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A Brézis–Oswald-Type Result for the Fractional (r,q)-Laplacian Problems and Its Application

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Abstract

This study derives the uniqueness of positive solutions to Brézis–Oswald-type problems involving the fractional (r,q)-Laplacian operator and discontinuous Kirchhoff-type coefficients. The Brézis–Oswald-type result and Ricceri's abstract global minimum principle are critical tools in identifying this uniqueness. We consider an eigenvalue problem associated with fractional (r,q)-Laplacian problems to confirm the existence of a positive solution for our problem without the Kirchhoff coefficient. Moreover, we establish the uniqueness result of the Brézis–Oswald type by exploiting a generalization of the discrete Picone inequality.

Keywords: fractional (r,q)-Laplacian; Kirchhoff-type function; uniqueness; global minima

MSC: 35B33; 35D30; 35J20; 35J60; 35J66

1. Introduction

This study is dedicated to Brézis–Oswald-type problems involving fractional (r,q)-Laplacian operators and discontinuous Kirchhoff-type coefficients as follows:

$$\begin{cases} M\left([\varrho]_{s,r,q}\right)\left((-\Delta)_r^s\varrho+(-\Delta)_q^s\varrho\right)=g(z,\varrho) & \text{in} \quad \Omega, \\ \varrho>0 & \text{in} \quad \Omega, \\ \varrho=0 & \text{on} \quad \mathbb{R}^N\setminus\Omega, \\ [\varrho]_{s,r,q}\in\Lambda, \end{cases} \tag{1}$$

where $s\in(0,1)$, $r,q\in(1,+\infty)$, $1<r<q$, $sq<N$, $\Omega\subset\mathbb{R}^N$ is a bounded open set with the Lipschitz boundary $\partial\Omega$; $\Lambda\subseteq(0,+\infty)$ is an open interval; M is an increasing Kirchhoff-type function on Λ; and a non-negative function g and $[\varrho]_{s,r,q}$ will be specified later. Here, $(-\Delta)_m^s$ $(m\in\{r,q\})$ is the fractional m-Laplacian operator, defined as follows:

$$(-\Delta)_m^s\,\varrho(z)=2\lim_{\delta\searrow 0}\int_{\mathbb{R}^N\setminus B_\delta(z)}\frac{|\varrho(z)-\varrho(y)|^{m-2}\left(\varrho(z)-\varrho(y)\right)}{|z-y|^{N+sm}}\,dy, \quad z\in\mathbb{R}^N,$$

where $B_\delta(z):=\{z\in\mathbb{R}^N:|z-y|\le\delta\}$.

As we know, the combination of the operators $(-\Delta)_r^s\varrho$ and $(-\Delta)_q^s\varrho$ given in problem (1) can be viewed as a class of fractional problems with unbalanced double-phase operators. Such problems are strictly connected to mathematical physics concepts, such as fractional quantum mechanics, the fractional white noise limit, and fractional super-diffusion; for

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more information, see [1]. This operator can also be regarded as the fractional analog of the (r,q)-Laplacian $-\Delta_r - \Delta_q$, which is derived from the study of general reaction–diffusion equations with non-homogeneous diffusion. These types of problems stem from varied important applications in physics and related sciences, such as elasticity theory, elementary particles, biophysics, reaction–diffusion equations, plasma physics, and chemical reaction design; see [2] for more details. Recently, the study of fractional (r,q)-Laplacian problems has drawn significant attention, since it is not only a natural extension of the (r,q)-Laplacian equations but also presents many new phenomena and numerous applications described by nonlinear integral structures. See [1,3–5] for more information about the the existence and multiplicity results for nonlinear elliptic fractional (r,q)-Laplacian problems.

The present study is devoted to Brézis–Oswald-type results for the fractional (r,q)-Laplacian problems. Our research is inspired by the pioneering study by Brézis and Oswald [6], which provided insights and results with important implications for the investigation of differential problems and their applications. Since then, there has been much interest in providing the sufficient and/or necessary conditions to solve more general problems, taking various operators and boundary conditions into account; see [7–14] for more details. The authors of [11] investigated the criteria required for ensuring the existence and uniqueness results of positive solutions to the p-Laplacian equations under certain conditions. In [10], the existence and uniqueness of solutions for a quasilinear elliptic problem were obtained by creating an improvement for the typical Dìaz–Saa and Brézis–Oswald results in the Orlicz–Sobolev framework. As we know, the Dìaz–Saa-type inequalities in [11,15], operating as the primary tool, effectively take into account the Brézis–Oswald result for p-Laplacian problems or more general problems. However, these inequalities are not applicable to elliptic equations with fractional order directly. For this reason, the main difficulty is deriving the Brézis–Oswald-type problem with the fractional p-Laplacian operator that has a unique positive weak solution. To overcome this difficulty caused by the absence of the Dìaz–Saa-type inequalities for fractional orders, the authors of [7,12–14] took the discrete Picone inequality in [8,16] into account. By exploiting this inequality, Mugnai–Pinamonti–Vecchi [14] dealt with the necessary and sufficient conditions required to assure the uniqueness of positive solutions for the fractional p-Laplacian Robin boundary value problem. Inspired by this work, [13] recently applied the global minimum principle of Ricceri [17] to present the existence, uniqueness, and localization of positive solutions to the Laplacian problems involving discontinuous Kirchhoff-type functions. The authors of [12] explored new discrete Picone inequalities associated with nonlocal elliptic operators, such as fractional and non-homogeneous operators. Using these inequalities, they identified significant applications, including the existence, non-existence, and uniqueness of weak positive solutions to equations involving fractional (p,q)-Laplacian operators. Moreover, we refer to [7] for mixed local and nonlocal Dirichlet problems.

