downarrow(\mu, x_1, x_p; y_p, y_1), \Psi^\downarrow(\mu, x_p, x_1; y_1, y_p), \Psi^\downarrow(\mu, x_p, x_1; y_p, y_1) \right\} \\ &\leq \Psi^\downarrow(\lambda, x_1, x_2; y_1, y_2) + \Psi^\downarrow(\lambda, x_3, x_2; y_3, y_2) + \Psi^\downarrow(\lambda, x_4, x_3; y_4, y_3) \\ &\quad + \cdots + \Psi^\downarrow(\lambda, x_p, x_{p-1}; y_p, y_{p-1}). \end{aligned}$$

(iii) Suppose that M satisfies the \triangleleft -triangle inequality. Then, there exists $\lambda \in (0, 1)$, satisfying:

$$\begin{aligned} &\max \left\{ \Psi^\downarrow(\mu, x_1, x_p; y_1, y_p), \Psi^\downarrow(\mu, x_1, x_p; y_p, y_1), \Psi^\downarrow(\mu, x_p, x_1; y_1, y_p), \Psi^\downarrow(\mu, x_p, x_1; y_p, y_1) \right\} \\ &\leq \Psi^\downarrow(\lambda, x_2, x_1; y_2, y_1) + \Psi^\downarrow(\lambda, x_2, x_3; y_2, y_3) + \Psi^\downarrow(\lambda, x_3, x_4; y_3, y_4) \\ &\quad + \cdots + \Psi^\downarrow(\lambda, x_{p-1}, x_p; y_{p-1}, y_p). \end{aligned}$$

(iv) Suppose that M satisfies the \diamond -triangle inequality. Then, there exists $\lambda \in (0, 1)$ such that the following inequalities are satisfied:

- If p is even, then:

$$\begin{aligned} \Psi^\downarrow(\mu, x_1, x_p; y_1, y_p) &\leq \Psi^\downarrow(\lambda, x_2, x_1; y_2, y_1) + \Psi^\downarrow(\lambda, x_3, x_2; y_3, y_2) + \Psi^\downarrow(\lambda, x_3, x_4; y_3, y_4) \\ &\quad + \Psi^\downarrow(\lambda, x_5, x_4; y_5, y_4) + \Psi^\downarrow(\lambda, x_5, x_6; y_5, y_6) + \Psi^\downarrow(\lambda, x_7, x_6; y_7, y_6) \\ &\quad + \cdots + \Psi^\downarrow(\lambda, x_{p-1}, x_p; y_{p-1}, y_p), \end{aligned} \quad (26)$$

$$\begin{aligned} \Psi^\downarrow(\mu, x_1, x_p; y_p, y_1) &\leq \Psi^\downarrow(\lambda, x_2, x_1; y_1, y_2) + \Psi^\downarrow(\lambda, x_3, x_2; y_2, y_3) + \Psi^\downarrow(\lambda, x_3, x_4; y_4, y_3) \\ &\quad + \Psi^\downarrow(\lambda, x_5, x_4; y_4, y_5) + \Psi^\downarrow(\lambda, x_5, x_6; y_6, y_5) + \Psi^\downarrow(\lambda, x_7, x_6; y_6, y_7) \\ &\quad + \cdots + \Psi^\downarrow(\lambda, x_{p-1}, x_p; y_p, y_{p-1}), \end{aligned} \quad (27)$$

$$\begin{aligned} \Psi^\downarrow(\mu, x_p, x_1; y_1, y_p) &\leq \Psi^\downarrow(\lambda, x_1, x_2; y_2, y_1) + \Psi^\downarrow(\lambda, x_2, x_3; y_3, y_2) + \Psi^\downarrow(\lambda, x_4, x_3; y_3, y_4) \\ &\quad + \Psi^\downarrow(\lambda, x_4, x_5; y_5, y_4) + \Psi^\downarrow(\lambda, x_6, x_5; y_5, y_6) + \Psi^\downarrow(\lambda, x_6, x_7; y_7, y_6) \\ &\quad + \cdots + \Psi^\downarrow(\lambda, x_p, x_{p-1}; y_{p-1}, y_p), \end{aligned} \quad (28)$$

$$\begin{aligned} \Psi^\downarrow(\mu, x_p, x_1; y_p, y_1) &\leq \Psi^\downarrow(\lambda, x_1, x_2; y_1, y_2) + \Psi^\downarrow(\lambda, x_2, x_3; y_2, y_3) + \Psi^\downarrow(\lambda, x_4, x_3; y_4, y_3) \\ &\quad + \Psi^\downarrow(\lambda, x_4, x_5; y_4, y_5) + \Psi^\downarrow(\lambda, x_6, x_5; y_6, y_5) + \Psi^\downarrow(\lambda, x_6, x_7; y_6, y_7) \\ &\quad + \cdots + \Psi^\downarrow(\lambda, x_p, x_{p-1}; y_p, y_{p-1}). \end{aligned} \quad (29)$$

- If p is odd, then:

$$\begin{aligned}\Psi^\downarrow(\mu, x_1, x_p; y_1, y_p) &\leq \Psi^\downarrow(\lambda, x_1, x_2; y_1, y_2) + \Psi^\downarrow(\lambda, x_2, x_3; y_2, y_3) + \Psi^\downarrow(\lambda, x_4, x_3; y_4, y_3) \\ &\quad + \Psi^\downarrow(\lambda, x_4, x_5; y_4, y_5) + \Psi^\downarrow(\lambda, x_6, x_5; y_6, y_5) + \Psi^\downarrow(\lambda, x_6, x_7; y_6, y_7) \\ &\quad + \cdots + \Psi^\downarrow(\lambda, x_p, x_{p-1}; y_p, y_{p-1}),\end{aligned}\quad (30)$$

$$\begin{aligned}\Psi^\downarrow(\mu, x_1, x_p; y_p, y_1) &\leq \Psi^\downarrow(\lambda, x_1, x_2; y_2, y_1) + \Psi^\downarrow(\lambda, x_2, x_3; y_3, y_2) + \Psi^\downarrow(\lambda, x_4, x_3; y_3, y_4) \\ &\quad + \Psi^\downarrow(\lambda, x_4, x_5; y_5, y_4) + \Psi^\downarrow(\lambda, x_6, x_5; y_5, y_6) + \Psi^\downarrow(\lambda, x_6, x_7; y_7, y_6) \\ &\quad + \cdots + \Psi^\downarrow(\lambda, x_p, x_{p-1}; y_{p-1}, y_p),\end{aligned}\quad (31)$$

$$\begin{aligned}\Psi^\downarrow(\mu, x_p, x_1; y_1, y_p) &\leq \Psi^\downarrow(\lambda, x_2, x_1; y_1, y_2) + \Psi^\downarrow(\lambda, x_3, x_2; y_2, y_3) + \Psi^\downarrow(\lambda, x_3, x_4; y_4, y_3) \\ &\quad + \Psi^\downarrow(\lambda, x_5, x_4; y_4, y_5) + \Psi^\downarrow(\lambda, x_5, x_6; y_6, y_5) + \Psi^\downarrow(\lambda, x_7, x_6; y_6, y_7) \\ &\quad + \cdots + \Psi^\downarrow(\lambda, x_{p-1}, x_p; y_p, y_{p-1}),\end{aligned}\quad (32)$$

$$\begin{aligned}\Psi^\downarrow(\mu, x_p, x_1; y_p, y_1) &\leq \Psi^\downarrow(\lambda, x_2, x_1; y_2, y_1) + \Psi^\downarrow(\lambda, x_3, x_2; y_3, y_2) + \Psi^\downarrow(\lambda, x_3, x_4; y_3, y_4) \\ &\quad + \Psi^\downarrow(\lambda, x_5, x_4; y_5, y_4) + \Psi^\downarrow(\lambda, x_5, x_6; y_5, y_6) + \Psi^\downarrow(\lambda, x_7, x_6; y_7, y_6) \\ &\quad + \cdots + \Psi^\downarrow(\lambda, x_{p-1}, x_p; y_{p-1}, y_p).\end{aligned}\quad (33)$$

Proof. To prove part (i), if $\mu = 1$, then $\Psi(1, x_1, x_p; y_1, y_p) = 0$. Therefore, the result is obvious. Now we assume $\mu \in (0, 1)$. Using Lemma 1, there exists $\lambda \in (0, 1)$, satisfying:

$$(1 - \lambda) * \cdots * (1 - \lambda) > 1 - \mu. \quad (34)$$

Given any $\epsilon > 0$, the first observation of Remark 1 says that:

$$\begin{aligned}M\left(x_1, x_p, \Psi^\downarrow(\lambda, x_1, x_2; y_1, y_2) + \Psi^\downarrow(\lambda, x_2, x_3; y_2, y_3) + \cdots + \Psi^\downarrow(\lambda, x_{p-1}, x_p; y_{p-1}, y_p) + (p-1)\epsilon\right) \\ \geq M\left(x_1, x_2, \Psi^\downarrow(\lambda, x_1, x_2; y_1, y_2) + \epsilon\right) * \cdots * M\left(x_{p-1}, x_p, \Psi^\downarrow(\lambda, x_{p-1}, x_p; y_{p-1}, y_p) + \epsilon\right),\end{aligned}\quad (35)$$

and:

$$\begin{aligned}M\left(y_1, y_p, \Psi^\downarrow(\lambda, x_1, x_2; y_1, y_2) + \Psi^\downarrow(\lambda, x_2, x_3; y_2, y_3) + \cdots + \Psi^\downarrow(\lambda, x_{p-1}, x_p; y_{p-1}, y_p) + (p-1)\epsilon\right) \\ \geq M\left(y_1, y_2, \Psi^\downarrow(\lambda, x_1, x_2; y_1, y_2) + \epsilon\right) * \cdots * M\left(y_{p-1}, y_p, \Psi^\downarrow(\lambda, x_{p-1}, x_p; y_{p-1}, y_p) + \epsilon\right).\end{aligned}\quad (36)$$

Now applying the increasing property and commutativity of t-norm to Equations (35) and (36), we obtain:

$$\begin{aligned}\zeta\left(x_1, x_p; y_1, y_p, \Psi^\downarrow(\lambda, x_1, x_2; y_1, y_2) + \Psi^\downarrow(\lambda, x_2, x_3; y_2, y_3) + \cdots + \Psi^\downarrow(\lambda, x_{p-1}, x_p; y_{p-1}, y_p) + (p-1)\epsilon\right) \\ \geq \zeta\left(x_1, x_2; y_1, y_2, \Psi^\downarrow(\lambda, x_1, x_2; y_1, y_2) + \epsilon\right) * \cdots * \zeta\left(x_{p-1}, x_p; y_{p-1}, y_p, \Psi^\downarrow(\lambda, x_{p-1}, x_p; y_{p-1}, y_p) + \epsilon\right) \\ \geq (1 - \lambda) * \cdots * (1 - \lambda) \text{ (by Equation (15) and the increasing property of t-norm)} \\ > 1 - \mu \text{ (by Equation (34)).}\end{aligned}$$

The definition of Ψ^\downarrow says that:

$$\begin{aligned}\Psi^\downarrow(\lambda, x_1, x_2; y_1, y_2) + \Psi^\downarrow(\lambda, x_2, x_3; y_2, y_3) + \cdots + \Psi^\downarrow(\lambda, x_{p-1}, x_p; y_{p-1}, y_p) + (p-1)\epsilon \\ \geq \Psi^\downarrow(\mu, x_1, x_p; y_1, y_p).\end{aligned}$$

By taking $\epsilon \rightarrow 0+$, we obtain the desired inequality (Equation (24)).

