proceedings

MDPI

Proceedings

Continuous Measurements for Advanced Quantum Metrology †

Francesco Albarelli [1,2,*], Matteo A. C. Rossi [2,3], Dario Tamascelli [2] and Marco G. Genoni [2]

1. Department of Physics, University of Warwick, Coventry CV4 7AL, UK
2. Quantum Technology Lab, Dipartimento di Fisica 'Aldo Pontremoli', Università degli Studi di Milano, IT-20133 Milan, Italy; matteo.rossi@utu.fi (M.A.C.R.); dario.tamascelli@unimi.it (D.T.); marco.genoni@fisica.unimi.it (M.G.G.)
3. QTF Centre of Excellence, Turku Centre for Quantum Physics, Department of Physics and Astronomy, University of Turku, FI-20014 Turun Yliopisto, Finland

* Correspondence: francesco.albarelli@gmail.com

† Presented at the 11th Italian Quantum Information Science Conference (IQIS2018), Catania, Italy, 17–20 September 2018.

Published: 4 December 2019

Abstract: We review some recent results regarding the use of time-continuous measurements for quantum-enhanced metrology. First, we present the underlying quantum estimation framework and elucidate the correct figures of merit to employ. We then report results from two previous works where the system of interest is an ensemble of two-level atoms (qubits) and the quantity to estimate is a magnetic field along a known direction (a frequency). In the first case, we show that, by continuously monitoring the collective spin observable transversal to the encoding Hamiltonian, we get Heisenberg scaling for the achievable precision (i.e., $1/N$ for N atoms); this is obtained for an uncorrelated initial state. In the second case, we consider independent noises acting separately on each qubit and we show that the continuous monitoring of all the environmental modes responsible for the noise allows us to restore the Heisenberg scaling of the precision, given an initially entangled GHZ state.

Keywords: quantum metrology; quantum estimation; quantum trajectories; continuous measurements

1. Introduction

The ability to measure physical quantities with extreme precision is vitally important for the advancement of both fundamental science and technology and the ultimate limits to the precision of the estimation of physical parameters are dictated by quantum mechanics. In particular this has been studied in the context of quantum technologies, since it is possible to take advantage of purely quantum mechanical phenomena to obtain a precision which would not be possible in the classical case with the same resources. This field of scientific enquiry goes under the name of quantum metrology [1].

Quantum systems are not isolated and they interact with an environment; this interaction usually leads to a loss of genuine quantum features, such as entanglement and coherence. This kind of dynamics is generically called "noisy". Here, we restrict to Markovian noisy dynamics, described by Lindblad master equations. Roughly speaking, this assumption means that the system interacts with a "new instance" of the environment at each time, always in the same state. The description of continuous measurements that we adopt assumes this Markovian limit. Specifically, the continuous monitoring of the system can be understood as the result of continuous "strong" measurements of the environment after it has (weakly and instantaneously) interacted with the system.

Mathematically, the evolution of the density operator of the system is governed by a so-called stochastic master equation [2]. A solution (also called a quantum trajectory) corresponds to a stream of observed measurement outcomes, which are of course stochastic in nature. This is also called the *conditional* evolution of the system, since it is conditioned on the observed outcomes. The average state of a continuously monitored system (i.e., the so called unconditional state, obtained by discarding measurement results) corresponds to the state undergoing a noisy Lindblad evolution.

This manuscript is a succinct review of some of our recent results, where we show that continuously monitored quantum systems are useful for high-precision quantum metrology. Even if continuously monitored quantum systems have been studied in a metrological context before, see e.g., [3–5], our approach and results represent a new contribution to the field.

First, in Section 2 we briefly review the statistical tools to quantify the achievable precision in this scenario. In Section 3 we present two applications of this metrological framework to ensembles of two-level quantum systems (qubits); in particular the results obtained in [6] for a collective measurement in the large number of atoms limit and the more recent results in [7] for independent environments acting on every qubit. We show that the continuous monitoring can either *give rise* or *restore* Heisenberg scaling.