Another interesting aspect of this problem is the appearance of discontinuous Kirchhoff-type coefficients. The study of Kirchhoff-type problems, which were first introduced by Kirchhoff [18], is instrumental in physical and biological applications. For this reason, considerable attention has recently been given to investigating the elliptic equations associated with Kirchhoff coefficients; for more details, see [19–24] and the references therein. Very recently, Ricceri [17] presented the existence of a unique positive solution to the Laplacian problem with discontinuous Kirchhoff functions, taking into account a new approach different from those of previous related studies [19,20,22,23,25,26] with a continuous Kirchhoff coefficient. The author of [27] recently extended the result of [17] to p-Laplacian problems; see also the paper [28] for elliptic problems with double-phase operators. The key tools used in deriving these results in [27,28] are the abstract global minimum principle given in [17] and the Brézis–Oswald-type result based on [6]. As

mentioned before, the inequalities of the Dìaz–Saa type in [11,15] play an effective role in obtaining the uniqueness of a positive solution to the problems considered in [27,28]; see [29] for nonlinear fractional Laplacian problems of the Brézis–Oswald type. As far as we are aware, Brézis-Oswald type results with the Kirchhoff coefficient have not been studied much, and we only know of the recent work of Biagi and Vecchi [25]. In particular, they provided the uniqueness result to Laplacian problems of the Brézis–Oswald type with continuous Kirchhoff functions.

In the present study, we aim to examine the existence and uniqueness of positive solutions to the fractional (r,q)-Laplacian problems involving discontinuous Kirchhoff-type functions. To do this, we first demonstrate the Brézis–Oswald-type result for our problem with $M \equiv 1$. This is based on the remarkable study in [12]. However, our approach is slightly different from that of [12] in two respects. The first is that we consider the eigenvalue problem associated with the fractional (r,q)-Laplacian to obtain the existence of the positive solution for our problem without the Kirchhoff coefficient. The other is that we identify the uniqueness result by using the truncation function given in [9,14] to apply several versions of the discrete Picone inequalities in [8,12] as a key tool. Through applying such existence and uniqueness results, we give our main result by exploiting the global minimum principle of Ricceri [17].

As far as we know, the uniqueness results of positive solutions to the Brézis–Oswald-type fractional (r,q)-Laplacian problems with the discontinuous Kirchhoff coefficient have not been reported. Although our results are motivated by the previous work in [13] for problems associated with the fractional p-Laplacian problem, (1) has more complex nonlinearities than [13] and must be analyzed carefully. In particular, the present study extends the results for the fractional equations in [13,29] to fractional problems with unbalanced double-phase operators.

The remainder of this study is organized as follows. In Section 2, we present some useful preliminary knowledge of function spaces. Section 3 presents the variational framework associated with problem (1) with $M \equiv 1$, and then we derive the existence and uniqueness of a positive solution to our problem under suitable assumptions. Section 4, applying the framework defined in Section 3, demonstrates the existence result of a unique nontrivial positive solution to problem (1) involving the discontinuous Kirchhoff coefficient.

2. Preliminaries

Let $s \in (0,1)$ and $r \in (1,\infty)$ be real numbers with $rs < N$ and let the fractional Sobolev space $W^{s,r}(\mathbb{R}^N)$ be defined as

$$W^{s,r}(\mathbb{R}^N) := \left\{ \rho \in L^r(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\rho(z)-\rho(y)|^r}{|z-y|^{N+rs}}\,dy\,dz < +\infty \right\},$$

which is endowed with the norm

$$\|\rho\|_{W^{s,r}(\mathbb{R}^N)} := \left(\|\rho\|^r_{L^r(\mathbb{R}^N)} + |\rho|^r_{W^{s,r}(\mathbb{R}^N)} \right)^{\frac{1}{r}},$$

where

$$\|\rho\|^r_{L^r(\mathbb{R}^N)} := \int_\Omega |\rho(z)|^r\,dz \qquad \text{and} \qquad |\rho|^r_{W^{s,r}(\mathbb{R}^N)} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\rho(z)-\rho(y)|^r}{|z-y|^{N+rs}}\,dy\,dz.$$

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a smooth boundary. The space $W^{s,r}(\Omega)$ is the set of functions defined by

$$W^{s,r}(\Omega) := \left\{ \rho \in W^{s,r}(\mathbb{R}^N) : \rho(z) = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega \right\}$$

with respect to the norm

$$\|\rho\|_{W^{s,r}(\Omega)} := \left(\|\rho\|^r_{L^r(\Omega)} + [\rho]^r_{s,r}\right)^{\frac{1}{r}},$$

where

$$[\rho]^r_{s,r} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\rho(z)-\rho(y)|^r}{|z-y|^{N+rs}}\,dy\,dz.$$

Then, the space $W^{s,r}(\Omega)$ is a reflexive and separable Banach space. The fractional Sobolev space $W^{s,r}_0(\Omega)$ is given by the closure of $C^\infty_0(\Omega)$ with respect to the norm $\|\cdot\|_{W^{s,r}(\Omega)}$. Moreover, the space $C^\infty_0(\Omega)$ is dense in $W^{s,r}(\Omega)$, i.e., $W^{s,r}_0(\Omega) = W^{s,r}(\Omega)$; for examples, see [30,31].