On the other hand, we also have:

$$\begin{aligned} & M\left(x_1, x_p, \Psi^\downarrow(\lambda, x_1, x_2; y_2, y_1) + \Psi^\downarrow(\lambda, x_2, x_3; y_3, y_2) + \cdots + \Psi^\downarrow(\lambda, x_{p-1}, x_p; y_p, y_{p-1}) + (p-1)\epsilon\right) \\ & \geq M\left(x_1, x_2, \Psi^\downarrow(\lambda, x_1, x_2; y_2, y_1) + \epsilon\right) * \cdots * M\left(x_{p-1}, x_p, \Psi^\downarrow(\lambda, x_{p-1}, x_p; y_p, y_{p-1}) + \epsilon\right), \end{aligned} \quad (37)$$

and:

$$\begin{aligned} & M\left(y_p, y_1, \Psi^\downarrow(\lambda, x_1, x_2; y_2, y_1) + \Psi^\downarrow(\lambda, x_2, x_3; y_3, y_2) + \cdots + \Psi^\downarrow(\lambda, x_{p-1}, x_p; y_p, y_{p-1}) + (p-1)\epsilon\right) \\ & \geq M\left(y_2, y_1, \Psi^\downarrow(\lambda, x_1, x_2; y_2, y_1) + \epsilon\right) * \cdots * M\left(y_p, y_{p-1}, \Psi^\downarrow(\lambda, x_{p-1}, x_p; y_p, y_{p-1}) + \epsilon\right). \end{aligned} \quad (38)$$

Now applying the increasing property and commutativity of t-norm to Equations (37) and (38), we also obtain:

$$\begin{aligned} & \zeta\left(x_1, x_p; y_p, y_1, \Psi^\downarrow(\lambda, x_1, x_2; y_2, y_1) + \Psi^\downarrow(\lambda, x_2, x_3; y_3, y_2) + \cdots + \Psi^\downarrow(\lambda, x_{p-1}, x_p; y_p, y_{p-1}) + (p-1)\epsilon\right) \\ & \geq \zeta\left(x_1, x_2; y_2, y_1, \Psi^\downarrow(\lambda, x_1, x_2; y_2, y_1) + \epsilon\right) * \cdots * \zeta\left(x_{p-1}, x_p; y_p, y_{p-1}, \Psi^\downarrow(\lambda, x_{p-1}, x_p; y_p, y_{p-1}) + \epsilon\right) \\ & \geq (1-\lambda) * \cdots * (1-\lambda) \text{ (by Equation (15) and the increasing property of t-norm)} \\ & > 1-\mu \text{ (by Equation (34)).} \end{aligned}$$

The definition of Ψ^\downarrow says that:

$$\begin{aligned} & \Psi^\downarrow(\lambda, x_1, x_2; y_2, y_1) + \Psi^\downarrow(\lambda, x_2, x_3; y_3, y_2) + \cdots + \Psi^\downarrow(\lambda, x_{p-1}, x_p; y_p, y_{p-1}) + (p-1)\epsilon \\ & \geq \Psi^\downarrow(\mu, x_1, x_p; y_p, y_1). \end{aligned}$$

By taking $\epsilon \rightarrow 0+$, we obtain the desired inequality (Equation (25)). Since the other inequalities can be similarly obtained, we omit the details.

The above argument is still valid to obtain part (ii) by referring the second observation of Remark 1. Further, we can use the third observation of Remark 1 to obtain part (iii). Finally, part (iv) can be obtained by referring to the fourth observation of Remark 1. This completes the proof. \square

Let (X, M) be a fuzzy semi-metric space, and let $\{x_n\}_{n=1}^\infty$ be a sequence in X . We write $x_n \xrightarrow{M^>} x$ as $n \rightarrow \infty$ if and only if:

$$\lim_{n \rightarrow \infty} M(x_n, x, t) = 1 \text{ for all } t > 0.$$

We also write $x_n \xrightarrow{M^<} x$ as $n \rightarrow \infty$ if and only if:

$$\lim_{n \rightarrow \infty} M(x, x_n, t) = 1 \text{ for all } t > 0.$$

The main convergence theorem is presented below. We first provide a useful lemma.

Lemma 2. Let $*$ be a t-norm. If $a * b > k$ then $a > k$ and $b > k$.

Proof. Since $b \leq 1$, the increasing property and boundary condition show that $b * k \leq 1 * k = k$. Suppose that $a \leq k$. Then we have $a * b \leq k * b$ and:

$$k < a * b \leq k * b \leq k.$$

A contradiction occurs. Therefore, we must have $a > k$. We can similarly show that $b > k$. This completes the proof. \square

Theorem 2. Let (X, M) be a fuzzy semi-metric space along with a t -norm $*$. We also assume that M satisfies the canonical condition, and that the t -norm $*$ is left-continuous at 1 with respect to the first or second argument. Let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be two sequences in X . Then we have the following properties:

- $x_n \xrightarrow{M^p} x$ and $y_n \xrightarrow{M^p} y$ as $n \rightarrow \infty$ if and only if $\Psi^\downarrow(\lambda, x_n, x; y_n, y) \rightarrow 0$ as $n \rightarrow \infty$ for all $\lambda \in (0, 1)$.
- $x_n \xrightarrow{M^p} x$ and $y_n \xrightarrow{M^q} y$ as $n \rightarrow \infty$ if and only if $\Psi^\downarrow(\lambda, x_n, x; y, y_n) \rightarrow 0$ as $n \rightarrow \infty$ for all $\lambda \in (0, 1)$.
- $x_n \xrightarrow{M^q} x$ and $y_n \xrightarrow{M^p} y$ as $n \rightarrow \infty$ if and only if $\Psi^\downarrow(\lambda, x, x_n; y_n, y) \rightarrow 0$ as $n \rightarrow \infty$ for all $\lambda \in (0, 1)$.
- $x_n \xrightarrow{M^q} x$ and $y_n \xrightarrow{M^q} y$ as $n \rightarrow \infty$ if and only if $\Psi^\downarrow(\lambda, x, x_n; y, y_n) \rightarrow 0$ as $n \rightarrow \infty$ for all $\lambda \in (0, 1)$.

Proof. For any fixed $\lambda \in (0, 1)$, using Lemma 1, it follows that there exists $\lambda_0 \in (0, 1)$, satisfying:

$$(1 - \lambda_0) * (1 - \lambda_0) > 1 - \lambda. \quad (39)$$

We just prove the first case, since the other cases can be similarly obtained. Suppose that $M(x_n, x, t) \rightarrow 1$ and $M(y_n, y, t) \rightarrow 1$ as $n \rightarrow \infty$ for all $t > 0$. Then, given any $t > 0$ and $\delta > 0$, there exists $n_{t,\delta}^{(1)}, n_{t,\delta}^{(2)} \in \mathbb{N}$, satisfying $|M(x_n, x, t) - 1| < \delta$ for $n \geq n_{t,\delta}^{(1)}$ and $|M(y_n, y, t) - 1| < \delta$ for $n \geq n_{t,\delta}^{(2)}$. Therefore, given any $\epsilon \in (0, 1)$, there exists $n_\epsilon \in \mathbb{N}$, satisfying:

$$\left| M\left(x_n, x, \frac{\epsilon}{2}\right) - 1 \right| < \lambda_0 \text{ and } \left| M\left(y_n, y, \frac{\epsilon}{2}\right) - 1 \right| < \lambda_0,$$

for $n \geq n_\epsilon$. We also have:

$$M\left(x_n, x, \frac{\epsilon}{2}\right) > 1 - \lambda_0 \text{ and } M\left(y_n, y, \frac{\epsilon}{2}\right) > 1 - \lambda_0,$$

for $n \geq n_\epsilon$. The increasing property of t -norm says that:

$$\zeta\left(x_n, x; y_n, y, \frac{\epsilon}{2}\right) = M\left(x_n, x, \frac{\epsilon}{2}\right) * M\left(y_n, y, \frac{\epsilon}{2}\right) \geq (1 - \lambda_0) * (1 - \lambda_0) > 1 - \lambda.$$

The definition of Ψ^\downarrow says that:

$$\Psi^\downarrow(\lambda, x_n, x; y_n, y) \leq \frac{\epsilon}{2} < \epsilon,$$

for $n \geq n_\epsilon$. This shows that $\Psi^\downarrow(\lambda, x_n, x; y_n, y) \rightarrow 0$ as $n \rightarrow \infty$.

Conversely, assume that $\Psi^\downarrow(\lambda, x_n, x; y_n, y) \rightarrow 0$ as $n \rightarrow \infty$ for all $\lambda \in (0, 1)$. Now, given any $\delta > 0$ and $\lambda \in (0, 1]$, there exists $n_{\delta,\lambda} \in \mathbb{N}$, satisfying $|\Psi^\downarrow(\lambda, x_n, x; y_n, y)| < \delta$ for all $n \geq n_{\delta,\lambda}$. Therefore, for any fixed $t > 0$ and given any $\epsilon \in (0, 1)$, there exists $n_\epsilon \in \mathbb{N}$, satisfying:

$$\Psi\left(\frac{\epsilon}{2}, x_n, x; y_n, y\right) = \left| \Psi\left(\frac{\epsilon}{2}, x_n, x; y_n, y\right) \right| < t,$$

for $n \geq n_\epsilon$, which implies:

$$\zeta(x_n, x; y_n, y, t) \geq 1 - \frac{\epsilon}{2} > 1 - \epsilon,$$

for $n \geq n_\epsilon$ by part (i) of Proposition 9, i.e.:

$$M(x_n, x, t) * M(y_n, y, t) > 1 - \epsilon,$$

for $n \geq n_\epsilon$. Lemma 2 says that:

$$M(x_n, x, t) > 1 - \epsilon \text{ and } M(y_n, y, t) > 1 - \epsilon,$$

for $n \geq n_\epsilon$. This shows that $x_n \xrightarrow{M^p} x$ and $y_n \xrightarrow{M^p} y$ as $n \rightarrow \infty$, and the proof is complete. \square

Example 5. From Example 1, we see that:

$$x_n \xrightarrow{M^>} x \text{ if and only if } \lim_{n \rightarrow \infty} d(x_n, x) = 0,$$

and:

$$x_n \xrightarrow{M^<} x \text{ if and only if } \lim_{n \rightarrow \infty} d(x, x_n) = 0.$$

From Example 4, we have:

$$\Psi^\downarrow(\lambda, x, y; u, v) = \frac{C + \sqrt{C^2 + D}}{2},$$

where:

$$C = \frac{(d(x, y) + d(u, v))(1 - \lambda)}{\lambda} \text{ and } D = \frac{d(x, y) \cdot d(u, v) \cdot (1 - \lambda)}{\lambda}.$$

It is clear to see that $x_n \xrightarrow{M^>} x$ and $y_n \xrightarrow{M^>} y$ as $n \rightarrow \infty$ if and only if $\Psi^\downarrow(\lambda, x_n, x; y_n, y) \rightarrow 0$ as $n \rightarrow \infty$ for all $\lambda \in (0, 1)$. The other convergence presented in Theorem 2 can be similarly verified.

Definition 4. Let (X, M) be a fuzzy semi-metric space, and let $\{x_n\}_{n=1}^\infty$ be a sequence in X .

- The sequence $\{x_n\}_{n=1}^\infty$ is said to be a $>$ -Cauchy sequence in a metric sense if and only if, given any pair (r, t) with $t > 0$ and $0 < r < 1$, there exists $n_{r,t} \in \mathbb{N}$, satisfying $M(x_m, x_n, t) > 1 - r$ for all pairs (m, n) of integers m and n with $m > n \geq n_{r,t}$.
- The sequence $\{x_n\}_{n=1}^\infty$ is said to be a $<$ -Cauchy sequence in a metric sense if and only if, given any pair (r, t) with $t > 0$ and $0 < r < 1$, there exists $n_{r,t} \in \mathbb{N}$, satisfying $M(x_n, x_m, t) > 1 - r$ for all pairs (m, n) of integers m and n with $m > n \geq n_{r,t}$.
- The sequence $\{x_n\}_{n=1}^\infty$ is said to be a Cauchy sequence in a metric sense if and only if, given any pair (r, t) with $t > 0$ and $0 < r < 1$, there exists $n_{r,t} \in \mathbb{N}$ satisfying $M(x_m, x_n, t) > 1 - r$ and $M(x_n, x_m, t) > 1 - r$ for all pairs (m, n) of integers m and n with $m, n \geq n_{r,t}$ and $m \neq n$.

