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Solutions to Variable-Order Fractional BVPs with Multipoint Data in $W^{s,p}$ Spaces

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Abstract

This study explores the existence of positive solutions within a Sobolev space for a boundary value problem that involves Riemann–Liouville fractional derivatives of variable order. The analysis utilizes the method of upper and lower solutions in combination with the Schauder fixed-point theorem. To illustrate the theoretical findings, a numerical example is included.

Keywords: fixed point theorem; upper and lower solutions; Riemann–Liouville derivative; variable order fractional operators

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1. Introduction

Fractional calculus has emerged as a prominent interdisciplinary field, expanding its influence well beyond mathematics into a wide range of scientific disciplines. In physics [1], it offers powerful methods for modeling complex systems, while in chemical kinetics [2], its nonlocal operators provide deeper insights into reaction dynamics. Applications in fluid dynamics [3] and viscoelasticity [4] exploit fractional derivatives to capture memory-dependent behaviors in materials effectively. This mathematical framework has also shown significant potential in areas such as electrochemistry [5], elasticity [6], and various branches of engineering [7]. In the social sciences [8], fractional calculus is increasingly used for modeling economic systems and analyzing financial processes [8,9], particularly those characterized by long-term memory. It also plays a vital role in biological and medical research [10,11], aiding in the study of growth patterns and the diffusion of drugs. Moreover, it enhances analytical methods in statistical mechanics [12] and image processing. Fractional models have proven especially effective in describing phenomena like nonlinear heat conduction [13], optimal control problems, and complex cosmic behaviors, where classical differential equations fall short [14].

The study of multipoint boundary value problems (BVPs) has attracted significant attention due to their broad applicability in physics and applied mathematics. Early work by Gupta [15] established solvability criteria for three-point nonlinear BVPs in second-order ODEs, while subsequent research extended these results to m-point problems under nonlin-

ear growth conditions (e.g., Feng et al. [16], Ma et al. [17]). Further advances include bifurcation methods for nodal solutions (Sun et al. [18]), positivity analysis (Zhang et al. [19]), and extensions to Banach spaces (Zhao et al. [20]) and fractional differential equations (Lv [21]). A system-based approach was later developed by Henderson et al. [22].

Despite its versatility, no single, universally effective method exists to address all classical problems related to arbitrary fractional differential equations. To study the existence, uniqueness, and qualitative properties of solutions, researchers employ a variety of analytical techniques, including the method of upper and lower solutions, Mawhin's continuation theory, decomposition approaches, variational iteration methods, and homotopy methods.

Recently, there has been significant attention on variable-order fractional operators. These operators have been rigorously defined and are increasingly being used to construct evolutionary systems of equations. They are invaluable tools for accurately modeling dynamic and complex real-world systems across various fields, including biology, mechanics, transport phenomena, and control theory. As the field continues to evolve, fractional calculus remains an active area of research, with new operators being developed to leverage their unique characteristics in addressing challenging scientific and engineering problems. The Caputo fractional derivative is commonly used because it works well with classical initial conditions. However, in this context, the Riemann–Liouville operator is preferred because it aligns more naturally with the integral formulation of the problem and offers analytical benefits for managing nonlocal boundary conditions. This selection is consistent with physical models, where singular kernel behaviors and memory effects play a significant role, as discussed in [23].

The scientific community has increasingly focused on variable order fractional calculus as a powerful tool for modeling complex engineering and physical systems. For those interested in a deeper understanding of this evolving field [24,25], several comprehensive studies can be found in [26–29]. The fundamental idea behind VOFC lies in generalizing classical fractional operators of fixed order w to operators with variable order, expressed as $w(\cdot)$. Notably, between 2021 and 2022, a number of influential works have advanced the theory and applications of VOFC, as highlighted in [26–29].

In the particular study [30], the authors investigate the existence of solutions for a boundary value problem that involves the Riemann–Liouville fractional derivative, subject to integral conditions:

$$\begin{cases} \mathcal{D}_{0+}^{w} v(\varepsilon) + \mathcal{B}\big(\varepsilon, v(\varepsilon), \mathcal{D}_{0+}^{\gamma} v(\varepsilon)\big) = 0, \ \ 0 < \varepsilon < 1, \\ \lim\limits_{\varepsilon \to 0^+} \varepsilon^{i-w} v(\varepsilon) = 0, \ i = 2, \ldots, n, \ \ v(1) = \sum\limits_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} I_{0+}^{\beta} v(\eta_{\mathfrak{c}}). \end{cases} \tag{1}$$

Building on the previous discussions, this paper introduces new qualitative findings related to the variable-order Riemann–Liouville fractional boundary value problem with integral conditions:

$$\begin{cases} \mathcal{D}_{0+}^{w(\varepsilon)} v(\varepsilon) + \mathcal{B}\Big(\varepsilon, v(\varepsilon), \mathcal{D}_{0+}^{\gamma(\varepsilon)} v(\varepsilon)\Big) = 0, \quad 0 < \varepsilon < 1, \\ \lim\limits_{\varepsilon \to 0^+} \varepsilon^{i-w(\varepsilon)} v(\varepsilon) = 0, \ i = 2, \ldots, n, \qquad v(1) = \sum\limits_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} I_{0+}^{\beta(\varepsilon)} v(\eta_{\mathfrak{c}}). \end{cases} \tag{2}$$

In this context, $\mathcal{D}_{0+}^{w(\varepsilon)}$ and $\mathcal{D}_{0+}^{\gamma(\varepsilon)}$ represent the variable-order Riemann–Liouville fractional derivatives of orders $w(\varepsilon)$ and $\gamma(\varepsilon)$, respectively. The expression $I_{0^+}^{\beta(\varepsilon)}$ denotes the variable-order Riemann–Liouville fractional integral of order $\beta(\varepsilon) > 0$, $n - 1 \leq w(\varepsilon) < n$,

$n \geq 4$, $0 < \gamma(\varepsilon) < 1$. The function $\mathcal{B} : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^+$ is a Caratheodory function that satisfies certain assumptions. Let us now define the setting more precisely. Set

$$\xi = 1 - \frac{\Gamma(w_k)}{\Gamma(w_k + \beta_k)} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \left(\eta_{\mathfrak{c}} - \Re_{k-1}^{+}\right)^{w_k+\beta_k-1},$$

with $0 < \eta_{\mathfrak{c}} < 1$, $\lambda_{\mathfrak{c}} > 0$, $\mathfrak{c} = 0, \ldots, m$, and

$$\sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \leq \frac{\Gamma(w_k + \beta_k)}{\Gamma(w_k)}.$$

Then, $0 < \xi < 1$.

Here, we explore a variable-order fractional differential equation (FDE) involving the Riemann–Liouville (RL) derivative and establish conditions for the existence of solutions within subintervals. Our study highlights the crucial role of piecewise-constant functions in converting the variable-order RL fractional boundary value problem into a standard RL fractional boundary value problem.

The primary objective of this work is to analyze a mathematical problem with variable order. The desired variable-order FDE has been introduced in Section 1, while essential definitions and preliminary results are provided in Section 2. Using an iterative sequential approach and a limit point technique, we derive the existence and uniqueness criteria in Section 3. Finally, Sections 4 and 5 present a demonstrative example and the corresponding numerical analysis, respectively, to illustrate our findings.

2. Essential Concepts and Preliminaries

This section gathers some essential concepts and preliminary results that will serve as a foundation for the analysis presented in the later sections of the paper.

Definition 1 ([31,32]). *Let $-\infty < a < b < +\infty$ and consider a function $w : [a,b] \to (0, +\infty)$. The left-sided Riemann–Liouville fractional integral of variable order $w(\cdot)$ for a function $\mathcal{Q}$ is defined as follows:*

$$I_{a+}^{w(\varepsilon)} \mathcal{Q}(\varepsilon) = \int_a^{\varepsilon} \frac{(\varepsilon - \tau)^{w(\tau)-1}}{\Gamma(w(\tau))} \mathcal{Q}(\tau) d\tau, \quad \varepsilon > a,$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2 ([31,32]). *For $-\infty < a < b < +\infty$, let $w : [a,b] \to (n-1, n)$, with $n \in \mathbb{N}$. The left-sided Riemann–Liouville fractional derivative of variable order $w(\cdot)$ for $\mathcal{Q}$ is given by*

$$\mathcal{D}_{a+}^{w(\varepsilon)} \mathcal{Q}(\varepsilon) = \left(\frac{d}{d\varepsilon}\right)^n I_{a+}^{n-w(\varepsilon)} \mathcal{Q}(\varepsilon) = \left(\frac{d}{d\varepsilon}\right)^n \int_a^{\varepsilon} \frac{(\varepsilon - \tau)^{n-w(\tau)-1}}{\Gamma(n - w(\tau))} \mathcal{Q}(\tau) d\tau, \quad \varepsilon > a. \tag{3}$$

The following properties are derived from the existing literature:

Lemma 1 ([33]). *Let $\gamma > 0$, $a \geq 0$, $\mathcal{Q} \in L^1(a,b)$, and $\mathcal{D}_{a+}^{\gamma} \mathcal{Q} \in L^1(a,b)$. If $\mathcal{D}_{a+}^{\gamma} \mathcal{Q} = 0$, then*

$$\mathcal{Q}(\varepsilon) = \sum_{k=1}^{n} c_k (\varepsilon - a)^{\gamma - k},$$

where $n = \lceil \gamma \rceil$ and $c_k \in \mathbb{R}$ for $k = 1, \ldots, n$.

Lemma 2 ([33]). *For $\gamma > 0$, $a \geq 0$, $\mathcal{Q} \in L^1(a,b)$, and $\mathcal{D}^{\gamma}_{a+}\mathcal{Q} \in L^1(a,b)$, we have*

$$I^{\gamma}_{a+}D^{\gamma}_{a+}\mathcal{Q}(\varepsilon) = \mathcal{Q}(\varepsilon) + \sum_{k=1}^{n} d_k(\varepsilon - a)^{\gamma-k},$$

where $n = \lceil \gamma \rceil$ and $d_k \in \mathbb{R}$ for $k = 1, \ldots, n$.

