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A New Caputo Fractional Differential Equation with Infinite-Point Boundary Conditions: Positive Solutions

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Abstract

This paper mainly studies a different infinite-point Caputo fractional differential equation, whose nonlinear term may be singular. Under some conditions, we first use spectral analysis and fixed-point index theorem to explore the existence of positive solutions of the equation, and then use Banach fixed-point theorem to prove the uniqueness of positive solutions. Finally, an interesting example is used to explain the main result.

Keywords: positive solutions; Caputo fractional derivative; spectral analysis; fixed-point index; Banach fixed-point theorem

1. Introduction

Recently, fractional differential equations (FDEs) have become increasingly important in mathematics. The fractional derivative serves as an effective tool for accurately describing the memory and heritability of various materials and processes. Compared to traditional integer differential equations, FDEs are crucial in many fields, including physics, engineering, mechanics, and biology [1–3]. This widespread application makes them valuable in areas such as control theory [4], viscoelastic theory [5,6], epidemiological modeling [7], and more, effectively addressing many complex real-life problems. The multi-point or infinite point boundary value problem (BVP) of FDEs is one of the research directions welcomed by many scholars. Multi-point boundary value problems originated from various fields of applied mathematics and applied physics. They can not only describe many important and complex physical phenomena more accurately, such as the theory of non-uniform electromagnetic field, but also have a broader practical application background, such as population growth. With the continuous in-depth study of many scholars, people began to have an interest in the infinite point boundary value condition. In 2011, Gao and Han [8] first considered the solution of FDEs with infinite point boundary value conditions. In 2016, Guo et al. [9] first studied the infinite-point Caputo FDE problem. Xu and Yang [10] studied FDEs in control theory in combination with infinite point boundary conditions. In 2024, Li et al. [11] discussed an infinite-point Hadamard FDE problem.

The existence and uniqueness of solutions has always been one of the hot issues in the study of FDEs. Commonly used tools for proving the existence of solutions include the fixed-point theorem in cones [12–15], the upper and lower solution method [16,17], and Leray–Schauder degree theory [18], etc. To establish the uniqueness of solutions, techniques such as the Banach fixed-point theorem [16,19,20], Gronwall's inequality, and the Laplace

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transform method are frequently employed. For nonlinear FDEs, additional methods such as variational techniques and spectral analysis may also be utilized.

In [21], the author considered the existence and multiplicity of positive solutions for the FDE problem

$$\begin{cases} D^{\alpha}_{0^+}u(t) = a(t)f(t,u(t)),\ 0 < t < 1, \\ u(0) = u'(0) = 0, \quad u(1) = \sum_{i=1}^{m} \beta_j u(\xi_j), \end{cases}$$

where $D^{\alpha}_{0^+}$ is the Riemann–Liouville differential operator of order $2 < \alpha \leq 3$. The existence and multiplicity of solutions are obtained by two fixed-point theorems on a cone in Banach spaces.

In [9], Guo et al. studied the following infinite-point FDE problem

$$\begin{cases} {}^{c}D^{\alpha}_{0^+}u(t) + f(t,u(t),u'(t)) = 0,\ 0 < t < 1, \\ u(0) = u''(0) = 0, \quad u'(1) = \sum_{j=1}^{\infty} \eta_j u(\xi_j), \end{cases}$$

where ${}^{c}D^{\alpha}_{0^+}$ is the Caputo derivative, $2 < \alpha \leq 3$. The existence of multiple positive solutions is obtained by Avery–Peterson's fixed-point theorem.

In [11], Li et al. discussed an infinite-point Hadamard FDE problem

$$\begin{cases} -{}^{H}D^{\flat}_{a^+}v(t) + \varrho(t)\ell(t,v(t),{}^{H}D^{\mu}_{a^+}v(t)) = 0,\ a < t < b, \\ {}^{H}D^{\mu}_{a^+}v(a) = {}^{H}D^{\mu+1}_{a^+}v(a) = 0, \quad {}^{H}D^{\mu}_{a^+}v(b) = \sum_{j=1}^{\infty} k_j {}^{H}D^{\mu}_{a^+}v(\varsigma_j), \end{cases}$$

where ${}^{H}D^{\flat}_{a^+}$, ${}^{H}D^{\mu}_{a^+}$ are the Hadamard derivatives, $2.5 < \flat \leq 3$, $0 < \mu < 0.5$. The existence of positive solutions is obtained by the spectral analysis method, Gelfand's formula, and the cones fixed-point theorem.

In [22], Zhai et al. analyzed the following form of Hadamard FDE problem on an infinite interval

$$\begin{cases} {}^{H}D^{\alpha}_{1^+}x(t) + a(t)f(t,x(t)) + b(t)g(t,x(t)) = 0,\ 1 < t < +\infty, \\ x(1) = x'(1) = 0, \quad {}^{H}D^{\alpha-1}_{1^+}x(+\infty) = \sum_{i=1}^{m} \alpha_i {}^{H}I^{\beta_i}_{1^+}x(\eta) + c\sum_{j=1}^{n} \sigma_j x(\xi_j), \end{cases}$$

where ${}^{H}D^{\alpha}_{1^+}$ is the Hadamard-type fractional derivative of order α, $2 < \alpha < 3$, and ${}^{H}I^{\beta_i}_{1^+}$ is the Hadamard-type fractional integral of order $\beta_i > 0$ $(i = 1,2,\ldots,m)$. The local existence and uniqueness of positive solutions are obtained by two fixed-point theorems of a sum operator in partial ordering Banach spaces.

In [23], Zhang considered following nonlinear fractional differential equation with infinite-point boundary value conditions

$$\begin{cases} D^{\alpha}_{0^+}u(t) + q(t)f(t,u(t)) = 0,\ 0 < t < 1, \\ u(0) = u'(0) = \ldots = u^{(n-2)}(0) = 0, \quad u^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j u(\xi_j), \end{cases}$$

where $D^{\alpha}_{0^+}$ is the Riemann–Liouville derivative, $\alpha > 2$, $n - 1 < \alpha \leq n$. The local existence and multiplicity of positive solutions are obtained by fixed-point theorems.

Inspired by these references, here we consider a new infinite-point Caputo FDE problem

$$\begin{cases} {}^{c}D^{\alpha}_{a^+}v(t) + e(t)f(t,v(t),{}^{c}D^{\mu}_{a^+}v(t)) = 0,\ 0 < a < t < b, \\ {}^{c}D^{\mu}_{a^+}v(a) = {}^{c}D^{\mu+1}_{a^+}v(a) = {}^{c}D^{\mu+2}_{a^+}v(a) = 0, \quad {}^{c}D^{\mu}_{a^+}v(b) = \sum_{j=1}^{\infty} k_j {}^{c}D^{\mu}_{a^+}v(\delta_j), \end{cases} \tag{1}$$

where ${}^cD^{\alpha}_{a+}$ and ${}^cD^{\mu}_{a+}$ are Caputo fractional derivatives of orders α, μ, $3.5 < \alpha \leq 4$, $0 < \mu < 0.5$, $a < \delta_1 < \delta_2 < \ldots < \delta_{j-1} < \delta_j < \ldots < b$ $(j = 1, 2, \ldots)$, $k_j > 0$ and $\sum_{j=1}^{\infty} k_j(\delta_j - s)^{\alpha-\mu-1} > (b - s)^{\alpha-\mu-1}$, $\sum_{j=1}^{\infty} k_j(\delta_j - a)^3 < (b - a)^3$, $e(t)$ may be singular at $t = a$ or $t = b$, f is a given continuous function.

This paper mainly consists of the following parts: In Section 2, some definitions and lemmas are given to provide some basic contents for the later proof. In Section 3, the existence and uniqueness of solutions are proved. Theorems 1 and 2 use the fixed-point index theorem to prove the existence of positive solutions, and Theorem 3 uses the Banach fixed-point theorem to prove the uniqueness of positive solutions. In Section 4, an example is used to verify the correctness of the conclusion. In Section 5, the main contents, characteristics, and further research directions of this kind of equation are summarized.

2. Preliminaries

For the following proofs, we need some important definitions and lemmas.

Definition 1 ([24,25])**.** *The Caputo fractional derivative of order $\alpha > 0$ for a function u is defined as*

$$ {}^cD^{\alpha}_{a+}u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds, $$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α.

Definition 2 ([24,25])**.** *The Riemann–Liouville fractional integral of order $\alpha > 0$ for a function u is defined as*

$$ I^{\alpha}_{a+}u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) ds. $$

Lemma 1 ([24,25])**.** *With the given notations, the following equality holds*

$$ I^{\alpha}_{a+}({}^cD^{\alpha}_{a+})u(t) = u(t) + c_0 + c_1(t-a) + c_2(t-a)^2 + \ldots + c_{n-1}(t-a)^{n-1}, $$

where n is the least integer greater than or equal to α and c_i $(i = 0, 1, \ldots, n-1)$ are arbitrary constants.

Let $u(t) =^c D^{\mu}_{a+}v(t)$, $v(t) \in C([a, b], [0, +\infty))$, then the BVP(1) can be equivalent to the following:

$$ \begin{cases} {}^cD^{\alpha-\mu}_{a+}u(t) + e(t)f(t, I^{\mu}_{a+}u(t), u(t)) = 0, \ a < t < b, \\ u(a) = u'(a) = u''(a) = 0, \quad u(b) = \sum_{j=1}^{\infty} k_j u(\delta_j), \end{cases} \tag{2} $$

where $3.5 < \alpha \leq 4$ and $0 < \mu < 0.5$.