Lemma 1 ([31]). *Let $s \in (0,1)$ and $r \in (1,+\infty)$. Then, the following continuous embeddings hold:*

$$\begin{aligned} &W^{s,r}(\Omega) \hookrightarrow L^\gamma(\Omega) && \text{for all } \gamma \in [1, r^*_s], && \text{if } sr < N;\\ &W^{s,r}(\Omega) \hookrightarrow L^\gamma(\Omega) && \text{for every } \gamma \in [1,\infty), && \text{if } sr = N. \end{aligned}$$

In particular, the embedding $W^{s,r}(\Omega) \hookrightarrow\hookrightarrow L^\ell(\Omega)$ is compact for any $\ell \in [1, r^*_s)$, where r^*_s is the fractional critical Sobolev exponent, namely

$$r^*_s := \begin{cases} \frac{Nr}{N-sr} & \text{if } sr < N,\\ +\infty & \text{if } sr \geq N. \end{cases}$$

With the help of Lemma 1, we can identify the following result at once.

Lemma 2 ([24]). *For any $\rho \in W^{s,r}(\Omega)$ and $1 \leq \gamma \leq r^*_s$, there is a positive constant $\mathfrak{C} = C(s,r,N)$ such that*

$$\|\rho\|^p_{L^\gamma(\Omega)} \leq \mathfrak{C}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\rho(z)-\rho(y)|^r}{|z-y|^{N+rs}}\,dy\,dz.$$

Let $1 < r < q < \infty$ and $sq < N$ and let us consider the problem (1) in the closed linear subspace defined as

$$\mathbb{W} := \left\{\rho \in W^{s,r}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N) : \rho(y) = 0 \text{ a.e. in } \mathbb{R}^N\backslash\Omega\right\}$$

with respect to the norm

$$\|\rho\|_{\mathbb{W}} := \left(\|\rho\|^r_{L^r(\Omega)} + [\rho]_{s,r,q}\right)^{\frac{1}{r}},$$

where

$$[\rho]_{s,r,q} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\rho(z)-\rho(y)|^r}{|z-y|^{N+rs}}\,dy\,dz + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\rho(z)-\rho(y)|^q}{|z-y|^{N+qs}}\,dy\,dz.$$

By virtue of Lemmas 1 and 2, we can obtain the following.

Lemma 3 ([24]). *If $\rho \in \mathbb{W}$, then $\rho \in W^{s,r}(\Omega)$. For any $\rho \in \mathbb{W}$ and $1 \leq \gamma \leq r^*_s$, there is a constant $\mathfrak{C}_0 = C_0(s,r,N) > 0$ such that*

$$\|\rho\|^p_{L^\gamma(\Omega)} \leq \mathfrak{C}_0\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\rho(z)-\rho(y)|^r}{|z-y|^{N+rs}}\,dy\,dz + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\rho(z)-\rho(y)|^q}{|z-y|^{N+qs}}\,dy\,dz\right).$$

As a consequence, the embedding $\mathbb{W} \hookrightarrow L^{\ell}(\Omega)$ *is continuous for any* $\ell \in [1, r_s^*]$. *Moreover, for any* $\ell \in (1, r_s^*)$, *the embedding*

$$\mathbb{W} \hookrightarrow L^{\ell}(\Omega)$$

is compact.

3. Variational Setting and a Brezis–Oswald-Type Result

In this section, we outline the existence and uniqueness of a positive solution to the fractional (r, q)-Laplacian problem of the Brézis–Oswald type:

$$\begin{cases} (-\Delta)_r^s \rho + (-\Delta)_q^s \rho = \lambda g(y, \rho) & \text{in} \quad \Omega, \\ \rho > 0 & \text{in} \quad \Omega, \\ \rho = 0 & \text{on} \quad \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2}$$

To begin with, we give the variational setting related to problem (2).

Definition 1. *We consider that* $\rho \in \mathbb{W}$ *is a weak solution of* (2) *if*

$$\begin{aligned} &\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\rho(z) - \rho(y)|^{r-2}(\rho(z) - \rho(y))}{|z - y|^{N+rs}} (\psi(z) - \psi(y))\, dy\, dz \\ &+ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\rho(z) - \rho(y)|^{q-2}(\rho(z) - \rho(y))}{|z - y|^{N+qs}} (\psi(z) - \psi(y))\, dy\, dz = \lambda \int_{\Omega} g(z, \rho)\psi(z)\, dz \end{aligned} \tag{3}$$

for any $\psi \in \mathbb{W}$.

