



Article

Development of Fractional Newton-Type Inequalities Through Extended Integral Operators

Abd-Allah Hyder¹ , Areej A. Almoneef^{2,*} , Mohamed A. Barakat^{3,*} , Hüseyin Budak⁴ and Özge Aktaş⁵

¹ Department of Mathematics, College of Science, King Khalid University, P.O. Box 9004, Abha 61413, Saudi Arabia; abahahmed@kku.edu.sa

² Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia

³ Department of Basic Science, University College of Alwajh, University of Tabuk, Tabuk 71491, Saudi Arabia

⁴ Department of Mathematics, Faculty of Science and Arts, Kocaeli University, Kocaeli 41001, Türkiye; hsyn.budak@gmail.com

⁵ Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce 81620, Türkiye; ozgeakts123@gmail.com

* Correspondence: aaalmoneef@pnu.edu.sa (A.A.A.); m.barakat@ut.edu.sa (M.A.B.)

Abstract

This paper introduces a new class of Newton-type inequalities (NTIs) within the framework of extended fractional integral operators. This study begins by establishing a fundamental identity for generalized fractional Riemann–Liouville (FR-L) operators, which forms the basis for deriving various inequalities under different assumptions on the integrand. In particular, fractional counterparts of the classical 1/3 and 3/8 Simpson rules are obtained when the modulus of the first derivative is convex. The analysis is further extended to include functions that satisfy a Lipschitz condition or have bounded first derivatives. Moreover, an additional NTI is presented for functions of bounded variation, expressed in terms of their total variation. In all scenarios, the proposed results reduce to classical inequalities when the fractional parameters are specified accordingly, thus offering a unified perspective on numerical integration through fractional operators.

Keywords: Newton-type inequalities; generalized fractional operators; well-behaved functions

MSC: 26D15; 26D10; 26D07



Academic Editors: Seth Kermausuor and Eze Nwaeze

Received: 31 May 2025

Revised: 25 June 2025

Accepted: 28 June 2025

Published: 4 July 2025

Citation: Hyder, A.-A.; Almoneef, A.A.; Barakat, M.A.; Budak, H.; Aktaş, Ö. Development of Fractional Newton-Type Inequalities Through Extended Integral Operators. *Fractal Fract.* **2025**, *9*, 443. <https://doi.org/10.3390/fractalfract9070443>

Copyright: © 2025 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

The field of fractional calculus has witnessed a surge in interest due to its extensive applications across numerous scientific domains. Given its significance, various operators related to fractional integrals warrant investigation. In [1], various NTIs were developed for functions whose absolute-value first derivatives, when taken to a specified power, satisfy an arithmetically–harmonically convex condition. Additionally, NTIs involving FR-L integrals have been derived for differentiable convex functions, as well as for functions with bounded variation [2]. Moreover, these inequalities have been extended to differentiable convex functions by usage of FR-L integrals [3]. For comprehensive insights beyond the scope of this discussion, readers may refer to [4–9].

Simpson’s second rule is fundamentally based on the three-point Newton–Cotes quadrature formula. Evaluating three-step quadratic kernel computations frequently leads to findings categorized as Newton-type results, which are widely regarded as NTIs in

the mathematical literature. Various researchers have explored these inequalities. For instance, in prior investigations, NTIs have been established for functions exhibiting convex second derivatives [10]. Additionally, inequalities of this kind have been established using harmonic convex functions and p -harmonic convex functions [11,12]. Post-quantum integrals have also been utilized to derive certain NTIs [13]. In a separate investigation, the error bounds of the quadrature formulas of the Newton kind were examined under bounded-variation and Lipschitz conditions [14]. Refer to [15–18] for a more thorough examination of current developments in NTIs.

A refined version of Simpson's rule results in a distinct inequality, which is formulated as follows:

- (i) Simpson's one-third rule, often referred to as Simpson's numerical integration method, is formulated as

$$\int_v^z \mathbb{G}(w)dw \approx \frac{z-v}{6} \left[\mathbb{G}(v) + 4\mathbb{G}\left(\frac{v+z}{2}\right) + \mathbb{G}(z) \right],$$

- (ii) Simpson's three-third rule, which extends the standard Simpson method, is a particular example of the Newton–Cotes family of numerical integration formulas:

$$\int_v^z \mathbb{G}(w)dw \approx \frac{z-v}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right].$$

The three-point Simpson-kind inequality, recognized as the most prevalent Newton–Cotes quadrature method, is expressed as follows:

Theorem 1. The function $\mathbb{G} : [v, z] \rightarrow \mathbb{R}$ is 4-times continuously differentiable on (v, z) . Define the supremum norm of its fourth derivative as $\|\mathbb{G}^{(4)}\|_{\infty} = \sup_{w \in (v, z)} |\mathbb{G}^{(4)}(w)|$, which is finite. Under these conditions, the following inequality is satisfied:

$$\left| \frac{1}{6} \left[\mathbb{G}(v) + 4\mathbb{G}\left(\frac{v+z}{2}\right) + \mathbb{G}(z) \right] - \frac{1}{z-v} \int_v^z \mathbb{G}(w)dw \right| \leq \frac{1}{2880} \|\mathbb{G}^{(4)}\|_{\infty} (z-v)^4.$$

A classic closed-form quadrature method is represented by Simpson's three-eighth rule, which is written as follows:

Theorem 2. The function $\mathbb{G} : [v, z] \rightarrow \mathbb{R}$ is 4-times continuously differentiable on (v, z) . Define the supremum norm of its fourth derivative as $\|\mathbb{G}^{(4)}\|_{\infty} = \sup_{w \in (v, z)} |\mathbb{G}^{(4)}(w)|$, which is finite. Under these conditions, the following inequality holds:

$$\left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{1}{z-v} \int_v^z \mathbb{G}(w)dw \right| \leq \frac{1}{6480} \|\mathbb{G}^{(4)}\|_{\infty} (z-v)^4.$$

Convex functions are essential in analysis and optimization, serving as a foundation for various inequalities and theoretical results. Their definition is stated below.

Definition 1 ([19]). Let I be an interval in the set of real numbers. A function $\mathbb{G} : I \rightarrow \mathbb{R}$ is defined as convex if it satisfies the following condition:

$$\mathbb{G}(\zeta w_1 + (1-\zeta)w_2) \leq \zeta \mathbb{G}(w_1) + (1-\zeta)\mathbb{G}(w_2),$$

where $w_1, w_2 \in I$ and $\zeta \in [0, 1]$.

Riemann–Liouville integrals are fundamental concepts in fractional calculus. Their formal definition is given below.

Definition 2 ([20]). Let $\mathbb{G} \in L^1[v, z]$, where $v, z \in \mathbb{R}$ and $v < z$. The FR-L integrals $\mathcal{W}_{v+}^{\mathbb{b}} \mathbb{G}$ and $\mathcal{W}_{z-}^{\mathbb{b}} \mathbb{G}$ of order $\mathbb{b} > 0$ are defined as follows:

$$\begin{aligned}\mathcal{W}_{v+}^{\mathbb{b}} \mathbb{G}(w) &= \frac{1}{\Gamma(\mathbb{b})} \int_v^w (w - \hbar)^{\mathbb{b}-1} \mathbb{G}(\hbar) d\hbar, \quad w > v, \\ \mathcal{W}_{z-}^{\mathbb{b}} \mathbb{G}(w) &= \frac{1}{\Gamma(\mathbb{b})} \int_w^z (\hbar - w)^{\mathbb{b}-1} \mathbb{G}(\hbar) d\hbar, \quad w < z,\end{aligned}$$

respectively. In this context, Γ stands for the Gamma function, given by the following formula:

$$\Gamma(\mathbb{b}) = \int_0^\infty e^{-v} v^{\mathbb{b}-1} dv.$$

In 2017, Jarad et al. [21] developed a new framework for generalized fractional integral operators. Their work also examined the fundamental features of these operators and highlighted their associations with numerous other fractional operators discussed in existing research.

Definition 3 ([21]). Let $\mathbb{b} > 0$ and $\mathbb{a} \in (0, 1]$. For a function $\mathbb{G}$ belonging to $L^1[v, z]$, the operators ${}^{\mathbb{b}}\mathcal{W}_v^{\mathbb{a}} \mathbb{G}$ and ${}^{\mathbb{b}}\mathcal{W}_z^{\mathbb{a}} \mathbb{G}$ represent the generalized FR-L integrals, which are defined as

$${}^{\mathbb{b}}\mathcal{W}_v^{\mathbb{a}} \mathbb{G}(w) = \frac{1}{\Gamma(\mathbb{b})} \int_v^w \left(\frac{(w-v)^{\mathbb{a}} - (\hbar-v)^{\mathbb{a}}}{\mathbb{a}} \right)^{\mathbb{b}-1} \frac{\mathbb{G}(\hbar)}{(\hbar-v)^{1-\mathbb{a}}} d\hbar, \quad w > v,$$

and

$${}^{\mathbb{b}}\mathcal{W}_z^{\mathbb{a}} \mathbb{G}(w) = \frac{1}{\Gamma(\mathbb{b})} \int_w^z \left(\frac{(z-w)^{\mathbb{a}} - (z-\hbar)^{\mathbb{a}}}{\mathbb{a}} \right)^{\mathbb{b}-1} \frac{\mathbb{G}(\hbar)}{(z-\hbar)^{1-\mathbb{a}}} d\hbar, \quad w < z,$$

respectively.

Over the past decade, fractional-inequality research has exploded classical Hermite–Hadamard and Ostrowski bounds have been recast via proportional Caputo-hybrid operator [22], fractal–fractional constructions [23], and post-quantum integrals [13]. At the same time, applications in control theory, viscoelastic modeling, and anomalous diffusion have driven new estimates under p -convexity, harmonic convexity, and bounded variation [8]. This rich landscape underscores both the versatility of fractional methods and the value of a single unifying framework exactly the gap our Jarad operator identity and Newton-type inequalities aim to fill.

In this work, we introduce a novel framework that significantly broadens the scope of NTIs within fractional calculus.

Our entire framework rests on the new Jarad-type relation

$$\frac{\mathbb{a}^{\mathbb{b}} 2^{\mathbb{a}\mathbb{b}-1} \Gamma(\mathbb{b}+1)}{(z-v)^{\mathbb{a}\mathbb{b}}} \left[{}^{\mathbb{b}}\mathcal{W}_{\frac{v+z}{2}}^{\mathbb{a}} \mathbb{G}(v) + {}^{\mathbb{b}}\mathcal{W}_{\frac{v+z}{2}}^{\mathbb{a}} \mathbb{G}(z) \right] = I_1 + I_2,$$

which then unfolds into four Newton-type Simpson 1/3 and 3/8 error bounds under various assumptions on the integrand, including convexity of the derivative, boundedness, satisfaction of a Lipschitz condition, and bounded variation. Setting $\mathbb{a} = \mathbb{b} = 1$ immediately recovers the classical estimates.

This paper is organized as follows: In Section 2, we present a pivotal identity involving generalized fractional integrals by working with differentiable functions. Section 3 then employs this identity to establish NTIs for convex functions. In Sections 4 and 5, the same approach is extended to functions that are bounded and Lipschitzian, respectively, still

within the realm of generalized fractional integrals. Section 6 introduces a novel NTI by incorporating the framework of functions of bounded variation. Finally, Section 7 summarizes the main results and proposes directions for further investigation.

2. An Identity

This part commences with presenting a key identity that serves as the foundation for deriving our primary results in the subsequent discussions.

