fractal and fractional

Editorial

Developments in the Symmetry and Solutions to Fractional Differential Equations

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1. Introduction

Fractional differential equations constitute an important research direction in modern mathematics and applied sciences. In contrast to traditional integer-order differential equations, fractional differential equations can describe complex systems with memory effects, heredity and nonlocal characteristics more accurately. So, these equations appear in a wide variety of fields, such as physics, biology, control theory, financial engineering, fluid mechanics, image processing and materials science [1–5]. Fractional models have become important and significant investigation tools, especially in characterizing the behavior of viscoelastic materials, abnormal diffusion phenomena, neural information transmission, biological tissue responses and the evolution of financial assets [6–10]. With the development of mathematical theories and computational methods, various definitions of fractional derivatives, such as Riemann–Liouville, Caputo, Hadamard, etc., have been proposed and continuously improved, providing rich expression forms and theoretical support for continuously promoting and deepening research on fractional models.

In the study of fractional differential equations, analyzing the properties of their solutions has always been one of the core issues, covering multiple aspects such as the existence, uniqueness, stability, regularity, monotonicity, asymptotic behavior and symmetry of the solutions. These properties not only represent the basis of theoretical research, but are also directly related to the reliability and computability of a model. Due to the nonlocality and memorability characteristics of fractional derivatives, the solutions to fractional differential equations often exhibit complex global behaviors, whereas such characteristics are not demonstrated by the solutions to integer-order differential equations. Therefore, researchers are continuously developing and applying nonlinear functional analysis tools, such as the variational method [11,12], fixed-point theory [13,14], upper and lower solution methods [15,16], topological degree theory [17,18], critical point theory [19,20], monotone iterative method [21,22], etc., to establish the existence and uniqueness conditions for solutions to fractional differential equations, and analyze their stability and long-term dynamic behavior. Symmetry analysis not only helps to reveal the structural characteristics of the solution among its many properties and reduce the complexity of the problem, but also plays a key role in unique determination, identification of the solution's branch structure and model simplification. The symmetry of the solution is often closely related to the nonlocal diffusion behavior, especially in models involving nonlocal operators such as fractional Laplacian and fractional p-Laplacian, [23–30]. Therefore, the in-depth study of

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properties of fractional differential equation solutions, especially symmetry, promotes the development of the theory of nonlocal partial differential equations, and provides important mathematical tools and new ideas for the modeling and simulation of complex systems.

In view of the foregoing discussion, this Special Issue showcases the diversity of studies focusing on the properties and applications of solutions to fractional differential equations. It contains fourteen articles, which are briefly described in the next section. The purpose of this editorial is to elaborate on each of the articles included in this Special Issue and encourage the reader to explore them.

2. An Overview of the Published Articles

Fangyuan Dong et al. (Contribution 1) study a class of nonlocal Schrödinger–Poisson–Slater equations. The authors establish the existence, stability and symmetry-breaking of solutions in both radial and nonradial cases. In the radial case, variational methods are used to prove the existence of a positive solution. In the nonradial case, the existence of ground-state solutions is proven using the Nehari manifold method. Their results demonstrate the stability of solutions in both radial and nonradial cases, identifying critical parameter regimes associated with stability and instability. Their work advances our understanding of nonlocal interactions in symmetry-breaking and stability while extending existing theories to multiparameter and higher-dimensional settings within the Schrödinger–Poisson–Slater model.

Keyu Zhang et al. (Contribution 2) investigate the solvability of a Riemann–Liouville-type fractional-impulsive integral boundary value problem. Under some conditions on the spectral radius corresponding to the related linear operator, several existence theorems of the problem are obtained using fixed-point methods. In particular, they determine the existence of multiple positive solutions via the Avery–Peterson fixed-point theorem. Note that the linear operator depends on the impulsive term and the integral boundary condition.

Yongqing Wang (Contribution 3) discusses the positive solutions to a class of semipositone boundary value problems of fractional differential equations. The nonlinearity $f(t,x)$ may be singular at $t=0,1$ and satisfies $f(t,x) \geq -\mathfrak{a}(t)x - \mathfrak{R}(t)$. The author derives some new properties of the Green's function of the auxiliary problems, and discovers the multiplicity and existence of positive solutions by utilizing the fixed-point index theory. Their research results enrich the study of semipositone FBVPs, and the proposed method can also be applied to other types of differential equations.

Xianchen Wang et al. (Contribution 4) use the theory of fractional calculus to present a detailed description and the calculation process of the Adomian decomposition algorithm for incommensurate fractional-order chaotic systems. On this basis, the phase diagrams, Poincaré section, coexistence bifurcation diagrams, and coexistence Lyapunov exponent spectrum are analyzed. In addition, a fixed-time synchronization control scheme is proposed. The results offer a new approach to the practical application of incommensurate fractional-order chaotic systems.

Tao Liu et al.'s (Contribution 5) introduce and implement a method based on the spectral element method that relies on interpolating scaling functions (ISFs). Using an orthonormal projection, the method maps the equation to scaling spaces identified from multi-resolution analysis. This is achieved by expressing the Caputo fractional derivative as a square matrix based on ISFs. The convergence of the scheme is established through theoretical analysis. Simplicity in its implementation, significant accuracy, and high efficiency make this method a candidate for solving fractional-type and nonfractional equations.