Lemma 3 ([33]). *Let $\gamma > 0$, $a \geq 0$, $\mathcal{Q} \in L^1(a,b)$, and $\mathcal{D}^{\gamma}_{a+}\mathcal{Q} \in L^1(a,b)$. Then*

$$\mathcal{D}^{\gamma}_{a+}I^{\gamma}_{a+}\mathcal{Q}(\varepsilon) = \mathcal{Q}(\varepsilon).$$

Lemma 4 ([33]). *For $\gamma_1, \gamma_2 > 0$, $a \geq 0$, and $\mathcal{B} \in L^1(a,b)$, the following holds:*

$$I^{\gamma_1}_{a+}I^{\gamma_2}_{a+}\mathcal{Q}(\varepsilon) = I^{\gamma_2}_{a+}I^{\gamma_1}_{a+}\mathcal{Q}(\varepsilon) = I^{\gamma_1+\gamma_2}_{a+}\mathcal{Q}(\varepsilon).$$

Remark 1 ([34,35]). *In general, for variable-order fractional operators with distinct functions $\gamma_1(\varepsilon)$ and $\gamma_2(\varepsilon)$, the semigroup property fails to hold:*

$$I^{\gamma_1(\varepsilon)}_{a+}I^{\gamma_2(\varepsilon)}_{a+}\mathcal{Q}(\varepsilon) \neq I^{\gamma_1(\varepsilon)+\gamma_2(\varepsilon)}_{a+}\mathcal{Q}(\varepsilon). \tag{4}$$

Definition 3. *A generalized interval $I \subseteq \mathbb{R}$ is defined as any of the following: A standard interval (open, closed, or half-open), a singleton set $\{a\}$ where $a \in \mathbb{R}$, the empty set $\emptyset$.*

This definition follows the conventions established in prior works [36].

Definition 4. *A partition $\mathcal{P}$ of I is a finite collection of disjoint generalized intervals such that every element $x \in I$ belongs to exactly one subset $E \in \mathcal{P}$.*

A function $g\colon I \to \mathbb{R}$ is called piecewise constant with respect to $\mathcal{P}$ if g remains constant on each subset E within the partition.

Lemma 5 ([37]). *Let F be a bounded set in $L^p(0,1)$, where $1 \leq p < \infty$. Assume that*

(i) $\lim_{|h|\to 0} \|\mathcal{T}_h\mathcal{Q} - \mathcal{Q}\|_p = 0$ *uniformly for $\mathcal{Q} \in F$,*

(ii) $\lim_{\epsilon\to 0} \int_{1-\epsilon}^{1} |\mathcal{Q}(\varepsilon)|^p d\varepsilon = 0$ *uniformly for $\mathcal{Q} \in F$,*

where $\mathcal{T}_h\mathcal{Q}(\varepsilon) = \mathcal{Q}(\varepsilon_h)$. Then F is relatively compact in $L^p(0,1)$.

In what follows, let $[a,b]$ be a nonempty interval of $\mathbb{R}$. Let us introduce the Sobolev Space

$$W^{1,1}(a,b) = \left\{ v \in L^1(a,b) \mid v' \in L^1(a,b) \right\},$$

equipped with the norm

$$\|v\|_{W^{1,1}} = \|v\|_{L^1} + \|v'\|_{L^1}.$$

Here, v' denotes the distributional derivative of v. The space of Absolutely Continuous Functions $\mathrm{AC}(a,b)$ coincides with the Sobolev Space $W^{1,1}(a,b)$.

Definition 5 ([38]). *The Riemann–Liouville fractional Sobolev Space is given by*

$$W^{s,1}_{\mathrm{RL},a^+} = \left\{ v \in L^1(a,b) \mid I^{1-s}_{a^+}v \in W^{1,1}(a,b) \right\}, \quad 0 < s < 1.$$

$W^{s,1}_{\mathrm{RL},a^+}$ is a Banach Space endowed with the norm

$$\|v\|_{W^{s,1}_{\mathrm{RL},a^+}} = \|v\|_{L^1} + \left\| I^{1-s}_{a^+} v \right\|_{W^{1,1}}.$$

In [38], Bergounioux et al. give more details on the Sobolev Space $W^{s,1}_{\mathrm{RL},a^+}$.

Theorem 1 ([34]). *Let Ω be a convex subset of a Banach space and $\mathfrak{N} : \Omega \to \Omega$ be a completely continuous map. Then $\mathfrak{N}$ has at least one fixed point in Ω.*

3. Problem Setup

Let us consider $n \in \mathbb{N}$ a positive integer and $\{\Re_k\}_{k=0}^n$ a finite sequence such that

$$0 = \Re_0 < \Re_{k-1} < \Re_k < \Re_n = 1, \quad k = 2, \ldots, n-1.$$

Denote $\mathcal{M}_k := (\Re_{k-1}, \Re_k]$, $k = 1, 2, \ldots, n$. Then, $\mathcal{P} = \{\Re_k : k = 1, 2, \ldots, n\}$ is a partition of $[0, 1]$.

Consider a piecewise function $w : \mathcal{M} \to [n-1, n)$, with respect to $\mathcal{P}$, given by

$$w(s) = \sum_{k=1}^{n} w_k I_k(s) = \begin{cases} w_1, & \text{if } s \in \mathcal{M}_1, \\ w_2, & \text{if } s \in \mathcal{M}_2, \\ \vdots & \\ w_n, & \text{if } s \in \mathcal{M}_n, \end{cases}$$

with

$$I_k(s) := \begin{cases} 1, & \text{if } s \in \mathcal{M}_k, \\ 0, & \text{otherwise,} \end{cases}$$

where $n - 1 \leq w_k \leq n$ are positive constants and I_k are indicators of $\mathcal{M}_k$, for $k = 1, 2, \ldots, n$.

Then, for any $s \in \mathcal{M}_k$, $k = 1, 2, \ldots, n$, the left Riemann–Liouville fractional-order operator of variable order $w(\cdot)$ for $v \in C(\mathcal{M}, \mathbb{R})$ could be the sum of left Riemann–Liouville fractional-order derivatives of w_k, $k = 1, 2, \ldots, n$, orders

$$\mathcal{D}^{w(s)}_{0+} v(s) = \frac{d}{ds}\left(\int_0^{\Re_1} \frac{(s-z)^{-w_1}}{\Gamma(1-w_1)} v(z)dz + \cdots + \int_{\Re_{k-1}}^{s} \frac{(s-z)^{-w_k}}{\Gamma(1-w_k)} v(z)dz \right). \tag{5}$$

Thus, according to (5), the equation of the R-fractional Equation (2) can be written for any $s \in \Re_k$, $k = 1, 2, \ldots, n$, as

$$\frac{d}{ds}\left(\int_0^{\Re_1} \frac{(s-z)^{-w_1}}{\Gamma(1-w_1)} v(z)dz + \cdots + \int_{\Re_{k-1}}^{s} \frac{(s-z)^{-w_k}}{\Gamma(1-w_k)} v(z)dz \right) = -\mathcal{B}\big(s, v(s), \mathcal{D}^{\gamma_k}_{0+} v(s)\big). \tag{6}$$

To solve the integral Equation (6), let $\tilde{v} \colon \mathcal{M}_k \to \mathbb{R}$ be a continuous function such that $\tilde{v}(\varepsilon) \equiv 0$ on $\varepsilon \in [0, \Re_{k-1}]$. Then (6) is transformed into

$$\mathcal{D}^{w_k}_{\Re^+_{k-1}} \tilde{v}(s) + \mathcal{B}\left(s, \tilde{v}(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} \tilde{v}(s) \right) = 0, \qquad s \in \mathcal{M}_k.$$

We shall deal with the following BVP:

$$\begin{cases} \mathcal{D}^{w_k}_{\Re^+_{k-1}} v(\varepsilon) + \mathcal{B}\Big(\varepsilon, v(\varepsilon), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(\varepsilon)\Big) = 0, & \Re_{k-1} < \varepsilon < 1, \\ \lim\limits_{\varepsilon \to \Re^+_{k-1}} \varepsilon^{i-w_k} v(\varepsilon) = 0,\ i = 2, \ldots, n, & v(1) = \sum\limits_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} I^{\beta_k}_{\Re^+_{k-1}} v(\eta_{\mathfrak{c}}). \end{cases} \tag{7}$$

Lemma 6. *Assume that $h \in C(0,1) \cap L^1(0,1)$ and $n-1 < w(\varepsilon) \leq n$, $n \geq 4$, then the unique solution of the BVP (7) is given by*

$$v(\varepsilon) = \int_{\Re_{k-1}}^{1} \mathcal{G}(\varepsilon, s) \mathcal{B}\Big(s, u(s), \mathcal{D}^{\gamma_k}_{\Re_{k-1}} v(\varepsilon)\Big) ds + \frac{\varepsilon^{w_k - 1}}{\xi} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \int_{\Re_{k-1}}^{1} \mathcal{H}(\eta_{\mathfrak{c}}, s) \mathcal{B}\Big(s, v(s), \mathcal{D}^{\gamma_k}_{\Re_{k-1}} v(\varepsilon)\Big) ds,$$

where

$$\mathcal{G}(\varepsilon, s) = \frac{1}{\Gamma(w_k)} \begin{cases} \varepsilon^{w_k - 1}(1-s)^{w_k - 1} - (\varepsilon - s)^{w_k - 1}, & \Re_{k-1} \leq s \leq \varepsilon \leq 1, \\ \varepsilon^{w_k - 1}(1-s)^{w_k - 1}, & \Re_{k-1} \leq \varepsilon \leq s \leq 1, \end{cases}$$

and

$$\mathcal{H}(\varepsilon, s) = \frac{\Gamma(w_k)\Gamma(\beta_k)}{\Gamma(w_k + \beta_k)} \begin{cases} \frac{1}{\Gamma(w_k)\Gamma(\beta_k)} (\varepsilon - \Re_{k-1})^{w_k + \beta_k - 1} (1-s)^{w_k - 1} - (\varepsilon - s)^{w_k + \beta_k - 1}, \\ \qquad\qquad \Re_{k-1} \leq s \leq \varepsilon \leq 1, \\ (\varepsilon - \Re_{k-1})^{w_k + \beta_k - 1} (1-s)^{w_k - 1}, \\ \qquad\qquad \Re_{k-1} \leq \varepsilon \leq s \leq 1. \end{cases}$$