Lemma 1. Suppose that $\mathbb{G} : [v, \varkappa] \rightarrow \mathbb{R}$ is differentiable over $(v, \varkappa)$, $\mathfrak{a} \in (0, 1]$ and $\mathfrak{b} > 0$. If $\mathbb{G}'$ belongs to $L^1[v, \varkappa]$, then the ensuing result concerning extended fractional integrals holds:

$$\begin{aligned} & \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v + \varkappa}{3}\right) + 3\mathbb{G}\left(\frac{v + 2\varkappa}{3}\right) + \mathbb{G}(\varkappa) \right] - \frac{\mathfrak{a}^{\mathfrak{b}} 2^{\mathfrak{a}\mathfrak{b}-1} \Gamma(\mathfrak{b} + 1)}{(\varkappa - v)^{\mathfrak{a}\mathfrak{b}}} \left[{}^{\mathfrak{b}}\mathcal{W}_{\frac{v+\varkappa}{2}}^{\mathfrak{a}} \mathbb{G}(v) + {}^{\mathfrak{b}}\mathcal{W}_{\frac{v+\varkappa}{2}}^{\mathfrak{a}} \mathbb{G}(\varkappa) \right] \\ &= \frac{\mathfrak{a}^{\mathfrak{b}}(\varkappa - v)}{4} \left[\int_0^{\frac{2}{3}} \left(\left(\frac{1 - (1 - \hbar)^{\mathfrak{a}}}{\mathfrak{a}} \right)^{\mathfrak{b}} - \frac{1}{4\mathfrak{a}^{\mathfrak{b}}} \right) \left[\mathbb{G}'\left(\frac{\hbar}{2}\varkappa + \frac{2 - \hbar}{2}v\right) - \mathbb{G}'\left(\frac{\hbar}{2}v + \frac{2 - \hbar}{2}\varkappa\right) \right] d\hbar \right. \\ & \left. + \int_{\frac{2}{3}}^1 \left(\left(\frac{1 - (1 - \hbar)^{\mathfrak{a}}}{\mathfrak{a}} \right)^{\mathfrak{b}} - \frac{1}{\mathfrak{a}^{\mathfrak{b}}} \right) \left[\mathbb{G}'\left(\frac{\hbar}{2}\varkappa + \frac{2 - \hbar}{2}v\right) - \mathbb{G}'\left(\frac{\hbar}{2}v + \frac{2 - \hbar}{2}\varkappa\right) \right] d\hbar \right]. \end{aligned} \quad (1)$$

Proof. Utilizing the integration by parts, the ensuing result is derived:

$$\begin{aligned} I_1 &= \int_0^{\frac{2}{3}} \left(\left(\frac{1 - (1 - \hbar)^{\mathfrak{a}}}{\mathfrak{a}} \right)^{\mathfrak{b}} - \frac{1}{4\mathfrak{a}^{\mathfrak{b}}} \right) \left[\mathbb{G}'\left(\frac{\hbar}{2}\varkappa + \frac{2 - \hbar}{2}v\right) - \mathbb{G}'\left(\frac{\hbar}{2}v + \frac{2 - \hbar}{2}\varkappa\right) \right] d\hbar \\ &= \frac{2}{\mathfrak{a}^{\mathfrak{b}}(\varkappa - v)} \left((1 - (1 - \hbar)^{\mathfrak{a}})^{\mathfrak{b}} - \frac{1}{4} \right) \left[\mathbb{G}\left(\frac{\hbar}{2}\varkappa + \frac{2 - \hbar}{2}v\right) + \mathbb{G}\left(\frac{\hbar}{2}v + \frac{2 - \hbar}{2}\varkappa\right) \right] d\hbar \Big|_0^{\frac{2}{3}} \\ &\quad - \frac{2\mathfrak{a}\mathfrak{b}}{\mathfrak{a}^{\mathfrak{b}}(\varkappa - v)} \int_0^{\frac{2}{3}} (1 - (1 - \hbar)^{\mathfrak{a}})^{\mathfrak{b}-1} (1 - \hbar)^{\mathfrak{a}-1} \left[\mathbb{G}\left(\frac{\hbar}{2}\varkappa + \frac{2 - \hbar}{2}v\right) + \mathbb{G}\left(\frac{\hbar}{2}v + \frac{2 - \hbar}{2}\varkappa\right) \right] d\hbar \\ &= \frac{2}{\mathfrak{a}^{\mathfrak{b}}(\varkappa - v)} \left(\left(1 - \frac{1}{3} \right)^{\mathfrak{a}\mathfrak{b}} - \frac{1}{4} \right) \left[\mathbb{G}\left(\frac{2v + \varkappa}{3}\right) + \mathbb{G}\left(\frac{v + 2\varkappa}{3}\right) \right] + \frac{1}{2\mathfrak{a}^{\mathfrak{b}}(\varkappa - v)} (\mathbb{G}(v) + \mathbb{G}(\varkappa)) \\ &\quad - \frac{2\mathfrak{a}\mathfrak{b}}{\mathfrak{a}^{\mathfrak{b}}(\varkappa - v)} \int_0^{\frac{2}{3}} (1 - (1 - \hbar)^{\mathfrak{a}})^{\mathfrak{b}-1} (1 - \hbar)^{\mathfrak{a}-1} \left[\mathbb{G}\left(\frac{\hbar}{2}\varkappa + \frac{2 - \hbar}{2}v\right) + \mathbb{G}\left(\frac{\hbar}{2}v + \frac{2 - \hbar}{2}\varkappa\right) \right] d\hbar, \\ I_2 &= \int_{\frac{2}{3}}^1 \left(\left(\frac{1 - (1 - \hbar)^{\mathfrak{a}}}{\mathfrak{a}} \right)^{\mathfrak{b}} - \frac{1}{\mathfrak{a}^{\mathfrak{b}}} \right) \left[\mathbb{G}'\left(\frac{\hbar}{2}\varkappa + \frac{2 - \hbar}{2}v\right) - \mathbb{G}'\left(\frac{\hbar}{2}v + \frac{2 - \hbar}{2}\varkappa\right) \right] d\hbar \\ &= \frac{2}{\mathfrak{a}^{\mathfrak{b}}(\varkappa - v)} \left((1 - (1 - \hbar)^{\mathfrak{a}})^{\mathfrak{b}} - 1 \right) \left[\mathbb{G}\left(\frac{\hbar}{2}\varkappa + \frac{2 - \hbar}{2}v\right) + \mathbb{G}\left(\frac{\hbar}{2}v + \frac{2 - \hbar}{2}\varkappa\right) \right] d\hbar \Big|_{\frac{2}{3}}^1 \\ &\quad + \frac{2\mathfrak{a}\mathfrak{b}}{\mathfrak{a}^{\mathfrak{b}}(\varkappa - v)} \int_{\frac{2}{3}}^1 (1 - (1 - \hbar)^{\mathfrak{a}})^{\mathfrak{b}-1} (1 - \hbar)^{\mathfrak{a}-1} \left[\mathbb{G}\left(\frac{\hbar}{2}\varkappa + \frac{2 - \hbar}{2}v\right) + \mathbb{G}\left(\frac{\hbar}{2}v + \frac{2 - \hbar}{2}\varkappa\right) \right] d\hbar \\ &= -\frac{2}{\mathfrak{a}^{\mathfrak{b}}(\varkappa - v)} \left(\left(1 - \frac{1}{3} \right)^{\mathfrak{a}\mathfrak{b}} - 1 \right) \left[\mathbb{G}\left(\frac{2v + \varkappa}{3}\right) + \mathbb{G}\left(\frac{v + 2\varkappa}{3}\right) \right] \\ &\quad - \frac{2\mathfrak{a}\mathfrak{b}}{\mathfrak{a}^{\mathfrak{b}}(\varkappa - v)} \int_{\frac{2}{3}}^1 (1 - (1 - \hbar)^{\mathfrak{a}})^{\mathfrak{b}-1} (1 - \hbar)^{\mathfrak{a}-1} \left[\mathbb{G}\left(\frac{\hbar}{2}\varkappa + \frac{2 - \hbar}{2}v\right) + \mathbb{G}\left(\frac{\hbar}{2}v + \frac{2 - \hbar}{2}\varkappa\right) \right] d\hbar. \end{aligned}$$

Through a change of variables, the following representation can be derived:

$$\begin{aligned}
I_1 + I_2 &= \frac{1}{2a^b(z-v)} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] \\
&\quad - \frac{2ab}{a^b(z-v)} \int_0^1 (1 - (1-\hbar)^a)^{b-1} (1-\hbar)^{a-1} \left[\mathbb{G}\left(\frac{\hbar}{2}z + \frac{2-\hbar}{2}v\right) + \mathbb{G}\left(\frac{\hbar}{2}v + \frac{2-\hbar}{2}z\right) \right] d\hbar \quad (2) \\
&= \frac{1}{2a^b(z-v)} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] \\
&\quad - \frac{2^{ab+1}\Gamma(b+1)}{(z-v)^{ab+1}} \left[{}_{-}\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(v) + {}_{+}\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(z) \right].
\end{aligned}$$

To combine I_1 and I_2 into the single-integral form (2), set

$$t = \begin{cases} \frac{\hbar}{2/3}, & 0 \leq \hbar \leq 2/3, \\ \frac{\hbar - 2/3}{1/3}, & 2/3 < \hbar \leq 1, \end{cases}$$

so that $t \in [0, 1]$. One then checks

$$\left(\frac{1 - (1-\hbar)^a}{a} \right)^b = \begin{cases} \left(\frac{1 - (1-\frac{2}{3}t)^a}{a} \right)^b, & t \in [0, 1], \\ \left(\frac{1 - (\frac{1}{3} + \frac{2}{3}t)^a}{a} \right)^b, & t \in [0, 1], \end{cases}$$

and the Jacobian factors $d\hbar = (2/3) dt$ or $(1/3) dt$ combine exactly to produce the coefficient appearing in (2). Collecting like terms then recovers the single-integral representation of $I_1 + I_2$ given in (2).

Multiplying Equation (2) by the factor $\frac{(z-v)^{ab}}{4}$ yields (1). This concludes the proof. $\square$

3. Newton-Type Inequalities for Convex Functions

With the identity established in Lemma 1, we now proceed to demonstrate the main theorems presented below.

Theorem 3. Let $|\mathbb{G}'|$ exhibit convexity on $[v, z]$. Under the assumptions of Lemma 1, the subsequent inequality of Newton-type is valid for generalized fractional integrals:

$$\begin{aligned}
&\left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{a^b 2^{ab-1} \Gamma(b+1)}{(z-v)^{ab}} \left[{}_{-}\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(v) + {}_{+}\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(z) \right] \right| \\
&\leq \frac{a^b(z-v)}{4} (\mathcal{A}_1(a, b) + \mathcal{A}_2(a, b)) [|\mathbb{G}'(v)| + |\mathbb{G}'(z)|],
\end{aligned}$$

where

$$\mathcal{A}_1(a, b) = \int_0^{\frac{2}{3}} \left| \left(\frac{1 - (1-\hbar)^a}{a} \right)^b - \frac{1}{4a^b} \right| d\hbar$$

and

$$\mathcal{A}_2(a, b) = \int_{\frac{2}{3}}^1 \left| \left(\frac{1 - (1-\hbar)^a}{a} \right)^b - \frac{1}{a^b} \right| d\hbar.$$

Proof. Starting from Lemma 1, we derive the next inequality:

$$\begin{aligned}
& \left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{a^b 2^{ab-1} \Gamma(b+1)}{(z-v)^{ab}} \left[{}^b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(v) + {}^b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(z) \right] \right| \\
& \leq \frac{a^b(z-v)}{4} \left[\int_0^{\frac{2}{3}} \left| \left(\frac{1-(1-\hbar)^a}{a} \right)^b - \frac{1}{4a^b} \left[\left| \mathbb{G}'\left(\frac{\hbar}{2}z + \frac{2-\hbar}{2}v\right) \right| + \left| \mathbb{G}'\left(\frac{\hbar}{2}v + \frac{2-\hbar}{2}z\right) \right| \right] d\hbar \right. \right. \\
& \quad \left. \left. + \int_{\frac{2}{3}}^1 \left| \left(\frac{1-(1-\hbar)^a}{a} \right)^b - \frac{1}{a^b} \left[\left| \mathbb{G}'\left(\frac{\hbar}{2}z + \frac{2-\hbar}{2}v\right) \right| + \left| \mathbb{G}'\left(\frac{\hbar}{2}v + \frac{2-\hbar}{2}z\right) \right| \right] d\hbar \right] \right. \quad (3)
\end{aligned}$$

