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Implementation of POVMs by Projective Measurements and Postselection: Optimal Strategies and Applications to Unambiguous State Discrimination †

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Abstract: We present new results concerning simulation of general quantum measurements (POVMs) by projective measurements (PMs) for the task of Unambiguous State Discrimination (USD). We formulate a problem of finding optimal strategy of simulation for given quantum measurement. The problem can be solved for qubit and qutrits measurements by Semi-Definite Programming (SDP) methods.

Keywords: POVM; quantum measurements; simulability of POVMs; simulation of quantum measurements; USD; Unambiguous State Discrimination; SDP

1. Introduction and preliminaries

In the recent work [1] it was proved that arbitrary quantum measurement (POVM) can be simulated by projective measurements if one allows for standard classical operations (randomization and post-processing), *followed by postselection*. The concrete algorithm of simulation was presented, with probability of success equal to the $\frac{1}{d}$ for d-dimensional quantum system. It has been studied what advantage POVMs offer over PMs for the task of Unambiguous State Discrimination (USD).

In this work we present new result concerning the applications of general scheme of simulation for the task of USD. First, we show what is the best possible projective-simulable measurement for USD *without postselection*. Then we present the strategy of simulation which in the task of USD performs better than $\frac{1}{d}$ bound from [1]. Finally, we formulate a problem of finding optimal strategy of simulation *with postselection* for a given quantum measurement. The problem can be numerically solved for qubit and qutrit measurements via SDP techniques. We provide exemplary solutions for two extremal qubit POVMs.

We will now introduce necessary mathematical concepts. Every quantum measurement can be associated with a vector $\mathbf{M}$ of positive-semidefinite operators that sum up to identity. Such a vector is called POVM (Positive-Operator Valued Measure), and the constituting operators, denoted by M_i, are called effects. To avoid confusion, we stress that we use boldface font to denote the whole POVM $\mathbf{M}$ ($\mathbf{P}$ for projective) and standard italic font for particular effects of $\mathbf{M}$, i.e., M_i. In the discussion in Section 2 we use upper indices $\mathbf{P}^{(\alpha)}$ to denote *different measurements*, not effects. If a measurement is performed on the quantum system in state ρ, the probability of obtaining outcome labeled by i is given by Born's rule—$p\left(i|\mathbf{M},\rho\right) = \mathrm{tr}\left(M_i\rho\right)$.

In [2] the notion of simulability of POVMs by PMs was introduced—the POVM $\mathbf{M}$ is said to be PM-simulable if sampling from statistics that it would generate for arbitrary quantum state can be achieved by classical randomization of some projective measurements $\left\{\mathbf{P}^{(\alpha)}\right\}$ (not necessarily on the same space as $\mathbf{M}$), followed by classical post-processing. We will denote set of n-outcome PM-simulable measurements on space of dimension d as $\mathbb{SP}\left(n,d\right)$. In recent work [1] the notion of PM-simulability was extended by the simulation of the POVM using projective measurements *and postselection*. To simulate n-outcome POVM $\mathbf{M} = \left(M_1, M_2, \ldots, M_n\right)$ with postselection, one needs to find $q \in (0,1]$, such that for $(n+1)$-outcome POVM defined as $\tilde{\mathbf{M}}(q) = \left(qM_1, qM_2, \ldots, qM_n, \mathbb{I}\left(1-q\right)\right)$, we have $\tilde{\mathbf{M}}(q) \in \mathbb{SP}\left(n+1,d\right)$. Furthermore, q can be interpreted as a success probability of simulation. In [1] a simulation scheme attaining $q = \frac{1}{d}$ was proposed and proved to be optimal. We note that a related protocol appeared in [3].