2. Materials and Methods

2.1. Stochastic and Lindblad Master Equations

The evolution of the conditional state of a continuously monitored quantum system is described by a stochastic master equation (SME). We restrict to Markovian SMEs, which physically correspond to *sequential* monitoring, i.e., the output modes (roughly speaking, the environment right after the interaction with the main system) are sequentially and instantly measured. As an example, we report here the stochastic master equation corresponding to homodyne detection (HD) of the environmental modes:

$$d\rho^{(c)} = -i[\hat{H}, \rho^{(c)}]\, dt + \sum_j \mathcal{D}[\hat{c}_j]\rho^{(c)}\, dt + \sqrt{\eta} \sum_j \mathcal{H}[\hat{c}_j]\rho^{(c)}\, dw_j \,, \tag{1}$$

here $dw_j = dy_j - \sqrt{\eta}\mathrm{Tr}[\varrho^{(c)}(\hat{c}_j + \hat{c}_j^\dagger)]$ represent independent Gaussian stochastic processes (Wiener increments satisfying $dw_j dw_k = \delta_{jk} dt$). The measurement results are given by dy_j and they represent the expectation value of th operator $(\hat{c}_j + \hat{c}_j^\dagger)$ plus Gaussian fluctuations. The operators $\hat{c}_j$ can be arbitrary operators on the system and they are determined by the interaction between the system and the environment; η represents the efficiency of the monitoring (assumed to be equal for all the environmental modes). We have also introduced the superoperator $\mathcal{D}[\hat{A}]\bullet = \hat{A} \bullet \hat{A}^\dagger - \frac{1}{2}\left\{\hat{A}^\dagger \hat{A}, \bullet\right\}$. The corresponding Lindblad master equation (ME) is obtained by averaging over the trajectories (since $\mathbb{E}\left[dw_j\right] = 0$):

$$\frac{d\rho(t)}{dt} = \mathcal{L}\rho(t) = -i\left[\hat{H}, \rho(t)\right] + \sum_i \mathcal{D}[\hat{c}_i]\rho(t) \,. \tag{2}$$

Different SMEs can correspond to the same Lindblad equation; these different SMEs are said to be different *unravellings* of the same ME. Another common and useful unravelling is obtained when the environment output modes are measured by a photo-detector (PD); in this case the stochastic part is given by Poisson increments [2], which are not real valued as dw_j, but binary valued (i.e., 0 or 1).

2.2. Cramér-Rao Bounds

We analyse the precision of an estimate of the value of a physical parameter in the context of local quantum estimation theory [8]. According to the Cramér-Rao bound of classical statistics, the precision (standard deviation) of any unbiased estimator for a parameter ω is lower bounded as $\delta\omega \geq \frac{1}{\sqrt{M\mathcal{F}[p(x|\omega)]}}$ where M is the number of repetitions of the experiment and $\mathcal{F}[p(x|\omega)]$ is the classical Fisher information (FI). For a quantum system the probability is given by the Born rule $p(x|\omega) = \mathrm{Tr}[\varrho_\omega \Pi_x]$, where ϱ_ω is a family of quantum states parametrized by ω, and Π_x is a POVM operator describing the measurement process. By optimizing over all the possible POVMs we obtain the quantum Cramér-Rao inequality $\delta\omega \geq \frac{1}{\sqrt{M\mathcal{F}[p(x|\omega)]}} \geq \frac{1}{\sqrt{M\mathcal{Q}[\varrho_\omega]}}$, where $\mathcal{Q}[\varrho_\omega]$ is the quantum Fisher information (QFI) [8] of the state ϱ_ω.

For a continuously monitored system there is a continuous (in time) stream of measurement outcomes, e.g. the photo-currents $y_j(t) = \int_0^t dy_j$ for the case of HD described by Equation (1). The probability of observing a particular stream of outcomes is also the probability of having a particular conditional state (a trajectory) and we generically call this probability p_{traj}. The statistics of these outcomes will in general give information about the parameter of interest, which usually appears in the Hamiltonian of the system. Furthermore, the final conditional state at the end of the monitoring will also depend on the parameter and it can be measured by an arbitrary POVM. We assume that it is possible to choose the optimal POVM on the final conditional state, so that the relevant figure of merit for the precision of the estimation is the QFI of such state.