Proof. Let v be a solution of the problem (7). Then, we have

$$v(\varepsilon) = c_1 \varepsilon^{w_k - 1} + c_2 \varepsilon^{w_k - 2} + c_3 \varepsilon^{w_k - 3} + \cdots + c_n \varepsilon^{w_k - n} - \frac{1}{\Gamma(w_k)} \int_{\Re^+_{k-1}}^{\varepsilon} (\varepsilon - s)^{w_k - 1} \mathcal{B}\Big(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(\varepsilon)\Big) ds.$$

Taking into consideration conditions (7), it yields

$$c_2 = c_3 = \cdots = c_n = 0,$$

and

$$c_1 = \frac{1}{\xi} \Bigg[\frac{1}{\Gamma(w_k)} \int_{\Re_{k-1}}^{1} (1-s)^{w_k - 1} \mathcal{B}\Big(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(\varepsilon)\Big) ds - \frac{\Gamma(w_k)\Gamma(\beta_k)}{\Gamma(w_k + \beta_k)} \sum_{\eta=0}^{m} \lambda_{\mathfrak{c}} \int_{\Re_{k-1}}^{\eta_{\mathfrak{c}}} (\eta_{\mathfrak{c}} - s)^{w_k + \beta_k - 1} \mathcal{B}\Big(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(\varepsilon)\Big) ds \Bigg].$$

Hence, the solution of problem (7) is

$$\begin{aligned} v(\varepsilon) = &- \frac{1}{\Gamma(w_k)} \int_{\Re_{k-1}}^{\varepsilon} (\varepsilon - s)^{w_k - 1} \mathcal{B}\Big(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(\varepsilon)\Big) ds \\ &+ \frac{\varepsilon^{w_k - 1}}{\xi} \Bigg[\frac{1}{\Gamma(w_k)} \int_{\Re_{k-1}}^{1} (1-s)^{w_k - 1} \mathcal{B}\Big(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(\varepsilon)\Big) ds \\ &- \frac{\Gamma(w_k)\Gamma(\beta_k)}{\Gamma(w_k + \beta_k)} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \int_{\Re_{k-1}}^{\eta_{\mathfrak{c}}} (\eta_{\mathfrak{c}} - s)^{w_k + \beta_k - 1} \mathcal{B}\Big(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(\varepsilon)\Big) ds \Bigg]. \end{aligned} \tag{8}$$

Now, by some calculations and the fact that

$$\xi = 1 - \frac{\Gamma(w_k)}{\Gamma(w_k+\beta_k)} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} (\eta_{\mathfrak{c}} - \Re_{k-1})^{w_k+\beta_k-1}.$$

We can rewrite the second term in (8) as

$$\begin{aligned}
&\frac{\varepsilon^{w_k-1}}{\Gamma(w_k)\xi} \int_{\Re_{k-1}}^{1} (1-s)^{w_k-1} \mathcal{B}\Big(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(\varepsilon)\Big) ds \\
&\quad = \frac{\varepsilon^{w_k-1}}{\Gamma(w_k)} \int_{\Re_{k-1}}^{1} (1-s)^{w_k-1} \mathcal{B}\Big(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(\varepsilon)\Big) ds \\
&\qquad + \frac{\varepsilon^{w_k-1}}{\Gamma(w_k+\beta_k)\xi} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} (\eta_{\mathfrak{c}} - \Re_{k-1})^{w_k+\beta_k-1} \int_{\Re_{k-1}}^{1} (1-s)^{w_k-1} \mathcal{B}\Big(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(\varepsilon)\Big) ds \\
&\quad = \frac{\varepsilon^{w_k-1}}{\Gamma(w_k)} \int_{\Re_{k-1}}^{\varepsilon} (1-s)^{w_k-1} \mathcal{B}\Big(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(\varepsilon)\Big) ds \\
&\qquad + \frac{\varepsilon^{w_k-1}}{\Gamma(w_k)} \int_{\varepsilon}^{1} (1-s)^{w_k-1} \mathcal{B}\Big(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(\varepsilon)\Big) ds \\
&\qquad + \frac{\varepsilon^{w_k-1}}{\Gamma(w_k+\beta_k)\xi} \sum_{=0}^{m} \lambda_{\mathfrak{c}} (\eta_{\mathfrak{c}} - \Re_{k-1})^{w_k+\beta_k-1} \int_{\Re_{k-1}}^{\eta_{\mathfrak{c}}} (1-s)^{w_k-1} \mathcal{B}\Big(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(\varepsilon)\Big) ds \\
&\qquad + \frac{\varepsilon^{w_k-1}}{\Gamma(w_k+\beta_k)\xi} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} (\eta_{\mathfrak{c}} - \Re_{k-1})^{w_k+\beta_k-1} \int_{\eta_{\mathfrak{c}}}^{1} (1-s)^{w_k-1} \mathcal{B}\Big(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(\varepsilon)\Big) ds.
\end{aligned}$$

Thus, (8) becomes the following:

$$\begin{aligned}
v(\varepsilon) &= \frac{1}{\Gamma(w_k)} \int_{\Re_{k-1}}^{\varepsilon} \Big[\varepsilon^{w_k-1}(1-s)^{w_k-1} - (\varepsilon-s)^{w_k-1}\Big] \mathcal{B}\Big(s, v(s), \mathcal{D}^{\gamma_k}_{\Re_{k-1}} v(\varepsilon)\Big) ds \\
&\quad + \frac{1}{\Gamma(w_k)} \int_{\varepsilon}^{1} \varepsilon^{w_k-1}(1-s)^{w_k-1} \mathcal{B}\Big(s, v(s), \mathcal{D}^{\gamma_k}_{\Re_{k-1}} v(\varepsilon)\Big) ds \\
&\quad + \frac{\Gamma(w_k)\Gamma(\beta_k)\varepsilon^{w_k-1}}{\Gamma(w_k+\beta_k)\xi} \\
&\quad \times \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \int_{\Re_{k-1}}^{\eta_{\mathfrak{c}}} \left[\frac{1}{\Gamma(w_k)\Gamma(\beta_k)} (\eta_{\mathfrak{c}} - \Re_{k-1})^{\omega_k+\beta_k-1}(1-s)^{w_k-1} - (\eta_{\mathfrak{c}}-s)^{w_k+\beta_k-1}\right] h(s) ds \\
&\quad + \frac{\varepsilon^{w_k-1}}{\Gamma(w_k+\beta_k)\xi} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \int_{\eta_{\mathfrak{c}}}^{1} (\eta_{\mathfrak{c}} - \Re_{k-1})^{w_k+\beta_k-1}(1-s)^{w_k-1} \mathcal{B}\Big(s, v(s), \mathcal{D}^{\gamma_k}_{\Re_{k-1}} v(\varepsilon)\Big) ds.
\end{aligned}$$

The proof is complete. □

Lemma 7. *The functions $\mathcal{G}$ and $\mathcal{H}$ are continuous, nonnegative, and satisfy*

$$\mathcal{G}(\varepsilon, s) \le \frac{1}{\Gamma(w_k)}, \quad \mathcal{H}(\varepsilon, s) \le \frac{\Gamma(w_k)\Gamma(\beta_k)}{\Gamma(w_k+\beta_k)}, \quad \Re_{k-1} \le \varepsilon, s \le 1.$$

Let $a, c \in \mathbb{R}^+$, $b, d \in \mathbb{R}$, and define the upper and lower control functions $\mathcal{U}(\varepsilon, u, v) : [\Re_{k-1}, 1] \times [a, \infty) \times [b, \infty) \to \mathbb{R}^+$ and $\mathfrak{L}(\varepsilon, u, v) : [\Re_{k-1}, 1] \times (-\infty, c] \times (-\infty, d] \to \mathbb{R}^+$, respectively, by

$$\begin{aligned}
\mathcal{U}(\varepsilon, u, v) &= \sup\{\mathcal{B}(\varepsilon, \lambda, \mu) : a \le \lambda \le u, b \le \mu \le v\}, \\
\mathfrak{L}(\varepsilon, u, v) &= \inf\{\mathcal{B}(\varepsilon, \lambda, \mu) : u \le \lambda \le c, v \le \mu \le d\}.
\end{aligned}$$

We have $\mathfrak{L}(\varepsilon, u, v) \le \mathcal{B}(\varepsilon, u, v) \le \mathcal{U}(\varepsilon, u, v)$, for $\Re_{k-1} \le \varepsilon \le 1$, $a \le u \le c$, $b \le v \le d$. Set the cone

$$K = \left\{ v \in W^{1-\gamma,1}_{RL,\Re^+_{k-1}} : v(\varepsilon) \geq 0, \Re_{k-1} \leq \varepsilon \leq 1 \right\}.$$

Note that the norm in the space $W^{1-\gamma,1}_{RL,\Re^+_{k-1}}$ is

$$\|v\|_{W^{1-\gamma,1}_{RL,\Re^+_{k-1}}} = \|v\|_{L^1} + \left\| I^{1-\gamma_k}_{\Re^+_{k-1}} v \right\|_{L^1} + \left\| \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v \right\|_{L^1}.$$

Let us consider the following hypotheses:

Hypothesis 1 (H1). *There exist $v_*, v^* \in K$, such that $a \leq v_*(\varepsilon) \leq v^*(\varepsilon) \leq c$, $b \leq \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v_*(\varepsilon) \leq \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v^*(\varepsilon) \leq d$, and:*

$$\begin{aligned}
v^*(\varepsilon) &\geq \int_{\Re_{k-1}}^{1} \mathcal{G}(\varepsilon, s) \mathcal{U}\left(s, v^*(s), \mathcal{D}^{\gamma}_{\Re^+_{k-1}} v^*(s)\right) ds \\
&\quad + \frac{\varepsilon^{w_k - 1}}{\xi} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \int_{\Re^+_{k-1}}^{1} \mathcal{H}(\eta_k, s) \mathcal{U}\left(s, v^*(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v^*(s)\right) ds, \\
v_*(t) &\leq \int_{\Re_{k-1}}^{1} \mathcal{G}(\varepsilon, s) \mathfrak{L}\left(s, v_*(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v_*(s)\right) ds \\
&\quad + \frac{\varepsilon^{w_k - 1}}{\xi} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \int_{\Re_{k-1}}^{1} \mathcal{H}(\eta_k, s) \mathfrak{L}\left(s, v_*(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v_*(s)\right) ds,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v^*(\varepsilon) &\geq -\frac{1}{\Gamma(w_k - \gamma_k)} \int_{\Re_{k-1}}^{\varepsilon} (\varepsilon - s)^{w_k - \gamma_k - 1} \mathfrak{L}\left(s, v^*(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v^*(s)\right) ds \\
&\quad + \frac{\varepsilon^{w_k - \gamma_k - 1}}{\xi \Gamma(w_k - \gamma_k)} \left[\int_0^1 (1 - s)^{w_k - 1} \mathcal{U}\left(s, v^*(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v^*(s)\right) ds \right. \\
&\quad \left. - \frac{\Gamma(w_k)\Gamma(\beta_k)}{\Gamma(w_k + \beta_k)} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \int_{\Re_{k-1}}^{\eta_{\mathfrak{c}}} (\eta_{\mathfrak{c}} - s)^{w_k + \beta_k - 1} \mathfrak{L}\left(s, v^*(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v^*(s)\right) ds \right], \\
\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v_*(\varepsilon) &\leq -\frac{1}{\Gamma(w_k - \gamma_k)} \int_{\Re_{k-1}}^{\varepsilon} (\varepsilon - s)^{w_k - \gamma_k - 1} \mathcal{U}\left(s, v_*(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v_*(s)\right) ds \\
&\quad + \frac{\varepsilon^{w_k - \gamma_k - 1}}{\xi \Gamma(w_k - \gamma_k)} \left[\int_{\Re_{k-1}}^{1} (1 - s)^{w_k - 1} \mathfrak{L}\left(s, v_*(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v_*(s)\right) ds \right. \\
&\quad \left. - \frac{\Gamma(w_k)\Gamma(\beta_k)}{\Gamma(w_k + \beta_k)} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \int_{\Re_{k-1}}^{\eta_{\mathfrak{c}}} (\eta_{\mathfrak{c}} - s)^{w_k + \beta_k - 1} \mathcal{U}\left(s, v_*(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v_*(s)\right) ds \right].
\end{aligned}$$

Hypothesis 2 (H2). *There exists a nonnegative function $g \in L^1[\Re_{k-1}, 1]$, two constants $C \geq 0$ and $R > 0$ such that*

$$\mathcal{B}(\varepsilon, u, v) \leq g(\varepsilon) + C(|u| + |v|), \quad \Re_{k-1} \leq \varepsilon \leq 1, \ u, v \in \mathbb{R},$$

and

$$(\|g\|_{L^1} + CR)\left[\frac{1 + \xi + \Gamma(w_k)\Gamma(\beta_k)}{\Gamma(w_k - \gamma_k)} + \frac{(\xi + \Gamma(w_k)\Gamma(\beta_k))(1 + \Gamma(2 - \gamma_k))}{\Gamma(w_k)\Gamma(2 - \gamma_k)} \right] \leq R. \tag{9}$$

Theorem 2. *Assume that hypotheses $(H1)$ and $(H2)$ hold. Then the boundary value problem (7) has at least one positive solution in $W^{1-\gamma_k,1}_{RL,\Re^+_{k-1}}$ such that $v_*(\varepsilon) \leq v(\varepsilon) \leq v^*(\varepsilon)$ and $\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v_*(\varepsilon) \leq \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(\varepsilon) \leq \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v^*(\varepsilon)$, for all $\varepsilon \in [\Re_{k-1}, 1]$.*

Proof. Denote by $\mathfrak{D}_R$ the set

$$\mathfrak{D}_R := \Bigg\{ u \in K, \|v\|_{W^{1-\gamma_k,1}_{RL,\Re^+_{k-1}}} \leq R, v_*(\varepsilon) \leq v(\varepsilon) \leq v^*(\varepsilon),$$
$$\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v_*(\varepsilon) \leq \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(\varepsilon) \leq \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v^*(\varepsilon), \Re_{k-1} \leq \varepsilon \leq 1 \Bigg\}.$$

It is clear that $\mathfrak{D}_R$ is a bounded, closed and convex subset of $W^{1-\gamma_k,1}_{RL,\Re^+_{k-1}}$. Define the operator $H : \mathfrak{D}_R \to W^{1-\gamma_k,1}_{RL,\,\Re^+_{k-1}}$ by

$$Hu(\varepsilon) = \int_{\Re_{k-1}}^{1} \mathcal{G}(\varepsilon, s)\mathcal{B}\Big(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(s)\Big)ds + \frac{\varepsilon^{w_k-1}}{\xi} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \int_{\Re_{k-1}}^{1} \mathcal{H}(\eta_{\mathfrak{c}}, s)\mathcal{B}\Big(s, v(s), \mathcal{D}^{\gamma_k}_{\Re_{k-1}+} v(s)\Big)ds.$$

We show that H satisfies the assumptions of Schauder's fixed point theorem. The proof will be done in some steps.

Claim 1: The operator H is continuous in $W^{1-\gamma_k,1}_{RL\,,\Re^+_{k-1}}$.

Let (v_n) be a sequence such that $v_n \to v$ in $W^{1-\gamma_k,1}_{RL\,,\Re^+_{k-1}}$. Taking Lemma 7 and condition $\sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \leq \frac{\Gamma(w_k+\beta_k)}{\Gamma(w_k)}$ into account, we obtain

$$\begin{aligned}
|Hv_n(\varepsilon) - Hv(\varepsilon)| &\leq \int_{\Re_{k-1}}^{1} \mathcal{G}(\varepsilon, s)\Big|\mathcal{B}\Big(s, v_n(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v_n(s)\Big) - \mathcal{B}\Big(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(s)\Big)\Big|ds \\
&\quad + \frac{\varepsilon^{w_k-1}}{\xi} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \int_{\Re_{k-1}}^{1} \mathcal{H}(\eta_{\mathfrak{c}}, s) \\
&\quad \times \Big|\mathcal{B}\Big(s, v_n(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v_n(s)\Big) - \mathcal{B}\Big(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(s)\Big)\Big|ds \\
&\leq \frac{\xi + \Gamma(w_k)\Gamma(\beta_k)}{\Gamma(w_k)} \times \Big\|\mathcal{B}\Big(\cdot, v_n(\cdot), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v_n(\cdot)\Big) - \mathcal{B}\Big(\cdot, v(\cdot), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(\cdot)\Big)\Big\|_{L^1}.
\end{aligned}$$

Consequently,

$$\|Hv_n - Hv\|_{L^1} \leq \frac{\xi + \Gamma(w_k)\Gamma(\beta_k)}{\Gamma(w_k)} \times \Big\|\mathcal{B}\Big(\cdot, v_n(\cdot), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v_n(\cdot)\Big) - \mathcal{B}\Big(\cdot, v(\cdot), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(\cdot)\Big)\Big\|_{L^1}.$$

Similarly, we obtain

$$\begin{aligned}
\Big|I^{\gamma_k}_{\Re^+_{k-1}} Hv_n(\varepsilon) - I^{\gamma_k}_{\Re^+_{k-1}} Hv(\varepsilon)\Big| &\leq \frac{1}{\Gamma(1-\gamma_k)} \int_{\Re_{k-1}}^{\varepsilon} (\varepsilon - s)^{-\gamma_k} |Hv_n(s) - Hv(s)|ds \\
&\leq \frac{\xi + \Gamma(w_k)\Gamma(\beta_k)}{\Gamma(w_k)\Gamma(2-\gamma_k)} \times \Big\|\mathcal{B}\Big(\cdot, v_n(\cdot), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v_n(\cdot)\Big) - \mathcal{B}\Big(\cdot, v(\cdot), \mathcal{D}^{\gamma_k}_{\Re_{k-1}+} v(\cdot)\Big)\Big\|_{L^1},
\end{aligned}$$

hence

$$\begin{aligned}
&\Big\|I^{\gamma_k}_{\Re^+_{k-1}} Hv_n - I^{\gamma_k}_{\Re^+_{k-1}} Hv\Big\|_{L^1} \\
&\leq \frac{\xi + \Gamma(w_k)\Gamma(\beta_k)}{\Gamma(w_k)\Gamma(2-\gamma_k)} \times \Big\|\mathcal{B}\Big(\cdot, v_n(\cdot), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v_n(\cdot)\Big) - \mathcal{B}\Big(\cdot, v(\cdot), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(\cdot)\Big)\Big\|_{L^1}.
\end{aligned} \tag{10}$$

From (8), Hv can be written as

$$Hv(\varepsilon) = -I^{w_k}\mathcal{B}\Big(\varepsilon, v(\varepsilon), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(\varepsilon)\Big) + \varepsilon^{w_k-1}\mathcal{L}v,$$

where

$$\mathcal{L}v = \frac{1}{\xi\Gamma(w_k)}\int_{\Re_{k-1}}^{1}(1-s)^{w_k-1}\mathcal{B}\left(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(s)\right)ds - \frac{\Gamma(w_k)\Gamma(\beta_k)}{\xi\Gamma(w_k+\beta_k)}\sum_{\mathfrak{c}=0}^{m}\lambda_{\mathfrak{c}}\int_{\Re_{k-1}}^{\eta_{\mathfrak{c}}}(\eta_{\mathfrak{c}}-s)^{w_k+\beta_k-1}\mathcal{B}\left(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(s)\right)ds.$$

Then, by hypothesis (H2) and condition $\sum_{\mathfrak{c}=0}^{m}\lambda_{\mathfrak{c}} \leq \frac{\Gamma(w_k+\beta_k)}{\Gamma(w_k)}$, the following estimate holds

$$\begin{aligned}|\mathcal{L}v| &\leq \left(\frac{1}{\xi\Gamma(w_k)} + \frac{\Gamma(w_k)\Gamma(\beta_k)}{\xi\Gamma(w_k+\beta_k)}\sum_{\mathfrak{c}=0}^{m}\lambda_{\mathfrak{c}}\right)\int_{\Re_{k-1}}^{1}\left[g(s) + C\left(|v(s)| + \left|\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(s)\right|\right)\right]ds \\ &\leq \frac{1+\Gamma(w_k)\Gamma(\beta_k)}{\xi\Gamma(w_k)}(\|g\|_{L^1} + CR).\end{aligned}$$