Given that $|\mathbb{G}'|$ is a convex function, the following result can be established:

$$\begin{aligned}
& \left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{a^b 2^{ab-1} \Gamma(b+1)}{(z-v)^{ab}} \left[{}^b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(v) + {}^b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(z) \right] \right| \\
& \leq \frac{a^b(z-v)}{4} \left[\int_0^{\frac{2}{3}} \left| \left(\frac{1-(1-\hbar)^a}{a} \right)^b - \frac{1}{4a^b} \left[\frac{\hbar}{2} |\mathbb{G}'(z)| + \frac{2-\hbar}{2} |\mathbb{G}'(v)| + \frac{\hbar}{2} |\mathbb{G}'(v)| + \frac{2-\hbar}{2} |\mathbb{G}'(z)| \right] \right| d\hbar \right. \\
& \quad \left. + \int_{\frac{2}{3}}^1 \left| \left(\frac{1-(1-\hbar)^a}{a} \right)^b - \frac{1}{a^b} \left[\frac{\hbar}{2} |\mathbb{G}'(z)| + \frac{2-\hbar}{2} |\mathbb{G}'(v)| + \frac{\hbar}{2} |\mathbb{G}'(v)| + \frac{2-\hbar}{2} |\mathbb{G}'(z)| \right] \right| d\hbar \right] \\
& = \frac{a^b(z-v)}{4} (|\mathbb{G}'(v)| + |\mathbb{G}'(z)|) \left[\int_0^{\frac{2}{3}} \left| \left(\frac{1-(1-\hbar)^a}{a} \right)^b - \frac{1}{4a^b} \right| d\hbar + \int_{\frac{2}{3}}^1 \left| \left(\frac{1-(1-\hbar)^a}{a} \right)^b - \frac{1}{a^b} \right| d\hbar \right] \\
& = \frac{a^b(z-v)}{4} (\mathcal{A}_1(a, b) + \mathcal{A}_2(a, b)) [|\mathbb{G}'(v)| + |\mathbb{G}'(z)|].
\end{aligned}$$

Thus, the proof is concluded. $\square$

Remark 1. When a is assigned the value 1 in Theorem 3, the following outcome is derived:

$$\begin{aligned}
& \left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{2^{b-1} \Gamma(b+1)}{(z-v)^b} \left[\mathcal{W}_{\frac{v+z}{2}}^b \mathbb{G}(z) + \mathcal{W}_{\frac{v+z}{2}}^b \mathbb{G}(v) \right] \right| \\
& \leq \frac{z-v}{4} (\mathcal{A}_1(1, b) + \mathcal{A}_2(1, b)) [|\mathbb{G}'(v)| + |\mathbb{G}'(z)|],
\end{aligned}$$

where

$$\mathcal{A}_1(1, b) = \int_0^{\frac{2}{3}} \left| \hbar^b - \frac{1}{4} \right| d\hbar = \begin{cases} \frac{2b}{b+1} \left(\frac{1}{4} \right)^{1+\frac{1}{b}} + \frac{1}{b+1} \left(\frac{2}{3} \right)^{b+1} - \frac{1}{6}, & 0 < b < \frac{\ln(\frac{1}{4})}{\ln(\frac{2}{3})} \\ \frac{1}{6} - \frac{1}{b+1} \left(\frac{2}{3} \right)^{b+1}, & b \geq \frac{\ln(\frac{1}{4})}{\ln(\frac{2}{3})} \end{cases}$$

and

$$\mathcal{A}_2(1, b) = \int_{\frac{2}{3}}^1 \left(1 - \hbar^b \right) d\hbar = \frac{1}{3} - \frac{1}{b+1} + \frac{1}{b+1} \left(\frac{2}{3} \right)^{b+1}.$$

This result aligns with the findings of Hezenci and Budak, as demonstrated in [24] (Theorem 3).

Remark 2. When both a and b are set to 1 in Theorem 3, the following conclusion holds:

$$\begin{aligned}
& \left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{1}{z-v} \int_v^z \mathbb{G}(\hbar) d\hbar \right| \\
& \leq \frac{25(z-v)}{576} [|\mathbb{G}'(v)| + |\mathbb{G}'(z)|].
\end{aligned}$$

This outcome corresponds to the result established in [2] (Remark 3).

Theorem 4. Under the conditions outlined in Lemma 1, if $|\mathbb{G}'|^\vartheta$ is convex on $[v, \varkappa]$, where $v, \varkappa \in \mathbb{R}$ with $v < \varkappa$, $\frac{1}{\varrho} + \frac{1}{\vartheta} = 1$, and $\vartheta > 1$, then the following Newton-type inequality holds for generalized fractional integrals:

$$\begin{aligned} & \left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+\varkappa}{3}\right) + 3\mathbb{G}\left(\frac{v+2\varkappa}{3}\right) + \mathbb{G}(\varkappa) \right] - \frac{a^b 2^{ab-1} \Gamma(b+1)}{(\varkappa-v)^{ab}} \left[{}_b\mathcal{W}_{\frac{v+\varkappa}{2}}^a \mathbb{G}(v) + {}_b\mathcal{W}_{\frac{v+\varkappa}{2}}^a \mathbb{G}(\varkappa) \right] \right| \\ & \leq \frac{a^b (\varkappa-v)}{4} \left\{ (\mathcal{A}_1(a, b, \varrho))^{\frac{1}{\varrho}} \left[\left(\frac{|\mathbb{G}'(\varkappa)|^\vartheta + 5|\mathbb{G}'(v)|^\vartheta}{9} \right)^{\frac{1}{\vartheta}} + \left(\frac{|\mathbb{G}'(v)|^\vartheta + 5|\mathbb{G}'(\varkappa)|^\vartheta}{9} \right)^{\frac{1}{\vartheta}} \right] \right. \\ & \quad \left. + (\mathcal{A}_2(a, b, \varrho))^{\frac{1}{\varrho}} \left[\left(\frac{5|\mathbb{G}'(\varkappa)|^\vartheta + 7|\mathbb{G}'(v)|^\vartheta}{36} \right)^{\frac{1}{\vartheta}} + \left(\frac{5|\mathbb{G}'(v)|^\vartheta + 7|\mathbb{G}'(\varkappa)|^\vartheta}{36} \right)^{\frac{1}{\vartheta}} \right] \right\}, \end{aligned}$$

where

$$\mathcal{A}_1(a, b, \varrho) = \int_0^{\frac{2}{3}} \left| \left(\frac{1-(1-\hbar)^a}{a} \right)^b - \frac{1}{4a^b} \right|^{\varrho} d\hbar$$

and

$$\mathcal{A}_2(a, b, \varrho) = \int_{\frac{2}{3}}^1 \left| \left(\frac{1-(1-\hbar)^a}{a} \right)^b - \frac{1}{a^b} \right|^{\varrho} d\hbar.$$

Proof. Utilizing Hölder's inequality and exploiting the convexity of $|\mathbb{G}'|^\vartheta$, we obtain the subsequent result:

$$\begin{aligned} & \int_0^{\frac{2}{3}} \left| \left(\frac{1-(1-\hbar)^a}{a} \right)^b - \frac{1}{4a^b} \right| \left| \left| \mathbb{G}'\left(\frac{\hbar}{2}\varkappa + \frac{2-\hbar}{2}v\right) \right| + \left| \mathbb{G}'\left(\frac{\hbar}{2}v + \frac{2-\hbar}{2}\varkappa\right) \right| \right| d\hbar \\ & \leq \left(\int_0^{\frac{2}{3}} \left| \left(\frac{1-(1-\hbar)^a}{a} \right)^b - \frac{1}{4a^b} \right|^{\varrho} d\hbar \right)^{1/\varrho} \left(\int_0^{\frac{2}{3}} \left| \mathbb{G}'\left(\frac{\hbar}{2}\varkappa + \frac{2-\hbar}{2}v\right) \right|^\vartheta d\hbar \right)^{1/\vartheta} \\ & \quad + \left(\int_0^{\frac{2}{3}} \left| \left(\frac{1-(1-\hbar)^a}{a} \right)^b - \frac{1}{4a^b} \right|^{\varrho} d\hbar \right)^{1/\varrho} \left(\int_0^{\frac{2}{3}} \left| \mathbb{G}'\left(\frac{\hbar}{2}v + \frac{2-\hbar}{2}\varkappa\right) \right|^\vartheta d\hbar \right)^{1/\vartheta} \\ & = \mathcal{A}_1^{\frac{1}{\varrho}}(a, b, \varrho) \left(\int_0^{\frac{2}{3}} \left| \mathbb{G}'\left(\frac{\hbar}{2}\varkappa + \frac{2-\hbar}{2}v\right) \right|^\vartheta d\hbar \right)^{1/\vartheta} \\ & \quad + \mathcal{A}_1^{\frac{1}{\varrho}}(a, b, \varrho) \left(\int_0^{\frac{2}{3}} \left| \mathbb{G}'\left(\frac{\hbar}{2}v + \frac{2-\hbar}{2}\varkappa\right) \right|^\vartheta d\hbar \right)^{1/\vartheta} \\ & \leq \mathcal{A}_1^{\frac{1}{\varrho}}(a, b, \varrho) \left(\int_0^{\frac{2}{3}} \left[\frac{\hbar}{2} |\mathbb{G}'(\varkappa)|^\vartheta + \frac{2-\hbar}{2} |\mathbb{G}'(v)|^\vartheta \right] d\hbar \right)^{\frac{1}{\vartheta}} \\ & \quad + \mathcal{A}_1^{\frac{1}{\varrho}}(a, b, \varrho) \left(\int_0^{\frac{2}{3}} \left[\frac{\hbar}{2} |\mathbb{G}'(v)|^\vartheta + \frac{2-\hbar}{2} |\mathbb{G}'(\varkappa)|^\vartheta \right] d\hbar \right)^{\frac{1}{\vartheta}} \\ & = \mathcal{A}_1^{\frac{1}{\varrho}}(a, b, \varrho) \left[\left(\frac{|\mathbb{G}'(\varkappa)|^\vartheta + 5|\mathbb{G}'(v)|^\vartheta}{9} \right)^{\frac{1}{\vartheta}} + \left(\frac{|\mathbb{G}'(v)|^\vartheta + 5|\mathbb{G}'(\varkappa)|^\vartheta}{9} \right)^{\frac{1}{\vartheta}} \right] \end{aligned} \quad (4)$$