In the task of USD, one is asked to unambiguously distinguish between quantum states ρ_i generated from the ensemble $\mathcal{E} = \{p_i, \rho_i\}_{i=1}^{n}$. If quantum states are measured by POVM $\mathbf{M}$, probability of success in this task is given by $p_{succ}\left(\mathcal{E},\mathbf{M}\right) = \sum_{i=1}^{n} p_i \mathrm{tr}\left(\rho_i M_i\right)$. The requirement of unambiguity means that $\mathrm{tr}\left(M_i\rho_j\right) = 0$ if $i \neq j$. Furthermore, we identify additional, M_{n+1} effect with "inconclusive answer". In what follows, we will consider the general case of ensemble consisting of non-commuting, pure states, i.e., $\rho_i = |\psi_i\rangle\langle\psi_i|$ and $\langle\psi_i|\psi_j\rangle \neq 0$.

2. Results

2.1. The Best PM-Simulable POVM for USD

It may be interesting to pose a question which particular *projective-simulable* measurement gives the highest probability of success in the task of USD. According to the definition, the PM-simulable POVM can be written as a convex combination of projective measurements $\mathbf{M}_{USD}^{\mathbb{SP}} = \sum_\alpha \lambda_\alpha \mathbf{P}^{(\alpha)}$ with $\lambda \in [0,1]$ and $\sum_\alpha \lambda_\alpha = 1$. For projective measurement $\mathbf{P}^{(\alpha)} = \left(P_1^{(\alpha)}, P_2^{(\alpha)}, \ldots, P_{n+1}^{(\alpha)}\right)$, the effects fulfill the idempotency and orthogonality conditions $P_i^{(\alpha)} P_j^{(\alpha)} = P_i^{(\alpha)}\delta_{ij}$. First, let us note that the unambiguity condition $\mathrm{tr}\left(P_i^{(\alpha)}\rho_j\right) = 0$ for $i \neq j$ forces a very special structure on every $\mathbf{P}^{(\alpha)}$. Namely, at least one effect of each of the $\mathbf{P}^{(\alpha)}$ must be the projection on the orthogonal complement of the subspace spanned by all except one of the states from the ensemble, i.e., $P_k^{(\alpha)} = \{\rho_i\}_{i\neq k}^{\perp}$. The excluded one labeled by k is identified unambiguously if the result of the measurement $\mathbf{P}^\alpha$ is k. In this case only single quantum state can be discriminated unambiguously by a particular projective measurement, while all the other states will yield the "inconclusive" result. Such a measurement has a structure $\mathbf{P}^{(\alpha)} = \left(0, \ldots, 0, P_\alpha^{(\alpha)}, 0, \ldots, 0, \left(\mathbb{I} - P_\alpha^{(\alpha)}\right)\right)$, where without loss of generality (indeed, the number of projective measurements for optimal simulation of the POVM for USD task, due to their structure, is at most n—every additional effect would correspond only to the "inconclusive" result.) we have identified $k = \alpha$ (see [1] for detailed proofs of the above statements).

The success probability in this case is equal to $p_{succ}\left(\mathcal{E},\mathbf{M}_{USD}^{\mathbb{SP}}\right) = \sum_\alpha \sum_{i=1}^{n} p_i \lambda_\alpha \mathrm{tr}\left(P_i^\alpha \rho_i\right) = \sum_\alpha \lambda_\alpha p_\alpha \mathrm{tr}\left(P_\alpha^{(\alpha)}\rho_\alpha\right)$. Since this is a strictly convex function on a set of $\mathbf{P}^{(\alpha)}$ it follows that it attains maximum at one of the extremal points, i.e., particular $\mathbf{P}^{(\alpha)}$. Finally, the optimal probability of success for PM-simulable POVM in the USD task is given by

$$p_{succ}^{*} = \max_\alpha p_\alpha \mathrm{tr}\left(P_\alpha^{(\alpha)}\rho_\alpha\right). \tag{1}$$

2.2. Better Strategy of Simulation for USD

In [1] the application of simulation with postselection for the task of USD was analysed. The authors focused on deriving bounds for probability of success, while simulating a POVM *optimal* for this problem. The simulation scheme requires performing randomized projective measurements proportional to effects of POVM. However, including in simulation protocol application of a projective measurement corresponding to "inconclusive" answer seems to be not particularly practical idea—at the end it is anyway glued with the last, additional effect M_{n+1}. Therefore, from experimental point of view, it may be more reasonable to simulate another measurement, consisting only of rescaled effects corresponding to "conclusive" answers. Here we derive such a strategy.