Thus,

$$\begin{aligned}&\left\|\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} Hv_n(\varepsilon) - \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} Hv(\varepsilon)\right\| \\ &= \left\|I^{w_k-\gamma_k}_{\Re^+_{k-1}}\mathcal{B}\left(\varepsilon, v_n(\varepsilon), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v_n(\varepsilon)\right) - I^{w_k-\gamma_k}_{\Re^+_{k-1}}\mathcal{B}\left(\varepsilon, v(\varepsilon), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(\varepsilon)\right)\right. \\ &\quad \left. + \frac{\Gamma(w_k)}{\Gamma(w_k-\gamma_k)}\varepsilon^{w_k-\gamma_k-1}(\mathcal{L}v_n - \mathcal{H}v)\right\| \\ &\leq \frac{1}{\Gamma(w_k-\gamma_k)}\int_{\Re_{k-1}}^{1}(1-s)^{w_k-\gamma_k-1} \\ &\quad \times \left|\mathcal{B}\left(s, v_n(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v_n(s)\right) - \mathcal{B}\left(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(s)\right)\right|ds \\ &\quad + \frac{\Gamma(w_k)}{\Gamma(w_k-\gamma_k)}\varepsilon^{w_k-\gamma_k-1}\left[\frac{1}{\xi\Gamma(w_k)}\int_{\Re_{k-1}}^{1}(1-s)^{w_k-1}\right. \\ &\quad \times \left|\mathcal{B}\left(s, v_n(s), \mathcal{B}^{\gamma_k}_{\Re^+_{k-1}} v_n(s)\right) - \mathcal{B}\left(s, v_n(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v_n(s)\right)\right|ds \\ &\quad + \frac{\Gamma(w_k)\Gamma(\beta_k)}{\xi\Gamma(w_k+\beta_k)}\sum_{\mathfrak{c}=0}^{m}\lambda_{\mathfrak{c}}\int_{\Re_{k-1}}^{\eta_{\mathfrak{c}}}(\eta_{\mathfrak{c}}-s)^{w_k+\beta_k-1} \\ &\quad \left.\times \left|\mathcal{B}\left(s, v_n(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v_n(s)\right) - \mathcal{B}\left(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(s)\right)\right|ds\right] \\ &\leq \frac{1+\xi+\Gamma(w_k)\Gamma(\beta_k)}{\Gamma(w_k-\gamma_k)}\left\|\mathcal{B}\left(s, v_n(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v_n(s)\right) - \mathcal{B}\left(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(s)\right)\right\|_{L^1}.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\left\|\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} Hv_n(\varepsilon) - \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} Hv(\varepsilon)\right\|_{L^1} &\leq \frac{1+\xi+\Gamma(w_k)\Gamma(\beta_k)}{\Gamma(\alpha_k-\gamma_k)}\times \\ &\left\|\mathcal{B}\left(\cdot, v_n(\cdot), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v_n(\cdot)\right) - \mathcal{B}\left(\cdot, v(\cdot), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(\cdot)\right)\right\|_{L^1}.\end{aligned} \tag{11}$$

Thanks to inequalities (9)–(11), the operator H is continuous in $W^{1-\gamma_k,1}_{RL,\Re^+_{k-1}}$.

Claim 2: $H(\mathfrak{D}_R) \subset \mathfrak{D}_R$. Let $u \in \mathfrak{D}_R$. Lemma 7 implies,

$$\|Hv\|_{L^1} \leq \frac{\xi+\Gamma(w_k)\Gamma(\beta_k)}{\Gamma(w_k)}(\|g\|_{L^1} + CR). \tag{12}$$

Similarly, we obtain

$$\left\| I^{1-\gamma_k}_{\Re^+_{k-1}} Hv \right\|_{L^1} \leq \frac{\xi + \Gamma(w_k)\Gamma(\beta_k)}{\Gamma(w_k)\Gamma(2-\gamma_k)} (\|g\|_{L^1} + CR), \tag{13}$$

and

$$\left\| \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} Hv \right\|_{L^1} \leq \frac{1 + \xi + \Gamma(w_k)\Gamma(\beta_k)}{\Gamma(w_k - \gamma_k)} (\|g\|_{L^1} + CR). \tag{14}$$

Taking (9) and (12)–(14) into account, we obtain $\|Hv\|_{W^{1-\gamma_k,1}_{RL,\Re^+_{k-1}}} \leq R$. Now, since $v \in \mathfrak{D}_R$, we obtain $v_*(\varepsilon) \leq v(\varepsilon) \leq v^*(\varepsilon)$. By the hypothesis (H1) and the concept of upper and lower control functions, we have

$$\begin{aligned}
Hv(\varepsilon) &\leq \int_{\Re_{k-1}}^{1} \mathcal{G}(\varepsilon, s)\mathcal{U}\left(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(s)\right) ds \\
&\quad + \frac{\varepsilon^{w_k-1}}{\xi} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \int_{\Re_{k-1}}^{1} \mathcal{H}(\eta_{\mathfrak{c}}, s)\mathcal{U}\left(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(s)\right) ds \\
&\leq \int_{\Re_{k-1}}^{1} \mathcal{G}(\varepsilon, s)\mathcal{U}\left(s, v^*(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v^*(s)\right) ds \\
&\quad + \frac{\varepsilon^{w_k-1}}{\xi} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \int_{\Re_{k-1}}^{1} \mathcal{H}(\eta_{\mathfrak{c}}, s)\mathcal{U}\left(s, v^*(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v^*(s)\right) ds \\
&\leq v^*(\varepsilon).
\end{aligned}$$

Similarly, we prove that $Hv(\varepsilon) \geq v_*(\varepsilon)$. Thus, $v_*(\varepsilon) \leq Hv(\varepsilon) \leq v^*(\varepsilon)$, for all $v \in \mathfrak{D}_R$. Now, with the help of (8), we see that $Hv(\varepsilon)$ can be written as

$$\begin{aligned}
\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} Hv(\varepsilon) &= -\frac{1}{\Gamma(w_k - \gamma_k)} \int_{\Re_{k-1}}^{\varepsilon} (\varepsilon - s)^{w_k-\gamma_k-1} \mathcal{B}\left(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(s)\right) ds \\
&\quad + \frac{\varepsilon^{w_k-\gamma_k-1}}{\xi\Gamma(w_k - \gamma_k)} \left[\int_{\Re_{k-1}}^{1} (1-s)^{w_k-1} \mathcal{B}\left(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(s)\right) ds \right. \\
&\quad \left. - \frac{\Gamma(w_k)\Gamma(\beta_k)}{\Gamma(w_k + \beta_k)} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \int_{\Re_{k-1}}^{\eta_{\mathfrak{c}}} (\eta_{\mathfrak{c}} - s)^{w_k+\beta_k-1} \mathcal{B}\left(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(s)\right) ds \right].
\end{aligned}$$

So,

$$\begin{aligned}
\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} Hv(\varepsilon) &\leq -\frac{1}{\Gamma(w_k - \gamma_k)} \int_{\Re_{k-1}}^{\varepsilon} (\varepsilon - s)^{w_k-\gamma_k-1} \mathfrak{L}\left(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(s)\right) ds \\
&\quad + \frac{\varepsilon^{w_k-\gamma_k-1}}{\xi\Gamma(w_k - \gamma_k)} \left[\int_{\Re_{k-1}}^{1} (1-s)^{w_k-1} \mathcal{U}\left(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(s)\right) ds \right. \\
&\quad \left. - \frac{\Gamma(w_k)\Gamma(\beta_k)}{\Gamma(w_k + \beta_k)} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \int_{\Re_{k-1}}^{\eta_{\mathfrak{c}}} (\eta_{\mathfrak{c}} - s)^{\alpha_k+\beta_k-1} \mathfrak{L}\left(s, v(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(s)\right) ds \right] \\
&\leq -\frac{1}{\Gamma(w_k - \gamma_k)} \int_{\Re_{k-1}}^{\varepsilon} (\varepsilon - s)^{w_k-\gamma_k-1} \mathfrak{L}\left(s, v^*(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v^*(s)\right) ds \\
&\quad + \frac{\varepsilon^{w_k-\gamma_k-1}}{\xi\Gamma(w_k - \gamma_k)} \left[\int_{\Re_{k-1}}^{1} (1-s)^{w_k-1} \mathcal{U}\left(s, v^*(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v^*(s)\right) ds \right. \\
&\quad \left. - \frac{\Gamma(w_k)\Gamma(\beta_k)}{\Gamma(w_k + \beta_k)} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \int_{\Re_{k-1}}^{\eta_{\mathfrak{c}}} (\eta_{\mathfrak{c}} - s)^{w_k+\beta_k-1} \mathfrak{L}\left(s, v^*(s), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v^*(s)\right) ds \right] \\
&\leq \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v^*(\varepsilon).
\end{aligned}$$

Similarly, we prove that $\mathcal{D}^{\gamma}_{\Re^+_{k-1}} Hv(\varepsilon) \geq \mathcal{D}^{\gamma}_{\Re^+_{k-1}} v_*(\varepsilon)$ and, then, $H(\mathfrak{D}_R) \subset \mathfrak{D}_R$.