Lemma 2. *Given $h \in C[a, b] \cap L^1(a, b)$, the fractional problem*

$$ \begin{cases} {}^cD^{\alpha-\mu}_{a+}u(t) + h(t) = 0, \ a < t < b, \\ u(a) = u'(a) = u''(a) = 0, u(b) = \sum_{j=1}^{\infty} k_j u(\delta_j), \end{cases} $$

has a unique solution

$$ u(t) = \int_a^b G(t, s)h(s) ds, $$

where

$$ G(t, s) = H(t, s) + \frac{(t-a)^3}{\Delta} \sum_{j=1}^{\infty} k_j H(\delta_j, s), $$

$$\Delta = (b-a)^3 - \sum_{j=1}^{\infty} k_j(\delta_j - a)^3,$$

$$H(t,s) = \frac{1}{\Gamma(\alpha-\mu)(b-a)^3} \begin{cases} (t-a)^3(b-s)^{\alpha-\mu-1} - (t-s)^{\alpha-\mu-1}(b-a)^3, & a \le s \le t \le b, \\ (t-a)^3(b-s)^{\alpha-\mu-1}, & a \le t \le s \le b. \end{cases}$$

Proof. According to Lemma 1, we can obtain

$$\begin{aligned} u(t) &= -I_{a^+}^{\alpha-\mu}h(t) + c_0 + c_1(t-a) + c_2(t-a)^2 + c_3(t-a)^3 \\ &= -\int_a^t \frac{(t-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}h(s)ds + c_0 + c_1(t-a) + c_2(t-a)^2 + c_3(t-a)^3, \end{aligned}$$

where $c_0,\ c_1,\ c_2$, and c_3 are arbitrary constants. From $u(a) = 0$, we have $c_0 = 0$. Then

$$u'(t) = -\int_a^t \frac{(t-s)^{\alpha-\mu-2}}{\Gamma(\alpha-\mu-1)}h(s)ds + c_1 + 2c_2(t-a) + 3c_3(t-a)^2,$$

$$u''(t) = -\int_a^t \frac{(t-s)^{\alpha-\mu-3}}{\Gamma(\alpha-\mu-2)}h(s)ds + 2c_2 + 6c_3(t-a),$$

from $u'(a) = u''(a) = 0$, we get $c_1 = c_2 = 0$. Therefore,

$$u(\delta_j) = -\int_a^{\delta_j} \frac{(\delta_j-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}h(s)ds + (\delta_j-a)^3c_3,$$

by $u(b) = \sum_{j=1}^{\infty} k_j u(\delta_j)$, we get

$$\begin{aligned} u(b) &= -\int_a^b \frac{(b-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}h(s)ds + (b-a)^3c_3 \\ &= \sum_{j=1}^{\infty} k_j \left[-\int_a^{\delta_j} \frac{(\delta_j-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}h(s)ds + (\delta_j-a)^3c_3 \right], \end{aligned}$$

then, we get

$$c_3 = \frac{1}{\Delta}\left[\int_a^b \frac{(b-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}h(s)ds - \sum_{j=1}^{\infty} k_j \int_a^{\delta_j} \frac{(\delta_j-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}h(s)ds \right],$$

where

$$\Delta = (b-a)^3 - \sum_{j=1}^{\infty} k_j(\delta_j - a)^3.$$

Hence,

$$\begin{aligned} u(t) = &-\frac{1}{\Gamma(\alpha-\mu)}\int_a^t (t-s)^{\alpha-\mu-1}h(s)ds + \frac{(t-a)^3}{\Delta\Gamma(\alpha-\mu)}\int_a^b (b-s)^{\alpha-\mu-1}h(s)ds \\ &- \frac{(t-a)^3}{\Delta\Gamma(\alpha-\mu)}\sum_{j=1}^{\infty} k_j \int_a^{\delta_j} (\delta_j-s)^{\alpha-\mu-1}h(s)ds \\ &+ \frac{1}{\Gamma(\alpha-\mu)(b-a)^3}\int_a^b (t-a)^3(b-s)^{\alpha-\mu-1}h(s)ds \\ &- \frac{1}{\Gamma(\alpha-\mu)(b-a)^3}\int_a^b (t-a)^3(b-s)^{\alpha-\mu-1}h(s)ds \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha-\mu)(b-a)^3}\int_a^t (t-a)^3(b-s)^{\alpha-\mu-1}h(s)ds \\
&+ \frac{1}{\Gamma(\alpha-\mu)(b-a)^3}\int_t^b (t-a)^3(b-s)^{\alpha-\mu-1}h(s)ds \\
&- \frac{1}{\Gamma(\alpha-\mu)(b-a)^3}\int_a^t (b-a)^3(t-s)^{\alpha-\mu-1}h(s)ds \\
&+ \left[\frac{1}{\Delta\Gamma(\alpha-\mu)} - \frac{1}{\Gamma(\alpha-\mu)(b-a)^3}\right]\int_a^b (t-a)^3(b-s)^{\alpha-\mu-1}h(s)ds \\
&- \frac{1}{\Delta\Gamma(\alpha-\mu)}\sum_{j=1}^{\infty} k_j \int_a^{\delta_j} (t-a)^3(\delta_j-s)^{\alpha-\mu-1}h(s)ds \\
&= \frac{1}{\Gamma(\alpha-\mu)(b-a)^3}\int_a^t \left[(t-a)^3(b-s)^{\alpha-\mu-1} - (b-a)^3(t-s)^{\alpha-\mu-1}\right]h(s)ds \\
&+ \frac{1}{\Gamma(\alpha-\mu)(b-a)^3}\int_t^b (t-a)^3(b-s)^{\alpha-\mu-1}h(s)ds \\
&+ \frac{(b-a)^3-\Delta}{\Delta\Gamma(\alpha-\mu)(b-a)^3}\int_a^b (t-a)^3(b-s)^{\alpha-\mu-1}h(s)ds \\
&- \frac{1}{\Delta\Gamma(\alpha-\mu)}\sum_{j=1}^{\infty} k_j \int_a^{\delta_j} (t-a)^3(\delta_j-s)^{\alpha-\mu-1}h(s)ds \\
&= \int_a^b H(t,s)h(s)ds + \frac{(t-a)^3}{\Delta}\left[\frac{1}{\Gamma(\alpha-\mu)(b-a)^3}\sum_{j=1}^{\infty} k_j \int_a^b (\delta_j-a)^3(b-s)^{\alpha-\mu-1}h(s)ds\right. \\
&\left.- \frac{1}{\Gamma(\alpha-\mu)(b-a)^3}\sum_{j=1}^{\infty} k_j \int_a^{\delta_j} (b-a)^3(\delta_j-s)^{\alpha-\mu-1}h(s)ds\right] \\
&= \int_a^b H(t,s)h(s)ds + \frac{(t-a)^3}{\Delta}\sum_{j=1}^{\infty} k_j \int_a^b H(\delta_j,s)h(s)ds \\
&= \int_a^b G(t,s)h(s)ds.
\end{aligned}
$$

The proof is now finished. □

Lemma 3. *For $s,\ t \in [a,b]$, we have*
(i) $H(t,s) \le \frac{w(s)}{\Gamma(\alpha-\mu)}$,
(ii) $H(t,s) \ge \frac{v(t)w(s)}{\Gamma(\alpha-\mu)(b-a)^3}$,
where $w(s) = (b-s)^{\alpha-\mu-1}$, $v(t) = (t-a)^3(b-t)$.

Proof. (i) By Lemma 2, we have

$$
\begin{aligned}
H(t,s) &= \frac{1}{\Gamma(\alpha-\mu)(b-a)^3}\left[(t-a)^3(b-s)^{\alpha-\mu-1} - (t-s)^{\alpha-\mu-1}(b-a)^3\right] \\
&\le \frac{1}{\Gamma(\alpha-\mu)(b-a)^3}(t-a)^3(b-s)^{\alpha-\mu-1} \\
&\le \frac{1}{\Gamma(\alpha-\mu)(b-a)^3}(b-a)^3(b-s)^{\alpha-\mu-1} \\
&= \frac{1}{\Gamma(\alpha-\mu)}(b-s)^{\alpha-\mu-1},
\end{aligned}
$$

we can get $H(t,s) \leq \frac{w(s)}{\Gamma(\alpha-\mu)}$, where $w(s) = (b-s)^{\alpha-\mu-1}$.
(ii) Case 1: $a \leq s \leq t \leq b$

$$\begin{aligned} H(t,s) &= \frac{1}{\Gamma(\alpha-\mu)(b-a)^3}\Big[(t-a)^3(b-s)^{\alpha-\mu-1} - (t-s)^{\alpha-\mu-1}(b-a)^3\Big] \\ &= \frac{1}{\Gamma(\alpha-\mu)(b-a)^4}\Big[(b-a)(t-a)^3(b-s)^{\alpha-\mu-1} - (b-a)(t-s)^{\alpha-\mu-1}(b-a)^3\Big] \\ &\geq \frac{1}{\Gamma(\alpha-\mu)(b-a)^4}\Big[(t-a)^3(b-s)^{\alpha-\mu} - (b-a)^4(t-s)^{\alpha-\mu-1}\Big] \\ &= \frac{1}{\Gamma(\alpha-\mu)(b-a)^4}(t-a)^3(b-s)^{\alpha-\mu-1}\left[(b-s) - \frac{(b-a)^4(t-s)^{\alpha-\mu-1}}{(t-a)^3(b-s)^{\alpha-\mu-1}}\right], \end{aligned}$$

let $l(s) = (b-s) - \frac{(b-a)^4(t-s)^{\alpha-\mu-1}}{(t-a)^3(b-s)^{\alpha-\mu-1}}$, then

$$l'(s) = \frac{(b-a)^4}{(t-a)^3}(\alpha-\mu-1)(t-s)^{\alpha-\mu-2}(b-s)^{\mu-\alpha}(b-t) - 1 \leq 0, \ (s \to t),$$

so we have

$$H(t,s) \geq \frac{1}{\Gamma(\alpha-\mu)(b-a)^4}(t-a)^3(b-s)^{\alpha-\mu-1}(b-t).$$