Let us denote by $M_? = M_{n+1}$ an effect corresponding to inconclusive answer and define $\mathrm{tr}\,(M_?) = \chi_?$. Since any POVM for the USD task, due to the very unambiguity condition, consists of n rescaled projectors on some subspaces, each of the first n effects can be written in the form $M_i = \alpha_i |\phi_i\rangle\langle\phi_i|$ (see [1] for details). Furthermore, we have $\sum_i^n M_i = \mathbb{I} - M_?$. By identifying $\mathrm{tr}\,(\sum_{i=1}^n M_i) = \sum_{i=1}^n \alpha_i$ and $\mathrm{tr}\,(\mathbb{I} - M_?) = d - \chi_?$, we obtain $1 = \sum_{i=1}^n \frac{\alpha_i}{d-\chi_?}$. Therefore numbers $q^*\alpha_i$, with $q^* = \frac{1}{d-\chi_?}$, form a probability distribution.

It follows from the direct computation that by randomization of the projectors of the form $\mathbf{P}^{(i)} = (0, \ldots, |\phi_i\rangle\langle\phi_i|, 0, \ldots, \mathbb{I} - |\phi_i\rangle\langle\phi_I|)$ with probabilities $\alpha_i q$ one simulates (without postselection) a POVM $\tilde{\mathbf{M}} = (q^* M_1, \ldots, q^* M_n, (1-q^*)\,\mathbb{I} + q^* M_?)$. The succes probability for the USD task is equal to

$$\tilde{p}_{succ} = \sum_{i=1}^{n} p_i \mathrm{tr}\,(q^* \rho_i M_i) = q^* p_{succ} = \frac{p_{succ}}{d - \chi_?} \geq \frac{p_{succ}}{d}, \tag{2}$$

which outperforms the bound from [1]. Furthermore the number q^* cannot be made higher. Indeed, we want to simulate POVM of the form $\tilde{\mathbf{M}} = (qM_1, \ldots, qM_n, (1-q)\,\mathbb{I} + qM_?)$ using a convex combination of projective measurements of the form (that is the only possible form, see previous subsection) $\mathbf{P}^{(i)} = (0, \ldots, |\phi_i\rangle\langle\phi_i|, 0, \ldots, \mathbb{I} - |\phi_i\rangle\langle\phi_I|)$. Since q does not (by assumption) depend on $\mathbf{P}^i$ we have $\sum_i q\alpha_i \leq 1$ from which follows $q \leq \frac{1}{\sum_i \alpha_i} = \frac{1}{d-\chi_?}$.

2.3. Optimal Strategy for Simulation with Postselection for a Given Measurement

The procedure of simulation with postselection requires construction from n-outcome POVM $\mathbf{M}$, an $(n+1)$-outcome POVM $\tilde{\mathbf{M}}\,(q) \in \mathbb{SP}\,(n+1, d)$. If we formally add the $(n+1)$th 'null' effect to the $\mathbf{M}$, meaning that $\mathbf{M} = (M_1, M_2, \ldots, M_n, 0)$, we can interpret a simulation protocol as introducing a noise channel parameterized by q. Namely, $\Phi_q^{\mathbb{SP}}\,(\mathbf{M}) = \tilde{\mathbf{M}}\,(q) = (qM_1, qM_2, \ldots, qM_n, \mathbb{I}\,(1-q))$. Such a frame makes the problem similar to the one introduced in [2], where the noise affecting measurement was depolarizing $\Phi_t^{dep}\,(\mathbf{M}) = t\mathbf{M} + (1-t)\,\frac{\mathrm{tr}(\mathbf{M})}{d}\mathbb{I}$ with t being a so called "visibility". The authors were interested in finding maximum t, for which given measurement $\mathbf{M}$ affected by depolarizing noise channel is projective-simulable *without postselection*. We will denote such a "critical visiblity" by $t^*\,(\mathbf{M}) = \max\left\{t | \Phi_t^{dep}\,(\mathbf{M}) \in \mathbb{SP}\,(n, d)\right\}$. Inspired by this, we can formulate a new problem of finding maximal q, such that $\tilde{\mathbf{M}}\,(q) = \Phi_q\,(\mathbf{M})$ is projective-simulable *with postselection*. Namely, we introduce

$$q^*\,(\mathbf{M}) = \max\left(q | \Phi_q^{\mathbb{SP}}\,(\mathbf{M}) \in \mathbb{SP}\,(n+1, d)\right). \tag{3}$$