Claim 3: $H(\mathfrak{D}_R)$ is relatively compact in $W^{1-\gamma_k,1}_{RL,\Re^+_{k-1}}$. To this end, we show that the two statements of Lemma 5 hold. Let $v \in \mathfrak{D}_R$. In fact,

$$
\begin{aligned}
|Hv(\varepsilon+h)-Hv(\varepsilon)| &\le \int_{\Re_{k-1}}^{1} |\mathcal{G}(\varepsilon+h,s)-\mathcal{G}(\varepsilon,s)|\mathcal{B}\Big(s,v(s),\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}}v(s)\Big)ds \\
&\quad+\frac{(\varepsilon+h)^{w_k-1}-\varepsilon^{w_k-1}}{\xi}\sum_{\mathfrak{c}=0}^{m}\lambda_{\mathfrak{c}}\int_{\Re_{k-1}}^{1}\mathcal{H}(\eta_{\mathfrak{c}},s)\mathcal{B}\Big(s,v(s),\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}}v(s)\Big)ds \\
&\le \frac{1}{\Gamma(w_k)}\Bigg[\Big((\varepsilon+h)^{w_k-1}-\varepsilon^{w_k-1}\Big)\int_{\Re_{k-1}}^{1}(1-s)^{w_k-1}\mathcal{B}\Big(s,v(s),\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}}v(s)\Big)ds \\
&\quad+\int_{\Re_{k-1}}^{\varepsilon}\Big((\varepsilon+h-s)^{w_k-1}-(\varepsilon-s)^{w_k-1}\Big)\mathcal{B}\Big(s,v(s),\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}}v(s)\Big)ds \\
&\quad+\int_{\varepsilon}^{\varepsilon+h}(\varepsilon+h-s)^{w_k-1}\mathcal{B}\Big(s,v(s),\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}}v(s)\Big)ds\Bigg] \\
&\quad+\frac{\Gamma(\beta_k)\big((\varepsilon+h)^{w_k-1}-\varepsilon^{w_k-1}\big)}{\xi}\int_{\Re_{k-1}}^{1}\mathcal{B}\Big(s,v(s),\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}}v(s)\Big)ds \\
&\le \left(\frac{(2w_k-1)h}{\Gamma(w_k)}+\frac{\Gamma(\beta_k)(w_k-1)h}{\xi}\right)(\|g\|_{L^1}+CR), \qquad (15)
\end{aligned}
$$

which goes to 0 as $h\to 0$. By virtue of (12), it yields

$$
\begin{aligned}
&\left|I^{1-\gamma_k}_{\Re^+_{k-1}}Hv(\varepsilon+h)-I^{1-\gamma_k}_{\Re^+_{k-1}}Hv(\varepsilon)\right| \qquad (16)\\
&\le \frac{1}{\Gamma(1-\gamma_k)}\Bigg[\int_{\Re_{k-1}}^{\varepsilon}\big((\varepsilon-s)^{-\gamma_k}-(\varepsilon+h-s)^{-\gamma_k}\big)|Hv(s)|ds \\
&\quad+\int_{\varepsilon}^{\varepsilon+h}(\varepsilon+h-s)^{-\gamma_k}|Hv(s)|ds\Bigg] \\
&\le \frac{(\xi+\Gamma(w_k)\Gamma(\beta_k))\big((\varepsilon+h)^{1-\gamma_k}-\varepsilon^{1-\gamma_k}+2h^{1-\gamma_k}\big)}{\Gamma(w_k)\Gamma(2-\gamma_k)}(\|g\|_{L^1}+CR), \qquad (17)
\end{aligned}
$$

which tends to 0 as $h\to 0$. Moreover, we have

$$
\begin{aligned}
&\left\|\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}}Hv(\varepsilon+h)-\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}}Hv(\varepsilon)\right\| \\
&=\Bigg\|I^{w_k-\gamma_k}_{\Re^+_{k-1}}\mathcal{B}\Big(\varepsilon+h,v(\varepsilon+h),\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}}v(\varepsilon+h)\Big) \\
&\quad-I^{w_k-\gamma_k}_{\Re^+_{k-1}}\mathcal{B}\Big(\varepsilon,v(\varepsilon),\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}}v(\varepsilon)\Big) \\
&\quad+\frac{\Gamma(w_k)}{\Gamma(w_k-\gamma_k)}\Big((\varepsilon+h)^{w_k-\gamma_k-1}-\varepsilon^{w_k-\gamma_k-1}\Big)\mathcal{H}v\Bigg\| \\
&\le \frac{(\|g\|_{L^1}+CR)}{\Gamma(w_k-\gamma_k)}\left(\frac{(1+\Gamma(w_k)\Gamma(\beta_k))(w_k-\gamma_k-2)h}{\xi}+(w_k-\gamma_k-2)h+h^{w_k-\gamma_k-1}\right), \qquad (18)
\end{aligned}
$$

which tends to 0 as $h\to 0$. From (15), (16) and (18), we obtain that

$$
\|\mathcal{T}_hHv-Hv\|_{W^{1-\gamma_k,1}_{RL,\Re^+_{k-1}}}\to 0 \quad \text{as } h\to 0,
$$

for any $v\in\mathfrak{D}_R$ and, then, the first statement of Lemma 5 is proved. Now, let us show the

second statement of Lemma 5. By the help of (12)–(14), it yields

$$\int_{1-\epsilon}^{1} |Hv(\varepsilon)|d\varepsilon + \int_{1-\epsilon}^{1} \left| I_{\Re_{k-1}^{+}}^{1-\gamma_k} Hv(\varepsilon) \right| d\varepsilon + \int_{1-\epsilon}^{1} \left| \mathcal{D}_{\Re_{k-1}^{+}}^{\gamma_k} Hv(\varepsilon) \right| d\varepsilon$$
$$\leq \epsilon(\|g\|_{L^1} + CR) \times \left[\frac{1+\xi+\Gamma(w_k)\Gamma(\beta_k)}{\Gamma(w_k-\gamma_k)} + \frac{(\xi+\Gamma(w_k)\Gamma(\beta_k))(1+\Gamma(2-\gamma_k))}{\Gamma(w_k)\Gamma(2-\gamma_k)} \right],$$

which tends to 0, uniformly on $\mathfrak{D}_R$ and, since all the hypotheses of Lemma 5 are satisfied, then H is relatively compact on $\mathfrak{D}_R$. By Schauder's fixed point theorem, H has a fixed point $v \in \mathfrak{D}_R$, which is a positive solution of the problem (7). The proof is now completed. □

Corollary 1. *Assume that there exist two positive constants l and $\mathfrak{L}$ such that*

$$\sup\{\mathcal{B}(\varepsilon, x, y) : \Re_{k-1} \leq \varepsilon \leq 1, x \geq 0, y \in \mathbb{R}\} \leq \mathfrak{L},$$
$$\inf\{\mathcal{B}(\varepsilon, x, y) : \Re_{k-1} \leq \varepsilon \leq 1, x \geq 0, y \in \mathbb{R}\} \geq l,$$

and

$$\frac{l}{\mathfrak{L}} \geq \xi + \frac{\Gamma(w_k+1)}{\Gamma(w_k+\beta_k+1)} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \eta_{\mathfrak{c}}^{w_k+\beta_k}.$$

Then, the problem (7) has at least one positive solution $v \in W_{RL,\Re_{k-1}^{+}}^{1-\gamma_k,1}$.

Proof. We begin by proving that hypotheses (H1) and (H2) hold. Given the notions of $\mathfrak{L}(\varepsilon, u, v)$ and $\mathcal{U}(\varepsilon, u, v)$, we obtain

$$l \leq \mathfrak{L}(\varepsilon, u, v) \leq \mathcal{U}(\varepsilon, u, v) \leq L, \quad \Re_{k-1}^{+} < \varepsilon < 1, x \geq 0, y \in \mathbb{R}.$$

Set

$$v_*(\varepsilon) = \left(-\frac{l(\varepsilon-\Re_{k-1})^{w_k}}{\Gamma(w_k+1)} + \frac{\mathfrak{L}(\varepsilon-\Re_{k-1})^{w_k-1}}{\Gamma(w_k+1)\xi} - \frac{l(\varepsilon-\Re_{k-1})^{w_k-1}}{\Gamma(w_k+\beta_k+1)\xi} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} (\eta_{\mathfrak{c}}-\Re_{k-1})^{w_k+\beta_k} \right),$$
$$v^*(\varepsilon) = \left(-\frac{\mathfrak{L}(\varepsilon-\Re_{k-1})^{w_k}}{\Gamma(w_k+1)} + \frac{l(\varepsilon-\Re_{k-1})^{w_k-1}}{\Gamma(w_k+1)\xi} - \frac{\mathfrak{L}(\varepsilon-\Re_{k-1})^{w_k-1}}{\Gamma(w_k+\beta_k+1)\xi} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} (\eta_{\mathfrak{c}}-\Re_{k-1})^{w_k+\beta_k} \right),$$

then

$$0 \leq v_*(\varepsilon) \leq v^*(\varepsilon),$$

$$v^*(\varepsilon) \geq \mathfrak{L}\left(\int_{\Re_{k-1}^{+}}^{1} \mathcal{G}(\varepsilon, s)ds + \frac{\varepsilon^{w_k-1}}{\xi} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \int_{\Re_{k-1}}^{1} \mathcal{H}(\eta_{\mathfrak{c}}, s)ds \right)$$
$$\geq \int_{\Re_{k-1}}^{1} \mathcal{G}(\varepsilon, s)\mathcal{U}\left(s, v^*(s), \mathcal{D}_{\Re_{k-1}^{+}}^{\gamma_k} v^*(s) \right) ds$$
$$+ \frac{\varepsilon^{w_k-1}}{\xi} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \int_{\Re_{k-1}}^{1} \mathcal{H}(\eta_{\mathfrak{c}}, s)\mathcal{U}\left(s, v^*(s), \mathcal{D}_{\Re_{k-1}^{+}}^{\gamma_k} v^*(s) \right) ds,$$

and

$$v_*(\varepsilon) \leq \int_{\Re_{k-1}}^{1} \mathcal{G}(\varepsilon, s)\mathfrak{L}\left(s, v_*(s), \mathcal{D}_{\Re_{k-1}^{+}}^{\gamma_k} v_*(s) \right) ds$$
$$+ \frac{\varepsilon^{w_k-1}}{\xi} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \int_{\Re_{k-1}}^{1} \mathcal{H}(\eta_{\mathfrak{c}}, s)\mathfrak{L}\left(s, u_*(s), \mathcal{D}_{\Re_{k-1}^{+}}^{\gamma_k} v_*(s) \right) ds.$$

Furthermore, computation gives

$$\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v^*(\varepsilon) = -\frac{l(\varepsilon-\Re_{k-1})^{w_k-\gamma_k}}{\Gamma(w_k-\gamma_k+1)} + \frac{\mathcal{L}(\varepsilon-\Re_{k-1})^{w_k-\gamma_k-1}}{w_k\Gamma(w_k-\gamma_k)\xi} - \frac{l\Gamma(w_k)\Gamma(\beta_k)(\varepsilon-\Re_{k-1})^{w_k-\gamma_k-1}}{\xi\Gamma(w_k-\gamma_k)\Gamma(w_k+\beta_k+1)} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}}(\eta_{\mathfrak{c}}-\Re_{k-1})^{w_k+\beta_k}.$$