Case 2: $a \leq t \leq s \leq b$

$$\begin{aligned} H(t,s) &= \frac{1}{\Gamma(\alpha-\mu)(b-a)^3}(t-a)^3(b-s)^{\alpha-\mu-1} \\ &= \frac{1}{\Gamma(\alpha-\mu)(b-a)^3(b-t)}(b-t)(t-a)^3(b-s)^{\alpha-\mu-1} \\ &\geq \frac{1}{\Gamma(\alpha-\mu)(b-a)^4}(b-t)(t-a)^3(b-s)^{\alpha-\mu-1}. \end{aligned}$$

Combining Cases 1 and 2, we have $w(s) = (b-s)^{\alpha-\mu-1}$, $v(t) = (t-a)^3(b-t)$ and we can get $H(t,s) \geq \frac{v(t)w(s)}{\Gamma(\alpha-\mu)(b-a)^4}$. □

Lemma 4. *The properties of the Green function $G(t,s)$ are as follows:*
(i) $G(t,s) : [a,b] \times [a,b] \to [0,+\infty)$ is continuous;
(ii) $G(t,s) \leqslant \frac{w(s)}{\Gamma(\alpha-\mu)}w_0$;
(iii) $G(t,s) \geqslant \frac{v(t)w(s)}{\Gamma(\alpha-\mu)(b-a)^4}$.
where $w(t) = (b-t)^{\alpha-\mu-1}$, $v(t) = (t-a)^3(b-t)$, $w_0 = 1 + \frac{(b-a)^3}{\Delta}\sum_{j=1}^{\infty} k_j$.

Proof. (i) According to the expressions of $H(t,s)$ and $G(t,s)$, we can get $G(t,s)$ is continuous in $[a,b] \times [a,b]$.
(ii) By Lemma 3, we have $H(t,s) \leq \frac{w(s)}{\Gamma(\alpha-\mu)}$. Then,

$$\begin{aligned} G(t,s) &= H(t,s) + \frac{(t-a)^3}{\Delta}\sum_{j=1}^{\infty} k_j H(\delta_j, s) \\ &\leq \frac{w(s)}{\Gamma(\alpha-\mu)} + \frac{(b-a)^3}{\Delta}\sum_{j=1}^{\infty} k_j \frac{w(s)}{\Gamma(\alpha-\mu)} \\ &= \frac{w(s)}{\Gamma(\alpha-\mu)}\left[1 + \frac{(b-a)^3}{\Delta}\sum_{j=1}^{\infty} k_j\right] \\ &= \frac{w(s)}{\Gamma(\alpha-\mu)}w_0. \end{aligned}$$

(iii) By Lemma 3, we have $H(t,s) \geq \frac{v(t)w(s)}{\Gamma(\alpha-\mu)(b-a)^4}$. Then,

$$G(t,s) \geq H(t,s) \geq \frac{v(t)w(s)}{\Gamma(\alpha-\mu)(b-a)^4}.$$

The proof is now finished. □

Let $E = C[a,b],\ \|u\| = \max_{a\leq t\leq b}|u(t)|$, then $(E, \|.\|)$ is Banach space. And we have

$$P = \{u \in E : u(t) \geq 0,\ t \in [a,b]\},$$

$$K = \left\{u \in P : u(t) \geq \frac{v(t)}{(b-a)^4 w_0}\|u\|,\ t \in [a,b]\right\},$$

where w_0 is given in Lemma 4. Obviously, K is a sub-cone of P. And we give some definitions: $K_r = \{u \in K : \|u\| < r\}$, $\partial K_r = \{u \in K : \|u\| = r\}$, $\overline{K_r} = \{u \in K : \|u\| \leq r\}$.

Next, assume the following hypotheses hold.

(H1) $e : (a,b) \to [0,+\infty)$ is non-negative, $e(t) \neq 0$ and $e(t)$ may be singular at $t = a,\ t = b$ and

$$\int_a^b w(s)e(s)ds < +\infty.$$

(H2) $f : [a,b] \times (0,+\infty) \times (0,+\infty) \to [0,+\infty)$ is continuous, and $\forall\ 0 < r < j < +\infty$,

$$\limsup_{m\to+\infty}\left\{\sup \int_{k(m)} w(s)e(s)f(s,x_1(s),x_2(s))ds \mid x_1 \in \overline{K}_{\bar{j}}\backslash K_r, x_2 \in \overline{K}_j\backslash K_r\right\} = 0,$$

where

$$k(m) = \left[a, a+\frac{1}{m}\right]\bigcup\left[b-\frac{1}{m}, b\right],\ \bar{j} = \frac{1}{\Gamma(\mu+1)}(b-a)^\mu j.$$

(H3) For any $t \in [a,b]$, $x_1,\ x_2,\ y_1,\ y_2 \in K$, there exist real number $l_1,\ l_2$ such that

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq l_1|x_1 - x_2| + l_2|y_1 - y_2|.$$

Nonlinear operator $A : K\backslash\{0\} \to P$ and linear operator $J : E \to E$ are defined as follows:

$$Au(t) = \int_a^b G(t,s)e(s)f(s, I_{a+}^\mu u(s), u(s))ds,\ t \in [a,b].$$

$$Ju(t) = \int_a^b G(t,s)e(s)u(s)ds,\ t \in [a,b]. \tag{3}$$

Lemma 5 (Arzela–Ascoli Theorem [26])**.** *Let (T,d) be a compact metric space and $A \subseteq C(T)$. Then the following assertions are equivalent:*
(i) A is relatively norm compact;
(ii) A is uniformly bounded, i.e., $\exists\ M > 0$ such that $\|f(t)\| \leq M$, $\forall\ f \in A$, $\forall\ t \in T$, and A is equicontinuous, i.e., $\forall\ \varepsilon > 0$, $\exists\ \delta_\varepsilon > 0$ such that $\forall\ x,\ y \in T$ with $d(x,y) < \delta_\varepsilon$ it follows that $|f(x) - f(y)| < \varepsilon$, $\forall\ f \in A$;
(iii) Any sequence $(f_n)_{n\in\mathbb{N}} \subseteq A$ contains a uniformly convergent subsequence.

Lemma 6 (Krein–Rutmann's Theorem [27])**.** *Assume $J : E \to E$, where J is a linear operator and is continuous, and $J(K) \subset K$, where K is a total cone. If there exist a positive constant d and $\zeta \in E\backslash(-K)$ that makes $dJ(\zeta) \geq \zeta$, then the spectral radius of J be greater than 0, and which has a positive eigenfunction in regard to its first eigenvalue $\lambda = r(J)^{-1}$.*

Remark 1 (Gelfand's formula [27])**.** *The spectral radius of J meets*

$$r(J) = \lim_{n \to +\infty} \|J^n\|^{\frac{1}{n}},$$

where J is a linear bounded operator, and $\|.\|$ is the norm of the operator.

Lemma 7. *$J : E \to E$ defined by (3) is a linear operator with complete continuity under (H1), and the spectral radius $r(J)$ of J is unequal zero; furthermore, J exists as a positive eigenfunction ζ in regard to its first eigenvalue $\lambda_1 = r(J)^{-1}$.*

Proof. Step 1. We need to verify operator $J : K \to K$.

For any $u \in K$, we have

$$\|Ju\| = \max_{t \in [a,b]} \int_a^b G(t,s)e(s)u(s)ds \leq \frac{w_0}{\Gamma(\alpha - \mu)} \int_a^b w(s)e(s)u(s)ds,$$

$$Ju(t) \geq \frac{v(t)}{\Gamma(\alpha - \mu)(b-a)^4} \int_a^b w(s)e(s)u(s)ds \geq \frac{v(t)}{(b-a)^4 w_0} \|Ju\|,$$

so we get $Ju \in K$.

Step 2. We need to verify that operator J has a completely continuous property from K to K.

According to (H1) and Lemma 4(i), operator J is uniform boundedness.

For any $t_1,\ t_2 \in [a,b],\ t_1 \leq t_2$,

$$|Ju(t_1) - Ju(t_2)| = \left| \int_a^b (G(t_1,s) - G(t_2,s))e(s)u(s)ds \right| \leq \int_a^b |(G(t_1,s) - G(t_2,s))| e(s)u(s)ds,$$

by Lemma 4(i), $G(t_1,s) \to G(t_2,s)$ when $t_1 \to t_2$, then,

$$\|Ju(t_1) - Ju(t_2)\| = \max_{t \in [a,b]} |Ju(t_1) - Ju(t_2)| \to 0.$$

According to the Arzela–Ascoli theorem, operator J has a completely continuous property from K to K.