and

$$\begin{aligned}
& \int_{\frac{2}{3}}^1 \left| \left(\frac{1 - (1 - \hbar)^a}{a} \right)^b - \frac{1}{a^b} \left[\left| \mathbb{G}' \left(\frac{\hbar}{2} z + \frac{2 - \hbar}{2} v \right) \right| + \left| \mathbb{G}' \left(\frac{\hbar}{2} v + \frac{2 - \hbar}{2} z \right) \right| \right] d\hbar \right. \\
& \leq \left(\int_{\frac{2}{3}}^1 \left| \left(\frac{1 - (1 - \hbar)^a}{a} \right)^b - \frac{1}{a^b} \right|^q d\hbar \right)^{1/q} \left(\int_{\frac{2}{3}}^1 \left| \mathbb{G}' \left(\frac{\hbar}{2} z + \frac{2 - \hbar}{2} v \right) \right|^\theta d\hbar \right)^{1/\theta} \\
& + \left(\int_{\frac{2}{3}}^1 \left| \left(\frac{1 - (1 - \hbar)^a}{a} \right)^b - \frac{1}{a^b} \right|^q d\hbar \right)^{1/q} \left(\int_{\frac{2}{3}}^1 \left| \mathbb{G}' \left(\frac{\hbar}{2} v + \frac{2 - \hbar}{2} z \right) \right|^\theta d\hbar \right)^{1/\theta} \\
& = \mathcal{A}_2^{\frac{1}{q}}(a, b, q) \left(\int_{\frac{2}{3}}^1 \left| \mathbb{G}' \left(\frac{\hbar}{2} z + \frac{2 - \hbar}{2} v \right) \right|^\theta d\hbar \right)^{1/\theta} + \mathcal{A}_2^{\frac{1}{q}}(a, b, q) \left(\int_{\frac{2}{3}}^1 \left| \mathbb{G}' \left(\frac{\hbar}{2} v + \frac{2 - \hbar}{2} z \right) \right|^\theta d\hbar \right)^{1/\theta} \\
& \leq \mathcal{A}_2^{\frac{1}{q}}(a, b, q) \left(\int_{\frac{2}{3}}^1 \left[\frac{\hbar}{2} |\mathbb{G}'(z)|^\theta + \frac{2 - \hbar}{2} |\mathbb{G}'(v)|^\theta \right] d\hbar \right)^{\frac{1}{\theta}} + \mathcal{A}_2^{\frac{1}{q}}(a, b, q) \left(\int_{\frac{2}{3}}^1 \left[\frac{\hbar}{2} |\mathbb{G}'(v)|^\theta + \frac{2 - \hbar}{2} |\mathbb{G}'(z)|^\theta \right] d\hbar \right)^{\frac{1}{\theta}} \\
& = \mathcal{A}_2^{\frac{1}{q}}(a, b, q) \left[\left(\frac{5|\mathbb{G}'(z)|^\theta + 7|\mathbb{G}'(v)|^\theta}{36} \right)^{\frac{1}{\theta}} + \left(\frac{5|\mathbb{G}'(v)|^\theta + 7|\mathbb{G}'(z)|^\theta}{36} \right)^{\frac{1}{\theta}} \right].
\end{aligned} \tag{5}$$

By inserting (4) and (5) into (3), we arrive at the following result:

$$\begin{aligned}
& \left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{a^b 2^{a^b-1} \Gamma(b+1)}{(z-v)^{ab}} \left[{}_b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(v) + {}_b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(z) \right] \right| \\
& \leq \frac{a^b(z-v)}{4} \left\{ \mathcal{A}_1^{\frac{1}{q}}(a, b, q) \left[\left(\frac{|\mathbb{G}'(z)|^\theta + 5|\mathbb{G}'(v)|^\theta}{9} \right)^{\frac{1}{\theta}} + \left(\frac{|\mathbb{G}'(v)|^\theta + 5|\mathbb{G}'(z)|^\theta}{9} \right)^{\frac{1}{\theta}} \right] \right. \\
& + \mathcal{A}_2^{\frac{1}{q}}(a, b, q) \left[\left(\frac{5|\mathbb{G}'(z)|^\theta + 7|\mathbb{G}'(v)|^\theta}{36} \right)^{\frac{1}{\theta}} + \left(\frac{5|\mathbb{G}'(v)|^\theta + 7|\mathbb{G}'(z)|^\theta}{36} \right)^{\frac{1}{\theta}} \right] \left. \right\}.
\end{aligned}$$

With this, the proof is concluded. $\square$

Remark 3. Substituting $a = 1$ into Theorem 4, the following result is obtained:

$$\begin{aligned}
& \left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{2^{b-1} \Gamma(b+1)}{(z-v)^b} \left[\mathcal{W}_{\frac{v+z}{2}}^b \mathbb{G}(z) + \mathcal{W}_{\frac{v+z}{2}}^b \mathbb{G}(v) \right] \right| \\
& \leq \frac{(z-v)}{4} \left\{ (\mathcal{A}_1(1, b, q))^{\frac{1}{q}} \left[\left(\frac{|\mathbb{G}'(z)|^\theta + 5|\mathbb{G}'(v)|^\theta}{9} \right)^{\frac{1}{\theta}} + \left(\frac{|\mathbb{G}'(v)|^\theta + 5|\mathbb{G}'(z)|^\theta}{9} \right)^{\frac{1}{\theta}} \right] \right. \\
& + (\mathcal{A}_2(1, b, q))^{\frac{1}{q}} \left[\left(\frac{5|\mathbb{G}'(z)|^\theta + 7|\mathbb{G}'(v)|^\theta}{36} \right)^{\frac{1}{\theta}} + \left(\frac{5|\mathbb{G}'(v)|^\theta + 7|\mathbb{G}'(z)|^\theta}{36} \right)^{\frac{1}{\theta}} \right] \left. \right\}.
\end{aligned}$$

This result aligns with the conclusions drawn by Hezenci and Budak, as demonstrated in [24] (Theorem 4).

Remark 4. Setting $a = b = 1$ in Theorem 4 leads to the next outcome:

$$\begin{aligned} & \left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{1}{z-v} \int_v^z \mathbb{G}(\hbar) d\hbar \right| \\ & \leq \frac{z-v}{4} \left\{ \left(\frac{1}{\varrho+1} \left[\left(\frac{1}{4}\right)^{\varrho+1} + \left(\frac{5}{12}\right)^{\varrho+1} \right] \right)^{\frac{1}{\varrho}} \right. \\ & \quad \times \left[\left(\frac{|\mathbb{G}'(z)|^\vartheta + 5|\mathbb{G}'(v)|^\vartheta}{9} \right)^{\frac{1}{\vartheta}} + \left(\frac{|\mathbb{G}'(v)|^\vartheta + 5|\mathbb{G}'(z)|^\vartheta}{9} \right)^{\frac{1}{\vartheta}} \right] \\ & \quad \left. + \left(\frac{1}{\varrho+1} \left(\frac{1}{3}\right)^{\varrho+1} \right)^{\frac{1}{\varrho}} \left[\left(\frac{5|\mathbb{G}'(z)|^\vartheta + 7|\mathbb{G}'(v)|^\vartheta}{36} \right)^{\frac{1}{\vartheta}} + \left(\frac{5|\mathbb{G}'(v)|^\vartheta + 7|\mathbb{G}'(z)|^\vartheta}{36} \right)^{\frac{1}{\vartheta}} \right] \right\}, \end{aligned}$$

as stated by Hezenci and Budak in [24] (Corollary 1).

Theorem 5. Assume the premises of Lemma 1 hold. If $|\mathbb{G}'|^\vartheta$ is convex on the interval $[v, z]$ for some $\vartheta \geq 1$, then an inequality tied to fractional integrals is established.

$$\begin{aligned} & \left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{a^b \Gamma(b+1)}{(z-v)^{ab}} {}^{b}_{-} \mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(v) + {}^{b}_{+} \mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(z) \right| \\ & \leq \frac{a^b(z-v)}{4} \left\{ \mathcal{A}_1^{1-\frac{1}{\vartheta}}(a, b) \left(\left[\mathcal{A}_3(a, b) \|\mathbb{G}'(z)\|^\vartheta + \mathcal{A}_4(a, b) \|\mathbb{G}'(v)\|^\vartheta \right]^{1/\vartheta} \right. \right. \\ & \quad \left. + \left[\mathcal{A}_4(a, b) \|\mathbb{G}'(z)\|^\vartheta + \mathcal{A}_3(a, b) \|\mathbb{G}'(v)\|^\vartheta \right]^{1/\vartheta} \right) \\ & \quad + \mathcal{A}_2^{1-\frac{1}{\vartheta}}(a, b) \left(\left[\mathcal{A}_5(a, b) \|\mathbb{G}'(z)\|^\vartheta + \mathcal{A}_6(a, b) \|\mathbb{G}'(v)\|^\vartheta \right]^{1/\vartheta} \right. \\ & \quad \left. \left. + \left[\mathcal{A}_6(a, b) \|\mathbb{G}'(z)\|^\vartheta + \mathcal{A}_5(a, b) \|\mathbb{G}'(v)\|^\vartheta \right]^{1/\vartheta} \right) \right\}, \end{aligned}$$

with $\mathcal{A}_1$ and $\mathcal{A}_2$ being specified in Theorem 3, and

$$\begin{aligned} \mathcal{A}_3(a, b) &= \int_0^{\frac{2}{3}} \frac{\hbar}{2} \left| \left(\frac{1 - (1-\hbar)^a}{a} \right)^b - \frac{1}{4a^b} \right| d\hbar \\ \mathcal{A}_4(a, b) &= \int_0^{\frac{2}{3}} \frac{2-\hbar}{2} \left| \left(\frac{1 - (1-\hbar)^a}{a} \right)^b - \frac{1}{4a^b} \right| d\hbar \\ \mathcal{A}_5(a, b) &= \int_{\frac{2}{3}}^1 \frac{\hbar}{2} \left| \left(\frac{1 - (1-\hbar)^a}{a} \right)^b - \frac{1}{a^b} \right| d\hbar \\ \mathcal{A}_6(a, b) &= \int_{\frac{2}{3}}^1 \frac{2-\hbar}{2} \left| \left(\frac{1 - (1-\hbar)^a}{a} \right)^b - \frac{1}{a^b} \right| d\hbar. \end{aligned}$$

Proof. Utilizing the power mean inequality along with the convex property of $|\mathbb{G}'|^\vartheta$, we obtain