The third inequality in hypothesis (H1) holds, in fact,

$$\begin{aligned}
&-\frac{1}{\Gamma(w_k-\gamma_k)}\int_{\Re_{k-1}}^{\varepsilon}(\varepsilon-s)^{w_k-\gamma_k-1}\mathfrak{L}\left(s,v_*(s),\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}}v_*(s)\right)ds\\
&\quad+\frac{\varepsilon^{w_k-\gamma_k-1}}{\xi\Gamma(w_k-\gamma_k)}\left[\int_{\Re_{k-1}}^{1}(1-s)^{w_k-1}\mathcal{U}\left(s,v_*(s),\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}}v_*(s)\right)ds\right.\\
&\quad\left.-\frac{\Gamma(w_k)\Gamma(\beta_k)}{\Gamma(w_k+\beta_k)}\sum_{\mathfrak{c}=0}^{m}\lambda_{\mathfrak{c}}\int_{\Re_{k-1}}^{\eta_{\mathfrak{c}}}(\eta_{\mathfrak{c}}-s)^{w_k+\beta_k-1}\mathfrak{L}\left(s,v_*(s),\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}}v_*(s)\right)ds\right]\\
&\leq -\frac{l(\varepsilon-\Re_{k-1})^{w_k-\gamma_k}}{\Gamma(w_k-\gamma_k+1)}+\frac{(\varepsilon-\Re_{k-1})^{w_k-\gamma_k-1}}{\xi\Gamma(w_k-\gamma_k)}\left[\int_{\Re_{k-1}}^{1}L(1-s)^{w_k-1}ds\right.\\
&\quad\left.-\frac{l\Gamma(w_k)\Gamma(\beta_k)}{\Gamma(w_k+\beta_k+1)}\sum_{\mathfrak{c}=0}^{m}\lambda_{\mathfrak{c}}(\eta_{\mathfrak{c}}-\Re_{k-1})^{w_k+\beta_k}ds\right]\\
&=\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}}v^*(\varepsilon).
\end{aligned}$$

Similarly, we prove that the fourth inequality in hypothesis (H1) is satisfied:

$$\begin{aligned}
\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}}v_*(\varepsilon) \leq &-\frac{1}{\Gamma(w_k-\gamma_k)}\int_{\Re_{k-1}}^{\varepsilon}(\varepsilon-s)^{k-\gamma_k-1}\mathcal{U}\left(s,v_*(s),\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}}v_*(s)\right)ds\\
&+\frac{(\varepsilon-\Re_{k-1})^{w_k-\gamma_k-1}}{\xi\Gamma(w_k-\gamma_k)}\left[\int_{\Re_{k-1}}^{1}(1-s)^{w_k-1}\mathcal{L}\left(s,v_*(s),\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}}v_*(s)\right)ds\right.\\
&\left.-\frac{\Gamma(w_k)\Gamma(\beta_k)}{\Gamma(w_k+\beta_k)}\sum_{\mathfrak{c}=0}^{m}\lambda_{\mathfrak{c}}\int_{\Re_{k-1}}^{\eta_{\mathfrak{c}}}(\eta_{\mathfrak{c}}-s)^{w_k+\beta_k-1}\mathcal{U}\left(s,v*(s),\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}}v_*(s)\right)ds\right].
\end{aligned}$$

Finally, choosing R such that

$$R \geq L\left[\frac{1+\xi+\Gamma(w_k)\Gamma(\beta_k)}{\Gamma(w_k-\gamma_k)}+\frac{(\xi+\Gamma(w_k)\Gamma(\beta_k))(1+\Gamma(2-\gamma_k))}{\Gamma(w_k)\Gamma(2-\gamma_k)}\right],$$

all the hypotheses in Theorem 2 are satisfied, hence the problem (7) has at least one positive solution $v \in \mathfrak{D}_R$ such that $v_*(\varepsilon) \leq v(\varepsilon) \leq v^*(\varepsilon)$, and $\mathcal{D}^{\gamma_k}_{\Re^+_{k-1}}v_*(\varepsilon) \leq \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}}v(\varepsilon) \leq \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}}v^*(\varepsilon)$, for all $\varepsilon \in [\Re_{k-1},1]$. The proof is achieved. □

Theorem 3. *Assume that conditions* (H1), (H2), *and Equation (7) are fulfilled for all* $k \in \{1,2,\ldots,n\}$. *Then, Equation (7) has a solution* $v \in W^{1-\gamma_k,1}_{RL,\Re^+_{k-1}}$.

Proof. By Theorem (2), Equation (7) admits at least one solution $u \in W^{1-\gamma_k,1}_{RL,\Re^+_{k-1}}$. We define the solution function, for each $k \in \{1,2,\ldots,n\}$, as

$$v(\varepsilon) = \begin{cases} 0, & \varepsilon \in [0,\Re_{k-1}],\\ v_k, & \varepsilon \in \mathcal{M}_k. \end{cases}$$

Then, the function

$$v(\varepsilon) = \begin{cases} v_1(\varepsilon), & \varepsilon \in \mathcal{M}_1, \\ v_2(\varepsilon), & \varepsilon \in \mathcal{M}_2, \\ \quad \vdots & \\ v_n(\varepsilon), & \varepsilon \in \mathcal{M}_n \end{cases}$$

is a solution of the given nonlinear BVP of RLFDEVO (7) in $W^{1-\gamma_k,1}_{RL,\Re^+_{k-1}}$. □

4. Example

Let $\mathcal{M} := [4,5]$. Consider the nonlinear variable-order BVP of RLFDE with

$$w(\varepsilon) = \begin{cases} \frac{17}{4}, & \mathcal{M}_1 := [4;4.5], \\ \frac{19}{4}, & \mathcal{M}_2 := [4.5;5], \end{cases} \quad \begin{cases} \mathcal{B}\left(\varepsilon, v(\varepsilon), \mathcal{D}^{\gamma_k}_{\Re^+_{k-1}} v(\varepsilon)\right) = l + (L-l)\varepsilon, \ \ \varepsilon \in \mathcal{M}_1, \mathcal{M}_2, \\ \lambda_1 = 0.5, \ \ \lambda_2 = 0.25, \ \ \eta_1 = 0.25, \ \ \eta_2 = 0.5, \ \ m = 1, \\ \gamma_1 = \gamma_2 = 0.75, \ \beta_1 = \beta_2 = 0.25, \ \Re_0 = \Re_1 = 0. \end{cases} \tag{19}$$

Using (19), we consider two auxiliary constant-order BVPs of RLFDEs:

$$\begin{cases} \mathcal{D}^{\frac{17}{4}}_{\Re^+_0} v(\varepsilon) + \mathcal{B}\left(\varepsilon, v(\varepsilon), \mathcal{D}^{\frac{3}{4}}_{\Re^+_0} v(\varepsilon)\right) = 0, & \Re_0 < \varepsilon < 1, \\ \lim\limits_{\varepsilon \to \Re^+_0} \varepsilon^{i-\frac{17}{4}} v(\varepsilon) = 0, \ i = 2, \ldots, n, & v(1) = \sum\limits_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} I^{\frac{1}{4}}_{\Re_0} v(\eta_{\mathfrak{c}}), \end{cases} \tag{20}$$

and

$$\begin{cases} \mathcal{D}^{\frac{19}{4}}_{\Re_1} v(\varepsilon) + \mathcal{B}\left(\varepsilon, v(\varepsilon), \mathcal{D}^{\frac{3}{4}}_{\Re_1} v(\varepsilon)\right) = 0, & \Re_1 < \varepsilon < 1, \\ \lim\limits_{\varepsilon \to \Re_1} \varepsilon^{i-\frac{19}{4}} v(\varepsilon) = 0, \ i = 2, \ldots, n, & v(1) = \sum\limits_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} I^{\frac{1}{4}}_{\Re_1} v(\eta_{\mathfrak{c}}). \end{cases} \tag{21}$$

We take the following conditions:

$$\begin{cases} \xi = 1 - \frac{\Gamma(w_1)}{\Gamma(w_1+\beta_1)} \sum\limits_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \left(\eta_{\mathfrak{c}} - \Re^+_0\right)^{w_1+\beta_1-1} = 0.99204, \ w_1(\varepsilon) \in \mathcal{M}_1, \\ \lambda_1 + \lambda_2 \le \frac{\Gamma(w_1+\beta_1)}{\Gamma(w_1)} = 1.4039, \\ \xi + \frac{\Gamma(w_1+1)}{\Gamma(w_1+\beta_1+1)} \sum\limits_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \eta_{\mathfrak{c}}^{w_1+\beta_1} = 0.996974 \ge \frac{l}{L}, \\ l \le \mathcal{B}(\varepsilon, x, y) \le L, \ L = 1, \end{cases}$$

$$\begin{cases} \xi = 1 - \frac{\Gamma(w_2)}{\Gamma(w_2+\beta_2)} \sum\limits_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \left(\eta_{\mathfrak{c}} - \Re^+_1\right)^{w_2+\beta_2-1} = 0.90622, \ w_2(\varepsilon) \in \mathcal{M}_2, \\ \lambda_1 + \lambda_2 \le \frac{\Gamma(w_2+\beta_2)}{\Gamma(w_2)} = 2.7567, \\ \xi + \frac{\Gamma(w_2+1)}{\Gamma(w_2+\beta_2+1)} \sum\limits_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} \eta_{\mathfrak{c}}^{w_2+\beta_2} = 0.976501 \ge \frac{l}{L}, \\ l \le \mathcal{B}(\varepsilon, x, y) \le L, \ L = 1. \end{cases}$$

The solutions are given by the following. For $k \in \{1,2\}$,

$$v^*(\varepsilon) = \left[-\frac{l(\varepsilon - \Re_{k-1})^{w_k}}{\Gamma(w_k+1)} + \frac{L(\varepsilon - \Re_{k-1})^{w_k-1}}{\xi\Gamma(w_k+1)} - \frac{l(\varepsilon - \Re_{k-1})^{w_k-1}}{\xi\Gamma(w_k+\beta_k+1)} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} (\eta_{\mathfrak{c}} - \Re_{k-1})^{w_k+\beta_k} \right],$$

$$v_*(\varepsilon) = \left[-\frac{L(\varepsilon - \Re_{k-1})^{w_k}}{\Gamma(w_k+1)} + \frac{l(\varepsilon - \Re_{k-1})^{k-1} w}{\xi\Gamma(w_k+1)} - \frac{L(\varepsilon - \Re_{k-1})^{w_k-1}}{\xi\Gamma(w_k+\beta_k+1)} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}} (\eta_{\mathfrak{c}} - \Re_{k-1})^{w_k+\beta_k} \right].$$