Definition 5. Let (X, M) be a fuzzy semi-metric space such that M satisfies the canonical condition, and let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be two sequences in X .

- Given any fixed $\lambda \in (0, 1)$, the sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are said to be the joint $(\lambda, >, >)$ -Cauchy sequences with respect to Ψ^\downarrow if and only if, given any $\epsilon > 0$, there exists $n_{\epsilon, \lambda} \in \mathbb{N}$ such that $m > n \geq n_{\epsilon, \lambda}$ implies $\Psi^\downarrow(\lambda, x_m, x_n; y_m, y_n) < \epsilon$.
- Given any fixed $\lambda \in (0, 1)$, the sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are said to be the joint $(\lambda, >, <)$ -Cauchy sequences with respect to Ψ^\downarrow if and only if, given any $\epsilon > 0$, there exists $n_{\epsilon, \lambda} \in \mathbb{N}$ such that $m > n \geq n_{\epsilon, \lambda}$ implies $\Psi^\downarrow(\lambda, x_m, x_n; y_n, y_m) < \epsilon$.
- Given any fixed $\lambda \in (0, 1)$, the sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are said to be the joint $(\lambda, <, >)$ -Cauchy sequences with respect to Ψ^\downarrow if and only if, given any $\epsilon > 0$, there exists $n_{\epsilon, \lambda} \in \mathbb{N}$ such that $m > n \geq n_{\epsilon, \lambda}$ implies $\Psi^\downarrow(\lambda, x_n, x_m; y_m, y_n) < \epsilon$.
- Given any fixed $\lambda \in (0, 1)$, the sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are said to be the joint $(\lambda, <, <)$ -Cauchy sequences with respect to Ψ^\downarrow if and only if, given any $\epsilon > 0$, there exists $n_{\epsilon, \lambda} \in \mathbb{N}$ such that $m > n \geq n_{\epsilon, \lambda}$ implies $\Psi^\downarrow(\lambda, x_n, x_m; y_n, y_m) < \epsilon$.

Theorem 3. Let (X, M) be a fuzzy semi-metric space along with a t -norm $*$. We also assume that M satisfies the canonical condition, and that the t -norm $*$ is left-continuous at 1 with respect to the first or second argument. Let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be two sequences in X . Then, we have the following properties:

- $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are two $>$ -Cauchy sequences in a metric sense if and only if $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are the joint $(\lambda, >, >)$ -Cauchy sequences with respect to Ψ^\downarrow for any $\lambda \in (0, 1)$.

- (ii) $\{x_n\}_{n=1}^{\infty}$ is a $>$ -Cauchy sequences and $\{y_n\}_{n=1}^{\infty}$ is a $<$ -Cauchy sequences in a metric sense if and only if $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are the joint $(\lambda, >, <)$ -Cauchy sequences with respect to Ψ^{\downarrow} for any $\lambda \in (0, 1)$.
- (iii) $\{x_n\}_{n=1}^{\infty}$ is a $<$ -Cauchy sequences and $\{y_n\}_{n=1}^{\infty}$ is a $>$ -Cauchy sequences in a metric sense if and only if $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are the joint $(\lambda, <, >)$ -Cauchy sequences with respect to Ψ^{\downarrow} for any $\lambda \in (0, 1)$.
- (iv) $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are two $<$ -Cauchy sequences if and only if $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are the joint $(\lambda, <, <)$ -Cauchy sequences in a metric sense with respect to Ψ^{\downarrow} for any $\lambda \in (0, 1)$.

Proof. It suffices to just prove part (i), since the other cases can be similarly obtained. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are $>$ -Cauchy sequences. Then, given any $t > 0$ and $\delta > 0$, there exists $n_{t,\delta} \in \mathbb{N}$ such that $m > n \geq n_{t,\delta}$ implies $M(x_m, x_n, t) > 1 - \delta$ and $M(y_m, y_n, t) > 1 - \delta$. Now, given any $\epsilon \in (0, 1)$, there exists $n_{\epsilon} \in \mathbb{N}$ such that $m > n \geq n_{\epsilon}$ implies:

$$M\left(x_m, x_n, \frac{\epsilon}{2}\right) > 1 - \lambda_0 \text{ and } M\left(y_m, y_n, \frac{\epsilon}{2}\right) > 1 - \lambda_0.$$

The increasing property of t-norm says that:

$$\begin{aligned} \zeta\left(x_m, x_n; y_m, y_n, \frac{\epsilon}{2}\right) &= M\left(x_m, x_n, \frac{\epsilon}{2}\right) * M\left(y_m, y_n, \frac{\epsilon}{2}\right) \\ &\geq (1 - \lambda_0) * (1 - \lambda_0) > 1 - \lambda \text{ (using Equation (39)).} \end{aligned}$$

Further, by referring to the definition of Ψ^{\downarrow} , we obtain:

$$\Psi^{\downarrow}(\lambda, x_m, x_n; y_m, y_n) \leq \frac{\epsilon}{2} < \epsilon,$$

for $m > n \geq n_{\epsilon}$.

Conversely, using the assumption, for any fixed $t > 0$ and given any $\epsilon \in (0, 1)$, there exists $n_{\epsilon} \in \mathbb{N}$ such that $m > n \geq n_{\epsilon}$ implies $\Psi(\epsilon/2, x_m, x_n; y_m, y_n) < t$. Using Proposition 9, we obtain:

$$\zeta(x_m, x_n; y_m, y_n, t) \geq 1 - \frac{\epsilon}{2} > 1 - \epsilon,$$

for $m > n \geq n_{\epsilon}$, i.e.:

$$M(x_m, x_n, t) * M(y_m, y_n, t) > 1 - \epsilon,$$

for $m > n \geq n_{\epsilon}$. Lemma 2 says that:

$$M(x_m, x_n, t) > 1 - \epsilon \text{ and } M(y_m, y_n, t) > 1 - \epsilon,$$

for $m > n \geq n_{\epsilon}$, which shows that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are $>$ -Cauchy sequences. This completes the proof. \square

5. Convergence Based on the Supremum

Using the infimum and assuming the canonical condition, the infimum type of dual double fuzzy semi-metric was proposed in the previous section. In this section, we shall consider the supremum to propose the so-called supremum type of dual double fuzzy semi-metric.

Recall that the purpose for considering the canonical condition is to guarantee the infimum type of dual fuzzy semi-metric space to be well-defined. Now, we shall consider the rational condition to guarantee the supremum type of dual fuzzy semi-metric space to be well-defined. The formal definition is given below.

Definition 6. Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the rational condition, and that the t-norm $*$ is right-continuous at 0 with respect to the first or second

argument. Given any fixed $x, y, u, v \in X$ with $x \neq y$ or $u \neq v$ and any fixed $\lambda \in [0, 1)$, we consider the following set:

$$\Pi^\uparrow(\lambda, x, y; u, v) = \{t > 0 : \zeta(x, y; u, v, t) \leq 1 - \lambda\},$$

which will be used to define a function $\Psi^\uparrow : X^4 \rightarrow [0, +\infty)$ by:

$$\Psi^\uparrow(\lambda, x, y; u, v) = \sup \Pi^\uparrow(\lambda, x, y; u, v) = \sup \{t > 0 : \zeta(x, y; u, v, t) \leq 1 - \lambda\}.$$

The mapping Π^\uparrow from $(0, 1] \times X^4$ into $[0, \infty)$ is called the supremum type of dual double fuzzy semi-metric.

Example 6. Continued from Example 1, we have:

$$\Pi^\uparrow(\lambda, x, y; u, v) = \left\{t > 0 : \frac{t}{t + d(x, y)} \cdot \frac{t}{t + d(u, v)} \leq 1 - \lambda\right\} = \left\{t > 0 : t \leq \frac{C + \sqrt{C^2 + D}}{2}\right\},$$

where:

$$C = \frac{(d(x, y) + d(u, v))(1 - \lambda)}{\lambda} \text{ and } D = \frac{d(x, y) \cdot d(u, v) \cdot (1 - \lambda)}{\lambda}.$$

We also have:

$$\Psi^\uparrow(\lambda, x, y; u, v) = \sup \Pi^\uparrow(\lambda, x, y; u, v) = \left\{t > 0 : t \leq \frac{C + \sqrt{C^2 + D}}{2}\right\} = \frac{C + \sqrt{C^2 + D}}{2}.$$

For any $x \neq y$ or $u \neq v$, we need to claim that the set $\Pi^\uparrow(\lambda, x, y; u, v)$ is nonempty. Suppose that $\Pi^\uparrow(\lambda, x, y; u, v) = \emptyset$. The definition says that $\zeta(x, y; u, v, t) > 1 - \lambda$ for all $t > 0$. Therefore, we obtain:

$$\lim_{t \rightarrow 0+} \zeta(x, y; u, v, t) \geq 1 - \lambda,$$

which contradicts Equation (9). This says that Definition 6 is well-defined, which also says that $\Psi^\uparrow(\lambda, x, y; u, v) > 0$. We also have:

$$\Psi^\uparrow(0, x, y; u, v) = \sup \{t > 0 : \zeta(x, y; u, v, t) \leq 0\} = \sup \{t > 0\} = +\infty.$$

Moreover, if $\lambda_1 > \lambda_2$, then:

$$\Pi^\uparrow(\lambda_1, x, y; u, v) \subseteq \Pi^\uparrow(\lambda_2, x, y; u, v) \text{ and } \Psi^\uparrow(\lambda_1, x, y; u, v) \leq \Psi^\uparrow(\lambda_2, x, y; u, v). \quad (40)$$

Proposition 12. Let (X, M) be a fuzzy semi-metric space along with a t -norm $*$. We also assume that M satisfies the rational condition, and that the t -norm $*$ is right-continuous at 0 with respect to the first or second argument. Given any fixed $x, y, u, v \in X$ with $x \neq y$ or $u \neq v$, suppose that $\Psi^\uparrow(\lambda, x, y; u, v) = +\infty$. Then, the following statements hold true:

- (i) Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Then we have $\zeta(x, y; u, v, t) \leq 1 - \lambda$ for all $t > 0$.
- (ii) Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft\}$. Then we have $\zeta(y, x; u, v, t) \leq 1 - \lambda$ and $\zeta(x, y; v, u, t) \leq 1 - \lambda$ for all $t > 0$.
- (iii) Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. Then we have $\zeta(y, x; v, u, t) \leq 1 - \lambda$ for all $t > 0$.

Proof. The fact $\Psi^\uparrow(\lambda, x, y; u, v) = +\infty$ says that $\zeta(x, y; u, v, t) \leq 1 - \lambda$ for sufficiently large $t > 0$ in the sense of $t \rightarrow \infty$. To prove part (i), we assume that there exists $t_0 > 0$, satisfying $\zeta(x, y; u, v, t_0) > 1 - \lambda$.

Parts (i) and (ii) of Proposition 3 say that the mapping $\zeta(x, y; u, v, \cdot)$ is nondecreasing. Therefore, if $t_1 > t_0$, then:

$$\zeta(x, y; u, v, t_1) \geq \zeta(x, y; u, v, t_0) > 1 - \lambda,$$

which contradicts $\zeta(x, y; u, v, t) \leq 1 - \lambda$ for sufficiently large $t > 0$.

To prove part (ii), we assume that there exists $t_0 > 0$, satisfying $\zeta(y, x; u, v, t_0) > 1 - \lambda$. Part (ii) of Proposition 3 says that the mapping $\zeta(x, y; u, v, \cdot)$ is \triangleleft -semisymmetrically nondecreasing. Therefore, if $t_1 > t_0$, then:

$$\zeta(x, y; u, v, t_1) \geq \zeta(y, x; u, v, t_0) > 1 - \lambda,$$

which contradicts $\zeta(x, y; u, v, t) \leq 1 - \lambda$ for sufficiently large $t > 0$. We can similarly obtain another inequality using the fact of the mapping $\zeta(x, y; u, v, \cdot)$ to be \triangleright -semisymmetrically nondecreasing.