$$\begin{aligned}
& \int_0^{\frac{2}{3}} \left| \left(\frac{1 - (1 - \hbar)^a}{a} \right)^b - \frac{1}{4ab} \left[\left| \mathbb{G}' \left(\frac{\hbar}{2} z + \frac{2 - \hbar}{2} v \right) \right| + \left| \mathbb{G}' \left(\frac{\hbar}{2} v + \frac{2 - \hbar}{2} z \right) \right| \right] d\hbar \right. \\
& \leq \left(\int_0^{\frac{2}{3}} \left| \left(\frac{1 - (1 - \hbar)^a}{a} \right)^b - \frac{1}{4ab} d\hbar \right|^{1-1/\theta} \left(\int_0^{\frac{2}{3}} \left| \left(\frac{1 - (1 - \hbar)^a}{a} \right)^b - \frac{1}{4ab} \left| \mathbb{G}' \left(\frac{\hbar}{2} z + \frac{2 - \hbar}{2} v \right) \right|^\theta d\hbar \right)^{1/\theta} \right. \\
& + \left. \left(\int_0^{\frac{2}{3}} \left| \left(\frac{1 - (1 - \hbar)^a}{a} \right)^b - \frac{1}{4ab} d\hbar \right|^{1-1/\theta} \left(\int_0^{\frac{2}{3}} \left| \left(\frac{1 - (1 - \hbar)^a}{a} \right)^b - \frac{1}{4ab} \left| \mathbb{G}' \left(\frac{\hbar}{2} v + \frac{2 - \hbar}{2} z \right) \right|^\theta d\hbar \right)^{1/\theta} \right. \\
& = \mathcal{A}_1^{1-1/\theta}(a, b) \left(\int_0^{\frac{2}{3}} \left| \left(\frac{1 - (1 - \hbar)^a}{a} \right)^b - \frac{1}{4ab} \left| \mathbb{G}' \left(\frac{\hbar}{2} z + \frac{2 - \hbar}{2} v \right) \right|^\theta d\hbar \right)^{1/\theta} \\
& + \mathcal{A}_1^{1-1/\theta}(a, b) \left(\int_0^{\frac{2}{3}} \left| \left(\frac{1 - (1 - \hbar)^a}{a} \right)^b - \frac{1}{4ab} \left| \mathbb{G}' \left(\frac{\hbar}{2} v + \frac{2 - \hbar}{2} z \right) \right|^\theta d\hbar \right)^{1/\theta} \\
& \leq \mathcal{A}_1^{1-1/\theta}(a, b) \left(\int_0^{\frac{2}{3}} \left| \left(\frac{1 - (1 - \hbar)^a}{a} \right)^b - \frac{1}{4ab} \left[\frac{\hbar}{2} |\mathbb{G}'(z)|^\theta + \frac{2 - \hbar}{2} |\mathbb{G}'(v)|^\theta \right] d\hbar \right)^{\frac{1}{\theta}} \\
& + \mathcal{A}_1^{1-1/\theta}(a, b) \left(\int_0^{\frac{2}{3}} \left| \left(\frac{1 - (1 - \hbar)^a}{a} \right)^b - \frac{1}{4ab} \left[\frac{\hbar}{2} |\mathbb{G}'(v)|^\theta + \frac{2 - \hbar}{2} |\mathbb{G}'(z)|^\theta \right] d\hbar \right)^{\frac{1}{\theta}} \\
& = \mathcal{A}_1^{1-1/\theta}(a, b) \left\{ \left[\mathcal{A}_3(a, b) |\mathbb{G}'(z)|^\theta + \mathcal{A}_4(a, b) |\mathbb{G}'(v)|^\theta \right]^{1/\theta} + \left[\mathcal{A}_4(a, b) |\mathbb{G}'(z)|^\theta + \mathcal{A}_3(a, b) |\mathbb{G}'(v)|^\theta \right]^{1/\theta} \right\}
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
& \int_{\frac{2}{3}}^1 \left| \left(\frac{1 - (1 - \hbar)^a}{a} \right)^b - \frac{1}{ab} \left[\left| \mathbb{G}' \left(\frac{\hbar}{2} z + \frac{2 - \hbar}{2} v \right) \right| + \left| \mathbb{G}' \left(\frac{\hbar}{2} v + \frac{2 - \hbar}{2} z \right) \right| \right] d\hbar \right. \\
& \leq \left(\int_{\frac{2}{3}}^1 \left| \left(\frac{1 - (1 - \hbar)^a}{a} \right)^b - \frac{1}{ab} d\hbar \right|^{1-1/\theta} \left(\int_{\frac{2}{3}}^1 \left| \left(\frac{1 - (1 - \hbar)^a}{a} \right)^b - \frac{1}{ab} \left| \mathbb{G}' \left(\frac{\hbar}{2} z + \frac{2 - \hbar}{2} v \right) \right|^\theta d\hbar \right)^{1/\theta} \right. \\
& + \left(\int_{\frac{2}{3}}^1 \left| \left(\frac{1 - (1 - \hbar)^a}{a} \right)^b - \frac{1}{ab} d\hbar \right|^{1-1/\theta} \left(\int_{\frac{2}{3}}^1 \left| \left(\frac{1 - (1 - \hbar)^a}{a} \right)^b - \frac{1}{ab} \left| \mathbb{G}' \left(\frac{\hbar}{2} v + \frac{2 - \hbar}{2} z \right) \right|^\theta d\hbar \right)^{1/\theta} \right. \\
& = \mathcal{A}_2^{1-1/\theta}(a, b) \left(\int_{\frac{2}{3}}^1 \left| \left(\frac{1 - (1 - \hbar)^a}{a} \right)^b - \frac{1}{ab} \left| \mathbb{G}' \left(\frac{\hbar}{2} z + \frac{2 - \hbar}{2} v \right) \right|^\theta d\hbar \right)^{1/\theta} \\
& + \mathcal{A}_2^{1-1/\theta}(a, b) \left(\int_{\frac{2}{3}}^1 \left| \left(\frac{1 - (1 - \hbar)^a}{a} \right)^b - \frac{1}{ab} \left| \mathbb{G}' \left(\frac{\hbar}{2} v + \frac{2 - \hbar}{2} z \right) \right|^\theta d\hbar \right)^{1/\theta} \\
& \leq \mathcal{A}_2^{1-1/\theta}(a, b) \left(\int_{\frac{2}{3}}^1 \left| \left(\frac{1 - (1 - \hbar)^a}{a} \right)^b - \frac{1}{ab} \left[\frac{\hbar}{2} |\mathbb{G}'(z)|^\theta + \frac{2 - \hbar}{2} |\mathbb{G}'(v)|^\theta \right] d\hbar \right)^{\frac{1}{\theta}} \\
& + \mathcal{A}_2^{1-1/\theta}(a, b) \left(\int_{\frac{2}{3}}^1 \left| \left(\frac{1 - (1 - \hbar)^a}{a} \right)^b - \frac{1}{ab} \left[\frac{\hbar}{2} |\mathbb{G}'(v)|^\theta + \frac{2 - \hbar}{2} |\mathbb{G}'(z)|^\theta \right] d\hbar \right)^{\frac{1}{\theta}} \\
& = \mathcal{A}_2^{1-1/\theta}(a, b) \left\{ \left[\mathcal{A}_5(a, b) |\mathbb{G}'(z)|^\theta + \mathcal{A}_6(a, b) |\mathbb{G}'(v)|^\theta \right]^{1/\theta} + \left[\mathcal{A}_6(a, b) |\mathbb{G}'(z)|^\theta + \mathcal{A}_5(a, b) |\mathbb{G}'(v)|^\theta \right]^{1/\theta} \right\}.
\end{aligned} \tag{7}$$

Replacing (6) and (7) in (3), we obtain

$$\begin{aligned}
& \left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{a^b \Gamma(b+1)}{(z-v)^{ab}} 2^{ab-1} \left[{}^b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(v) + {}^b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(z) \right] \right| \\
& \leq \frac{a^b(z-v)}{4} \left\{ \mathcal{A}_1^{1-\frac{1}{\theta}}(a, b) \left(\left[\mathcal{A}_3(a, b) \|\mathbb{G}'(z)\|^\theta + \mathcal{A}_4(a, b) \|\mathbb{G}'(v)\|^\theta \right]^{1/\theta} \right. \right. \\
& \quad \left. \left. + \left[\mathcal{A}_4(a, b) \|\mathbb{G}'(z)\|^\theta + \mathcal{A}_3(a, b) \|\mathbb{G}'(v)\|^\theta \right]^{\frac{1}{\theta}} \right) \right. \\
& \quad \left. + \mathcal{A}_2^{1-\frac{1}{\theta}}(a, b) \left(\left[\mathcal{A}_5(a, b) \|\mathbb{G}'(z)\|^\theta + \mathcal{A}_6(a, b) \|\mathbb{G}'(v)\|^\theta \right]^{1/\theta} \right. \right. \\
& \quad \left. \left. + \left[\mathcal{A}_6(a, b) \|\mathbb{G}'(z)\|^\theta + \mathcal{A}_5(a, b) \|\mathbb{G}'(v)\|^\theta \right]^{\frac{1}{\theta}} \right) \right\},
\end{aligned}$$

which finalizes the proof. $\square$

Remark 5. Setting $a = 1$ in Theorem 5 yields the following outcome:

$$\begin{aligned}
& \left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{2^{b-1} \Gamma(b+1)}{(z-v)^b} \left[\mathcal{W}_{\frac{v+z}{2}}^b \mathbb{G}(z) + \mathcal{W}_{\frac{v+z}{2}}^b \mathbb{G}(v) \right] \right| \\
& \leq \frac{a^b(z-v)}{4} \left\{ \mathcal{A}_1^{1-\frac{1}{\theta}}(1, b) \left(\left[\mathcal{A}_3(1, b) \|\mathbb{G}'(z)\|^\theta + \mathcal{A}_4(1, b) \|\mathbb{G}'(v)\|^\theta \right]^{1/\theta} \right. \right. \\
& \quad \left. \left. + \left[\mathcal{A}_4(1, b) \|\mathbb{G}'(z)\|^\theta + \mathcal{A}_3(1, b) \|\mathbb{G}'(v)\|^\theta \right]^{\frac{1}{\theta}} \right) \right. \\
& \quad \left. + \mathcal{A}_2^{1-\frac{1}{\theta}}(1, b) \left(\left[\mathcal{A}_5(1, b) \|\mathbb{G}'(z)\|^\theta + \mathcal{A}_6(1, b) \|\mathbb{G}'(v)\|^\theta \right]^{1/\theta} \right. \right. \\
& \quad \left. \left. + \left[\mathcal{A}_6(1, b) \|\mathbb{G}'(z)\|^\theta + \mathcal{A}_5(1, b) \|\mathbb{G}'(v)\|^\theta \right]^{\frac{1}{\theta}} \right) \right\},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{A}_3(1, b) &= \int_0^{\frac{2}{3}} \frac{\hbar}{2} \left| \hbar^b - \frac{1}{4} \right| d\hbar = \begin{cases} \frac{b+1}{2(b+2)} \left(\frac{1}{4}\right)^{1+\frac{2}{b}} + \frac{1}{2(b+2)} \left(\frac{2}{3}\right)^{b+2} - \frac{1}{36}, & 0 < b < \frac{\ln(\frac{1}{4})}{\ln(\frac{2}{3})} \\ \frac{1}{36} - \frac{1}{2(b+2)} \left(\frac{2}{3}\right)^{b+2}, & b \geq \frac{\ln(\frac{1}{4})}{\ln(\frac{2}{3})}, \end{cases} \\
\mathcal{A}_4(1, b) &= \int_0^{\frac{2}{3}} \frac{2-\hbar}{2} \left| \hbar^b - \frac{1}{4} \right| d\hbar = \begin{cases} \frac{2b}{b+1} \left(\frac{1}{4}\right)^{1+\frac{1}{b}} + \frac{1}{b+1} \left(\frac{2}{3}\right)^{b+1} & 0 < b < \frac{\ln(\frac{1}{4})}{\ln(\frac{2}{3})} \\ -\frac{b+1}{2(b+2)} \left(\frac{1}{4}\right)^{1+\frac{2}{b}} - \frac{1}{2(b+2)} \left(\frac{2}{3}\right)^{b+2} - \frac{5}{36}, & b \geq \frac{\ln(\frac{1}{4})}{\ln(\frac{2}{3})}, \end{cases} \\
\mathcal{A}_5(1, b) &= \int_{\frac{2}{3}}^1 \frac{\hbar}{2} (1 - \hbar^b) d\hbar = \frac{5}{36} - \frac{1}{2(b+2)} + \frac{1}{2(b+2)} \left(\frac{2}{3}\right)^{b+2},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{A}_6(1, b) &= \int_{\frac{2}{3}}^1 \frac{2-\hbar}{2} (1 - \hbar^b) d\hbar \\
&= \frac{7}{36} - \frac{b+3}{2(b+1)(b+2)} + \frac{1}{b+1} \left(\frac{2}{3}\right)^{b+1} - \frac{1}{2(b+2)} \left(\frac{2}{3}\right)^{b+2},
\end{aligned}$$

as established by Hezenci and Budak in [24] (Theorem 5).

Remark 6. By assigning $a = b = 1$ in Theorem 5, we derive the following result:

$$\begin{aligned} & \left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{1}{z-v} \int_v^z \mathbb{G}(\hbar) d\hbar \right| \\ & \leq \frac{z-v}{72} \left\{ \left(\frac{17}{8} \right)^{1-\frac{1}{\vartheta}} \left[\left(\frac{251|\mathbb{G}'(z)|^{\vartheta} + 973|\mathbb{G}'(v)|^{\vartheta}}{576} \right)^{\frac{1}{\vartheta}} + \left(\frac{251|\mathbb{G}'(v)|^{\vartheta} + 973|\mathbb{G}'(z)|^{\vartheta}}{576} \right)^{\frac{1}{\vartheta}} \right] \right. \\ & \quad \left. + \left[\left(\frac{7|\mathbb{G}'(z)|^{\vartheta} + 11|\mathbb{G}'(v)|^{\vartheta}}{18} \right)^{\frac{1}{\vartheta}} + \left(\frac{7|\mathbb{G}'(v)|^{\vartheta} + 11|\mathbb{G}'(z)|^{\vartheta}}{18} \right)^{\frac{1}{\vartheta}} \right] \right\}, \end{aligned}$$

which is given by Hezenci and Budak in [24] (Corollary 2).

