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Abstract

Functionals of Harmonics Functions †

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Let $\mathcal{C}_T^s$, with $T > 0$, be the set of continuous, T-periodic functions $\mathbf{f}:\mathbb{R}\to\mathbb{R}^s$, and let $\Gamma:\mathcal{C}_T^s\to\mathbb{R}$ be a real functional on $\mathcal{C}_T^s$. If Γ is n times Fréchet differentiable on $\mathcal{C}_T^s$, then it has an n-th order Taylor expansion around $\mathbf{0}$ (see e.g., [1]). Such a Taylor expansion can be obtained as the n-th order truncation of the series

$$\begin{aligned}\Gamma[\mathbf{f}] = \sum_{n_1=0}^{\infty}\cdots\sum_{n_s=0}^{\infty} &\langle c_{\mathbf{n}}(t_{11},\ldots,t_{1n_1},\ldots,t_{s1},\ldots,t_{sn_s}) \\ &\times f_1(t_{11})\cdots f_1(t_{1n_1})\cdots f_s(t_{s1})\cdots f_s(t_{sn_s})\rangle,\end{aligned} \tag{1}$$

where $\mathbf{n}=(n_1,\ldots,n_s)$ and we have introduced the notation

$$\langle\Omega(t_1,\ldots,t_r)\rangle = \frac{1}{T^r}\int_0^T dt_1\cdots\int_0^T dt_r\,\Omega(t_1,\ldots,t_r). \tag{2}$$

The kernels $c_{n_1,\ldots,n_s}(t_{11},\ldots,t_{sn_s})$ are all real, T-periodic, and symmetric in all their arguments. In this contribution we will prove the following theorem.

Theorem 1. *Let Γ be a functional with Taylor series (1), and take*

$$\mathbf{f}(t) = \big(\epsilon_1\cos(q_1\omega t+\phi_1),\ldots,\epsilon_s\cos(q_s\omega t+\phi_s)\big), \tag{3}$$

where $\mathbf{q}\equiv(q_1,\ldots,q_s)\in\mathbb{N}^s$ is such that $\gcd(q_1,\ldots,q_s)=1$ and $\omega=2\pi/T$. Then,

$$\Gamma[\mathbf{f}] = C_{\mathbf{0}}(\boldsymbol{\epsilon}) + \sum_{\mathbf{x}\in S_+}\epsilon_1^{|x_1|}\cdots\epsilon_s^{|x_s|}C_{\mathbf{x}}(\boldsymbol{\epsilon})\cos\big(\mathbf{x}\cdot\boldsymbol{\phi}+\theta_{\mathbf{x}}(\boldsymbol{\epsilon})\big), \tag{4}$$

where, $\boldsymbol{\phi}\equiv(\phi_1,\ldots,\phi_s)$, $\boldsymbol{\epsilon}\equiv(\epsilon_1,\ldots,\epsilon_s)$, and functions $C_{\mathbf{x}}(\boldsymbol{\epsilon})$ and $\theta_{\mathbf{x}}(\boldsymbol{\epsilon})$ do not depend on $\boldsymbol{\phi}$ and are even in each ϵ_i, $i=1,\ldots,s$, for every $\mathbf{x}\in S_+$. $\mathbf{x}\in S_+$ is the set of vectors $\mathbf{x}$ whose leftmost nonzero component is positive.

In the special case when Γ is invariant under time-shift, i.e., $\Gamma[\mathbf{f}(t+\tau)]=\Gamma[\mathbf{f}(t)]$ for all $0<\tau<T$, we recover the results in [2]

$$\Gamma[\mathbf{f}] = C_{\mathbf{0}}(\boldsymbol{\epsilon}) + \sum_{\mathbf{x}\in\mathcal{D}_+}\epsilon_1^{|x_1|}\cdots\epsilon_s^{|x_s|}C_{\mathbf{x}}(\boldsymbol{\epsilon})\cos\big(\mathbf{x}\cdot\boldsymbol{\phi}+\theta_{\mathbf{x}}(\boldsymbol{\epsilon})\big), \tag{5}$$

where $\mathcal{D}_+$ denote the set of nonzero solutions of the Diophantine equation $\mathbf{q} \cdot \mathbf{x} = q_1 x_1 + \cdots + q_s x_s = 0$, whose leftmost nonzero component is positive.

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