To prove part (iii), we assume that there exists $t_0 > 0$, satisfying $\zeta(y, x; v, u, t_0) > 1 - \lambda$. Parts (ii) and (iii) of Proposition 3 say that the mapping $\zeta(x, y; u, v, \cdot)$ is symmetrically nondecreasing. Therefore, if $t_1 > t_0$, then:

$$\zeta(x, y; u, v, t_1) \geq \zeta(y, x; v, u, t_0) > 1 - \lambda,$$

which contradicts $\zeta(x, y; u, v, t) \leq 1 - \lambda$ for sufficiently large $t > 0$. This completes the proof. \square

Proposition 13. Let (X, M) be a fuzzy semi-metric space along with a t -norm $*$. We also assume that M satisfies the rational and canonical conditions, and that the t -norm $*$ is right-continuous at 0 and left-continuous at 1 with respect to the first or second argument. Then, given any fixed $x, y, u, v \in X$ with $x \neq y$ or $u \neq v$, we have $\Psi^\uparrow(\lambda, x, y; u, v) < +\infty$ for $\lambda \in (0, 1)$.

Proof. We assume that $\Psi^\uparrow(\lambda, x, y; u, v) = +\infty$, which means that $\zeta(x, y; u, v, t) \leq 1 - \lambda$ for sufficiently large t in the sense of $t \rightarrow \infty$. Using Equation (8), we obtain

$$1 = \lim_{t \rightarrow \infty} \zeta(x, y; u, v, t) \leq 1 - \lambda,$$

which leads to a contradiction for $0 < \lambda < 1$. This completes the proof. \square

Proposition 14. Let (X, M) be a fuzzy semi-metric space along with a t -norm $*$. Assume that M satisfies the canonical and rational conditions. We also assume that the t -norm $*$ is left-continuous at 1 and right-continuous at 0 with respect to the first or second argument, and that the t -norm $*$ also satisfies the strictly increasing property. For any fixed $x, y, u, v \in X$ with $x \neq y$ or $u \neq v$, the following statements hold true:

(i) Suppose that M satisfies the strict \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Then we have:

$$\Psi^\uparrow(\lambda, x, y; u, v) \leq \Psi^\downarrow(\lambda, x, y; u, v)$$

for each $\lambda \in (0, 1)$.

(ii) Suppose that M satisfies the strict \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft\}$. Then we have:

$$\Psi^\uparrow(\lambda, x, y; u, v) \leq \Psi^\downarrow(\lambda, y, x; u, v) \text{ and } \Psi^\uparrow(\lambda, x, y; u, v) \leq \Psi^\downarrow(\lambda, x, y; v, u)$$

for each $\lambda \in (0, 1)$.

(iii) Suppose that M satisfies the strict \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. Then we have:

$$\Psi^\uparrow(\lambda, x, y; u, v) \leq \Psi^\downarrow(\lambda, y, x; v, u)$$

for each $\lambda \in (0, 1)$.

Proof. Proposition 13 says that $\Psi^\uparrow(\lambda, x, y; u, v) < +\infty$ for all $\lambda \in (0, 1)$. According to the concept of supremum, given any $\epsilon > 0$, there exists $t_\epsilon > 0$, satisfying $\zeta(x, y; u, v, t_\epsilon) \leq 1 - \lambda$

and $\Psi^\uparrow(\lambda, x, y; u, v) - \epsilon < t_\epsilon$. To prove part (i), parts (i) and (ii) of Proposition 10 say that $t_\epsilon \leq \Psi^\downarrow(\lambda, x, y; u, v)$, which implies $\Psi^\uparrow(\lambda, x, y; u, v) - \epsilon < \Psi^\downarrow(\lambda, x, y; u, v)$. Since ϵ can be any positive real number, we obtain the desired inequality.

To prove part (ii), parts (i) and (ii) of Proposition 10 say that $t_\epsilon \leq \Psi^\downarrow(\lambda, y, x; u, v)$, which implies $\Psi^\uparrow(\lambda, x, y; u, v) - \epsilon < \Psi^\downarrow(\lambda, y, x; u, v)$. Since ϵ can be any positive real number, we obtain $\Psi^\uparrow(\lambda, x, y; u, v) \leq \Psi^\downarrow(\lambda, y, x; u, v)$. Another inequality can be similarly obtained.

To prove part (iii), parts (ii) and (iii) of Proposition 10 say that $t_\epsilon \leq \Psi^\downarrow(\lambda, y, x; v, u)$, which implies $\Psi^\uparrow(\lambda, x, y; u, v) - \epsilon < \Psi^\downarrow(\lambda, y, x; v, u)$. Since ϵ can be any positive real number, we obtain the desired inequality. This completes the proof. \square

Proposition 15. Let (X, M) be a fuzzy semi-metric space along with a t -norm $*$. We also assume that M satisfies the rational condition, and that the t -norm $*$ is right-continuous at 0 with respect to the first or second argument. For any fixed $x, y, u, v \in X$ with $x \neq y$ or $u \neq v$, and any fixed $\lambda \in (0, 1)$, we assume $\Psi^\uparrow(\lambda, x, y; u, v) < +\infty$.

(i) For any $\epsilon > 0$, we have the following inequality:

$$\zeta(x, y; u, v, \Psi^\uparrow(\lambda, x, y; u, v) + \epsilon) > 1 - \lambda \quad (41)$$

(ii) If $\epsilon > 0$ is sufficiently small satisfying $\Psi^\uparrow(\lambda, x, y; u, v) > \epsilon$, then the following statements hold true:

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Then we have:

$$\zeta(x, y; u, v, \Psi^\uparrow(\lambda, x, y; u, v) - \epsilon) \leq 1 - \lambda. \quad (42)$$

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft\}$. Then we have:

$$\zeta(y, x; u, v, \Psi^\uparrow(\lambda, x, y; u, v) - \epsilon) \leq 1 - \lambda \text{ and } \zeta(x, y; v, u, \Psi^\uparrow(\lambda, x, y; u, v) - \epsilon) \leq 1 - \lambda. \quad (43)$$

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. Then we have:

$$\zeta(y, x; v, u, \Psi^\uparrow(\lambda, x, y; u, v) - \epsilon) \leq 1 - \lambda. \quad (44)$$

Proof. To prove part (i), given any $\epsilon > 0$, we assume that $\zeta(x, y, \Psi^\uparrow(\lambda, x, y; u, v) + \epsilon) \leq 1 - \lambda$. The definition of Ψ^\uparrow says that $\Psi^\uparrow(\lambda, x, y; u, v) \geq \Psi^\uparrow(\lambda, x, y; u, v) + \epsilon$. This contradiction shows that $\zeta(x, y; u, v, \Psi^\uparrow(\lambda, x, y; u, v) + \epsilon) > 1 - \lambda$.

To prove part (ii), according to the concept of supremum for $\Psi^\uparrow(\lambda, x, y; u, v)$, given any $\epsilon > 0$ with $\Psi^\uparrow(\lambda, x, y; u, v) > \epsilon$, there exists $t_\epsilon > 0$, satisfying $\zeta(x, y; u, v, t_\epsilon) \leq 1 - \lambda$ and $t_\epsilon > \Psi^\uparrow(\lambda, x, y; u, v) - \epsilon$. Therefore, we consider three cases below:

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Parts (i) and (ii) of Proposition 3 say that the mapping $\zeta(x, y; u, v, \cdot)$ is nondecreasing. Therefore, we have:

$$\zeta(x, y; u, v, \Psi^\uparrow(\lambda, x, y; u, v) - \epsilon) \leq \zeta(x, y; u, v, t_\epsilon) \leq 1 - \lambda.$$

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft\}$. Part (ii) of Proposition 3 says that the mapping $\zeta(x, y; u, v, \cdot)$ is \triangleleft -semisymmetrically nondecreasing. Therefore, we have:

$$\zeta(y, x; u, v, \Psi^\uparrow(\lambda, x, y; u, v) - \epsilon) \leq \zeta(x, y; u, v, t_\epsilon) \leq 1 - \lambda.$$

We can similarly obtain another inequality using the fact of the mapping $\zeta(x, y; u, v, \cdot)$ to be \triangleright -semisymmetrically nondecreasing.

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. Parts (ii) and (iii) of Proposition 3 say that the mapping $\zeta(x, y; u, v, \cdot)$ is symmetrically nondecreasing. Therefore, we have:

$$\zeta(y, x; v, u, \Psi^\uparrow(\lambda, x, y; u, v) - \epsilon) \leq \zeta(x, y; u, v, t_\epsilon) \leq 1 - \lambda.$$

This completes the proof. \square

Proposition 16. Let (X, M) be a fuzzy semi-metric space along with a t -norm $*$. We also assume that M satisfies the rational condition, and that the t -norm $*$ is right-continuous at 0 with respect to the first or second argument. Given any fixed $x, y, u, v \in X$ with $x \neq y$ or $u \neq v$, and any fixed $\lambda \in (0, 1)$, the following statements hold true:

(i) Suppose that $t > \Psi^\uparrow(\lambda, x, y; u, v)$. Then, we have the following properties:

- If M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$, then $\zeta(x, y; u, v, t) > 1 - \lambda$.
- If M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft\}$, then $\zeta(y, x; u, v, t) > 1 - \lambda$ and $\zeta(x, y; v, u, t) > 1 - \lambda$.
- If M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$, then $\zeta(y, x; v, u, t) > 1 - \lambda$.

(ii) We have the following properties:

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. If $0 < t < \Psi^\uparrow(\lambda, x, y; u, v)$, then $\zeta(x, y; u, v, t) \leq 1 - \lambda$.
- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft\}$. If $\Psi^\uparrow(\lambda, y, x; u, v) = +\infty$ or $\Psi^\uparrow(\lambda, x, y; v, u) = +\infty$ or $0 < t < \Psi^\uparrow(\lambda, x, y; u, v) < +\infty$, then $\zeta(x, y; u, v, t) \leq 1 - \lambda$.
- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. If $\Psi^\uparrow(\lambda, y, x; v, u) = +\infty$ or $0 < t < \Psi^\uparrow(\lambda, x, y; u, v) < +\infty$, then $\zeta(x, y; u, v, t) \leq 1 - \lambda$.

Proof. To prove part (i), the fact $t > \Psi^\uparrow(\lambda, x, y; u, v)$ says that there exists $\epsilon > 0$, satisfying $t \geq \Psi^\uparrow(\lambda, x, y; u, v) + \epsilon$. We consider three cases below:

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Parts (i) and (ii) of Proposition 3 say that the mapping $\zeta(x, y; u, v, \cdot)$ is nondecreasing. Therefore, using Equation (41), we obtain:

$$\zeta(x, y; u, v, t) \geq \zeta(x, y; u, v, \Psi^\uparrow(\lambda, x, y; u, v) + \epsilon) > 1 - \lambda.$$

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft\}$. Part (ii) of Proposition 3 says that the mapping $\zeta(x, y; u, v, \cdot)$ is both \triangleleft -semisymmetrically nondecreasing and \triangleright -semisymmetrically nondecreasing. Therefore, using Equation (41), we obtain:

$$\zeta(y, x; u, v, t) \geq \zeta(x, y; u, v, \Psi^\uparrow(\lambda, x, y; u, v) + \epsilon) > 1 - \lambda,$$

and:

$$\zeta(x, y; v, u, t) \geq \zeta(x, y; u, v, \Psi^\uparrow(\lambda, x, y; u, v) + \epsilon) > 1 - \lambda.$$

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. Parts (ii) and (iii) of Proposition 3 say that the mapping $\zeta(x, y; u, v, \cdot)$ is symmetrically nondecreasing. Therefore, using Equation (41), we obtain:

$$\zeta(y, x; v, u, t) \geq \zeta(x, y; u, v, \Psi^\uparrow(\lambda, x, y; u, v) + \epsilon) > 1 - \lambda.$$

To prove part (ii), we consider three cases below:

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Using part (i) of Proposition 12, if $\Psi^\uparrow(\lambda, x, y; u, v) = +\infty$, then it is done. Now, for $\Psi^\uparrow(\lambda, x, y; u, v) < +\infty$, the fact $t < \Psi^\uparrow(\lambda, x, y; u, v)$ says that there exists $\epsilon > 0$, satisfying $0 < t \leq \Psi^\uparrow(\lambda, x, y; u, v) - \epsilon$. Using Equation (42), we obtain:

$$\zeta(x, y; u, v, t) \leq \zeta(x, y; u, v, \Psi^\uparrow(\lambda, x, y; u, v) - \epsilon) \leq 1 - \lambda.$$

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft\}$. Using part (ii) of Proposition 12, if $\Psi^\uparrow(\lambda, y, x; u, v) = +\infty$ or $\Psi^\uparrow(\lambda, x, y; v, u) = +\infty$, then it is done. Now, for $\Psi^\uparrow(\lambda, x, y; u, v) < +\infty$, using Equation (43), we obtain:

$$\zeta(x, y; u, v, t) \leq \zeta(y, x; u, v, \Psi^\uparrow(\lambda, x, y; u, v) - \epsilon) \leq 1 - \lambda,$$

and:

$$\zeta(x, y; u, v, t) \leq \zeta(x, y; v, u, \Psi^\uparrow(\lambda, x, y; u, v) - \epsilon) \leq 1 - \lambda.$$

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. Using part (iii) of Proposition 12, if $\Psi^\uparrow(\lambda, y, x; v, u) = +\infty$, then it is done. Now, for $\Psi^\uparrow(\lambda, x, y; u, v) < +\infty$, using Equation (44), we obtain:

$$\zeta(x, y; u, v, t) \leq \zeta(y, x; v, u, \Psi^\uparrow(\lambda, x, y; u, v) - \epsilon) \leq 1 - \lambda.$$

This completes the proof. \square

Proposition 17. Let (X, M) be a fuzzy semi-metric space along with a t -norm $*$. We also assume that M satisfies the rational condition, and that the t -norm $*$ is right-continuous at 0 with respect to the first or second argument. Given any fixed $x, y, u, v \in X$ with $x \neq y$ or $u \neq v$, and any fixed $\lambda \in (0, 1)$, the following statements hold true:

- Suppose that $\zeta(x, y; u, v, t) \leq 1 - \lambda$ for $t > 0$. Then, we have the following properties:
 - If M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$, then $t \leq \Psi^\uparrow(\lambda, x, y; u, v)$.
 - If M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft\}$, then $t \leq \Psi^\uparrow(\lambda, y, x; u, v)$ and $t \leq \Psi^\uparrow(\lambda, x, y; v, u)$.
 - If M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$, then $t \leq \Psi^\uparrow(\lambda, y, x; v, u)$.
- We have the following properties:
 - Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. If $\zeta(x, y; u, v, t) > 1 - \lambda$, then $\Psi^\uparrow(\lambda, x, y; u, v) < +\infty$ and $t \geq \Psi^\uparrow(\lambda, x, y; u, v)$.
 - Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft\}$. If $\zeta(x, y; u, v, t) > 1 - \lambda$, then $\Psi^\uparrow(\lambda, y, x; u, v) < +\infty$, $\Psi^\uparrow(\lambda, x, y; v, u) < +\infty$ and $\Psi^\uparrow(\lambda, y, x; v, u) < +\infty$.
 - Suppose that M satisfies the \diamond -triangle inequality:
 - If $\zeta(x, y; u, v, t) > 1 - \lambda$, then $\Psi^\uparrow(\lambda, y, x; v, u) < +\infty$.
 - If $\zeta(x, y; u, v, t) > 1 - \lambda$ and $\Psi^\uparrow(\lambda, x, y; u, v) < +\infty$, then $t \geq \Psi^\uparrow(\lambda, x, y; u, v)$.

Proof. To prove part (i), we consider three cases below:

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. It is clear to see that the fact $\Psi^\uparrow(\lambda, x, y; u, v) = +\infty$ implies $t \leq \Psi^\uparrow(\lambda, x, y; u, v)$. Now, for $\Psi^\uparrow(\lambda, x, y; u, v) < +\infty$, using the contraposition of first property of part (i) of Proposition 16, we see that if $\zeta(x, y; u, v, t) \leq 1 - \lambda$, then $t \leq \Psi^\uparrow(\lambda, x, y; u, v)$.
- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft\}$. It is clear to see that the fact $\Psi^\uparrow(\lambda, y, x; u, v) = +\infty$ implies $t \leq \Psi^\uparrow(\lambda, y, x; u, v)$. Now, for $\Psi^\uparrow(\lambda, y, x; u, v) < +\infty$, using the

contraposition of second property of part (i) of Proposition 16, we see that if $\zeta(x, y; u, v, t) \leq 1 - \lambda$, then $t \leq \Psi^\uparrow(\lambda, y, x; u, v)$. We can similarly show that if $\zeta(x, y; u, v, t) \leq 1 - \lambda$, then $t \leq \Psi^\uparrow(\lambda, x, y; v, u)$.

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. It is clear to see that the fact $\Psi^\uparrow(\lambda, y, x; v, u) = +\infty$ implies $t \leq \Psi^\uparrow(\lambda, y, x; v, u)$. Now, for $\Psi^\uparrow(\lambda, y, x; v, u) < +\infty$, using the contraposition of third property of part (i) of Proposition 16, we see that if $\zeta(y, x; v, u, t) \leq 1 - \lambda$, then $t \leq \Psi^\uparrow(\lambda, y, x; v, u)$.

To prove part (ii), we consider three cases below:

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Using the contraposition of part (i) of Proposition 12 and the contraposition of first property of part (ii) of Proposition 16, we can obtain the desired result.
- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft\}$. Using part (ii) of Proposition 12, if $\zeta(x, y; u, v, t) > 1 - \lambda$, then $\Psi^\uparrow(\lambda, y, x; u, v) < +\infty$ and $\Psi^\uparrow(\lambda, x, y; v, u) < +\infty$. Using part (iii) of Proposition 12, if $\zeta(x, y; u, v, t) > 1 - \lambda$, then $\Psi^\uparrow(\lambda, y, x; v, u) < +\infty$.
- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$. Using part (iii) of Proposition 12, if $\zeta(x, y; u, v, t) > 1 - \lambda$, then $\Psi^\uparrow(\lambda, y, x; v, u) < +\infty$. Using the contraposition of third property of part (ii) of Proposition 16, if $\zeta(x, y; u, v, t) > 1 - \lambda$ and $\Psi^\uparrow(\lambda, x, y; u, v) < +\infty$ then $t \geq \Psi^\uparrow(\lambda, x, y; u, v)$.

This completes the proof. \square

Proposition 18. Let (X, M) be a fuzzy semi-metric space along with a t -norm $*$. We also assume that M satisfies the rational condition, and that the t -norm $*$ is right-continuous at 0 with respect to the first or second argument. Given any fixed $x, y, u, v \in X$ with $x \neq y$ or $u \neq v$, and any fixed $\lambda \in (0, 1)$, the following statements hold true:

- (i) Suppose that $\Psi^\uparrow(\lambda, x, y; u, v) < +\infty$, and that the mapping $\zeta(x, y; u, v, \cdot) : (0, \infty) \rightarrow [0, 1]$ is right-continuous on $(0, \infty)$. Then we have:

$$\zeta(x, y; u, v, \Psi^\uparrow(\lambda, x, y; u, v)) \geq 1 - \lambda. \quad (45)$$

- (ii) Suppose that the mapping $\zeta(x, y; u, v, \cdot) : (0, \infty) \rightarrow [0, 1]$ is left-continuous on $(0, \infty)$. Then, the following statements hold true:

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$.

$$\text{If } \Psi^\uparrow(\lambda, x, y; u, v) < +\infty, \text{ then } \zeta(x, y; u, v, \Psi^\uparrow(\lambda, x, y; u, v)) \leq 1 - \lambda.$$

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft\}$.

$$\text{If } \Psi^\uparrow(\lambda, y, x; u, v) < +\infty, \text{ then } \zeta(x, y; u, v, \Psi^\uparrow(\lambda, y, x; u, v)) \leq 1 - \lambda,$$

and:

$$\text{if } \Psi^\uparrow(\lambda, x, y; v, u) < +\infty, \text{ then } \zeta(x, y; u, v, \Psi^\uparrow(\lambda, x, y; v, u)) \leq 1 - \lambda.$$

- Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\triangleright, \triangleleft, \diamond\}$.

$$\text{If } \Psi^\uparrow(\lambda, y, x; v, u) < +\infty, \text{ then } \zeta(x, y; u, v, \Psi^\uparrow(\lambda, y, x; v, u)) \leq 1 - \lambda.$$

- (iii) Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$, and that the mapping $\zeta(x, y; u, v, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous on $(0, \infty)$.

$$\text{If } \Psi^\uparrow(\lambda, x, y; u, v) < +\infty, \text{ then } \zeta\left(x, y; u, v, \Psi^\uparrow(\lambda, x, y; u, v)\right) = 1 - \lambda.$$

Proof. To prove part (i), by taking the limit $\epsilon \rightarrow 0+$ to the inequality (Equation (41)), we obtain Equation (45). To prove part (ii), by taking the limit $\epsilon \rightarrow 0+$ to the inequalities (Equations (42)–(44)), we also obtain the desired results. Part (iii) follows from parts (i) and (ii) immediately. This completes the proof. \square

Theorem 4. (Triangle Inequalities for Dual Double Fuzzy Semi-Metric). Let (X, M) be a fuzzy semi-metric space along with a t -norm $*$. We also assume that M satisfies the rational condition, and that the t -norm $*$ is right-continuous at 0 and left-continuous at 1 with respect to the first or second argument. Given any distinct fixed $x_1, x_2, \dots, x_p, y_1, y_2, \dots, y_p \in X$ and any fixed $\mu \in (0, 1]$, we have the following properties:

- (i) Suppose that M satisfies the \bowtie -triangle inequality. There exists $\lambda \in (0, 1)$, satisfying:

$$\begin{aligned} \Psi^\uparrow(\mu, x_1, x_p; y_1, y_p) &\leq \Psi^\uparrow(\lambda, x_1, x_2; y_1, y_2) + \Psi^\uparrow(\lambda, x_2, x_3; y_2, y_3) + \dots \\ &\quad + \Psi^\uparrow(\lambda, x_{p-2}, x_{p-1}; y_{p-2}, y_{p-1}) + \Psi^\uparrow(\lambda, x_{p-1}, x_p; y_{p-1}, y_p), \end{aligned} \quad (46)$$

$$\begin{aligned} \Psi^\uparrow(\mu, x_1, x_p; y_p, y_1) &\leq \Psi^\uparrow(\lambda, x_1, x_2; y_2, y_1) + \Psi^\uparrow(\lambda, x_2, x_3; y_3, y_2) + \dots \\ &\quad + \Psi^\uparrow(\lambda, x_{p-2}, x_{p-1}; y_{p-1}, y_{p-2}) + \Psi^\uparrow(\lambda, x_{p-1}, x_p; y_p, y_{p-1}), \end{aligned} \quad (47)$$

$$\begin{aligned} \Psi^\uparrow(\mu, x_p, x_1; y_1, y_p) &\leq \Psi^\uparrow(\lambda, x_p, x_{p-1}; y_{p-1}, y_p) + \Psi^\uparrow(\lambda, x_{p-1}, x_{p-2}; y_{p-2}, y_{p-1}) \\ &\quad + \dots + \Psi^\uparrow(\lambda, x_3, x_2; y_2, y_3) + \Psi^\uparrow(\lambda, x_2, x_1; y_1, y_2), \end{aligned}$$

$$\begin{aligned} \Psi^\uparrow(\mu, x_p, x_1; y_p, y_1) &\leq \Psi^\uparrow(\lambda, x_p, x_{p-1}; y_p, y_{p-1}) + \Psi^\uparrow(\lambda, x_{p-1}, x_{p-2}; y_{p-1}, y_{p-2}) \\ &\quad + \dots + \Psi^\uparrow(\lambda, x_3, x_2; y_3, y_2) + \Psi^\uparrow(\lambda, x_2, x_1; y_2, y_1). \end{aligned}$$