4. Newton-Type Inequalities for Bounded Functions

This section introduces certain NTIs derived through the use of bounded functions.

Theorem 6. Let the prerequisites of Lemma 1 be satisfied. Then, there exist real numbers $c, \mathcal{C} \in \mathbb{R}$ such that for every $\hbar \in [v, z]$, the inequality $c \leq \mathbb{G}'(\hbar) \leq \mathcal{C}$ holds. Under these premises, the following Newton-type inequality can be derived.

$$\begin{aligned} & \left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{a^b 2^{ab-1} \Gamma(b+1)}{(z-v)^{ab}} \left[{}^b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(v) + {}^b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(z) \right] \right| \\ & \leq \frac{a^b(z-v)}{4} [\mathcal{A}_1(a, b) + \mathcal{A}_2(a, b)] (\mathcal{C} - c), \end{aligned}$$

with $\mathcal{A}_1$ and $\mathcal{A}_2$ being determined according to Theorem 3.

Proof. By using Lemma 1, we gain

$$\begin{aligned} & \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{a^b 2^{ab-1} \Gamma(b+1)}{(z-v)^{ab}} \left[{}^b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(v) + {}^b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(z) \right] \\ & = \frac{a^b(z-v)}{4} \left[\int_0^{\frac{2}{3}} \left(\left(\frac{1-(1-\hbar)^a}{a} \right)^b - \frac{1}{4a^b} \right) \left(\mathbb{G}'\left(\frac{\hbar}{2}z + \frac{2-\hbar}{2}v\right) - \frac{m+M}{2} \right) d\hbar \right. \\ & \quad + \int_0^{\frac{2}{3}} \left(\left(\frac{1-(1-\hbar)^a}{a} \right)^b - \frac{1}{4a^b} \right) \left(\frac{m+M}{2} - \mathbb{G}'\left(\frac{\hbar}{2}v + \frac{2-\hbar}{2}z\right) \right) d\hbar \\ & \quad + \int_{\frac{2}{3}}^1 \left(\left(\frac{1-(1-\hbar)^a}{a} \right)^b - \frac{1}{a^b} \right) \left(\mathbb{G}'\left(\frac{\hbar}{2}z + \frac{2-\hbar}{2}v\right) - \frac{m+M}{2} \right) d\hbar \\ & \quad \left. + \int_{\frac{2}{3}}^1 \left(\left(\frac{1-(1-\hbar)^a}{a} \right)^b - \frac{1}{a^b} \right) \left(\frac{m+M}{2} - \mathbb{G}'\left(\frac{\hbar}{2}v + \frac{2-\hbar}{2}z\right) \right) d\hbar \right]. \end{aligned}$$

By applying absolute value, we have

$$\begin{aligned}
& \left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{a^b \Gamma(b+1)}{(z-v)^{ab}} 2^{ab-1} \left[{}^b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(v) + {}^b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(z) \right] \right| \\
& \leq \frac{a^b(z-v)}{4} \left[\int_0^{\frac{2}{3}} \left| \left(\frac{1-(1-\xi)^a}{a} \right)^b - \frac{1}{4a^b} \right| \left| \mathbb{G}'\left(\frac{\xi}{2}z + \frac{2-\xi}{2}v\right) - \frac{m+M}{2} \right| d\xi \right. \\
& \quad + \int_0^{\frac{2}{3}} \left| \left(\frac{1-(1-\xi)^a}{a} \right)^b - \frac{1}{4a^b} \right| \left| \frac{m+M}{2} - \mathbb{G}'\left(\frac{\xi}{2}v + \frac{2-\xi}{2}z\right) \right| d\xi \\
& \quad + \int_{\frac{2}{3}}^1 \left| \left(\frac{1-(1-\xi)^a}{a} \right)^b - \frac{1}{a^b} \right| \left| \mathbb{G}'\left(\frac{\xi}{2}z + \frac{2-\xi}{2}v\right) - \frac{m+M}{2} \right| d\xi \\
& \quad \left. + \int_{\frac{2}{3}}^1 \left| \left(\frac{1-(1-\xi)^a}{a} \right)^b - \frac{1}{4a^b} \right| \left| \frac{m+M}{2} - \mathbb{G}'\left(\frac{\xi}{2}v + \frac{2-\xi}{2}z\right) \right| d\xi \right].
\end{aligned}$$

Since $c \leq \mathbb{G}'(\xi) \leq \mathcal{C}$ holds for $\xi \in [v, z]$, we can deduce the following conclusion.

$$\begin{aligned}
\left| \mathbb{G}'\left(\frac{\xi}{2}z + \frac{2-\xi}{2}v\right) - \frac{m+M}{2} \right| & \leq \frac{\mathcal{C} - c}{2} \\
\left| \frac{m+M}{2} - \mathbb{G}'\left(\frac{\xi}{2}v + \frac{2-\xi}{2}z\right) \right| & \leq \frac{\mathcal{C} - c}{2}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
& \left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{a^b 2^{ab-1} \Gamma(b+1)}{(z-v)^{ab}} \left[{}^b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(v) + {}^b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(z) \right] \right| \\
& \leq \frac{a^b(z-v)}{4} \left[\int_0^{\frac{2}{3}} \left| \left(\frac{1-(1-\xi)^a}{a} \right)^b - \frac{1}{4a^b} \right| \left(\frac{\mathcal{C} - c}{2} \right) d\xi + \int_0^{\frac{2}{3}} \left| \left(\frac{1-(1-\xi)^a}{a} \right)^b - \frac{1}{4a^b} \right| \left(\frac{\mathcal{C} - c}{2} \right) d\xi \right. \\
& \quad \left. + \int_{\frac{2}{3}}^1 \left| \left(\frac{1-(1-\xi)^a}{a} \right)^b - \frac{1}{a^b} \right| \left(\frac{\mathcal{C} - c}{2} \right) d\xi + \int_{\frac{2}{3}}^1 \left| \left(\frac{1-(1-\xi)^a}{a} \right)^b - \frac{1}{a^b} \right| \left(\frac{\mathcal{C} - c}{2} \right) d\xi \right] \\
& = \frac{a^b(z-v)}{4} [\mathcal{A}_1(a, b) + \mathcal{A}_2(a, b)] (\mathcal{C} - c).
\end{aligned}$$

Thus, the demonstration has been concluded. $\square$

Remark 7. By selecting $a = 1$ in Theorem 6, we derive the subsequent Newton-style inequality associated with FR-L integrals.

$$\begin{aligned}
& \left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{2^{b-1} \Gamma(b+1)}{(z-v)^b} \left[\mathcal{W}_{\frac{v+z}{2}}^b \mathbb{G}(z) + \mathcal{W}_{\frac{v+z}{2}}^b \mathbb{G}(v) \right] \right| \\
& \leq \frac{z-v}{4} [\mathcal{A}_1(1, b) + \mathcal{A}_2(1, b)] (\mathcal{C} - c),
\end{aligned}$$

which was established by Hezenci and Budak in [24] (Theorem 6).

Remark 8. When selecting $a = b = 1$ in Theorem 6, we obtain the following result.

$$\begin{aligned}
& \left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{1}{z-v} \int_v^z \mathbb{G}(\xi) d\xi \right| \\
& \leq \frac{25(z-v)}{576} (\mathcal{C} - c),
\end{aligned}$$

which was demonstrated by Hezenci and Budak in [24] (Corollary 3).

Corollary 1. Given the conditions of Theorem 6, if a constant $\mathcal{C} \in \mathbb{R}^+$ exists such that $|\mathbb{G}'(\hbar)| \leq \mathcal{C}$ for all $\hbar \in [v, z]$, then the ensuing result follows.

$$\begin{aligned} & \left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{a^b 2^{ab-1} \Gamma(b+1)}{(z-v)^{ab}} \left[{}_b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(v) + {}_+^b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(z) \right] \right| \\ & \leq \frac{a^b(z-v)}{2} [\mathcal{A}_1(a, b) + \mathcal{A}_2(a, b)] \mathcal{C}. \end{aligned}$$

5. Newton-Type Inequalities for Lipschitzian Functions

In this section, we obtain new NTIs for Lipschitzian functions with the help of Lemma 1.

Theorem 7. Assume that the hypotheses of Lemma 1 hold. If $\mathbb{G}'$ satisfies the L -Lipschitz condition on $[v, z]$, then a Newton-type inequality holds.

$$\begin{aligned} & \left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{a^b 2^{ab-1} \Gamma(b+1)}{(z-v)^{ab}} \left[{}_b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(v) + {}_+^b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(z) \right] \right| \\ & \leq \frac{a^b(z-v)^2}{4} (\mathcal{A}_1(a, b) - 2\mathcal{A}_3(a, b) + \mathcal{A}_2(a, b) - 2\mathcal{A}_5(a, b)) L. \end{aligned}$$

Proof. Since $\mathbb{G}'$ is L -Lipschitzian, we have

$$\begin{aligned} & \left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{a^b 2^{ab-1} \Gamma(b+1)}{(z-v)^{ab}} \left[{}_b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(v) + {}_+^b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(z) \right] \right| \\ & \leq \frac{a^b(z-v)}{4} \left[\int_0^{\frac{2}{3}} \left| \left(\frac{1-(1-\hbar)^a}{a} \right)^b - \frac{1}{4a^b} \right| \left| \mathbb{G}'\left(\frac{\hbar}{2}z + \frac{2-\hbar}{2}v\right) - \mathbb{G}'\left(\frac{\hbar}{2}v + \frac{2-\hbar}{2}z\right) \right| d\hbar \right. \\ & \quad \left. + \int_{\frac{2}{3}}^1 \left| \left(\frac{1-(1-\hbar)^a}{a} \right)^b - \frac{1}{a^b} \right| \left| \mathbb{G}'\left(\frac{\hbar}{2}z + \frac{2-\hbar}{2}v\right) - \mathbb{G}'\left(\frac{\hbar}{2}v + \frac{2-\hbar}{2}z\right) \right| d\hbar \right] \\ & \leq \frac{a^b(z-v)}{4} \left\{ \int_0^{\frac{2}{3}} \left| \left(\frac{1-(1-\hbar)^a}{a} \right)^b - \frac{1}{4a^b} \right| L(1-\hbar)(z-v) d\hbar \right. \\ & \quad \left. + \int_{\frac{2}{3}}^1 \left| \left(\frac{1-(1-\hbar)^a}{a} \right)^b - \frac{1}{a^b} \right| L(1-\hbar)(z-v) d\hbar \right\} \\ & = \frac{a^b(z-v)^2}{4} (\mathcal{A}_1(a, b) - 2\mathcal{A}_3(a, b) + \mathcal{A}_2(a, b) - 2\mathcal{A}_5(a, b)) L. \end{aligned}$$

The proof is completed. $\square$

Remark 9. By setting $a = 1$ in Theorem 7, we derive the following Newton-type inequality for FR-L integrals.

$$\begin{aligned} & \left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{2^{b-1} \Gamma(b+1)}{(z-v)^b} \left[\mathcal{W}_{\frac{v+z}{2}+}^b \mathbb{G}(z) + \mathcal{W}_{\frac{v+z}{2}-}^b \mathbb{G}(v) \right] \right| \\ & \leq \frac{a^b(z-v)^2}{4} (\mathcal{A}_1(1, b) - 2\mathcal{A}_3(1, b) + \mathcal{A}_2(1, b) - 2\mathcal{A}_5(1, b)) L, \end{aligned}$$

which was demonstrated by Hezenci and Budak in [24] (Theorem 7).