Step 3. By Krein–Rutmann's theorem, we prove that J has the first eigenvalue λ_1 and $\lambda_1 > 0$.

By the process of proof of Lemma 4, there exists $t_0 \in (a,b)$ that makes $G(t_0,t_0) > 0$. Therefore, there exists $[m,n] \subset (a,b)$ such that $t_0 \in [m,n]$ and $G(t,s) > 0$ for any $t,\ s \in [m,n]$. Choose $u \in K$ that makes $u(t_0) > 0$ and $u(t) = 0$ for all $t \notin [m,n]$.

For any $t \in [m,n]$, we have

$$Ju(t) = \int_a^b G(t,s)e(s)u(s)ds \geq \int_m^n G(t,s)e(s)u(s)ds > 0,$$

then, there exists $d > 0$ that makes $d(Ju)(t) \geq u(t),\ t \in [a,b]$.

By Lemma 6, we have that $r(J)$ is unequal to zero. For the given first eigenvalue $\lambda_1 = r(J)^{-1}$, there exists a positive eigenfunction $\varphi*$ for J that makes $\lambda_1 J\varphi* = \varphi*$. □

Lemma 8. *Suppose (H1) and (H2) hold, then operator $A : \overline{K}_j \backslash K_r \to K$ is completely continuous.*

Proof. Step 1. We need to verify operator $A : \overline{K}_j \backslash K_r \to K$.

For any $u \in \overline{K}_j \backslash K_r$, we have

$$\|Au\| = \max_{t\in[a,b]} \int_a^b G(t,s)e(s)f(s, I_{a^+}^{\mu}u(s), u(s))ds$$
$$\leq \frac{w_0}{\Gamma(\alpha-\mu)} \int_a^b w(s)e(s)f(s, I_{a^+}^{\mu}u(s), u(s))ds,$$

and

$$Au(t) \geq \frac{v(t)}{\Gamma(\alpha-\mu)(b-a)^4} \int_a^b w(s)e(s)f(s, I_{a^+}^{\mu}u(s), u(s))ds$$
$$\geq \frac{v(t)}{\Gamma(\alpha-\mu)(b-a)^4} \frac{\Gamma(\alpha-\mu)}{w_0}\|Au\|$$
$$= \frac{v(t)}{(b-a)^4 w_0}\|Au\|,$$

so operator A from $\overline{K}_j \backslash K_r$ to K.

Step 2. We prove $A : \overline{K}_j \backslash K_r \to K$ is well defined. This implies that we need to prove

$$\sup_{u\in\overline{K}_j\backslash K_r} \int_a^b w(s)e(s)f(s, I_{a^+}^{\mu}u(s), u(s))ds < +\infty, \ \forall\, r > 0.$$

For $\forall\, u \in \overline{K}_j \backslash K_r$,

$$I_{a^+}^{\mu}u(t) = \frac{1}{\Gamma(\mu)} \int_a^t (t-s)^{\mu-1}u(s)ds$$
$$\leq \frac{1}{\Gamma(\mu)}\|u\| \int_a^t (t-s)^{\mu-1}ds$$
$$= \frac{1}{\Gamma(\mu+1)}\|u\|(t-a)^{\mu}$$
$$\leq \frac{1}{\Gamma(\mu+1)}\|u\|(b-a)^{\mu}$$
$$\leq \frac{1}{\Gamma(\mu+1)}(b-a)^{\mu}j,$$

that means $\|I_{a^+}^{\mu}u\| \leq \frac{1}{\Gamma(\mu+1)}(b-a)^{\mu}j = \bar{j}$.

By (H2), there must be a non-negative integer $v_0 > 1$ such that

$$\sup_{u\in\overline{K}_j\backslash K_r} \int_{k(v_0)} w(s)e(s)f(s, I_{a^+}^{\mu}u(s), u(s))ds < \frac{\Gamma(\alpha-\mu)}{w_0}.$$

Choosing

$$\varrho = \min\left\{\frac{1}{(b-a)^3 w_0}, \frac{1}{\Gamma(\mu)w_0}\right\},$$
$$\overline{\varrho} = \max\left\{1, \frac{(b-a)^{\mu}}{\Gamma(\mu+1)}\right\}.$$

$\forall\, u \in \overline{K}_j \backslash K_r$, we have

$$u(t) \leq \|u\| \leq j \leq \overline{\varrho}j,$$
$$u(t) \geq \frac{v(t)}{(b-a)^4 w_0}\|u\| = \frac{(t-a)^3(b-t)\|u\|}{(b-a)^4 w_0},$$

then,

$$u(t) \geq \frac{(t-a)^3(b-a)\|u\|}{(b-a)^4 w_0} = \frac{(t-a)^3\|u\|}{(b-a)^3 w_0} \geq (t-a)^3 \varrho \|u\| \geq (t-a)^\mu \varrho \|u\|.$$

For $I_{a^+}^{\mu} u(t)$, we have

$$I_{a^+}^{\mu} u(t) \leq \frac{1}{\Gamma(\mu+1)} \|u\| (b-a)^\mu \leq \overline{\varrho} \|u\| \leq \overline{\varrho} j,$$

and

$$\begin{aligned} I_{a^+}^{\mu} u(t) &= \frac{1}{\Gamma(\mu)} \int_a^t (t-s)^{\mu-1} u(s) ds \\ &\geq \frac{1}{\Gamma(\mu)} \int_a^t (t-s)^{\mu-1} \frac{(s-a)^3(b-s)\|u\|}{(b-a)^4 w_0} ds \\ &= \frac{\|u\|}{\Gamma(\mu)(b-a)^4 w_0} \int_a^t (t-s)^{\mu-1} (s-a)^3 (b-s) ds \\ &\geq \frac{\|u\|}{\Gamma(\mu)(b-a)^4 w_0} \int_a^t (t-a)^{\mu-1} (s-a)^3 (b-s) ds, \end{aligned}$$

then,

$$I_{a^+}^{\mu} u(t) \geq \frac{\|u\|}{\Gamma(\mu)(b-a)^4 w_0} \int_a^t (t-a)^{\mu-1} (b-a)^3 (b-a) ds = \frac{\|u\|}{\Gamma(\mu) w_0} (t-a)^\mu \geq \|u\| (t-a)^\mu \varrho.$$

$\forall\, t \in \left[a + \frac{1}{\nu_0}, b - \frac{1}{\nu_0}\right]$, we have

$$(\frac{1}{\nu_0})^\mu \varrho r \leq u(t) \leq \overline{\varrho} j,$$

$$(\frac{1}{\nu_0})^\mu \varrho r \leq I_{a^+}^{\mu} u(t) \leq \overline{\varrho} j.$$

Thus,

$$\begin{aligned} &\sup_{u \in \overline{K}_j \setminus K_r} \int_a^b \frac{w_0}{\Gamma(\alpha-\mu)} w(s) e(s) f(s, I_{a^+}^{\mu} u(s), u(s)) ds \\ &\leq \sup_{u \in \overline{K}_j \setminus K_r} \int_{k(\nu_0)} \frac{w_0}{\Gamma(\alpha-\mu)} w(s) e(s) f(s, I_{a^+}^{\mu} u(s), u(s)) ds \\ &+ \sup_{u \in \overline{K}_j \setminus K_r} \int_{a+\frac{1}{\nu_0}}^{b-\frac{1}{\nu_0}} \frac{w_0}{\Gamma(\alpha-\mu)} w(s) e(s) f(s, I_{a^+}^{\mu} u(s), u(s)) ds \\ &\leq 1 + D_1 \int_a^b \frac{w_0}{\Gamma(\alpha-\mu)} w(s) e(s) ds \\ &< +\infty, \end{aligned}$$

where

$$D_1 = \max\left\{ f(t, x_1, x_2) : (t, x_1, x_2) \in \left[a + \frac{1}{\nu_0}, b - \frac{1}{\nu_0}\right] \times \left[(\frac{1}{\nu_0})^\mu \varrho r, \overline{\varrho} j\right] \times \left[(\frac{1}{\nu_0})^\mu \varrho r, \overline{\varrho} j\right] \right\}.$$

So $A : \overline{K}_j \setminus K_r \to K$ is well defined, and A has the uniformly bounded property on any bounded set.