The exact solution is given by

$$\begin{aligned} v(\varepsilon) = &-\frac{l(\varepsilon-\Re_{k-1})^{w_k}}{\Gamma(w_k+1)} - \frac{(L-l)(\varepsilon-\Re_{k-1})^{w_k+1}}{\Gamma(w_k+2)} + \frac{l(\varepsilon-\Re_{k-1})^{w_k-1}}{\Gamma(w_k+1)\xi} \\ &+ \frac{(L-l)(\varepsilon-\Re_{k-1})^{w_k-1}}{\Gamma(w_k+2)\xi} - \frac{l(\varepsilon-\Re_{k-1})^{w_k-1}}{\Gamma(w_k+\beta_k+1)} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}}(\eta_{\mathfrak{c}}-\Re_{k-1})^{w_k+\beta_k} \\ &- \frac{(L-l)(\varepsilon-\Re_{k-1})^{w_k-1}}{\Gamma(w_k+\beta_k+2)} \sum_{\mathfrak{c}=0}^{m} \lambda_{\mathfrak{c}}(\eta_{\mathfrak{c}}-\Re_{k-1})^{w_k+\beta_k+1}. \end{aligned}$$

By computations, we obtain

$$\begin{aligned} v^*(\varepsilon) &= 0.015997742619(\varepsilon-\Re_{k-1})^{3.25} - 0.01601(\varepsilon-\Re_{k-1})^{4.25}, \\ v_*(\varepsilon) &= 0.0159977426(\varepsilon-\Re_{k-1})^{3.5} - 1.9105\times10^{-1}(\varepsilon-\Re_{k-1})^{4.5}, \\ v(\varepsilon) &= 0.012373(\varepsilon-\Re_{k-1})^{3.25} + 0.01239973(\varepsilon-\Re_{k-1})^{4.25} - 0.000039113(\varepsilon-\Re_{k-1})^{5.5}. \end{aligned}$$

In addition, we have

$$\begin{aligned} \mathcal{D}^{\gamma}_{\Re^{+}_{k-1}} v^*(\varepsilon) &= 10^{-2}(\varepsilon-\Re_{k-1})^3(3.6014 - 4.00198(\varepsilon-\Re_{k-1})), \\ \mathcal{D}^{\gamma}_{\Re^{+}_{k-1}} v_*(\varepsilon) &= 10^{-2}(\varepsilon-\Re_{k-1})^3(3.00387 - 3.85667(\varepsilon-\Re_{k-1})), \\ \mathcal{D}^{\gamma}_{\Re^{+}_{k-1}} v(\varepsilon) &= -9.3834\times10^{-6}(\varepsilon-\Re_{k-1})^5 \\ &\quad - 4.1198\times10^{-3}(\varepsilon-\Re_{k-1})^4 + 3.7117\times10^{-2}(\varepsilon-\Re_{k-1})^3. \end{aligned}$$

Hence, from Corollary 1, we conclude that the problem has at least one positive solution $u \in W^{1,1}$ such that $0 \le v_*(\varepsilon) \le v(\varepsilon) \le v^*(\varepsilon)$, and $\mathcal{D}^{\gamma}_{+} v_*(\varepsilon) \le \mathcal{D}^{\gamma}_{+} v(\varepsilon) \le \mathcal{D}^{\gamma}_{+} v^*(\varepsilon)$.

5. Numerical Analysis

In this section, we took 100 points for $\alpha(t) = 5 - t$. See Figures 1–3.

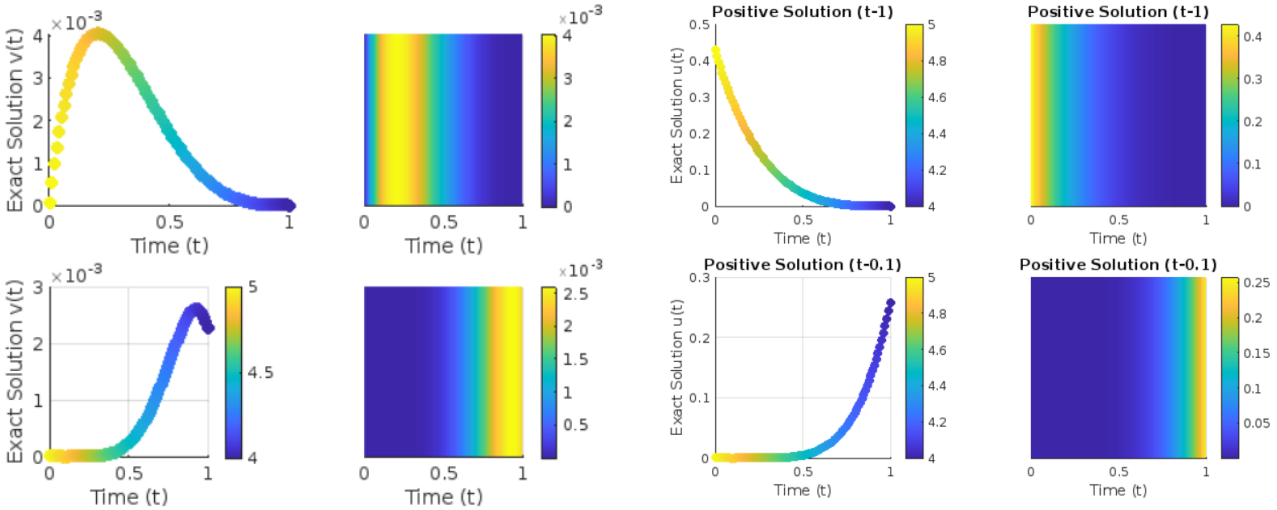


Figure 1. Exact solution for $\alpha(t) = 5 - t$.

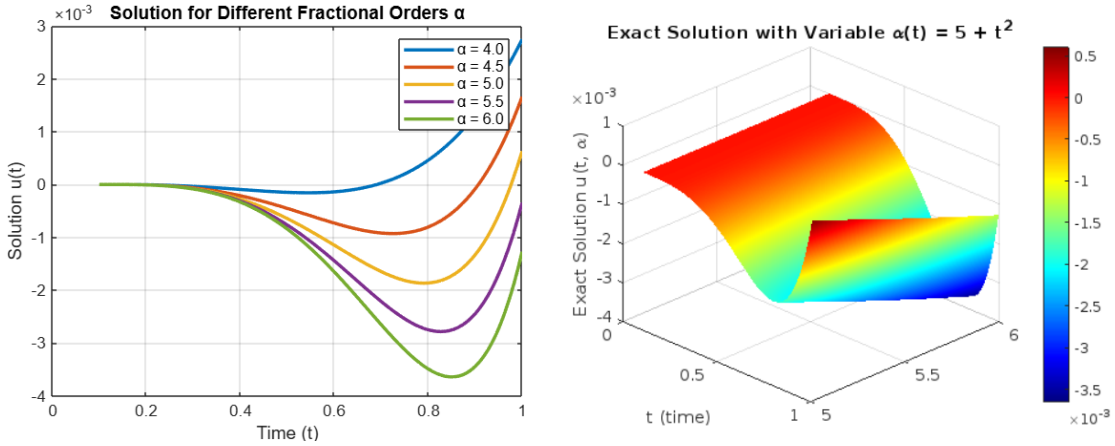


Figure 2. Exact solution for $\alpha(t) = 5 + t^2$ with negative coefficient.

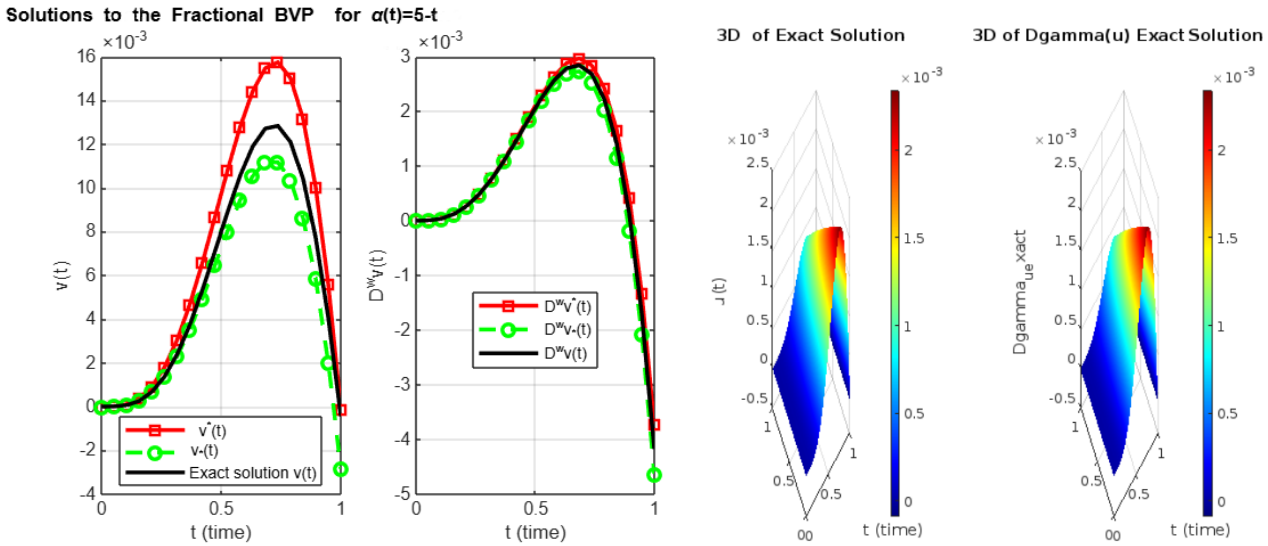


Figure 3. Solution for $\alpha(t) = 5 - t$.

6. Conclusions

This study demonstrates the existence of solutions for variable-order Riemann–Liouville (RL) fractional differential equations under multipoint boundary conditions. By employing a piecewise constant approximation technique, we transform the problem into an equivalent system of constant-order fractional differential equations (FDEs), which allows for both analytical and numerical analysis. The existence of solutions is rigorously proven using Schauder's fixed point theorem under suitable mathematical conditions.

These findings open up new research opportunities, particularly for extensions involving ψ-Caputo derivatives.

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