Let the functional $\Upsilon : \mathbb{W} \to \mathbb{R}$ be defined as

$$\Upsilon(\rho) := \frac{1}{r} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\rho(z) - \rho(y)|^r}{|z - y|^{N+rs}}\, dy\, dz + \frac{1}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\rho(z) - \rho(y)|^q}{|z - y|^{N+qs}}\, dz\, dy. \tag{4}$$

Then, it is necessary to identify that Υ is well defined on $\mathbb{W}$, and the below result is ensured from an analogous argument as in the proof of Lemma 2 in [23].

Lemma 4. *The functional* Υ *is convex and weakly lower semicontinuous on* $\mathbb{W}$.

This section aims to give the existence result of a unique nontrivial positive solution to problem (2), which is the first main result of this study. To do this, we assume that the following assumptions are satisfied.

(F1) $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

(F2) $0 \leq g(\cdot, \xi) \in L^{\infty}(\Omega)$ for every $\xi \geq 0$, and there is a positive constant σ_1 such that

$$g(z, \xi) \leq \sigma_1 \left(1 + |\xi|^{r-1}\right)$$

for all $\xi \geq 0$ and for almost all $z \in \Omega$.

(F3) $\lim_{\xi \to +\infty} \frac{g(z,\xi)}{\xi^{r-1}} = 0$ and $\lim_{\xi \to 0^+} \frac{g(z,\xi)}{\xi^{r-1}} = +\infty$, being uniform in $z \in \Omega$.

(F4) The function $\xi \mapsto \frac{g(z,\xi)}{\xi^{r-1}}$ is strictly decreasing in $(0, +\infty)$ for almost all $z \in \Omega$.

Under conditions (F1) and (F2), the functional $\Phi : \mathbb{W} \to \mathbb{R}$ is defined by

$$\Phi(\rho) := \int_{\Omega} G(z, \rho(z))\, dz$$

for any $\rho \in \mathbb{W}$, where $G(z,\eta) = \int_0^\eta g(z,t)dt$. Then, it is necessary to identify that $\Phi \in C^1(\mathbb{W},\mathbb{R})$, and its Fréchet derivative is

$$\langle \Phi'(\rho), \psi \rangle = \lambda \int_\Omega g(z,\rho)\psi \, dz$$

for any $\rho, \psi \in \mathbb{W}$. Next, the functional $\Gamma : \mathbb{W} \to \mathbb{R}$ is defined as

$$\Gamma(\rho) = \Upsilon(\rho) - \lambda\Phi(\rho).$$

Then, $\Gamma \in C^1(\mathbb{W},\mathbb{R})$ and its Fréchet derivative are

$$\langle \Gamma'(\rho), \psi \rangle = \langle \Upsilon'(\rho), \psi \rangle - \lambda \langle \Phi'(\rho), \psi \rangle \quad \text{for any } \rho, \psi \in \mathbb{W}.$$

The following is a discrete version of the notable Picone inequality; see [8], Proposition 4.2, and [16], Lemma 2.6, for proofs.

Lemma 5. *(Discrete Picone inequality). Let $1 < r < \infty$ and let $a, b, c, d \in [0, +\infty)$, with $a, b > 0$. Then,*

$$\phi_r(a-b)\left[\frac{c^r}{a^{r-1}} - \frac{d^r}{b^{r-1}}\right] \leq |c-d|^r, \tag{5}$$

where $\phi_r(\xi) = |\xi|^{r-2}\xi$ for $\xi \in \mathbb{R}$. Moreover, if the equality holds in (5), then

$$\frac{a}{b} = \frac{c}{d}.$$

The following assertion is a generalization of Lemma 5, which is introduced in [12].

Lemma 6. *Let $1 < q < \infty$ and $1 < r \leq q$. Let u, v be two non-negative Lebesgue-measurable functions such that $u > 0$ in Ω and is non-constant. Then,*

$$\phi_q(u(z)-u(y))\left[\frac{v(z)^r}{f(u(z))} - \frac{v(y)^r}{f(u(y))}\right] \leq |v(z)-v(y)|^r |u(z)-u(y)|^{q-r}, \tag{6}$$

where $f(\xi) = \alpha\xi^{q-1} + \beta\xi^{r-1}$, with $\alpha \geq 0$ and $\beta \geq 1$

For any $\varphi_i \in \mathbb{W}$ and $\delta > 0$, we define the truncation

$$\varphi_{i,\delta} := \min\{\varphi_i, \delta^{-1}\}. \tag{7}$$

Now, we present a technical lemma that will be very practical hereinafter. The proof of the following consequence is entirely the same as that of Lemma 2.3 in [14]; see [13] for more details. Thus, we omit it.

Lemma 7. *Let $\varphi_1, \varphi_2 \in \mathbb{W}$ with $\varphi_1, \varphi_2 \geq 0$ and set*

$$v := \frac{\varphi_{2,\delta}^r}{(\varphi_1 + \delta)^{r-1}} - \varphi_{1,\delta},$$

where $\varphi_{1,\delta}$ and $\varphi_{2,\delta}$ are as in (7). Then, we obtain $v \in \mathbb{W}$.