Step 3. We prove that $A : \overline{K}_j \backslash K_r \to K$ is continuous.
$\forall\, \varepsilon > 0$, by (H2), there must be a non-negative integer $\nu_0 > 1$ that makes

$$\sup \int_{k(\nu_0)} w(s)e(s)f(s, I^{\mu}_{a^+}u(s), u(s))ds < \frac{\varepsilon\Gamma(\alpha-\mu)}{4w_0},$$

where

$$x_1 \in \overline{K}_{\bar{j}} \backslash K_r,\ x_2 \in \overline{K}_j \backslash K_r.$$

$\forall\, u_k,\ u_0 \in \overline{K}_j \backslash K_r$, we have $\|u_k - u_0\| \to 0, (k \to \infty)$. By $f(t, x_1, x_2)$ has uniformly continuous property on

$$\left[a + \frac{1}{\nu_0}, b - \frac{1}{\nu_0}\right] \times \left[(\frac{1}{\nu_0})^{\mu} \varrho r, \overline{\varrho} j\right] \times \left[(\frac{1}{\nu_0})^{\mu} \varrho r, \overline{\varrho} j\right],$$

we have

$$\lim_{k \to +\infty} |f(s, I^{\mu}_{a^+}u_k(s), u_k(s)) - f(s, I^{\mu}_{a^+}u_0(s), u_0(s))| = 0.$$

By the Lebesgue control convergence theorem,

$$\lim_{k \to +\infty} \int_{a+\frac{1}{\nu_0}}^{b-\frac{1}{\nu_0}} w(s)e(s)|f(s, I^{\mu}_{a^+}u_k(s), u_k(s)) - f(s, I^{\mu}_{a^+}u_0(s), u_0(s))|ds = 0,$$

and thus, $\forall\, N > 0$, for $k > N$,

$$\int_{a+\frac{1}{\nu_0}}^{b-\frac{1}{\nu_0}} w(s)e(s)|f(s, I^{\mu}_{a^+}u_k(s), u_k(s)) - f(s, I^{\mu}_{a^+}u_0(s), u_0(s))|ds < \frac{\varepsilon\Gamma(\alpha-\mu)}{2w_0}.$$

For $k > N$,

$$\begin{aligned}
\|Au_k - Au_0\| \leq & \sup_{u \in \overline{K}_j \backslash K_r} \int_{k(\nu_0)} \frac{w_0}{\Gamma(\alpha-\mu)} w(s)e(s)f(s, I^{\mu}_{a^+}u_k(s), u_k(s))ds \\
& + \sup_{u \in \overline{K}_j \backslash K_r} \int_{k(\nu_0)} \frac{w_0}{\Gamma(\alpha-\mu)} w(s)e(s)f(s, I^{\mu}_{a^+}u_0(s), u_0(s))ds \\
& + \int_{a+\frac{1}{\nu_0}}^{b-\frac{1}{\nu_0}} \frac{w_0}{\Gamma(\alpha-\mu)} w(s)e(s)|f(s, I^{\mu}_{a^+}u_k(s), u_k(s)) - f(s, I^{\mu}_{a^+}u_0(s), u_0(s))|ds \\
& < 2\frac{w_0}{\Gamma(\alpha-\mu)} \frac{\varepsilon\Gamma(\alpha-\mu)}{4w_0} + \frac{\varepsilon\Gamma(\alpha-\mu)}{2w_0} \frac{w_0}{\Gamma(\alpha-\mu)} \\
& = \varepsilon.
\end{aligned}$$

Thus, $A : \overline{K}_j \backslash K_r \to K$ is continuous.
Step 4. For any bounded set Ω and $\Omega \subset \overline{K}_j \backslash K_r$, we prove that $A(\Omega)$ is equicontinuous.
By (H2), $\forall\, \varepsilon > 0$, there is a natural number $\omega_1 > 1$ so that

$$\sup_{x_2 \in \overline{K}_j \backslash K_r} \int_{k(\omega_1)} w(s)e(s)f(s, I^{\mu}_{a^+}u(s), u(s))ds < \frac{\varepsilon\Gamma(\alpha-\mu)}{4w_0},$$

$$D_2 = \max\left\{ f(t, x_1, x_2) : (t, x_1, x_2) \in \left[a + \frac{1}{\omega_1}, b - \frac{1}{\omega_1}\right] \times \left[(\frac{1}{\omega_1})^{\mu} \varrho r, \overline{\varrho} j\right] \times \left[(\frac{1}{\omega_1})^{\mu} \varrho r, \overline{\varrho} j\right] \right\}.$$

By Lemma 4(i), $G(t,s)$ is uniformly continuous on $[a,b]\times[a,b]$.
$\forall\,\varepsilon>0, \exists\,\delta>0, \forall\,s\in[a+\frac{1}{\omega_1}, b-\frac{1}{\omega_1}]$, $|t-t'|<\delta$, $t,\ t'\in[a,b]$,

$$|G(t,s)-G(t',s)|\le\frac{\varepsilon}{2}\left(D_2\int_{a+\frac{1}{\omega_1}}^{b-\frac{1}{\omega_1}}e(s)ds\right)^{-1},$$

then, $\forall\,|t-t'|<\delta,\ t,\ t'\in[a,b],\ u\in\Omega,$

$$\begin{aligned}\|Au(t)-Au(t')\| &\le 2\sup_{u\in\overline{K}_j\setminus K_r}\int_{k(\omega_1)}\frac{w_0}{\Gamma(\alpha-\mu)}w(s)e(s)f(s,I_{a+}^{\mu}u(s),u(s))ds\\ &+\int_{a+\frac{1}{\omega_1}}^{b-\frac{1}{\omega_1}}|G(t,s)-G(t',s)|e(s)f(s,I_{a+}^{\mu}u(s),u(s))ds\\ &<2\frac{w_0}{\Gamma(\alpha-\mu)}\frac{\varepsilon\Gamma(\alpha-\mu)}{4w_0}+\frac{\varepsilon}{2}\\ &<\varepsilon,\end{aligned}$$

so $A(\Omega)$ is equicontinuous. According to the Arzela–Ascoli theorem, operator $A:\overline{K}_j\setminus K_r\to K$ is completely continuous. □

3. Main Results

Now, we need to prove the existence of solutions. In this part, we first give the following lemmas.

Lemma 9 ([27])**.** *Suppose K is a cone within Banach space E. Let $A:\overline{K_r}\to K$ be a completely continuous operator. In the case of $u_0\in K\setminus\theta$ so that $u-Au\neq\lambda u_0$ for an arbitrary $u\in\partial K_r$ and $\lambda\ge 0$, and thus $i(A,K_r,K)=0$. In the case of $Au=\lambda u$ for an arbitrary $u\in\partial K_r$ and $\lambda\ge 1$, and thus $i(A,K_r,K)=1$.*

Lemma 10. *Assume that (H1) holds, and then J has an eigenvalue $\tilde{\lambda}_1$ that makes*

$$\lim_{\varepsilon\to 0+}\lambda_\varepsilon=\tilde{\lambda}_1.$$

Proof. Let $\ldots\le\varepsilon_n\le\ldots\le\varepsilon_2\le\varepsilon_1$ and $\varepsilon_n\to 0$ $(n\to+\infty)$. Then for any $n<m$, $\zeta\in E$ and $t\in[a,b]$, we get

$$J_{\varepsilon_n}\zeta(t)\le J_{\varepsilon_m}\zeta(t)\le J_\varepsilon\zeta(t),$$

$$J_{\varepsilon_n}^k\zeta(t)\le J_{\varepsilon_m}^k\zeta(t)\le J_\varepsilon^k\zeta(t),\ k=2,3,\ldots,$$

where $J_{\varepsilon_n}^k=J(J_{\varepsilon_n}^{k-1}),\ k=2,3,\ldots$, so we have

$$\|J_{\varepsilon_n}^k\|\le\|J_{\varepsilon_m}^k\|\le\|J_\varepsilon^k\|,\ k=1,2,\ldots.$$

By the Remark 1, we get

$$r(J_{\varepsilon_n})\le r(J_{\varepsilon_m})\le r(J_\varepsilon),$$

$$\lambda_{\varepsilon_n}\ge\lambda_{\varepsilon_m}\ge\lambda_1,$$

where λ_1 is the first eigenvalue of J. Since λ_{ε_n} is monotonous and has a lower bound λ_1, we can obtain

$$\lim_{n\to+\infty}\lambda_{\varepsilon_n}=\tilde{\lambda}_1.$$

Then, we will prove $\tilde{\lambda}_1$ is an eigenvalue of J.

Suppose ζ_{ε_n} is one of the positive eigenfunctions of J_{ε_n} in regard to λ_{ε_n} with $\|\zeta_{\varepsilon_n}\| = 1$, that is,

$$\zeta_{\varepsilon_n}(t) = \lambda_{\varepsilon_n} \int_{a-\varepsilon_n}^{b+\varepsilon_n} G(t,s)e(s)\zeta_{\varepsilon_n}(s)ds = \lambda_{\varepsilon_n} J_{\varepsilon_n}\zeta_{\varepsilon_n}(t),\ n = 1,2,\ldots.$$

It is worth noting that

$$\|J_{\varepsilon_n}\zeta_{\varepsilon_n}\| = \max_{a\le t\le b} \int_{a-\varepsilon_n}^{b+\varepsilon_n} G(t,s)e(s)\zeta_{\varepsilon_n}(s)ds \le \int_a^b \frac{w_0}{\Gamma(\alpha-\mu)} w(s)e(s)ds,$$

so $\{J_{\varepsilon_n}\zeta_{\varepsilon_n}\} \subset E$ is uniform boundedness. For any $n \in N,\ t_1,\ t_2 \in [a,b]$, we have

$$|J_{\varepsilon_n}\zeta_{\varepsilon_n}(t_1) - J_{\varepsilon_n}\zeta_{\varepsilon_n}(t_2)| \le \int_{a-\varepsilon_n}^{b+\varepsilon_n} |G(t_1,s) - G(t_2,s)|e(s)\zeta_{\varepsilon_n}(s)ds \to 0,\ t_1 \to t_2,$$

so $\{J_{\varepsilon_n}\zeta_{\varepsilon_n}\} \subset E$ is equicontinuous. According to Lemma 5 and $\lim_{n\to+\infty} \lambda_{\varepsilon_n} = \tilde{\lambda}_1$, we have $\zeta_{\varepsilon_n} \to \zeta_0\ (n \to +\infty),\ \|\zeta_0\| = 1$. Then we get

$$\zeta_0(t) = \tilde{\lambda}_1 \int_a^b G(t,s)e(s)\zeta_0(s)ds = \tilde{\lambda}_1 J\zeta_0(t),\ t \in [a,b].$$

The proof is now finished. □

Theorem 1. *Assume the conditions (H1), (H2) are satisfied, and*

$$\liminf_{x_1,x_2\to 0^+} \min_{t\in[a,b]} \frac{f(t,x_1,x_2)}{x_1+x_2} > \lambda_1, \tag{4}$$

$$\limsup_{x_2\to+\infty} \max_{t\in[a,b]} \frac{f(t,x_1,x_2)}{x_2} < \lambda_1. \tag{5}$$

the BVP(1) has at least one positive solution, where λ_1 is the first eigenvalue of J defined by (3).