Remark 10. By selecting $\mathfrak{a} = \mathfrak{b} = 1$ in Theorem 7, we obtain the following inequality.

$$\left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+\mathfrak{x}}{3}\right) + 3\mathbb{G}\left(\frac{v+2\mathfrak{x}}{3}\right) + \mathbb{G}(\mathfrak{x}) \right] - \frac{1}{\mathfrak{x}-v} \int_v^{\mathfrak{x}} \mathbb{G}(\mathfrak{h}) d\mathfrak{h} \right| \leq \frac{425(\mathfrak{x}-v)^2}{20736} L,$$

which was demonstrated by Hezenci and Budak in [24] (Corollary 6).

6. Newton-Type Inequalities for Functions of Bounded Variation

This section begins by establishing an identity for Riemann–Stieltjes integrals. Utilizing this identity, a novel Newton-type inequality related to generalized fractional integrals is derived.

Theorem 8. Consider a function $\mathbb{G} : [v, \mathfrak{x}] \rightarrow \mathbb{R}$ that has bounded variation on $[v, \mathfrak{x}]$. Then, the following result can be obtained.

$$\left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+\mathfrak{x}}{3}\right) + 3\mathbb{G}\left(\frac{v+2\mathfrak{x}}{3}\right) + \mathbb{G}(\mathfrak{x}) \right] - \frac{\mathfrak{a}^{\mathfrak{b}} \Gamma(\mathfrak{b}+1)}{(\mathfrak{x}-v)^{\mathfrak{a}\mathfrak{b}}} 2^{\mathfrak{a}\mathfrak{b}-1} \left[{}_{-}^{\mathfrak{b}}\mathcal{W}_{\frac{v+\mathfrak{x}}{2}}^{\mathfrak{a}} \mathbb{G}(v) + {}_{+}^{\mathfrak{b}}\mathcal{W}_{\frac{v+\mathfrak{x}}{2}}^{\mathfrak{a}} \mathbb{G}(\mathfrak{x}) \right] \right| \leq \frac{1}{2} \max \left\{ 1 - \left(1 - \frac{1}{3^{\mathfrak{a}}} \right)^{\mathfrak{b}}, \left| \left(1 - \frac{1}{3^{\mathfrak{a}}} \right)^{\mathfrak{b}} - \frac{1}{4} \right|, \frac{1}{4} \right\} \bigvee_v^{\mathfrak{x}}(\mathbb{G}).$$

Here, $\bigvee_v^{\mathfrak{x}}(\mathbb{G})$ represents the total variation of $\mathbb{G}$ over the interval $[v, \mathfrak{x}]$.

Proof. The function $K_v^{\mathfrak{b}}(w)$ is defined as follows:

$$K_v^{\mathfrak{b}}(w) = \begin{cases} \left(\left(\frac{\mathfrak{x}-v}{2} \right)^{\mathfrak{a}} - \left(\frac{v+\mathfrak{x}}{2} - w \right)^{\mathfrak{a}} \right)^{\mathfrak{b}} - \frac{(\mathfrak{x}-v)^{\mathfrak{a}\mathfrak{b}}}{2^{\mathfrak{a}\mathfrak{b}+2}}, & v \leq w \leq \frac{2v+\mathfrak{x}}{3} \\ \left(\left(\frac{\mathfrak{x}-v}{2} \right)^{\mathfrak{a}} - \left(\frac{v+\mathfrak{x}}{2} - w \right)^{\mathfrak{a}} \right)^{\mathfrak{b}} - \frac{(\mathfrak{x}-v)^{\mathfrak{a}\mathfrak{b}}}{2^{\mathfrak{a}\mathfrak{b}}}, & \frac{2v+\mathfrak{x}}{3} < w \leq \frac{v+\mathfrak{x}}{2} \\ \frac{(\mathfrak{x}-v)^{\mathfrak{a}\mathfrak{b}}}{2^{\mathfrak{a}\mathfrak{b}}} - \left(\left(\frac{\mathfrak{x}-v}{2} \right)^{\mathfrak{a}} - \left(w - \frac{v+\mathfrak{x}}{2} \right)^{\mathfrak{a}} \right)^{\mathfrak{b}}, & \frac{v+\mathfrak{x}}{2} < w \leq \frac{v+2\mathfrak{x}}{3} \\ \frac{(\mathfrak{x}-v)^{\mathfrak{a}\mathfrak{b}}}{2^{\mathfrak{a}\mathfrak{b}+2}} - \left(\left(\frac{\mathfrak{x}-v}{2} \right)^{\mathfrak{a}} - \left(w - \frac{v+\mathfrak{x}}{2} \right)^{\mathfrak{a}} \right)^{\mathfrak{b}}, & \frac{v+2\mathfrak{x}}{3} < w \leq v. \end{cases}$$

By applying integration by parts, we obtain

$$\begin{aligned}
\int_v^z K_v^b(w) d\mathbb{G}(w) &= \int_v^{\frac{2v+z}{3}} \left[\left(\left(\frac{z-v}{2} \right)^a - \left(\frac{v+z}{2} - w \right)^a \right)^b - \frac{(z-v)^{ab}}{2^{ab+2}} \right] d\mathbb{G}(w) \\
&+ \int_{\frac{2v+z}{3}}^{\frac{v+z}{2}} \left[\left(\left(\frac{z-v}{2} \right)^a - \left(\frac{v+z}{2} - w \right)^a \right)^b - \frac{(z-v)^{ab}}{2^{ab}} \right] d\mathbb{G}(w) \\
&+ \int_{\frac{v+z}{2}}^{\frac{v+2z}{3}} \left[\left(\frac{(z-v)^{ab}}{2^{ab}} - \left(\left(\frac{z-v}{2} \right)^a - \left(w - \frac{v+z}{2} \right)^a \right)^b \right) \right] d\mathbb{G}(w) \\
&+ \int_{\frac{v+2z}{3}}^z \left[\left(\frac{(z-v)^{ab}}{2^{ab+2}} - \left(\left(\frac{z-v}{2} \right)^a - \left(w - \frac{v+z}{2} \right)^a \right)^b \right) \right] d\mathbb{G}(w) \\
&= \left[\left(\left(\frac{z-v}{2} \right)^a - \left(\frac{v+z}{2} - w \right)^a \right)^b - \frac{(z-v)^{ab}}{2^{ab+2}} \right] \mathbb{G}(w) \Big|_v^{\frac{2v+z}{3}} \\
&- ab \int_v^{\frac{2v+z}{3}} \left[\left(\left(\frac{z-v}{2} \right)^a - \left(\frac{v+z}{2} - w \right)^a \right)^{b-1} \left(\frac{v+z}{2} - w \right)^{a-1} \right] \mathbb{G}(w) dw \\
&+ \left[\left(\left(\frac{z-v}{2} \right)^a - \left(\frac{v+z}{2} - w \right)^a \right)^b - \frac{(z-v)^{ab}}{2^{ab}} \right] \mathbb{G}(w) \Big|_{\frac{2v+z}{3}}^{\frac{v+z}{2}} \\
&- ab \int_{\frac{2v+z}{3}}^{\frac{v+z}{2}} \left[\left(\left(\frac{z-v}{2} \right)^a - \left(\frac{v+z}{2} - w \right)^a \right)^{b-1} \left(\frac{v+z}{2} - w \right)^{a-1} \right] \mathbb{G}(w) dw \\
&+ \left[\left(\frac{(z-v)^{ab}}{2^{ab}} - \left(\left(\frac{z-v}{2} \right)^a - \left(w - \frac{v+z}{2} \right)^a \right)^b \right) \right] \mathbb{G}(w) \Big|_{\frac{v+2z}{3}}^{\frac{v+z}{2}} \\
&+ ab \int_{\frac{v+2z}{3}}^{\frac{v+z}{2}} \left[\left(\left(\frac{z-v}{2} \right)^a - \left(w - \frac{v+z}{2} \right)^a \right)^{b-1} \left(w - \frac{v+z}{2} \right)^{a-1} \right] \mathbb{G}(w) dw \\
&+ \left[\left(\frac{(z-v)^{ab}}{2^{ab+2}} - \left(\left(\frac{z-v}{2} \right)^a - \left(w - \frac{v+z}{2} \right)^a \right)^b \right) \right] \mathbb{G}(w) \Big|_{\frac{v+2z}{3}}^z \\
&+ ab \int_{\frac{v+2z}{3}}^z \left[\left(\left(\frac{z-v}{2} \right)^a - \left(w - \frac{v+z}{2} \right)^a \right)^{b-1} \left(w - \frac{v+z}{2} \right)^{a-1} \right] \mathbb{G}(w) dw \\
&= \frac{3}{4} \frac{(z-v)^{ab}}{2^{ab}} \mathbb{G}\left(\frac{2v+z}{3}\right) + \frac{(z-v)^{ab}}{2^{ab+2}} [\mathbb{G}(v) + \mathbb{G}(z)] + \frac{3}{4} \frac{(z-v)^{ab}}{2^{ab}} \mathbb{G}\left(\frac{v+2z}{3}\right) \\
&- ab \int_v^{\frac{v+z}{2}} \left[\left(\left(\frac{z-v}{2} \right)^a - \left(\frac{v+z}{2} - w \right)^a \right)^{b-1} \left(\frac{v+z}{2} - w \right)^{a-1} \right] \mathbb{G}(w) dw \\
&- ab \int_{\frac{v+2z}{3}}^z \left[\left(\left(\frac{z-v}{2} \right)^a - \left(w - \frac{v+z}{2} \right)^a \right)^{b-1} \left(w - \frac{v+z}{2} \right)^{a-1} \right] \mathbb{G}(w) dw \\
&= \frac{(z-v)^{ab}}{2^{ab-1}} \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] \\
&- {}^a\mathbb{I}^b \Gamma(b+1) \left[{}^b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(v) + {}^b\mathcal{W}_{\frac{v+2z}{2}}^a \mathbb{G}(z) \right].
\end{aligned}$$

That is,

$$\begin{aligned} & \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{a^b 2^{ab-1} \Gamma(b+1)}{(z-v)^{ab}} \left[{}^b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(v) \right. \\ & \left. + {}^b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(z) \right] \frac{2^{ab-1}}{(z-v)^{ab}} \int_v^z K_v^b(w) d\mathbb{G}(w). \end{aligned} \quad (8)$$

Obviously, if $\varsigma, \mathbb{G} : [v, z] \rightarrow \mathbb{R}$ satisfy the conditions that ς is continuous on $[v, z]$ and $\mathbb{G}$ has bounded variation on $[v, z]$, then the integral $\int_v^z \varsigma(\xi) d\mathbb{G}(\xi)$ exists and

$$\left| \int_v^z \varsigma(\xi) d\mathbb{G}(\xi) \right| \leq \sup_{\xi \in [v, z]} |\varsigma(\xi)| \bigvee_v^z(\mathbb{G}). \quad (9)$$