In [6] we have shown that for systems undergoing this kind of evolution the correct bound which combines the information coming from the continuous monitoring and the final measurement on the conditional state is $\delta\omega \geq \frac{1}{\sqrt{M\widetilde{\mathcal{Q}}_{\text{unr}}}}$, where we introduced the effective QFI (eQFI), defined as

$$\widetilde{\mathcal{Q}}_{\text{unr}} = \mathcal{F}[p_{\text{traj}}] + \sum_{\text{traj}} p_{\text{traj}} \mathcal{Q}[\rho^{(c)}]\,. \tag{3}$$

The classical Fisher information $\mathcal{F}[p_{\text{traj}}]$ corresponding to the continuous measurement itself can be obtained by sampling trajectories from the SME, as proposed in [9] (and in [10] for Gaussian states). Building on these reults, in [7] we have proposed an algorithm to reliably obtain the eQFI, by adapting numerical approach of [11] based on Kraus operators.

A more fundamental quantity, which we dubbed "ultimate" QFI (uQFI) is the QFI $\overline{\mathcal{Q}}_{\mathcal{L}}$ of the whole system and environment state (considering all the output modes). Under our Markovianity assumption this quantity can be obtained by solving a modified master equation [12]. The uQFI depends only on the Lindbladian $\mathcal{L}$ and not on the particular unravelling, while the eQFI $\widetilde{\mathcal{Q}}_{\text{unr}}$ pertains to a particular choice of measurement on the environmental modes, i.e., to a particular unravelling, hence the subscript. We stress that, while stochastic master equations like Equation (1) correspond to *sequential* monitoring schemes, the uQFI is optimized over all possible measurements on the system plus the environmental modes that have interacted with it. This includes schemes where the measurement of the environmental modes is performed in more complicated ways and we have no a priori reason to believe that the eQFI of sequential schemes can saturate the uQFI. Nonetheless in the relevant cases we have considered we have always found optimal sequential schemes where the two quantities are equal.

We can summarize the situation with the following inequalities

$$\mathcal{Q}[\varrho_{\text{unc}}] \leq \widetilde{\mathcal{Q}}_{\text{unr},\eta} \leq \overline{\mathcal{Q}}_{\mathcal{L}}\,, \tag{4}$$

where $\mathcal{Q}[\varrho_{\text{unc}}]$ is the QFI of the unconditional state given by the Lindblad ME (2) and we have also stressed that the eQFI depends on the efficiency η of the detection. The first inequality is given by the extended convexity of the QFI [13] and supports the intuitive idea that one can only gain information about the parameter by monitoring the environment.

3. Results

We consider an ensemble of N two level atoms interacting with a constant external magnetic field along a direction. The Hamiltonian of the system is $\hat{H}_\omega = \omega \hat{J}_z$, where $\hat{J}_\alpha$ are collective spin operators, defined as $\hat{J}_\alpha = \frac{1}{2}\sum_{i=0}^N \sigma_\alpha^{(i)}$ for $\alpha = x, y, z$ (here $\sigma_\alpha^{(i)}$ denotes Pauli matrices acting on the i-th spin), while the frequency ω is proportional to the intensity of the magnetic field. We have studied this paradigmatic metrological problem of frequency estimation in two very different conditions.

3.1. Collective Coupling with Initial Separable State

In [6] we assumed an initial pure separable state $|\psi(0)\rangle = \bigotimes_{k=0}^{N} \frac{1}{\sqrt{2}} (|0\rangle_k - i|1\rangle_k)$, where all the two-level atoms have a spin directed in the positive y direction (the state is a product of eigenstates of σ_y). We considered the monitoring of a single collective operator $\hat{c} = \sqrt{\kappa}\hat{J}_x$, transversal to the Hamiltonian, by performing homodyne detection on the environment; this amounts to a continuous non-demolition measurement of the collective spin $\hat{J}_x$ and it is described by Equation (1). For short times and small frequency $\omega t \ll 1$ and for large N the system can be described by continuous variables and we can reduce to study the dynamics of a Gaussian state (essentially we only need first and second statistical moments of $\hat{J}_z$ and $\hat{J}_x$) and obtain analytical results.