Proof. According to (4), there exists $r > 0$, for $t \in [a,b]$ such that

$$f(t,x_1,x_2) \ge \lambda_1(x_1+x_2),\ 0 < x_i \le r,\ i = 1,2, \tag{6}$$

let $r_0 = \min\left\{r, \frac{r}{\bar{\varrho}}\right\}$, for any $u \in \partial K_{r_0}$, since

$$0 < I_{a^+}^{\mu} u(s) \le \bar{\varrho} r_0 \le r,\ 0 < u(s) \le r_0 \le r,\ s \in [a,b], \tag{7}$$

we have from (6), (7) that

$$\begin{aligned} Au(t) &= \int_a^b G(t,s)e(s)f(s, I_{a^+}^{\mu}u(s), u(s))ds \\ &\ge \lambda_1 \int_a^b G(t,s)e(s)(I_{a^+}^{\mu}u(s) + u(s))ds \\ &\ge \lambda_1 \int_a^b G(t,s)e(s)u(s)ds \\ &= \lambda_1 Ju(t),\ t \in [a,b]. \end{aligned}$$

By Lemma 7, J has a positive eigenfunction ζ corresponding to λ_1, that is $\zeta = \lambda_1 J\zeta$.
Step 1. We will show

$$u - Au \ne d\zeta,\ u \in \partial K_{r_0},\ d \ge 0. \tag{8}$$

If not, there exist $u_0 \in \partial K_{r_0}$ and $d_0 \geq 0$ such that $u_0 - Au_0 \neq d_0\zeta$, then $d_0 > 0$, we have $u_0 = Au_0 + d_0\zeta \geq d_0\zeta$.

Let $\tilde{d} = \sup\{d | u_0 \geq d\zeta\}$, then $\tilde{d} \geq d,\ u_0 \geq \tilde{d}\zeta,\ \lambda_1 Ju_0 \geq \lambda_1 \tilde{d} J\zeta = \tilde{d}\zeta$. Thus, we have

$$u_0 = Au_0 + d_0\zeta \geq \lambda_1 Ju_0 + d_0\zeta \geq \tilde{d}\zeta + d_0\zeta = (\tilde{d} + d_0)\zeta,$$

which contradicts the definition of $\tilde{d}$. So (8) holds and by Lemma 9, we get

$$i(A, K_{r_0}, K) = 0.$$

According to (5), we choose a constant $0 < \gamma < 1$ makes

$$\limsup_{x_2 \to +\infty} \max_{t \in [a,b]} \frac{f(t, x_1, x_2)}{x_2} < \gamma\lambda_1. \tag{9}$$

Let the linear operator $\bar{J}$ satisfy $\bar{J}u = \gamma\lambda_1 Ju$, then $\bar{J} : E \to E$ is a bounded linear operator and $\bar{J}(K) \subset K$.

Further, we have

$$\bar{J}\zeta = \gamma\lambda_1 J\zeta = \gamma\zeta,$$

which means the spectral radius of $\bar{J}$ is $r(\bar{J}) = \gamma$ and $\bar{J}$ has the first eigenvalue $\gamma^{-1} > 1$. By Remark 1, we have

$$\gamma = \lim_{n \to +\infty} \|\bar{J}^n\|^{\frac{1}{n}}.$$

For the above equation, let $\varepsilon_0 = \frac{1}{2}(1 - \gamma)$ and there is a large enough natural number N that when $n \geq N$, we get $\|\bar{J}^n\| \leq [\gamma + \varepsilon_0]^n$. For any $u \in E$, we define

$$\|u\|_* = \sum_{i=1}^{N} [\gamma + \varepsilon_0]^{N-i} \|\bar{J}^{i-1}u\|,$$

where $\bar{J}^0 = I$ is the unit operator. $\|.\|_*$ is another norm of E.

Combining (5) and (9), there is $j_1 > r$, we have

$$f(t, x_1, x_2) \leq \gamma\lambda_1 x_2,\ x_1 > 0,\ x_2 \geq j_1,\ t \in [a, b].$$

Choosing $j > \max\left\{ j_1, \frac{2(\gamma + \varepsilon_0^{N-1})^{-1}}{\varepsilon_0} B_* \right\}$, where $B_* = \|B\|_*$ and

$$B = \sup_{u \in K_{j_1}} \int_a^b \frac{w_0}{\Gamma(\alpha - \mu)} w(s)e(s)f(s, I^{\mu}_{a^+}u(s), u(s))ds < +\infty.$$

Step 2. We will show

$$Au \neq du,\ u \in \partial K_j,\ d \geq 1. \tag{10}$$

If not, there exist $u_1 \in \partial K_j$ and $d_1 \geq 1$ such that $Au_1 = d_1 u_1$. Let $\bar{u}(t) = \min\{u_1(t), j_1\}$ and $D(u_1) = \{t \in [a, b] : u_1(t) > j_1\}$.

For $\bar{u} \in C([a, b], [0, +\infty))$, we get $\frac{v(t)}{(b-a)^4 w_0} j \leq u_1(t) \leq \|u_1\| = j$, and hence, combining with $j_1 < j$, there exists $a < t_0 < b$ such that $u(t_0) = j_1$. For $t \in [a, b]$, we have $\bar{u}(t) \leq j$, and $\bar{u}(t_0) = \min\{u_1(t_0), j_1\} = j_1$, then we have $\|u(t)\| = j_1$, that means $u \in \partial K_{j_1}$.

For any $t \in D(u_1)$, $u_1(t) \geq j_1,\ I^{\mu}_{a^+}u(t) \geq 0$, we have

$$
\begin{aligned}
Au_1(t) &= \int_a^b G(t,s)e(s)f(s, I_{a^+}^{\mu}u_1(s), u_1(s))ds \\
&\le \int_{D(u_1)} G(t,s)e(s)f(s, I_{a^+}^{\mu}u_1(s), u_1(s))ds + \int_{[a,b]\setminus D(u_1)} G(t,s)e(s)f(s, I_{a^+}^{\mu}u_1(s), u_1(s))ds \\
&\le \gamma\lambda_1 \int_a^b G(t,s)e(s)u_1(s)ds + \int_a^b \frac{w_0}{\Gamma(\alpha-\mu)} w(s)e(s)f(s, I_{a^+}^{\mu}u_1(s), u_1(s))ds \\
&\le \bar{J}u_1(t) + B.
\end{aligned}
$$

By $\bar{J} : E \to E$ is a bounded linear operator and $\bar{J}(K) \subset K$, we have

$$
0 \le \bar{J}^k(Au_1)(t) \le \bar{J}^k(\bar{J}u_1 + B)(t),\ k = 0, 1, \ldots, n-1.
$$

Further, we get

$$
\|\bar{J}^k(Au_1)\| \le \|\bar{J}^k(\bar{J}u_1 + B)\|,\ k = 0, 1, \ldots, n-1,
$$

so

$$
\begin{aligned}
\|Au_1\|_* &= \sum_{i=1}^{N} [\gamma + \varepsilon_0]^{N-i} \|\bar{J}^{i-1}(Au_1)\| \\
&\le \sum_{i=1}^{N} [\gamma + \varepsilon_0]^{N-i} \|\bar{J}^{i-1}(\bar{J}u_1 + B)\| \\
&= \|\bar{J}u_1 + B\|_*.
\end{aligned}
$$

By $u_1 \in \partial K_j$, we have

$$
\|u_1\|_* > [\gamma + \varepsilon_0]^{N-1}\|u_1\| = [\gamma + \varepsilon_0]^{N-1} j > \frac{2}{\varepsilon_0} B_*,
$$

which means

$$
B_* < \frac{\varepsilon_0}{2}\|u_1\|_*.
$$

Then, we get

$$
\begin{aligned}
d_1\|u_1\|_* &= \|Au_1\|_* \le \|\bar{J}u_1\|_* + \|B\|_* \\
&= \sum_{i=1}^{N} [\gamma + \varepsilon_0]^{N-i} \|\bar{J}^i u_1\| + B_* \\
&= [\gamma + \varepsilon_0] \sum_{i=1}^{N-1} [\gamma + \varepsilon_0]^{N-i-1} \|\bar{J}^i u_1\| + \|\bar{J}^N u_1\| + B_* \\
&\le [\gamma + \varepsilon_0] \sum_{i=1}^{N-1} [\gamma + \varepsilon_0]^{N-i-1} \|\bar{J}^i u_1\| + [\gamma + \varepsilon_0]^N \|u_1\| + B_* \\
&= [\gamma + \varepsilon_0] \sum_{i=1}^{N-1} [\gamma + \varepsilon_0]^{N-i} \|\bar{J}^{i-1} u_1\| + B_* \\
&= [\gamma + \varepsilon_0]\|u_1\|_* + B_* \\
&\le [\gamma + \varepsilon_0]\|u_1\|_* + \frac{\varepsilon_0}{2}\|u_1\|_* \\
&= \left[\frac{1}{4}\gamma + \frac{3}{4}\right]\|u_1\|_*.
\end{aligned}
$$