On account of Lemmas 4 and 7, we find that problem (2) possesses at least one positive solution. Before delving into this consequence, let us consider the eigenvalue problem

$$\begin{cases} (-\Delta)_r^s \varrho + (-\Delta)_q^s \varrho = \mu |\varrho|^{r-2}\varrho & \text{in} \quad \Omega, \\ \varrho = 0 & \text{on} \quad \mathbb{R}^N \setminus \Omega. \end{cases} \tag{8}$$

Define the C^1-functional

$$\Gamma_\mu(\varrho) := \frac{1}{r}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho(z)-\varrho(y)|^r}{|z-y|^{N+rs}}\,dy\,dz + \frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho(z)-\varrho(y)|^q}{|z-y|^{N+qs}}\,dy\,dz - \frac{\mu}{r}\int_\Omega |\varrho|^r\,dz$$

for any $\varrho \in \mathbb{W}$. In light of Lemma 2.1 in [32], we note that the first eigenvalue $\mu_{1,r}$ of the fractional r-Laplacian is characterized by

$$\mu_{1,r}\int_\Omega |\varrho(z)|^r dz = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho(z)-\varrho(y)|^r}{|z-y|^{N+rs}}\,dy\,dz.$$

The basic idea of the proof of the following assertions is based on the recent work [28].

Lemma 8. *Define the quantity*

$$\mu_{1,r,q} = \min_{\varrho\in\mathbb{W}\setminus\{0\}} \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho(z)-\varrho(y)|^r}{|z-y|^{N+rs}}\,dy\,dz + \frac{r}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho(z)-\varrho(y)|^q}{|z-y|^{N+qs}}\,dy\,dz}{\int_\Omega |\varrho|^r\,dz}.$$

Then, we have $\mu_{1,r,q} = \mu_{1,r}$. For any $\mu \in (-\infty, \mu_{1,r}]$, there exists no eigenvalue of problem (8).

Proof. From the definitions of $\mu_{1,r,q}$ and $\mu_{1,r}$, one has $\mu_{1,r,q} \geq \mu_{1,r}$. Let ϱ_0 be the positive eigenfunction corresponding to $\mu_{1,r}$ such that $\|\varrho_0\|_{L^r(\Omega)} = 1$ and

$$\mu_{1,r} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho_0(z)-\varrho_0(y)|^r}{|z-y|^{N+rs}}\,dy\,dz.$$

By using the homogeneity and letting $\tau \to 0^+$ in the inequality

$$\begin{aligned} \mu_{1,r,q} &\leq \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\tau\varrho_0(z)-\tau\varrho_0(y)|^r}{|z-y|^{N+rs}}\,dy\,dz + \frac{r}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\tau\varrho_0(z)-\tau\varrho_0(y)|^q}{|z-y|^{N+qs}}\,dy\,dz}{\int_\Omega |\tau\varrho_0|^r\,dz} \\ &= \mu_{1,r} + \tau^{q-r}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho_0(z)-\varrho_0(y)|^q}{|z-y|^{N+qs}}\,dy\,dz, \end{aligned}$$

we find that $\mu_{1,r,q} \leq \mu_{1,r}$ since $r < q$.

Now, let us check that there is no eigenvalue of problem (8) for any $\mu \in (-\infty, \mu_{1,r}]$. We suppose, to the contrary, that there is an eigenpair $(\mu, \varrho_\mu) \in (-\infty, \mu_{1,r}) \times \mathbb{W} \setminus \{0\}$. Then, based on the definition of $\mu_{1,r,q}$, it follows that

$$\begin{aligned} \mu_{1,r} > \mu &= \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho_\mu(z)-\varrho_\mu(y)|^r}{|z-y|^{N+rs}}\,dy\,dz + \frac{r}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho_\mu(z)-\varrho_\mu(y)|^q}{|z-y|^{N+qs}}\,dy\,dz}{\int_\Omega |\varrho_\mu|^r\,dz} \\ &\geq \mu_{1,r,q} = \mu_{1,r}, \end{aligned}$$

which implies a contradiction. On the other hand, we may assume that $\mu_{1,r}$ is an eigenvalue of problem (8). In this situation, there is a function ϱ_1 in $\mathbb{W} \setminus \{0\}$ such that $\|\varrho_1\|_{L^r(\Omega)} > 0$ and

$$\mu_{1,r} = \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho_1(z)-\varrho_1(y)|^r}{|z-y|^{N+rs}}\,dy\,dz + \frac{r}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho_1(z)-\varrho_1(y)|^q}{|z-y|^{N+qs}}\,dy\,dz}{\int_\Omega |\varrho_1|^r\,dz}.$$

Then, since $\varrho_1 \neq 0$, the definition of $\mu_{1,r}$ implies that

$$\mu_{1,r} = \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho_1(z)-\varrho_1(y)|^r}{|z-y|^{N+rs}}\,dy\,dz + \frac{r}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho_1(z)-\varrho_1(y)|^q}{|z-y|^{N+qs}}\,dy\,dz}{\int_\Omega |\varrho_1|^r\,dz}$$
$$\geq \mu_{1,r} + \frac{r\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho_1(z)-\varrho_1(y)|^q}{|z-y|^{N+qs}}\,dy\,dz}{q\int_\Omega |\varrho_1|^r\,dz} > \mu_{1,r}.$$

This is a contradiction. □

Lemma 9. *Every $\mu > \mu_{1,r}$ is an eigenvalue of problem (8).*

Proof. It is obvious that the functional Γ_μ is coercive and weakly lower semicontinuous on $\mathbb{W}$. Thus, there is a global minimizer ϱ_* in $\mathbb{W}$ for Γ_μ. From Lemma 8, we infer $\mu_{1,r,q} = \mu_{1,r}$. Since $\mu > \mu_{1,r,q}$, we derive $\Gamma_\mu(\varrho_*) \leq \Gamma_\mu(\varrho_0) < 0$ for some $\varrho_0 \in \mathbb{W}\setminus\{0\}$. This implies that $\varrho_* \neq 0$. In addition, $\Gamma'_\mu(\varrho_*) = 0$, so ϱ_* is an eigenfunction of problem (8) corresponding to the eigenvalue μ. □

Now, we are in a position to establish the existence of a nontrivial positive solution to problem (2).