- (ii) Suppose that M satisfies the \triangleright -triangle inequality. There exists $\lambda \in (0, 1)$, satisfying:

$$\begin{aligned} &\max \left\{ \Psi^\uparrow(\mu, x_1, x_p; y_1, y_p), \Psi^\uparrow(\mu, x_1, x_p; y_p, y_1), \Psi^\uparrow(\mu, x_p, x_1; y_1, y_p), \Psi^\uparrow(\mu, x_p, x_1; y_p, y_1) \right\} \\ &\leq \Psi^\uparrow(\lambda, x_1, x_2; y_1, y_2) + \Psi^\uparrow(\lambda, x_3, x_2; y_3, y_2) + \Psi^\uparrow(\lambda, x_4, x_3; y_4, y_3) \\ &\quad + \dots + \Psi^\uparrow(\lambda, x_p, x_{p-1}; y_p, y_{p-1}). \end{aligned}$$

- (iii) Suppose that M satisfies the \triangleleft -triangle inequality. There exists $\lambda \in (0, 1)$, satisfying:

$$\begin{aligned} &\max \left\{ \Psi^\uparrow(\mu, x_1, x_p; y_1, y_p), \Psi^\uparrow(\mu, x_1, x_p; y_p, y_1), \Psi^\uparrow(\mu, x_p, x_1; y_1, y_p), \Psi^\uparrow(\mu, x_p, x_1; y_p, y_1) \right\} \\ &\leq \Psi^\uparrow(\lambda, x_2, x_1; y_2, y_1) + \Psi^\uparrow(\lambda, x_2, x_3; y_2, y_3) + \Psi^\uparrow(\lambda, x_3, x_4; y_3, y_4) \\ &\quad + \dots + \Psi^\uparrow(\lambda, x_{p-1}, x_p; y_{p-1}, y_p). \end{aligned}$$

- (iv) Suppose that M satisfies the \diamond -triangle inequality. There exists $\lambda \in (0, 1)$ such that the following inequalities are satisfied:

- If p is even and $\Psi^\uparrow(\mu, x_1, x_p; y_1, y_p) < +\infty$, then:

$$\begin{aligned} \Psi^\uparrow(\mu, x_1, x_p; y_1, y_p) &\leq \Psi^\uparrow(\lambda, x_2, x_1; y_2, y_1) + \Psi^\uparrow(\lambda, x_3, x_2; y_3, y_2) + \Psi^\uparrow(\lambda, x_3, x_4; y_3, y_4) \\ &\quad + \Psi^\uparrow(\lambda, x_5, x_4; y_5, y_4) + \Psi^\uparrow(\lambda, x_5, x_6; y_5, y_6) + \Psi^\uparrow(\lambda, x_7, x_6; y_7, y_6) \\ &\quad + \dots + \Psi^\uparrow(\lambda, x_{p-1}, x_p; y_{p-1}, y_p). \end{aligned} \quad (48)$$

- If p is even and $\Psi^\uparrow(\mu, x_1, x_p; y_p, y_1) < +\infty$, then:

$$\begin{aligned} \Psi^\downarrow(\mu, x_1, x_p; y_p, y_1) &\leq \Psi^\downarrow(\lambda, x_2, x_1; y_1, y_2) + \Psi^\downarrow(\lambda, x_3, x_2; y_2, y_3) + \Psi^\downarrow(\lambda, x_3, x_4; y_4, y_3) \\ &\quad + \Psi^\downarrow(\lambda, x_5, x_4; y_4, y_5) + \Psi^\downarrow(\lambda, x_5, x_6; y_6, y_5) + \Psi^\downarrow(\lambda, x_7, x_6; y_6, y_7) \\ &\quad + \cdots + \Psi^\downarrow(\lambda, x_{p-1}, x_p; y_p, y_{p-1}). \end{aligned} \quad (49)$$

- If p is even and $\Psi^\uparrow(\mu, x_p, x_1; y_1, y_p) < +\infty$, then:

$$\begin{aligned} \Psi^\downarrow(\mu, x_p, x_1; y_1, y_p) &\leq \Psi^\downarrow(\lambda, x_1, x_2; y_2, y_1) + \Psi^\downarrow(\lambda, x_2, x_3; y_3, y_2) + \Psi^\downarrow(\lambda, x_4, x_3; y_3, y_4) \\ &\quad + \Psi^\downarrow(\lambda, x_4, x_5; y_5, y_4) + \Psi^\downarrow(\lambda, x_6, x_5; y_5, y_6) + \Psi^\downarrow(\lambda, x_6, x_7; y_7, y_6) \\ &\quad + \cdots + \Psi^\downarrow(\lambda, x_p, x_{p-1}; y_{p-1}, y_p). \end{aligned} \quad (50)$$

- If p is even and $\Psi^\uparrow(\mu, x_p, x_1; y_p, y_1) < +\infty$, then:

$$\begin{aligned} \Psi^\uparrow(\mu, x_p, x_1; y_p, y_1) &\leq \Psi^\uparrow(\lambda, x_1, x_2; y_1, y_2) + \Psi^\uparrow(\lambda, x_2, x_3; y_2, y_3) + \Psi^\uparrow(\lambda, x_4, x_3; y_4, y_3) \\ &\quad + \Psi^\uparrow(\lambda, x_4, x_5; y_4, y_5) + \Psi^\uparrow(\lambda, x_6, x_5; y_6, y_5) + \Psi^\uparrow(\lambda, x_6, x_7; y_6, y_7) \\ &\quad + \cdots + \Psi^\uparrow(\lambda, x_p, x_{p-1}; y_p, y_{p-1}). \end{aligned} \quad (51)$$

- If p is odd and $\Psi^\uparrow(\mu, x_1, x_p; y_1, y_p) < +\infty$, then:

$$\begin{aligned} \Psi^\uparrow(\mu, x_1, x_p; y_1, y_p) &\leq \Psi^\uparrow(\lambda, x_1, x_2; y_1, y_2) + \Psi^\uparrow(\lambda, x_2, x_3; y_2, y_3) + \Psi^\uparrow(\lambda, x_4, x_3; y_4, y_3) \\ &\quad + \Psi^\uparrow(\lambda, x_4, x_5; y_4, y_5) + \Psi^\uparrow(\lambda, x_6, x_5; y_6, y_5) + \Psi^\uparrow(\lambda, x_6, x_7; y_6, y_7) \\ &\quad + \cdots + \Psi^\uparrow(\lambda, x_p, x_{p-1}; y_p, y_{p-1}). \end{aligned} \quad (52)$$

- If p is odd and $\Psi^\uparrow(\mu, x_1, x_p; y_p, y_1) < +\infty$, then:

$$\begin{aligned} \Psi^\downarrow(\mu, x_1, x_p; y_p, y_1) &\leq \Psi^\downarrow(\lambda, x_1, x_2; y_2, y_1) + \Psi^\downarrow(\lambda, x_2, x_3; y_3, y_2) + \Psi^\downarrow(\lambda, x_4, x_3; y_3, y_4) \\ &\quad + \Psi^\downarrow(\lambda, x_4, x_5; y_5, y_4) + \Psi^\downarrow(\lambda, x_6, x_5; y_5, y_6) + \Psi^\downarrow(\lambda, x_6, x_7; y_7, y_6) \\ &\quad + \cdots + \Psi^\downarrow(\lambda, x_p, x_{p-1}; y_{p-1}, y_p). \end{aligned} \quad (53)$$

- If p is odd and $\Psi^\uparrow(\mu, x_p, x_1; y_1, y_p) < +\infty$, then:

$$\begin{aligned} \Psi^\downarrow(\mu, x_p, x_1; y_1, y_p) &\leq \Psi^\downarrow(\lambda, x_2, x_1; y_1, y_2) + \Psi^\downarrow(\lambda, x_3, x_2; y_2, y_3) + \Psi^\downarrow(\lambda, x_3, x_4; y_4, y_3) \\ &\quad + \Psi^\downarrow(\lambda, x_5, x_4; y_4, y_5) + \Psi^\downarrow(\lambda, x_5, x_6; y_6, y_5) + \Psi^\downarrow(\lambda, x_7, x_6; y_6, y_7) \\ &\quad + \cdots + \Psi^\downarrow(\lambda, x_{p-1}, x_p; y_p, y_{p-1}). \end{aligned} \quad (54)$$

- If p is odd and $\Psi^\uparrow(\mu, x_p, x_1; y_p, y_1) < +\infty$, then:

$$\begin{aligned} \Psi^\uparrow(\mu, x_p, x_1; y_p, y_1) &\leq \Psi^\uparrow(\lambda, x_2, x_1; y_2, y_1) + \Psi^\uparrow(\lambda, x_3, x_2; y_3, y_2) + \Psi^\uparrow(\lambda, x_3, x_4; y_3, y_4) \\ &\quad + \Psi^\uparrow(\lambda, x_5, x_4; y_5, y_4) + \Psi^\uparrow(\lambda, x_5, x_6; y_5, y_6) + \Psi^\uparrow(\lambda, x_7, x_6; y_7, y_6) \\ &\quad + \cdots + \Psi^\uparrow(\lambda, x_{p-1}, x_p; y_{p-1}, y_p). \end{aligned} \quad (55)$$

Proof. Lemma 1 says that there exists $\lambda \in (0, 1)$, satisfying:

$$(1 - \lambda) * \cdots * (1 - \lambda) > 1 - \mu. \quad (56)$$

To prove part (i), we assume that $\Psi^\uparrow(\lambda, x_i, x_{i+1}; y_i, y_{i+1}) < +\infty$ for all $i = 1, \dots, p-1$. Given any $\epsilon > 0$, the first observation of Remark 1 says that:

$$\begin{aligned} M(x_1, x_p, \Psi^\uparrow(\lambda, x_1, x_2; y_1, y_2) + \Psi^\uparrow(\lambda, x_2, x_3; y_2, y_3) + \dots + \Psi^\uparrow(\lambda, x_{p-1}, x_p; y_{p-1}, y_p) + (p-1)\epsilon) \\ \geq M(x_1, x_2, \Psi^\uparrow(\lambda, x_1, x_2; y_1, y_2) + \epsilon) * \dots * M(x_{p-1}, x_p, \Psi^\uparrow(\lambda, x_{p-1}, x_p; y_{p-1}, y_p) + \epsilon), \end{aligned} \quad (57)$$

and:

$$\begin{aligned} M(y_1, y_p, \Psi^\uparrow(\lambda, x_1, x_2; y_1, y_2) + \Psi^\uparrow(\lambda, x_2, x_3; y_2, y_3) + \dots + \Psi^\uparrow(\lambda, x_{p-1}, x_p; y_{p-1}, y_p) + (p-1)\epsilon) \\ \geq M(y_1, y_2, \Psi^\uparrow(\lambda, x_1, x_2; y_1, y_2) + \epsilon) * \dots * M(y_{p-1}, y_p, \Psi^\uparrow(\lambda, x_{p-1}, x_p; y_{p-1}, y_p) + \epsilon). \end{aligned} \quad (58)$$

Now, applying the increasing property and commutativity of t-norm to Equations (57) and (58), we obtain:

$$\begin{aligned} \zeta(x_1, x_p, y_1, y_p, \Psi^\uparrow(\lambda, x_1, x_2; y_1, y_2) + \Psi^\uparrow(\lambda, x_2, x_3; y_2, y_3) + \dots + \Psi^\uparrow(\lambda, x_{p-1}, x_p; y_{p-1}, y_p) + (p-1)\epsilon) \\ \geq \zeta(x_1, x_2, y_1, y_2, \Psi^\uparrow(\lambda, x_1, x_2; y_1, y_2) + \epsilon) * \dots * \zeta(x_{p-1}, x_p, y_{p-1}, y_p, \Psi^\uparrow(\lambda, x_{p-1}, x_p; y_{p-1}, y_p) + \epsilon) \\ \geq (1-\lambda) * \dots * (1-\lambda) \text{ (by Equation (41) and the increasing property of t-norm)} \\ > 1-\mu \text{ (by Equation (56)).} \end{aligned} \quad (59)$$