By $1 \le d_1 \le [\frac{1}{4}\gamma + \frac{3}{4}]$, we know $\gamma \ge 1$, which contradict with $0 < \gamma < 1$. So (10) holds and

$$
i(A, K_j, K) = 1.
$$

Finally, we get

$$i(A, K_j \backslash \bar{K}_{r_0}, K) = i(A, K_j, K) - i(A, K_{r_0}, K) = 1.$$

Hence, A has at least one fixed point in $K_j \backslash \bar{K}_{r_0}$, that is to say, the BVP (1) has at least one positive solution. It is worth noting that when for any small enough $0 < \varepsilon < 1$, we define a linear operator J_ε

$$J_\varepsilon u(t) = \int_{a+\varepsilon}^{b-\varepsilon} G(t,s)e(s)u(s)ds,\ t \in [a,b].$$

By Lemma 8, we have that $J_\varepsilon : K \to K$ denotes a linear operator with complete continuity, and the spentral radius $r(J_\varepsilon)$ of J_ε is unequal to zero, and furthermore, J_ε has a positive eigenfunction ζ_ϵ in regard to its first eigenvalue $\lambda_\varepsilon = r(J_\varepsilon)^{-1}$. □

Theorem 2. *Suppose the conditions (H1), (H2) are satisfied, and*

$$\liminf_{x_1+x_2 \to +\infty} \min_{t\in[a,b]} \frac{f(t,x_1,x_2)}{x_1+x_2} > \tilde{\lambda}_1, \tag{11}$$

$$\limsup_{x_1,x_2 \to 0^+} \max_{t\in[a,b]} \frac{f(t,x_1,x_2)}{x_2} < \lambda_1, \tag{12}$$

then BVP(1) has at least one positive solution, where λ_1 is the first eigenvalue of J defined by (3), and $\tilde{\lambda}_1$ is another eigenvalue of J.

Proof. According to (11), there exists $r_0 > 0$, for $t \in [a,b]$ such that

$$f(t,x_1,x_2) \le \lambda_1 x_2,\ 0 < x_1 \le \frac{(b-a)^\mu}{\Gamma(\mu+1)} r_0,\ 0 < x_1 \le r_0, \tag{13}$$

for any $u \in \partial K_{r_0}$, since

$$0 < I_{a^+}^\mu u(s) \le \frac{(b-a)^\mu}{\Gamma(\mu+1)} r_0 \le r,\ 0 < u(s) \le r_0,\ s \in [a,b], \tag{14}$$

we have from (13), (14) that

$$\begin{aligned} Au(t) &= \int_a^b G(t,s)e(s)f(s, I_{a^+}^\mu u(s), u(s))ds \\ &\le \lambda_1 \int_a^b G(t,s)e(s)u(s)ds \\ &= \lambda_1 Ju(t),\ t \in [a,b]. \end{aligned}$$

Step 1. By Lemma 9, we will show

$$Au \ne du,\ u \in \partial K_{r_0},\ d \ge 1 \tag{15}$$

If not, there exist $u_0 \in \partial K_{r_0}$ and $d_0 \ge 1$ such that $Au_0 = d_0 u_0$, then $d_0 > 1$ and

$$d_0 u_0 = Au_0 \le \lambda_1 J u_0. \tag{16}$$

By summarizing (16), we obtain

$$d_0^n u_0 \le \lambda_1^n J^n u_0,\ n = 1, 2, \ldots,$$

then,

$$\|J^n\| \geq \frac{\|J^n u_0\|}{\|u_0\|} \geq \frac{d_0^n \|u_0\|}{\lambda_1 \|u_0\|} = \frac{d_0^n}{\lambda_1^n}.$$

By the Remark 1, we have

$$r(J) = \lim_{n \to +\infty} \|J^n\|^{\frac{1}{n}} \geq \frac{d_0}{\lambda_1} > \frac{1}{\lambda_1},$$

which contradict with $r(J) = \frac{1}{\lambda_1}$. So (15) holds and

$$i(A, K_{r_0}, K) = 1.$$

According to (12) and $\lambda_\varepsilon \to \tilde{\lambda}_1(\varepsilon \to 0^+)$, there exist a small enough $\varepsilon \in (0, \frac{1}{2})$ and $j > r$, and we get

$$f(t, x_1, x_2) \geq \lambda_\varepsilon(x_1 + x_2) \geq 2\varrho\varepsilon^\mu j,$$

where λ_ε represents the first appeared eigenvalue J_ε. Assume that ζ_ε is the positive eigenfunction of J_ε with respect to λ_ε, then $\zeta_\varepsilon = \lambda_\varepsilon J_\varepsilon \zeta_\varepsilon$. For any $u \in \partial K_j,\ t \in [a+\varepsilon, b-\varepsilon]$, by Lemma 8, we have

$$\begin{aligned} I_{a^+}^\mu u(t) + u(t) &\geq \|u\|(t-a)^\mu \varrho + \|u\|(t-a)^\mu \varrho \\ &\geq 2\varrho\varepsilon^\mu j. \end{aligned}$$

Then,

$$\begin{aligned} Au(t) &= \int_a^b G(t,s)e(s)f(s, I_{a^+}^\mu u(s), u(s))ds \\ &\geq \int_{a+\varepsilon}^{b-\varepsilon} G(t,s)e(s)f(s, I_{a^+}^\mu u(s), u(s))ds \\ &\geq \lambda_\varepsilon \int_{a+\varepsilon}^{b-\varepsilon} G(t,s)e(s)(I_{a^+}^\mu u(s) + u(s))ds \\ &\geq \lambda_\varepsilon \int_{a+\varepsilon}^{b-\varepsilon} G(t,s)e(s)u(s)ds \\ &= \lambda_\varepsilon J_\varepsilon u(t). \end{aligned}$$

Step 2. We will show

$$u - Au \neq d\zeta_\varepsilon,\ u \in \partial K_j,\ d \geq 0. \tag{17}$$

Similar to the proof of Theorem 1, (17) holds and

$$i(A, K_j, K) = 0.$$

Finally, we get

$$i(A, K_j \backslash \bar{K}_{r_0}, K) = i(A, K_j, K) - i(A, K_{r_0}, K) = -1.$$

Hence, A has at least one fixed point in $K_j \backslash \bar{K}_{r_0}$, that is to say, the BVP (1) has at least one positive solution. □

Next, we need to use the Banach fixed-point theorem to prove the uniqueness of the solution.

Theorem 3. *Suppose the conditions (H1)-(H3) are satisfied, and*

$$k = \frac{w_0 M}{\Gamma(\alpha-\mu)}\left[\frac{l_1(b-a)^\mu}{\Gamma(\mu+1)} + l_2\right] < 1$$

then the BVP(1) has a unique positive solution.

Proof. By Theorems 1 and 2, we get the BVP (1) has at least one positive solution. Now we just need to prove that A is a contractive mapping.

According to (H1), there exists $M > 0$ such that $\int_a^b w(s)e(s)ds \le M < +\infty$. For any $u,\ u_0 \in K_j \backslash \bar{K}_{r_0},\ t \in [a,b]$,

$$\begin{aligned}
\left|I_{a+}^{\mu}u(t) - I_{a+}^{\mu}u_0(t)\right| &\le \frac{1}{\Gamma(\mu)}\int_a^t (t-s)^{\mu}|u(s)-u_0(s)|ds \\
&\le \frac{1}{\Gamma(\mu)}\|u-u_0\|\int_a^t (t-s)^{\mu-1}ds \\
&= \frac{1}{\Gamma(\mu)}\frac{(t-a)^{\mu}}{\mu}\|u-u_0\| \\
&\le \frac{(b-a)^{\mu}}{\Gamma(\mu+1)}\|u-u_0\|,
\end{aligned}$$

then, we get

$$\begin{aligned}
|Au(t) - Au_0(t)| &\le \int_a^b G(t,s)e(s)|f(s, I_{a+}^{\mu}u(s), u(s)) - f(s, I_{a+}^{\mu}u_0(s), u_0(s))|ds \\
&\le \frac{w_0}{\Gamma(\alpha-\mu)}\int_a^b w(s)e(s)[l_1|I_{a+}^{\mu}u(s) - I_{a+}^{\mu}u_0(s)| + l_2|u(s)-u_0(s)|]ds \\
&\le \frac{w_0 M}{\Gamma(\alpha-\mu)}\left[\frac{l_1(b-a)^{\mu}}{\Gamma(\mu+1)}\|u-u_0\| + l_2\|u-u_0\|\right] \\
&= k\|u-u_0\|,
\end{aligned}$$

where $k = \frac{w_0 M}{\Gamma(\alpha-\mu)}\left[\frac{l_1(b-a)^\mu}{\Gamma(\mu+1)} + l_2\right] < 1$, then

$$\|Au - Au_0\| \le k\|u-u_0\|,$$

according to Banach fixed-point theorem, the BVP(1) has a unique positive solution. □

4. An Example

Example 1 ([11]). *Consider a FDE problem*

$$\begin{cases} {}^cD_{1^+}^{\frac{15}{4}}v(t) + e(t)f(t, v(t), {}^cD_{1^+}^{\frac{1}{4}}v(t)) = 0, \quad 1 < t < 2, \\ {}^cD_{1^+}^{\frac{1}{4}}v(1) = {}^cD_{1^+}^{\frac{5}{4}}v(1) = {}^cD_{1^+}^{\frac{9}{4}}v(1) = 0, \quad {}^cD_{1^+}^{\frac{1}{4}}v(2) = \sum_{j=1}^{\infty} k_j {}^cD_{1^+}^{\frac{1}{4}}v(\delta_j), \end{cases}$$

where $\alpha = \frac{15}{4},\ \mu = \frac{1}{4},\ a = 1,\ b = 2,\ e(t) = \frac{1}{\sqrt{(2-t)(t-1)}},\ k_j = \frac{1}{2j^2},\ \delta_j = \frac{1}{j^2}$, *$e(t)$ is singular at* $t = 1$ *or* $t = 2$, $f(t,x,y) = (x+y)^{-\frac{1}{5}} + |\ln y|$.