Yujun Cui et al. (Contribution 6) study high-order nonlinear fractional elastic equations that depend on low-order derivatives in the nonlinearity. Under suitable weaker assumptions, the uniqueness is established using Perov's fixed-point theorem and matrix analysis, while the existence of solutions is proven via the Leray–Schauder alternative theorem and matrix analysis. Examples are provided to illustrate the key results.

Ayub Samadi et al. (Contribution 7) investigate a nonlocal fractional coupled system of (k,ψ)-Hilfer fractional differential equations, where the boundary conditions involve (k,ψ)-Hilfer fractional derivatives and (k,ψ)-Riemann–Liouville fractional integrals. The existence and uniqueness of solutions are established for the considered coupled system by using standard tools from fixed-point theory. More precisely, Banach and Krasnosel'skil"s fixed-point theorems are used, along with Leray–Schauder alternative. The results are illustrated through constructed numerical examples.

Weiwei Liu et al. (Contribution 8) investigate the existence of solutions to the initial value problem associated with a Hadamard-type fractional-order differential equation on an infinite interval. The equation's nonlinear term incorporates lower-order derivatives of the unknown functions. To establish the global existence criteria, the authors first verify that there exists a unique positive solution to an integral equation based on a class of new integral inequality. Next, a metrizable and complete locally convex space is constructed. On this space, the existence of at least one solution to the initial value problem is established by applying Schäuder's fixed-point theorem.

Nasser H. Sweilam et al. (Contribution 9) improve a mathematical model of monkeypox disease with a time delay to a crossover model by incorporating variable-order and fractional differential equations, along with stochastic fractional derivatives, in three different time intervals. They discuss the stability and positivity of the solutions to the proposed model. To analyze the model's behavior, they construct two numerical methods: the nonstandard modified Euler–Maruyama technique and the nonstandard Caputo proportional constant Adams–Bashforth five-step method. Most importantly, their work opens up new avenues for understanding the monkeypox epidemic.

Mengru Liu and Lihong Zhang (Contribution 10) study the monotonicity of the positive solution to the double-index logarithmic nonlinear fractional g-Laplacian parabolic equations with Marchaud fractional time derivatives by using the direct moving plane method. They successfully overcome the difficulties caused by the double nonlocality in space and time, as well as the nonlinearity of the fractional g-Laplacian. The results provide important tools and methods for investigating the qualitative properties of solutions, particularly for unbounded solutions to fractional elliptic and parabolic problems.

James Abah Ugboh et al. (Contribution 11) consider a faster iterative method for approximating the fixed points of generalized α-nonexpansive mappings. They prove several weak and strong convergence theorems for the considered method under mild conditions on the control parameters. Furthermore, the authors demonstrate that the class of mappings under consideration is more general than certain nonexpansive-type mappings. These results generalize and improve upon many existing results in the literature.

Anthony Torres-Hernandez et al. (Contribution 12) construct a family of radial functions to emulate thin-plate splines and propose methods for applying partial and full fractional derivatives in interpolation problems. They use QR decomposition to precondition matrices and introduce two types of abelian groups for fractional operators, including the Riemann–Liouville and Caputo derivatives. A radial interpolant is also proposed for solving fractional differential equations via the asymmetric collocation method, with illustrative examples provided. This work highlights the innovative integration of fractional operators with abelian group theory and radial basis functions.

Junjie Wang et al. (Contribution 13) develop finite difference and finite volume schemes for the fractional Laplacian operator and apply them to solve fractional diffusion equations. By combining fractional and classical interpolation functions, the authors handle boundary singularities and construct discrete schemes with provable properties. Numerical experiments confirm the accuracy and efficiency of the proposed methods. Moreover, it is easy to extend the numerical scheme to nonuniform dissection using a similar approach.

Tingting Guan and Lihong Zhang (Contribution 14) investigate solution properties of space–time fractional variable-order conformable nonlinear differential equations involving a generalized tempered fractional Laplace operator. The authors establish new conformable fractional inequalities and prove a maximum principle in this context. They further derive comparison results and analyze qualitative properties of solutions based on the maximum principle.

3. Conclusions

The Special Issue, "Developments in the Symmetry and Solutions to Fractional Differential Equations", presents a series studies that demonstrate the profound influence of fractional differential equations in modern mathematical analysis and applied sciences. The articles presented in this Special Issue focus on the structure and symmetry analysis of solutions, exploring the unique advantages of fractional models in depicting characteristics such as memorability, heredity and nonlocality in complex systems, and improving the traditional integer-order models describing such phenomena. For nonlocal Schrödinger–Poisson systems and impulsive boundary value problems incommensurate with chaotic systems and fractional Laplacian operators, many studies have proposed cutting-edge analytical methods, providing solid support for understanding core issues such as the existence, uniqueness, stability and symmetry-breaking of solutions to fractional differential equations.

This Special Issue includes fourteen original research papers demonstrating a range of innovative achievements from theoretical analysis to numerical methods. The salient features of this Special Issue include the establishment of a new maximum principle, the construction of an efficient numerical interpolation technique, the design of control strategies based on fractional systems, etc. The content of this Special Issue not only promotes the theoretical development of fractional differential equations but also provides new ideas for solving practical scientific and engineering problems. This Special Issue is expected to serve as a significant reference and inspiration for the continuous development of fractional differential equations and their potential applications in a wide range of scientific disciplines.

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List of Contributions

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