Under these assumptions, we have shown that by continuously monitoring $\hat{J}_x$ the eQFI about the parameter ω shows Heisenberg (quadratic) scaling in the number of atoms N. This remains true for every value of the efficiency, at the expense of increasing the number of atoms and the observation time, see also [14]. We have also shown the optimality of the scheme in the case of perfect efficiency, by proving that that the eQFI is equal to the uQFI. In this setting, the continuous monitoring creates a spin squeezed (multipartite-entangled) conditional state, which is a resource for quantum metrology. As a matter of fact *both* terms in the eQFI (3) show Heisenberg scaling, but the sum is needed to achieve optimal precision.

3.2. Independent Noises with Initial Entangled State

In [7] we considered the prototypical case of noisy quantum metrology: N independent environments acting on the qubits, given by N n different noise operators $\hat{c}_j = \sqrt{\kappa/2}\sigma_\alpha^{(j)}$; we restricted our study to $\alpha = z$ (parallel-noise, or pure dephasing) and $\alpha = x$ (transverse noise). In this situation the metrological resource is an initial multipartite-entangled state, we considered a GHZ state of the N qubits: $|\psi_{\mathsf{GHZ}}\rangle = (|0\rangle^{\otimes N} + |1\rangle^{\otimes N})/\sqrt{2}$. It is well known that in the noiseless case ($\kappa = 0$), the corresponding QFI is Heisenberg limited, $\mathcal{Q}_{\mathsf{HL}} = N^2 t^2$. When independent noise acts on the qubits, Heisenberg scaling is in general not possible [15–17]; however an intermediate scaling can be obtained for transverse noise, by optimizing over the observation time [18].

We considered the continuous monitoring of all the N environments responsible for the noisy evolution of the system. Differently from the previous case, the point of the the metrological scheme is now to optimally exploit the *initial* entanglement, counteracting the effect of the noise by measuring the environment.

For parallel noise we have proved analytically that for perfect efficiency $\eta = 1$ we can restore the QFI of the noiseless case, i.e., $\widetilde{\mathcal{Q}}_{\mathrm{unr},\eta=1} = \overline{\mathcal{Q}}_{\mathcal{L}} = \mathcal{Q}_{\mathsf{HL}}$, both with PD [7] and HD [19]. However, when the efficiency is not perfect the eQFI reduces to the QFI of the unconditional state with a rescaled noise coupling $\kappa(1-\eta)$, thus losing Heisenberg scaling. Furthermore, for parallel noise some of the assumptions on the monitoring of the N independent channels can be relaxed; due to the permutational symmetry of the initial state the same result is obtained by using only a single detector for all the channels.

For transverse noise we have shown analytically that the uQFI shows a quadratic scaling in N, but it is always smaller than the QFI of the noiseless case. The effect of the noise cannot be completely restored by continuous monitoring, due to the non-commutativity with the encoding Hamiltonian. Again for for perfect efficiency $\eta = 1$ the scheme is optimal (for both HD and PD) and the eQFI saturates the uQFI; overall we have $\widetilde{\mathcal{Q}}_{\mathrm{unr},\eta=1} = \overline{\mathcal{Q}}_{\mathcal{L}} < \mathcal{Q}_{\mathsf{HL}}$. For non-unit efficiency we see a monotonic increase of the the eQFI with η, but the problem has to be studied numerically (with the novel algorithm we developed) and we have no conclusive statements about the scaling in N.