Theorem 1. *If* (F1)–(F3) *hold, then problem* (2) *admits a positive solution for any $\lambda > 0$.*

Proof. By means of the subcritical growth of g, Φ is sequentially weakly continuous on $\mathbb{W}$. Fix $\lambda > 0$. Then, Lemma 4 implies that $\mu\Upsilon - \Phi$ is sequentially weakly lower semicontinuous on $\mathbb{W}$. Let us choose

$$\delta \in \left(0, \frac{\mathfrak{C}_0 + 1}{2q\lambda\mathfrak{C}_0}\right),$$

where $\mathfrak{C}_0$ is given in Lemma 3. Since $\lim_{\xi\to+\infty} \frac{G(z,\xi)}{\xi^r} = 0$, we choose a constant $C(\delta) > 0$ such that

$$G(z,\xi) \leq \delta|\xi|^r + C(\delta)$$

for almost all $z \in \Omega$ and for any $\xi \in \mathbb{R}$. Thus, we have

$$\Phi(\varrho) \leq \delta\int_\Omega |\varrho(z)|^r dz + C(\delta)|\Omega| \tag{9}$$

for any $\varrho \in \mathbb{W}$, where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^N$. This, together with Lemma 3 and the estimate (9), shows that, for any $\varrho \in \mathbb{W}$ with $\|\varrho\|_{\mathbb{W}} \geq 1$,

$$\begin{aligned}\Upsilon(\varrho) - \lambda\Phi(\varrho) &\geq \frac{1}{r}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho(z)-\varrho(y)|^r}{|z-y|^{N+rs}}\,dy\,dz + \frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho(z)-\varrho(y)|^q}{|z-y|^{N+qs}}\,dy\,dz \\ &\quad - \lambda\delta\int_\Omega |\varrho(z)|^r dz - \lambda C(\delta)|\Omega| \\ &\geq \frac{1}{q}\left(\frac{1}{2} + \frac{1}{2\mathfrak{C}_0}\right)\|\varrho\|^r_{\mathbb{W}} - \lambda\delta\int_\Omega |\varrho(z)|^r\,dz - \lambda C(\delta)|\Omega| \\ &\geq \left(\frac{\mathfrak{C}_0+1}{2q\mathfrak{C}_0} - \lambda\delta\right)\|\varrho\|^r_{\mathbb{W}} - \lambda C(\delta)|\Omega|. \end{aligned} \tag{10}$$

Hence, given the choice of δ, we know that

$$\lim_{\|\varrho\|_{\mathbb{W}}\to+\infty} (\Upsilon(\varrho) - \lambda\Phi(\varrho)) = +\infty.$$

Let us define the energy functional $\Gamma : \mathbb{W} \to \mathbb{R}$ as

$$\Gamma(\varrho) := \frac{1}{r}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho(z)-\varrho(y)|^r}{|z-y|^{N+rs}}\,dy\,dz + \frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho(z)-\varrho(y)|^q}{|z-y|^{N+qs}}\,dy\,dz - \lambda\int_{\mathbb{R}^N} G(z,\varrho)\,dz, \quad \varrho \in \mathbb{W},$$

and define the modified energy functional $\widetilde{\Gamma} : \mathbb{W} \to \mathbb{R}$ as

$$\widetilde{\Gamma}(\varrho) := \frac{1}{r}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho(z)-\varrho(y)|^r}{|z-y|^{N+rs}}\,dy\,dz + \frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho(z)-\varrho(y)|^q}{|z-y|^{N+qs}}\,dy\,dz - \lambda\int_{\mathbb{R}^N} G^+(z,\varrho)\,dz, \quad \varrho \in \mathbb{W},$$

where

$$G^+(z,\zeta) := \int_0^\zeta g^+(z,\xi)\,d\xi \ \text{ and } \ g^+(z,\zeta) := \begin{cases} g(z,\zeta), & \zeta \geq 0, \\ 0, & \zeta < 0 \end{cases}$$

for almost all $z \in \mathbb{R}^N$ and all $\zeta \in \mathbb{R}$. Based on Lemma 4 and the argument in (9) and (10), the functional $\widetilde{\Gamma}$ is also coercive and sequentially weakly lower semicontinuous on $\mathbb{W}$. Hence, there is an element $\varrho_0 \in \mathbb{W}$ such that

$$\widetilde{\Gamma}(\varrho_0) = \inf\{\widetilde{\Gamma}(\varrho) : \varrho \in \mathbb{W}\}.$$