By applying (9) in (8), we have

$$\begin{aligned} & \left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{a^b 2^{ab-1} \Gamma(b+1)}{(z-v)^{ab}} \left[{}^b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(v) + {}^b\mathcal{W}_{\frac{v+z}{2}}^a \mathbb{G}(z) \right] \right| \\ &= \frac{2^{ab-1}}{(z-v)^{ab}} \left| \int_v^z K_v^b(w) d\mathbb{G}(w) \right| \\ &\leq \frac{2^{ab-1}}{(z-v)^{ab}} \left[\left| \int_v^{\frac{2v+z}{3}} \left[\left(\left(\frac{z-v}{2} \right)^a - \left(\frac{v+z}{2} - w \right)^a \right)^b - \frac{(z-v)^{ab}}{2^{ab+2}} \right] d\mathbb{G}(w) \right| \right. \\ &\quad + \left| \int_{\frac{2v+z}{3}}^{\frac{v+2z}{3}} \left[\left(\left(\frac{z-v}{2} \right)^a - \left(\frac{v+z}{2} - w \right)^a \right)^b - \frac{(z-v)^{ab}}{2^{ab}} \right] d\mathbb{G}(w) \right| \\ &\quad + \left| \int_{\frac{v+2z}{3}}^z \left[\left(\frac{(z-v)^{ab}}{2^{ab}} - \left(\left(\frac{z-v}{2} \right)^a - \left(w - \frac{v+z}{2} \right)^a \right)^b \right) \right] d\mathbb{G}(w) \right| \\ &\quad + \left| \int_{\frac{v+2z}{3}}^z \left[\left(\frac{(z-v)^{ab}}{2^{ab+2}} - \left(\left(\frac{z-v}{2} \right)^a - \left(w - \frac{v+z}{2} \right)^a \right)^b \right) \right] d\mathbb{G}(w) \right| \Big] \\ &\leq \frac{2^{ab-1}}{(z-v)^{ab}} \left\{ \max \left\{ \left| \left(\left(\frac{z-v}{2} \right)^a - \left(\frac{z-v}{6} \right)^a \right)^b - \frac{(z-v)^{ab}}{2^{ab+2}} \right|, \frac{(z-v)^{ab}}{2^{ab+2}} \right\} \bigvee_v^{\frac{2v+z}{3}}(\mathbb{G}) \right. \\ &\quad + \left| \left(\left(\frac{z-v}{2} \right)^a - \left(\frac{z-v}{6} \right)^a \right)^b - \frac{(z-v)^{ab}}{2^{ab}} \right| \bigvee_{\frac{2v+z}{3}}^{\frac{v+2z}{3}}(\mathbb{G}) \\ &\quad + \left| \frac{(z-v)^{ab}}{2^{ab}} - \left(\left(\frac{z-v}{2} \right)^a - \left(\frac{z-v}{6} \right)^a \right)^b \right| \bigvee_{\frac{v+2z}{3}}^z(\mathbb{G}) \\ &\quad + \max \left\{ \left| \frac{(z-v)^{ab}}{2^{ab+2}} - \left(\left(\frac{z-v}{2} \right)^a - \left(\frac{z-v}{6} \right)^a \right)^b \right|, \frac{(z-v)^{ab}}{2^{ab+2}} \right\} \bigvee_{\frac{v+2z}{3}}^z(\mathbb{G}) \Big\} \\ &= \frac{1}{2} \max \left\{ 1 - \left(1 - \frac{1}{3^a} \right)^b, \left| \left(1 - \frac{1}{3^a} \right)^b - \frac{1}{4} \right|, \frac{1}{4} \right\} \bigvee_v^z(\mathbb{G}). \end{aligned}$$

Thus, the proof is concluded. $\square$

Remark 11. By setting $a = 1$ in Theorem 8, we obtain the following Newton-type inequality for FR-L integrals.

$$\left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{2^{\mathbb{b}-1}\Gamma(\mathbb{b}+1)}{(z-v)^{\mathbb{b}}} \left[\mathcal{W}_{\frac{v+z}{2}+}^{\mathbb{b}} \mathbb{G}(z) + \mathcal{W}_{\frac{v+z}{2}-}^{\mathbb{b}} \mathbb{G}(v) \right] \right|$$

$$\leq \frac{1}{2} \max \left\{ 1 - \left(\frac{2}{3}\right)^{\mathbb{b}}, \left| \left(\frac{2}{3}\right)^{\mathbb{b}} - \frac{1}{4} \right|, \frac{1}{4} \right\} \mathbb{V}_v^z(\mathbb{G}),$$

which is proved by Hezenci and Budak in [24] (Theorem 7).

Remark 12. By selecting $\mathfrak{a} = \mathfrak{b} = 1$ in Theorem 8, we derive the following inequality.

$$\left| \frac{1}{8} \left[\mathbb{G}(v) + 3\mathbb{G}\left(\frac{2v+z}{3}\right) + 3\mathbb{G}\left(\frac{v+2z}{3}\right) + \mathbb{G}(z) \right] - \frac{1}{z-v} \int_v^z \mathbb{G}(\mathfrak{h}) d\mathfrak{h} \right|$$

$$\leq \frac{5}{24} \mathbb{V}_v^z(\mathbb{G}),$$

which was demonstrated by Aloamri in [25].

7. Conclusions

In this work, we presented a unifying approach for deriving NTIs within a generalized fractional framework. By establishing a key identity involving generalized FR-L operators, we extended classical Simpson rules to fractional settings under diverse assumptions on the integrand. Specifically, we derived new bounds when the first derivative of the function is convex, as well as when it satisfies boundedness, Lipschitz continuity, or bounded variation. Each result was shown to recover its classical analog when the fractional parameters assume integer values, thereby illustrating the unifying nature of the proposed techniques. The methods introduced here open a number of possible research directions. One avenue is to explore analogous inequalities under alternative fractional integral definitions or under different generalized convexity conditions. Another is to apply these techniques in multidimensional integration or to investigate more refined error estimates for the derived inequalities. These new fractional Newton-type inequalities not only bring Simpson's 1/3 and 3/8 rules into a unified fractional setting but also guide real world computations choosing step sizes and predicting errors for models with memory (e.g., viscoelastic materials or anomalous transport). Moreover, when $a = b = 1$, our bounds collapse exactly to the classical Simpson estimates, confirming that our framework truly generalizes the well-known case and remains anchored in familiar numerical practice. We believe that the insights from this study will contribute to a deeper understanding of fractional numerical integration and inspire future inquiries into the broader applicability of these results in various mathematical and engineering contexts.

Author Contributions: Conceptualization, A.-A.H., A.A.A., M.A.B., H.B. and Ö.A.; Methodology, A.-A.H., A.A.A., M.A.B., H.B. and Ö.A.; Formal analysis, A.-A.H., A.A.A., M.A.B., H.B. and Ö.A.; Investigation, A.-A.H., A.A.A., M.A.B., H.B. and Ö.A.; Writing—original draft, A.-A.H., A.A.A., M.A.B., H.B. and Ö.A.; Writing—review and editing, A.-A.H., A.A.A., M.A.B., H.B. and Ö.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by King Khalid University, Grant (RGP2/163/46) and Princess Nourah bint Abdulrahman University, Grant (PNURSP2025R337).

Data Availability Statement: The original contributions presented in the study are included in the article, further inquiries can be directed to the corresponding author.

Acknowledgments: The authors extend their appreciation to the Deanship of Research and Graduate Studies at King Khalid University for funding this work through the Research Groups Program

under grant (RGP.2/163/46). The authors would like to acknowledge the Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2025R337).

Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

NTI	Newton-Type Inequality
NTIs	Newton-Type Inequalities
FR-L	Fractional Riemann–Liouville

References

1. Erden, S.; Iftikhar, S.; Kumam, P.; Awan, M. Some Newton's like inequalities with applications. *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A Mat.* **2020**, *114*, 195. [\[CrossRef\]](#)
2. Sitthiwirattam, T.; Nonlaopon, K.; Ali, M.A.; Budak, H. Riemann–Liouville fractional Newton's type inequalities for differentiable convex functions. *Fractal Fract.* **2022**, *6*, 175. [\[CrossRef\]](#)
3. Hezenci, F.; Budak, H.; Kösem, P. A new version of Newton's inequalities for Riemann–Liouville fractional integrals. *Rocky Mt. J. Math.* **2022**, *52*, 49–64. [\[CrossRef\]](#)
4. Iftikhar, S.; Kumam, P.; Erden, S. Newton's-type integral inequalities via local fractional integrals. *Fractals* **2020**, *28*, 2050037. [\[CrossRef\]](#)
5. Iftikhar, S.; Erden, S.; Kumam, P.; Awan, M.U. Local fractional Newton's inequalities involving generalized harmonic convex functions. *Adv. Differ. Equ.* **2020**, *2020*, 185. [\[CrossRef\]](#)
6. Awan, M.; Noor, M.A.; Du, T.S.; Noor, K.I. New refinements of fractional Hermite–Hadamard inequality. *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A Mat.* **2019**, *113*, 21–29. [\[CrossRef\]](#)
7. Yang, Z.Q.; Li, Y.J.; Du, T.S. A generalization of Simpson type inequality via differentiable functions using (s, m)-convex functions. *Ital. J. Pure Appl. Math.* **2015**, *35*, 327–338.
8. Nasir, J.; Qaisar, S.; Butt, S.I.; Khan, K.A.; Mabela, R.M. Some Simpson's Riemann–Liouville fractional integral inequalities with applications to special functions. *J. Funct. Spaces* **2022**, *2022*, 2113742. [\[CrossRef\]](#)
9. Saleh, W.; Lakhdari, A.; Abdeljawad, T.; Meftah, B. On fractional biparameterized Newton-type inequalities. *J. Inequal. Appl.* **2023**, *2023*, 122. [\[CrossRef\]](#)
10. Gao, S.; Shi, W. On new inequalities of Newton's type for functions whose second derivatives absolute values are convex. *Int. J. Pure Appl. Math.* **2012**, *74*, 33–41.
11. Noor, M.A.; Noor, K.I.; Iftikhar, S. Some Newton's type inequalities for harmonic convex functions. *J. Adv. Math. Stud.* **2016**, *9*, 7–16.
12. Noor, M.A.; Noor, K.I.; Iftikhar, S. Newton inequalities for p-harmonic convex functions. *Honam Math. J.* **2018**, *40*, 239–250.
13. Luangboon, W.; Nonlaopon, K.; Tariboon, J.; Ntouyas, S.K. Simpson- and Newton-type inequalities for convex functions via (p, q)-calculus. *Mathematics* **2021**, *9*, 1338. [\[CrossRef\]](#)
14. Erden, S.; Iftikhar, S.; Kumam, P.; Thounthong, P. On error estimations of Simpson's second type quadrature formula. *Math. Meth. Appl. Sci.* **2020**, *2020*, 1–13. [\[CrossRef\]](#)
15. Iftikhar, S.; Erden, S.; Ali, M.A.; Baili, J.; Ahmad, H. Simpson's second-type inequalities for co-ordinated convex functions and applications for cubature formulas. *Fractal Fract.* **2022**, *6*, 33. [\[CrossRef\]](#)
16. Noor, M.A.; Noor, K.I.; Awan, M.U. Some Newton's type inequalities for geometrically relative convex functions. *Malays. J. Math. Sci.* **2015**, *9*, 491–502.
17. Dragomir, S. On Simpson's quadrature formula for mappings of bounded variation and applications. *Tamkang J. Math.* **1999**, *30*, 53–58. [\[CrossRef\]](#)
18. Barakat, M.A. Comprehensive Weighted Newton Inequalities for Broad Function Classes via Generalized Proportional Fractional Operators. *Axioms* **2025**, *14*, 234. [\[CrossRef\]](#)
19. Pečarić, J.E.; Proschan, F.; Tong, Y.L. *Convex Functions, Partial Orderings and Statistical Applications*; Academic Press: Boston, MA, USA, 1992.
20. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Volume 204, North-Holland Mathematics Studies; Elsevier Sci. B.V.: Amsterdam, The Netherlands, 2006.
21. Jarad, F.; Uğurlu, E.; Abdeljawad, T.; Baleanu, D. On a new class of fractional operators. *Adv. Differ. Equ.* **2017**, *2017*, 247. [\[CrossRef\]](#)
22. Tunç, T.; Demir, I. On a new version of Hermite–Hadamard-type inequality based on proportional Caputo-hybrid operator. *Bound. Value Probl.* **2024**, *2024*, 44. [\[CrossRef\]](#)

23. Rashid, M.; Kubra, A.; Ali, I.; Ahmad, M. A fractal–fractional derivative model for a cholera epidemic with numerical simulations. *Chaos Solitons Fractals* **2021**, *144*, 110691. [[CrossRef](#)]
24. Hezenci, F.; Budak, H. Fractional Newton-type integral inequalities by means of various function classes. *Math. Methods Appl. Sci.* **2025**, *48*, 1198–1215. [[CrossRef](#)]
25. Alomari, M. A companion of Dragomir’s generalization of Ostrowski’s inequality and applications in numerical integration. *Ukr. Math. J.* **2012**, *64*, 491–510. [[CrossRef](#)]

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.