Since q is a success probability of simulation, the q^* is associated with POVM $\tilde{\mathbf{M}}\,(q^*)$ for which this probability is greatest—therefore with the optimal simulation strategy for a given measurement $\mathbf{M}$. In [2], the criteria of PM-simulability for qubit and qutrit measurements were formulated as semi-definite programming (SDP) problems. Therefore (3) can be solved by standard convex programming solvers and with this work we provide a proper code in Python for qubit measurements [4].

We will now give some exemplary solutions of (3). From the existence of critical visibility t^*, it follows that if a POVM is not PM-simulable, its depolarized version with $t \in [1, t^*[$ is also not PM-simulable. Therefore, by changing parameter t through that interval, we can generate a family of non-PM-simulable measurements with varying non-projective "character". In order to get some intuition, we calculate values of such a function $q^*\,(\mathbf{M}, t) = q^*\left(\Phi_t^{dep}\,(\mathbf{M})\right)$ for some particularly interesting qubit POVMs, namely 3-outcome 'trine' and 4-outcome 'tetrahedral' measurements (names come from the fact that Bloch vertices associated with their effects form an equilateral triangle and a tetrahedron, respectively). Both measurements are extremal and symmetric, while tetrahedral is also informationally complete. Figure 1 presents of such plots for 100 different $t \in [1, t^*]$. For clarity, $1 - t$ is plotted. Note that for $t = 1$ (non-depolarized extremal qubit POVM), the optimal success probability is equal to $q = \frac{1}{2} = \frac{1}{d}$.

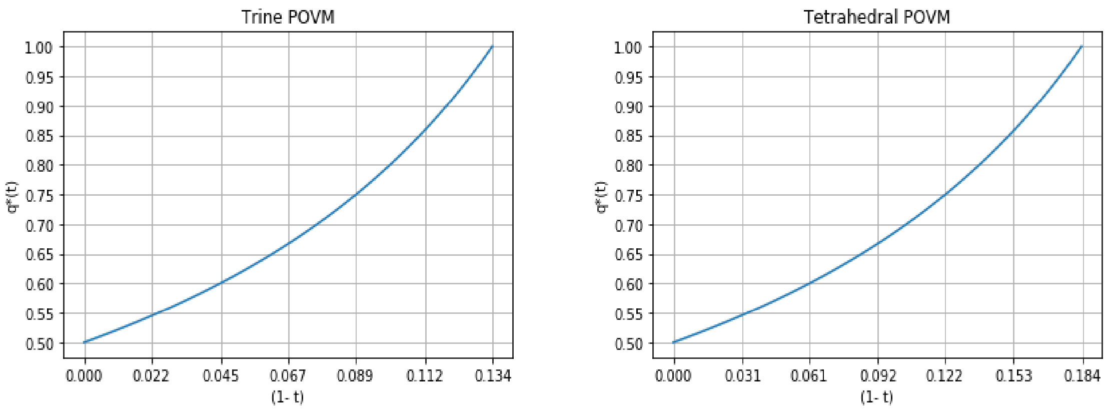

Figure 1. Plot of optimal "$\mathbb{SP}$-noise" parameter $q^*(\mathbf{M}, t)$ vs $(1 - t)$ for depolarized versions of two extremal qubit measurements: trine (**left**) and tetrahedral (**right**).

3. Discussion

We have presented new results concerning strategies for optimal simulation of POVMs by projective measurements for their applications to the task of USD. We have also formulated a task of finding the best strategy of simulation with postselection for a given measurement. Such strategies can be found numerically for qubit and qutrit measurements via SDP methods. Finally, we have provided examples of such strategies for two chosen qubit measurements.

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