Now, we show that it is possible to assume that $\varrho_0 \geq 0$. To this end, we assume that ϱ_0 is sign-changing. Based on Lemma 7, one has $\varrho_0^+ \in \mathbb{W}$, so $\widetilde{\Gamma}(\varrho_0) \leq \widetilde{\Gamma}(\varrho_0^+)$, where $\varrho_0^+ := \max\{\varrho_0, 0\}$. Since $\widetilde{\Gamma}(\varrho) = \Gamma(\varrho)$ when $\varrho(z) \geq 0$ for almost all $z \in \Omega$, we assert that

$$\begin{aligned}
\widetilde{\Gamma}(\varrho_0^+) &= \Gamma(\varrho_0^+) \\
&= \frac{1}{r}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho_0^+(z)-\varrho_0^+(y)|^r}{|z-y|^{N+rs}}\,dzdy \\
&\quad + \frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho_0^+(z)-\varrho_0^+(y)|^q}{|z-y|^{N+qs}}\,dzdy - \lambda\int_\Omega G(z,\varrho_0^+)\,dz \\
&\leq \frac{1}{r}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho_0(z)-\varrho_0(y)|^r}{|z-y|^{N+rs}}\,dzdy \\
&\quad + \frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho_0(z)-\varrho_0(y)|^q}{|z-y|^{N+qs}}\,dzdy - \lambda\int_\Omega G(z,\varrho_0^+)\,dz \\
&= \widetilde{\Gamma}(\varrho_0).
\end{aligned}$$

Therefore, ϱ_0^+ is a non-negative solution of (2). For simplicity, let us directly write ϱ_0 in place of ϱ_0^+. We claim that $\varrho_0 > 0$. Since $\varrho_0(z) \geq 0$ for almost all $z \in \mathbb{R}^N$, we find that either $\varrho_0(z) = 0$ or $\varrho_0(z) > 0$ for almost all $z \in \mathbb{R}^N$. In fact, let us suppose that $\varrho_0 \not\equiv 0$ in Ω. Then, it suffices that $\varrho_0 \not\equiv 0$ in all connected components of Ω. However, we may assume, to the contrary, that there is a connected component Σ of Ω such that $\varrho_0(z) = 0$ for almost all $z \in \Sigma$. Let us take any non-negative function $v \in C_0^\infty(\Sigma)$ as a test function in (3). Then, one has $\int_\Omega g(z,\varrho_0)v(z)\,dz = 0$ because $v(z) = 0$ for almost all $z \in \Sigma^c$ and $g(z,\varrho_0(z)) = g(z,0) = 0$ for almost all $z \in \Sigma$. Hence, we obtain

$$
\begin{aligned}
0 &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varrho_0(z)-\varrho_0(y)|^{r-2}(\varrho_0(z)-\varrho_0(y))(v(z)-v(y))|z-y|^{-(N+rs)}\,dy\,dz \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varrho_0(z)-\varrho_0(y)|^{q-2}(\varrho_0(z)-\varrho_0(y))(v(z)-v(y))|z-y|^{-(N+qs)}\,dy\,dz \\
&\quad - \lambda\int_{\Omega} g(z,\varrho_0)v(z)\,dz \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varrho_0(z)-\varrho_0(y)|^{r-2}(\varrho_0(z)-\varrho_0(y))(v(z)-v(y))|z-y|^{-(N+rs)}\,dy\,dz \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varrho_0(z)-\varrho_0(y)|^{q-2}(\varrho_0(z)-\varrho_0(y))(v(z)-v(y))|z-y|^{-(N+qs)}\,dy\,dz \\
&= 2\int_{\Sigma}\int_{\Sigma^c} |\varrho_0(z)-\varrho_0(y)|^{r-2}(\varrho_0(z)-\varrho_0(y))(v(z)-v(y))|z-y|^{-(N+rs)}\,dy\,dz \\
&\quad + 2\int_{\Sigma}\int_{\Sigma^c} |\varrho_0(z)-\varrho_0(y)|^{q-2}(\varrho_0(z)-\varrho_0(y))(v(z)-v(y))|z-y|^{-(N+qs)}\,dy\,dz \\
&= -2\Bigg[\int_{\Sigma}\int_{\Sigma^c} (\varrho_0(z))^{r-1}v(y)|z-y|^{-(N+rs)}\,dy\,dz \\
&\qquad\qquad + \int_{\Sigma}\int_{\Sigma^c} (\varrho_0(z))^{q-1}v(y)|z-y|^{-(N+qs)}\,dy\,dz\Bigg].
\end{aligned}
$$

From this, we assert that $\varrho_0(z) = 0$ for almost all $z \in \Sigma^c$, i.e., $\varrho_0(z) = 0$ for almost all $z \in \mathbb{R}^N$. This yields a contradiction to the fact that $\varrho_0(z) \neq 0$ for almost all $z \in \Omega$.