Therefore, we consider the following cases:

- Suppose that $\Psi^\uparrow(\mu, x_1, x_p; y_1, y_p) = +\infty$. We want to show that there exists i_0 , satisfying $\Psi^\uparrow(\lambda, x_{i_0}, x_{i_0+1}; y_{i_0}, y_{i_0+1}) = +\infty$. Assume that $\Psi^\uparrow(\lambda, x_i, x_{i+1}; y_i, y_{i+1}) < +\infty$ for all $i = 1, \dots, p-1$. Using Equation (59) and part (ii) of Proposition 17, it follows that $\Psi^\uparrow(\mu, x_1, x_p; y_1, y_p) < +\infty$. This contradiction says that there exists i_0 , satisfying $\Psi^\uparrow(\lambda, x_{i_0}, x_{i_0+1}; y_{i_0}, y_{i_0+1}) = +\infty$. In this case, the inequality (Equation (46)) holds true.
- Suppose that $\Psi^\uparrow(\mu, x_1, x_p; y_1, y_p) < +\infty$. We also consider the following cases:
 - If there exists i_0 , satisfying $\Psi^\uparrow(\lambda, x_{i_0}, x_{i_0+1}; y_{i_0}, y_{i_0+1}) = +\infty$, then the inequality (Equation (46)) also holds true.
 - We assume that $\Psi^\uparrow(\lambda, x_i, x_{i+1}; y_i, y_{i+1}) < +\infty$ for all $i = 1, \dots, p-1$. Using Equation (59) and part (ii) of Proposition 17 again, it follows that:

$$\begin{aligned} \Psi^\uparrow(\lambda, x_1, x_2; y_1, y_2) + \Psi^\uparrow(\lambda, x_2, x_3; y_2, y_3) + \dots + \Psi^\uparrow(\lambda, x_{p-1}, x_p; y_{p-1}, y_p) + (p-1)\epsilon \\ \geq \Psi^\uparrow(\mu, x_1, x_p; y_1, y_p). \end{aligned}$$

By taking the limit $\epsilon \rightarrow 0+$, we obtain the desired inequality (Equation (46)).

On the other hand, we also have:

$$\begin{aligned} M(x_1, x_p, \Psi^\uparrow(\lambda, x_1, x_2; y_2, y_1) + \Psi^\uparrow(\lambda, x_2, x_3; y_3, y_2) + \dots + \Psi^\uparrow(\lambda, x_{p-1}, x_p; y_p, y_{p-1}) + (p-1)\epsilon) \\ \geq M(x_1, x_2, \Psi^\uparrow(\lambda, x_1, x_2; y_2, y_1) + \epsilon) * \dots * M(x_{p-1}, x_p, \Psi^\uparrow(\lambda, x_{p-1}, x_p; y_p, y_{p-1}) + \epsilon), \end{aligned} \quad (60)$$

and:

$$\begin{aligned} M(y_p, y_1, \Psi^\uparrow(\lambda, x_1, x_2; y_2, y_1) + \Psi^\uparrow(\lambda, x_2, x_3; y_3, y_2) + \dots + \Psi^\uparrow(\lambda, x_{p-1}, x_p; y_p, y_{p-1}) + (p-1)\epsilon) \\ \geq M(y_2, y_1, \Psi^\uparrow(\lambda, x_1, x_2; y_2, y_1) + \epsilon) * \dots * M(y_p, y_{p-1}, \Psi^\uparrow(\lambda, x_{p-1}, x_p; y_p, y_{p-1}) + \epsilon). \end{aligned} \quad (61)$$

Now, applying the increasing property and commutativity of t-norm to Equations (60) and (61), we obtain:

$$\begin{aligned} & \zeta \left(x_1, x_p; y_p, y_1, \Psi^\uparrow(\lambda, x_1, x_2; y_2, y_1) + \Psi^\uparrow(\lambda, x_2, x_3; y_3, y_2) + \cdots + \Psi^\uparrow(\lambda, x_{p-1}, x_p; y_p, y_{p-1}) + (p-1)\epsilon \right) \\ & \geq \zeta \left(x_1, x_2; y_2, y_1, \Psi^\uparrow(\lambda, x_1, x_2; y_2, y_1) + \epsilon \right) * \cdots * \zeta \left(x_{p-1}, x_p; y_p, y_{p-1}, \Psi^\uparrow(\lambda, x_{p-1}, x_p; y_p, y_{p-1}) + \epsilon \right) \\ & \geq (1-\lambda) * \cdots * (1-\lambda) \text{ (by Equation (41) and the increasing property of t-norm)} \\ & > 1-\mu \text{ (by Equation (56)).} \end{aligned}$$

The inequality (Equation (47)) can be similarly obtained using the above argument. Further, the other inequalities can be similarly obtained.

The above argument is still valid by applying the second observation of Remark 1 to obtain part (ii). We can also apply the third observation of Remark 1 to obtain part (iii). Finally, part (iv) can be obtained using the fourth observation of Remark 1. This completes the proof. \square

Theorem 5. Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$. We also assume that M satisfies the rational condition, and that the t-norm $*$ is right-continuous at 0 with respect to the first or second argument. Let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be two sequences in X . Then we have the following properties:

(i) Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Then the following statements hold true:

- $x_n \xrightarrow{M^\triangleright} x$ and $y_n \xrightarrow{M^\triangleright} y$ as $n \rightarrow \infty$ if and only if $\Psi^\uparrow(\lambda, x_n, x; y_n, y) \rightarrow 0$ as $n \rightarrow \infty$ for all $\lambda \in (0, 1)$.
- $x_n \xrightarrow{M^\triangleright} x$ and $y_n \xrightarrow{M^\triangleleft} y$ as $n \rightarrow \infty$ if and only if $\Psi^\uparrow(\lambda, x_n, x; y, y_n) \rightarrow 0$ as $n \rightarrow \infty$ for all $\lambda \in (0, 1)$.
- $x_n \xrightarrow{M^\triangleleft} x$ and $y_n \xrightarrow{M^\triangleright} y$ as $n \rightarrow \infty$ if and only if $\Psi^\uparrow(\lambda, x, x_n; y_n, y) \rightarrow 0$ as $n \rightarrow \infty$ for all $\lambda \in (0, 1)$.
- $x_n \xrightarrow{M^\triangleleft} x$ and $y_n \xrightarrow{M^\triangleleft} y$ as $n \rightarrow \infty$ if and only if $\Psi^\uparrow(\lambda, x, x_n; y, y_n) \rightarrow 0$ as $n \rightarrow \infty$ for all $\lambda \in (0, 1)$.

(ii) Suppose that M satisfies the \diamond -triangle inequality. Then the following statements hold true:

- If $x_n \xrightarrow{M^\triangleright} x$ and $y_n \xrightarrow{M^\triangleright} y$ as $n \rightarrow \infty$, given any fixed $\lambda \in (0, 1)$, we have $\Psi^\uparrow(\lambda, x_n, x; y_n, y) < +\infty$ for all $n \in \mathbb{N}$ imply $\Psi^\uparrow(\lambda, x_n, x; y_n, y) \rightarrow 0$ as $n \rightarrow \infty$.
- If $x_n \xrightarrow{M^\triangleright} x$ and $y_n \xrightarrow{M^\triangleleft} y$ as $n \rightarrow \infty$, given any fixed $\lambda \in (0, 1)$, we have $\Psi^\uparrow(\lambda, x_n, x; y, y_n) < +\infty$ for all $n \in \mathbb{N}$ imply $\Psi^\uparrow(\lambda, x_n, x; y, y_n) \rightarrow 0$ as $n \rightarrow \infty$.
- If $x_n \xrightarrow{M^\triangleleft} x$ and $y_n \xrightarrow{M^\triangleright} y$ as $n \rightarrow \infty$, given any fixed $\lambda \in (0, 1)$, we have $\Psi^\uparrow(\lambda, x, x_n; y_n, y) < +\infty$ for all $n \in \mathbb{N}$ imply $\Psi^\uparrow(\lambda, x, x_n; y_n, y) \rightarrow 0$ as $n \rightarrow \infty$.
- If $x_n \xrightarrow{M^\triangleleft} x$ and $y_n \xrightarrow{M^\triangleleft} y$ as $n \rightarrow \infty$, given any fixed $\lambda \in (0, 1)$, we have $\Psi^\uparrow(\lambda, x, x_n; y, y_n) < +\infty$ for all $n \in \mathbb{N}$ imply $\Psi^\uparrow(\lambda, x, x_n; y, y_n) \rightarrow 0$ as $n \rightarrow \infty$.
- If $\Psi^\uparrow(\lambda, x_n, x; y_n, y) \rightarrow 0$ as $n \rightarrow \infty$ for all $\lambda \in (0, 1)$, then $x_n \xrightarrow{M^\triangleleft} x$ and $y_n \xrightarrow{M^\triangleleft} y$ as $n \rightarrow \infty$.
- If $\Psi^\uparrow(\lambda, x, x_n; y_n, y) \rightarrow 0$ as $n \rightarrow \infty$ for all $\lambda \in (0, 1)$, then $x_n \xrightarrow{M^\triangleright} x$ and $y_n \xrightarrow{M^\triangleleft} y$ as $n \rightarrow \infty$.
- If $\Psi^\uparrow(\lambda, x_n, x; y, y_n) \rightarrow 0$ as $n \rightarrow \infty$ for all $\lambda \in (0, 1)$, then $x_n \xrightarrow{M^\triangleleft} x$ and $y_n \xrightarrow{M^\triangleright} y$ as $n \rightarrow \infty$.
- If $\Psi^\uparrow(\lambda, x, x_n; y, y_n) \rightarrow 0$ as $n \rightarrow \infty$ for all $\lambda \in (0, 1)$, then $x_n \xrightarrow{M^\triangleright} x$ and $y_n \xrightarrow{M^\triangleright} y$ as $n \rightarrow \infty$.

Proof. For any fixed $\lambda \in (0, 1)$, using Lemma 1, it follows that there exists $\lambda_0 \in (0, 1)$, satisfying:

$$(1 - \lambda_0) * (1 - \lambda_0) > 1 - \lambda.$$

To prove part (i), we just prove the first case, since the other cases can be similarly obtained. Suppose that $M(x_n, x, t) \rightarrow 1$ and $M(y_n, y, t) \rightarrow 1$ as $n \rightarrow \infty$ for all $t > 0$. Then, given any $t > 0$ and $\delta > 0$, there exists $n_{t,\delta}^{(1)}, n_{t,\delta}^{(2)} \in \mathbb{N}$, satisfying $|M(x_n, x, t) - 1| < \delta$ for $n \geq n_{t,\delta}^{(1)}$ and $|M(y_n, y, t) - 1| < \delta$ for $n \geq n_{t,\delta}^{(2)}$. Given any $\epsilon \in (0, 1)$, there exists $n_\epsilon \in \mathbb{N}$, satisfying

$$\left| M\left(x_n, x, \frac{\epsilon}{2}\right) - 1 \right| < \lambda_0 \text{ and } \left| M\left(y_n, y, \frac{\epsilon}{2}\right) - 1 \right| < \lambda_0,$$

for $n \geq n_\epsilon$. We also have:

$$M\left(x_n, x, \frac{\epsilon}{2}\right) > 1 - \lambda_0 \text{ and } M\left(y_n, y, \frac{\epsilon}{2}\right) > 1 - \lambda_0,$$

for $n \geq n_\epsilon$. The increasing property of t-norm says that:

$$\zeta\left(x_n, x; y_n, y, \frac{\epsilon}{2}\right) = M\left(x_n, x, \frac{\epsilon}{2}\right) * M\left(y_n, y, \frac{\epsilon}{2}\right) \geq (1 - \lambda_0) * (1 - \lambda_0) > 1 - \lambda.$$

The first result of part (ii) of Proposition 17 says that:

$$\Psi^\uparrow(\lambda, x_n, x; y_n, y) \leq \frac{\epsilon}{2} < \epsilon,$$

for $n \geq n_\epsilon$. This shows that $\Psi^\uparrow(\lambda, x_n, x; y_n, y) \rightarrow 0$ as $n \rightarrow \infty$.