Letting $u(t) = {}^c D_{a^+}^{\mu}v(t)$, we can get

$$\begin{cases} {}^cD_{1^+}^{\frac{7}{2}}u(t) + e(t)f(t, I_{1^+}^{\frac{1}{4}}u(t), u(t)) = 0, \quad 1 < t < 2, \\ u(1) = u'(1) = u''(1) = 0, \quad u(2) = \sum_{j=1}^{\infty} k_j u(\delta_j), \end{cases} \tag{18}$$

then,

$$G(t,s) = H(t,s) + \frac{(t-1)^3}{\Delta}\sum_{j=1}^{\infty} k_j H(\delta_j, s),$$

$$\Delta = (b-a)^3 - \sum_{j=1}^{\infty} k_j(\delta_j - a)^3 \approx 1.2239,$$

$$H(t,s) = \frac{1}{\Gamma(\frac{7}{2})}\begin{cases} (t-1)^3(2-s)^{\frac{5}{2}} - (t-s)^{\frac{5}{2}}, & 1 \le s \le t \le 2, \\ (t-1)^3(2-s)^{\frac{5}{2}}, & 1 \le t \le s \le 2, \end{cases}$$

$$w_0 = 1 + \frac{1}{\Delta}\sum_{j=1}^{\infty}\frac{1}{2j^2} \approx 1.6718,$$

$$w(s) = (2-s)^{\frac{5}{2}},\ v(t) = (t-1)^3(2-t),$$

and the cone

$$K = \{u \in P : u(t) \ge 0.5981 v(t)\|u\|,\ t \in [1,2]\}.$$

Now, for any $0 < r < j < +\infty$, $u(t) \in \overline{K}_j \backslash K_r$,

$$\bar{j} = \frac{1}{\Gamma(\mu+1)}(b-a)^{\mu} j = \frac{1}{\Gamma(\frac{5}{4})} j,$$

$$\varrho = \min\left\{\frac{1}{(b-a)^3 w_0}, \frac{1}{\Gamma(\mu) w_0}\right\} = \min\{0.5982, 0.1650\} = 0.5982,$$

$$\overline{\varrho} = \max\left\{1, \frac{(b-a)^{\mu}}{\Gamma(\mu+1)}\right\} = \max\{1, 1.1033\} = 1.1033.$$

Since

$$\int_a^b w(s)e(s)ds = \int_1^2 \frac{(2-s)^{\frac{5}{2}}}{\sqrt{(2-s)(s-1)}}ds = \frac{16}{15} = M < +\infty,$$

so (H1) is satisfied. By

$$0.1650r \le (t-1)^{\frac{1}{4}}\varrho r \le u(t) \le j \le \overline{\varrho} j \le 1.1033j, \tag{19}$$

$$0.1650r \le (t-1)^{\frac{1}{4}}\varrho r \le I_{1^+}^{\frac{1}{4}} u(t) \le \overline{\varrho} j \le 1.1033j, \tag{20}$$

we have

$$|\ln u(t)| \le |\ln(t-1)^{\frac{1}{4}}\varrho r| + |\ln j| \le |\ln(t-1)^{\frac{1}{4}}| + |\ln j| + |\ln \varrho r|,\ t \in [1,2],$$

$$[u(t) + I_{1^+}^{\frac{1}{4}} u(t)]^{-\frac{1}{5}} \le (2\varrho r)^{-\frac{1}{5}}(t-1)^{-\frac{1}{20}},\ t \in [1,2],$$

then

$$\int_1^2 |\ln(t-1)^{\frac{1}{4}}|dt = \frac{1}{4} < +\infty,$$

$$\int_1^2 (t-1)^{-\frac{1}{20}} dt = \frac{20}{19} < +\infty.$$

By considering the absolute continuity of the obtained integral, one can derive that

$$\lim_{m \to +\infty}\int_{k(m)} |\ln(s-1)^{\frac{1}{4}}|ds = 0,$$

$$\lim_{m\to+\infty}\int_{k(m)}(s-1)^{-\frac{1}{20}}ds=0,$$

then,

$$\begin{aligned}
&\limsup_{m\to+\infty}\sup_{\substack{x_1\in\overline{K}_j\setminus K_r\\ x_2\in\overline{K}_j\setminus K_r}}\int_{k(m)}w(s)e(s)f(s,x_1(s),x_2(s))ds\\
=&\limsup_{m\to+\infty}\sup_{\substack{x_1\in\overline{K}_j\setminus K_r\\ x_2\in\overline{K}_j\setminus K_r}}\int_{k(m)}\frac{(2-s)^{\frac{5}{2}}}{\sqrt{(2-s)(s-1)}}\left[(I_{1^+}^{\frac{1}{4}}u(s)+u(s))^{-\frac{1}{5}}+|\ln u(s)|\right]ds\\
\le&\limsup_{m\to+\infty}\sup_{\substack{x_1\in\overline{K}_j\setminus K_r\\ x_2\in\overline{K}_j\setminus K_r}}\int_{k(m)}\frac{(2-s)^{\frac{5}{2}}}{\sqrt{(2-s)(s-1)}}\left[|\ln(s-1)^{\frac{1}{4}}|+|\ln j|+|\ln\varrho r|+(2\varrho r)^{-\frac{1}{5}}(s-1)^{-\frac{1}{20}}\right]ds\\
=&0,
\end{aligned}$$

so (H2) is satisfied. By

$$\liminf_{x_1,x_2\to0^+}\min_{t\in[1,2]}\frac{(x_1+x_2)^{-\frac{1}{5}}+|\ln(x_2)|}{x_1+x_2}=+\infty,$$

$$\limsup_{x_2\to+\infty}\max_{t\in[1,2]}\frac{(x_1+x_2)^{-\frac{1}{5}}+|\ln(x_2)|}{x_2}=0.$$

So far, all the conditions proposed in Theorem 1 have been meet, so there must be one or more positive solutions to (18).

Let $x_1,\ y_1,\ x_2,\ y_2$ satisfy (19), (20), and $x_1>x_2,\ y_1>y_2>1$, combined with the Lagrange mean value theorem, and we have

$$\begin{aligned}
|f(t,x_1,y_1)-f(t,x_2,y_2)|&=|(x_1+y_1)^{-\frac{1}{5}}-(x_2+y_2)^{-\frac{1}{5}}+|\ln y_1|-|\ln y_2||\\
&\le|(x_1+y_1)^{-\frac{1}{5}}-(x_2+y_2)^{-\frac{1}{5}}|+||\ln y_1|-|\ln y_2||\\
&\le\frac{1}{5}\xi^{-\frac{6}{5}}(|x_1-x_2|+|y_1-y_2|)+\frac{1}{\delta}|y_1-y_2|\\
&=\frac{1}{5}\xi^{-\frac{6}{5}}|x_1-x_2|+\left(\frac{1}{5}\xi^{-\frac{6}{5}}+\frac{1}{\delta}\right)|y_1-y_2|\\
&\le\frac{1}{5}\xi^{-\frac{6}{5}}|x_1-x_2|+\left(\frac{1}{5}\xi^{-\frac{6}{5}}+1\right)|y_1-y_2|,
\end{aligned}$$

where $\xi\in[x_1+y_1,x_2+y_2],\ \delta\in[y_1,y_2]$. Select $\xi=0.56$, then we have

$$l_1=\frac{1}{5}\xi^{-\frac{6}{5}}=0.4008,$$

$$l_2=\frac{1}{5}\xi^{-\frac{6}{5}}+1=1.4008,$$

so (H3) is satisfied.

$$k=\frac{w_0M}{\Gamma(\alpha-\mu)}\left[\frac{l_1(b-a)^\mu}{\Gamma(\mu+1)}+l_2\right]=\frac{1.6718\times\frac{16}{15}}{\Gamma(\frac{7}{2})}\left[\frac{0.4008}{\Gamma(1.25)}+1.4008\right]=0.9886<1.$$

By Theorem 3, we get that (18) has a unique positive solution.

5. Summary

This paper primarily investigates a class of Caputo FDEs with infinitely many points. Under certain conditions, the existence of positive solutions for the equation is established by employing the spectral analysis of related operators on cones and the fixed-point index theory. Meanwhile, the uniqueness of the positive solution is proved using the Banach fixed-point theorem. Compared with the existing literature, this study focuses on higher-order Caputo fractional singular nonlinear differential equations, where the nonlinear term involves the Caputo fractional derivative. The boundary conditions include infinitely many points, and the uniqueness of the positive solution is also discussed, which are the distinctive features of this work. However, the practical application of such Caputo FDEs requires further investigation.

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