Conflicts of Interest: The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

References

1. Giovannetti, V.; Lloyd, S.; Maccone, L. Advances in quantum metrology. *Nat. Photonics* **2011**, *5*, 222–229. doi:10.1038/nphoton.2011.35.
2. Wiseman, H.M.; Milburn, G.J. *Quantum Measurement and Control*; Cambridge University Press: New York, NY, USA, 2010.
3. Mabuchi, H. Dynamical identification of open quantum systems. *Quantum Semiclass. Opt.* **1996**, *8*, 1103–1108. doi:10.1088/1355-5111/8/6/002.
4. Geremia, J.M.; Stockton, J.K.; Doherty, A.C.; Mabuchi, H. Quantum Kalman Filtering and the Heisenberg Limit in Atomic Magnetometry. *Phys. Rev. Lett.* **2003**, *91*, 250801. doi:10.1103/PhysRevLett.91.250801.
5. Kiilerich, A.H.; Mølmer, K. Bayesian parameter estimation by continuous homodyne detection. *Phys. Rev. A* **2016**, *94*, 032103. doi:10.1103/PhysRevA.94.032103.
6. Albarelli, F.; Rossi, M.A.C.; Paris, M.G.A.; Genoni, M.G. Ultimate limits for quantum magnetometry via time-continuous measurements. *New J. Phys.* **2017**, *19*, 123011. doi:10.1088/1367-2630/aa9840.
7. Albarelli, F.; Rossi, M.A.C.; Tamascelli, D.; Genoni, M.G. Restoring Heisenberg scaling in noisy quantum metrology by monitoring the environment. *Quantum* **2018**, *2*, 110. doi:10.22331/q-2018-12-03-110.
8. Paris, M.G.A. Quantum estimation for quantum technology. *Int. J. Quantum Inf.* **2009**, *07*, 125–137. doi:10.1142/S0219749909004839.
9. Gammelmark, S.; Mølmer, K. Bayesian parameter inference from continuously monitored quantum systems. *Phys. Rev. A* **2013**, *87*, 032115. doi:10.1103/PhysRevA.87.032115.
10. Cramér-Rao bound for time-continuous measurements in linear Gaussian quantum systems. *Phys. Rev. A* **2017**, *95*, 012116.
11. Rouchon, P.; Ralph, J.F. Efficient quantum filtering for quantum feedback control. *Phys. Rev. A* **2015**, *91*, 012118. doi:10.1103/PhysRevA.91.012118.
12. Gammelmark, S.; Mølmer, K. Fisher Information and the Quantum Cramér-Rao Sensitivity Limit of Continuous Measurements. *Phys. Rev. Lett.* **2014**, *112*, 170401. doi:10.1103/PhysRevLett.112.170401.
13. Alipour, S.; Rezakhani, A.T. Extended convexity of quantum Fisher information in quantum metrology. *Phys. Rev. A* **2015**, *91*, 042104. doi:10.1103/PhysRevA.91.042104.
14. Albarelli, F.; Rossi, M.A.C.; Paris, M.G.A.; Genoni, M.G. The Role of Monitoring Time and Detectors Efficiencies in Time-Continuous Quantum Magnetometry. In *Toward a Science Campus in Milan*; Springer International Publishing: Cham, Germany, 2018; pp. 127–139. doi:10.1007/978-3-030-01629-6_11.
15. Huelga, S.F.; Macchiavello, C.; Pellizzari, T.; Ekert, A.; Plenio, M.B.; Cirac, J.I. Improvement of Frequency Standards with Quantum Entanglement. *Phys. Rev. Lett.* **1997**, *79*, 3865–3868. doi:10.1103/PhysRevLett.79.3865.
16. Escher, B.M.; de Matos Filho, R.L.; Davidovich, L. General framework for estimating the ultimate precision limit in noisy quantum-enhanced metrology. *Nat. Phys.* **2011**, *7*, 406–411. doi:10.1038/nphys1958.
17. Demkowicz-Dobrzański, R.; Kołodyński, J.; Guţă, M. The elusive Heisenberg limit in quantum-enhanced metrology. *Nat. Commun.* **2012**, *3*, 1063. doi:10.1038/ncomms2067.
18. Chaves, R.; Brask, J.B.; Markiewicz, M.; Kołodyński, J.; Acín, A. Noisy Metrology beyond the Standard Quantum Limit. *Phys. Rev. Lett.* **2013**, *111*, 120401. doi:10.1103/PhysRevLett.111.120401.
19. Albarelli, F. Continuous Measurements and Nonclassicality as Resources for Quantum Technologies. Ph.D. Thesis, Università degli Studi di Milano, Milan, Italy, 2018. doi:10.13130/albarelli-francesco_phd2018-11-19.