To prove the converse, suppose that $\Psi^\uparrow(\lambda, x_n, x; y_n, y) \rightarrow 0$ as $n \rightarrow \infty$ for all $\lambda \in (0, 1)$. Given any $\delta > 0$ and $\lambda \in (0, 1]$, there exists $n_{\delta,\lambda} \in \mathbb{N}$, satisfying $|\Psi^\uparrow(\lambda, x_n, x; y_n, y)| < \delta$ for all $n \geq n_{\delta,\lambda}$. For any fixed $t > 0$ and given any $\epsilon \in (0, 1)$, there exists $n_\epsilon \in \mathbb{N}$, satisfying:

$$\Psi^\uparrow(\epsilon, x_n, x; y_n, y) = \left| \Psi^\uparrow(\epsilon, x_n, x; y_n, y) \right| < t, \quad (62)$$

for $n \geq n_\epsilon$, which implies:

$$\zeta(x_n, x; y_n, y, t) > 1 - \epsilon,$$

for $n \geq n_\epsilon$ by the first result of part (i) of Proposition 16, i.e.,:

$$M(x_n, x, t) * M(y_n, y, t) > 1 - \epsilon,$$

for $n \geq n_\epsilon$. Lemma 2 says that:

$$M(x_n, x, t) > 1 - \epsilon \text{ and } M(y_n, y, t) > 1 - \epsilon,$$

for $n \geq n_\epsilon$. This shows that the sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ in X converge to x and y , respectively.

To prove part (ii), the first result to the fourth result can be similarly obtained by the third result of part (ii) of Proposition 17. For proving the fifth result, the fact $\Psi^\uparrow(\lambda, x_n, x; y_n, y) \rightarrow 0$ implies the inequality (Equation (62)). The third result of part (i) of Proposition 16 says that $\zeta(x, x_n; y, y_n, t) > 1 - \epsilon$, which implies $M(x, x_n, x, t) > 1 - \epsilon$ and $M(y, y_n, y, t) > 1 - \epsilon$. In other words, we have $x_n \xrightarrow{M^\triangleleft} x$ and $y_n \xrightarrow{M^\triangleleft} y$ as $n \rightarrow \infty$. The remaining three results can be similarly obtained. This completes the proof. \square

Example 7. From Example 1, we see that:

$$x_n \xrightarrow{M^\triangleright} x \text{ if and only if } \lim_{n \rightarrow \infty} d(x_n, x) = 0,$$

and:

$$x_n \xrightarrow{M^\triangleleft} x \text{ if and only if } \lim_{n \rightarrow \infty} d(x, x_n) = 0.$$

From Example 6, we have:

$$\Psi^\uparrow(\lambda, x, y; u, v) = \frac{C + \sqrt{C^2 + D}}{2},$$

where:

$$C = \frac{(d(x, y) + d(u, v))(1 - \lambda)}{\lambda} \text{ and } D = \frac{d(x, y) \cdot d(u, v) \cdot (1 - \lambda)}{\lambda}.$$

It is clear to see that $x_n \xrightarrow{M^p} x$ and $y_n \xrightarrow{M^p} y$ as $n \rightarrow \infty$ if and only if $\Psi^\uparrow(\lambda, x_n, x; y_n, y) \rightarrow 0$ as $n \rightarrow \infty$ for all $\lambda \in (0, 1)$. The other convergence presented in Theorem 5 can be similarly verified.

According to Definition 5, we can similarly define the concepts of four types of the joint Cauchy sequences with respect to Ψ^\uparrow . We omit the details.

Theorem 6. Let (X, M) be a fuzzy semi-metric space along with a t -norm $*$. We also assume that M satisfies the rational condition, and that the t -norm $*$ is right-continuous at 0 with respect to the first or second argument. Let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be two sequences in X . Then we have the following properties:

(i) Suppose that M satisfies the \circ -triangle inequality for $\circ \in \{\bowtie, \triangleright, \triangleleft\}$. Then the following statements hold true:

- $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are two $>$ -Cauchy sequences in a metric sense if and only if $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are the joint $(\lambda, >, >)$ -Cauchy sequences with respect to Ψ^\uparrow for any $\lambda \in (0, 1)$.
- $\{x_n\}_{n=1}^\infty$ is a $>$ -Cauchy sequences in a metric sense and $\{y_n\}_{n=1}^\infty$ is a $<$ -Cauchy sequences in a metric sense if and only if $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are the joint $(\lambda, >, <)$ -Cauchy sequences with respect to Ψ^\uparrow for any $\lambda \in (0, 1)$.
- $\{x_n\}_{n=1}^\infty$ is a $<$ -Cauchy sequences in a metric sense and $\{y_n\}_{n=1}^\infty$ is a $>$ -Cauchy sequences in a metric sense if and only if $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are the joint $(\lambda, <, >)$ -Cauchy sequences with respect to Ψ^\uparrow for any $\lambda \in (0, 1)$.
- $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are two $<$ -Cauchy sequences in a metric sense if and only if $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are the joint $(\lambda, <, <)$ -Cauchy sequences with respect to Ψ^\uparrow for any $\lambda \in (0, 1)$.

(ii) Suppose that M satisfies the \diamond -triangle inequality. Then the following statements hold true:

- Let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be two $>$ -Cauchy sequences in a metric sense. Given any fixed $\lambda \in (0, 1)$, if $\Psi^\uparrow(\lambda, x_m, x_n; y_m, y_n) < +\infty$ for all $m > n$, then $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are the joint $(\lambda, >, >)$ -Cauchy sequences with respect to Ψ^\uparrow for any $\lambda \in (0, 1)$.
- Let $\{x_n\}_{n=1}^\infty$ be a $>$ -Cauchy sequence in a metric sense and let $\{y_n\}_{n=1}^\infty$ be a $<$ -Cauchy sequence in a metric sense. Given any fixed $\lambda \in (0, 1)$, if $\Psi^\uparrow(\lambda, x_m, x_n; y_m, y_n) < +\infty$ for all $m > n$, then $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are the joint $(\lambda, >, <)$ -Cauchy sequences with respect to Ψ^\uparrow for any $\lambda \in (0, 1)$.
- Let $\{x_n\}_{n=1}^\infty$ be a $<$ -Cauchy sequence in a metric sense and let $\{y_n\}_{n=1}^\infty$ be a $>$ -Cauchy sequence in a metric sense. Given any fixed $\lambda \in (0, 1)$, if $\Psi^\uparrow(\lambda, x_m, x_n; y_m, y_n) < +\infty$ for all $m > n$, then $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are the joint $(\lambda, <, >)$ -Cauchy sequences with respect to Ψ^\uparrow for any $\lambda \in (0, 1)$.
- Let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be two $<$ -Cauchy sequences in a metric sense. Given any fixed $\lambda \in (0, 1)$, if $\Psi^\uparrow(\lambda, x_m, x_n; y_m, y_n) < +\infty$ for all $m > n$, then $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are the joint $(\lambda, <, <)$ -Cauchy sequences with respect to Ψ^\uparrow for any $\lambda \in (0, 1)$.
- Suppose that $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are the joint $(\lambda, >, >)$ -Cauchy sequences with respect to Ψ^\uparrow for any $\lambda \in (0, 1)$. Then $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are two $>$ -Cauchy sequences in a metric sense.
- Suppose that $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are the joint $(\lambda, >, <)$ -Cauchy sequences with respect to Ψ^\uparrow for any $\lambda \in (0, 1)$. Then $\{x_n\}_{n=1}^\infty$ is a $<$ -Cauchy sequences in a metric sense and $\{y_n\}_{n=1}^\infty$ is a $<$ -Cauchy sequences in a metric sense.

- Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are the joint $(\lambda, <, >)$ -Cauchy sequences with respect to Ψ^{\dagger} for any $\lambda \in (0, 1)$. Then $\{x_n\}_{n=1}^{\infty}$ is a $<$ -Cauchy sequences in a metric sense and $\{y_n\}_{n=1}^{\infty}$ is a $>$ -Cauchy sequences in a metric sense.
- Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are the joint $(\lambda, <, <)$ -Cauchy sequences with respect to Ψ^{\dagger} for any $\lambda \in (0, 1)$. Then $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are two $<$ -Cauchy sequences in a metric sense.

Proof. For any fixed $\lambda \in (0, 1)$, using Lemma 1, it follows that there exists $\lambda_0 \in (0, 1)$, satisfying:

$$(1 - \lambda_0) * (1 - \lambda_0) > 1 - \lambda.$$

To prove part (i), we just prove the first case, since the other cases can be similarly obtained. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are $>$ -Cauchy sequences in a metric sense. Therefore, given any $t > 0$ and $\delta > 0$, there exists $n_{t,\delta} \in \mathbb{N}$ such that $m > n \geq n_{t,\delta}$ implies $M(x_m, x_n, t) > 1 - \delta$ and $M(y_m, y_n, t) > 1 - \delta$. Now, given any $\epsilon \in (0, 1)$, there exists $n_{\epsilon} \in \mathbb{N}$ such that $m > n \geq n_{\epsilon}$ implies:

$$M\left(x_m, x_n, \frac{\epsilon}{2}\right) > 1 - \lambda_0 \text{ and } M\left(y_m, y_n, \frac{\epsilon}{2}\right) > 1 - \lambda_0.$$

The increasing property of t-norm says that:

$$\begin{aligned} \zeta\left(x_m, x_n; y_m, y_n, \frac{\epsilon}{2}\right) &= M\left(x_m, x_n, \frac{\epsilon}{2}\right) * M\left(y_m, y_n, \frac{\epsilon}{2}\right) \\ &\geq (1 - \lambda_0) * (1 - \lambda_0) > 1 - \lambda. \end{aligned}$$

Further, the first result of part (ii) of Proposition 17 says that:

$$\Psi^{\dagger}(\lambda, x_m, x_n; y_m, y_n) \leq \frac{\epsilon}{2} < \epsilon,$$

for $m > n \geq n_{\epsilon}$.

To prove the converse, from the assumption, we see that for any fixed $t > 0$ and given any $\epsilon \in (0, 1)$, there exists $n_{\epsilon} \in \mathbb{N}$ such that $m > n \geq n_{\epsilon}$ implies $\Psi^{\dagger}(\epsilon, x_m, x_n; y_m, y_n) < t$. Therefore, using the first result of part (i) of Proposition 16, we obtain $\zeta(x_m, x_n; y_m, y_n, t) > 1 - \epsilon$ for $m > n \geq n_{\epsilon}$, i.e.,:

$$M(x_m, x_n, t) * M(y_m, y_n, t) > 1 - \epsilon,$$

for $m > n \geq n_{\epsilon}$. Lemma 2 says that:

$$M(x_m, x_n, t) > 1 - \epsilon \text{ and } M(y_m, y_n, t) > 1 - \epsilon,$$

for $m > n \geq n_{\epsilon}$. This shows that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are $>$ -Cauchy sequences in a metric sense.

To prove part (ii), the first result to the fourth result can be similarly obtained by the third result of part (ii) of Proposition 17. For proving the fifth result, using the assumption, we see that for any fixed $t > 0$ and given any $\epsilon \in (0, 1)$, there exists $n_{\epsilon} \in \mathbb{N}$ such that $m > n \geq n_{\epsilon}$ implies $\Psi^{\dagger}(\epsilon, x_m, x_n; y_m, y_n) < t$. The third result of part (i) of Proposition 16 says that $\zeta(x_n, x_m; y_n, y_m, t) > 1 - \epsilon$ for $m > n \geq n_{\epsilon}$. Therefore, we obtain:

$$M(x_n, x_m, t) * M(y_n, y_m, t) > 1 - \epsilon,$$

for $m > n \geq n_{\epsilon}$. This shows that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are $<$ -Cauchy sequences in a metric sense. The remaining three results can be similarly obtained. This completes the proof. \square

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