Thus, to prove $\varrho_0 > 0$, it suffices to verify that $\widetilde{\Gamma}(\varrho_0) < 0$. Now, in view of Lemma 9, we fix any eigenpair $(\mu_*, \varrho) \in (\mu_{1,r}, \infty) \times \mathbb{W} \setminus \{0\}$, such that

$$
\mu_*\int_{\Omega}|\varrho(z)|^r dz = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho(z)-\varrho(y)|^r}{|z-y|^{N+rs}}\,dy\,dz + \frac{r}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho(z)-\varrho(y)|^q}{|z-y|^{N+qs}}\,dy\,dz.
$$

On account of Theorem 3.2 in [32], one has $\varrho \in L^\infty(\mathbb{R}^N)$. Let $\eta_0 \in L^\infty(\Omega)$ with $\eta_0 > 0$ and let $\zeta_0 \in (0, \|\eta_0\|_{L^\infty(\Omega)})$ be fixed. Then, the set

$$
\Omega_{\zeta_0} := \{z \in \Omega : \eta_0(z) \geq \zeta_0\}
$$

has a positive measure. Furthermore, we fix $\mathfrak{D} > 0$ so that

$$
\mathfrak{D} > \frac{\mu_*\int_{\Omega}|\varrho(z)|^r\,dz}{r\lambda\zeta_0\int_{\Omega_{\zeta_0}}|\varrho(z)|^r\,dz}.
$$

Due to the second condition in (F3), we choose a $\xi_0 > 0$ such that

$$
\frac{G(z,\xi)}{\xi^r} \geq \eta_0(z)\mathfrak{D}
$$

for all $\xi \in (0, \xi_0]$, and for almost all $z \in \Omega$. Then, for $\delta > 0$ that is small enough, we infer

$$
\begin{aligned}
&\lambda\int_{\Omega}\frac{G(z,\delta\varrho)}{\delta^r}\,dz \\
&\geq \lambda\mathfrak{D}\int_{\Omega}\eta_0(z)|\varrho(z)|^r\,dz \\
&\geq \lambda\mathfrak{D}\zeta_0\int_{\Omega_{\zeta_0}}|\varrho(z)|^r\,dz \\
&> \frac{\mu_*}{r}\int_{\Omega}|\varrho(z)|^r\,dz \\
&= \frac{1}{r}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varrho(z)-\varrho(y)|^r}{|z-y|^{N+rs}}\,dy\,dz + \frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varrho(z)-\varrho(y)|^q}{|z-y|^{N+qs}}\,dy\,dz. \qquad (11)
\end{aligned}
$$

Thus, due to (11), we find that

$$\begin{aligned}
&\lambda \int_{\mathbb{R}^N} G(z, \delta\varrho)\, dz \\
&> \frac{\delta^r}{r} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\varrho(z) - \varrho(y)|^r}{|z-y|^{N+rs}}\, dy\, dz + \frac{\delta^r}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\varrho(z) - \varrho(y)|^q}{|z-y|^{N+qs}}\, dy\, dz \\
&> \frac{1}{r} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\delta\varrho(z) - \delta\varrho(y)|^r}{|z-y|^{N+rs}}\, dy\, dz + \frac{1}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\delta\varrho(z) - \delta\varrho(y)|^q}{|z-y|^{N+qs}}\, dy\, dz \qquad (12)
\end{aligned}$$

for any $\delta > 0$ that is small enough, which is $\Gamma(\delta\varrho) < 0$, and the conclusion is satisfied. Hence, problem (2) has at least one positive solution for any $\lambda > 0$. □

With the help of Lemmas 5 and 6 and Theorem 1, we are ready to obtain our first main consequence. The elementary idea of the following result comes from [9,14].

Theorem 2. *If* (F1)–(F4) *hold, then, for any* $\lambda > 0$, *problem* (2) *has a unique positive solution.*

Proof. In view of Theorem 1, suppose that φ_1 and φ_2 are two weak positive solutions of (2). For any $\delta > 0$, we define the truncations $\varphi_{i,\delta}$ as in (7) for $i = 1, 2$. Let us define the functions

$$v_{1,\delta} := \frac{\varphi_{2,\delta}^r}{(\varphi_1 + \delta)^{r-1}} - \varphi_{1,\delta}$$

and

$$v_{2,\delta} := \frac{\varphi_{1,\delta}^r}{(\varphi_2 + \delta)^{r-1}} - \varphi_{2,\delta}.$$

Using Lemma 7, we deduce that $v_{i,\delta} \in \mathbb{W}$ for $i = 1, 2$. Now, set

$$\phi_m(\xi) := |\xi|^{m-2}\xi,$$

where $m \in \{r, q\}$. Taking into account the weak formulation (3) of φ_i, by selecting $v = v_{i,\delta}$ for $i = 1, 2$, we infer

$$\begin{aligned}
&\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \phi_r(\varphi_1(z) - \varphi_1(y))(v_{1,\delta}(z) - v_{1,\delta}(y))|z-y|^{-(N+rs)}\, dy\, dz \\
&\quad + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \phi_q(\varphi_1(z) - \varphi_1(y))(v_{1,\delta}(z) - v_{1,\delta}(y))|z-y|^{-(N+qs)}\, dy\, dz \\
&\quad = \lambda \int_\Omega g(z, \varphi_1) v_{1,\delta}(z)\, dz \qquad (13)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \phi_r(\varphi_2(z) - \varphi_2(y))(v_{2,\delta}(z) - v_{2,\delta}(y))|z-y|^{-(N+rs)}\, dy\, dz \\
&\quad + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \phi_q(\varphi_2(z) - \varphi_2(y))(v_{2,\delta}(z) - v_{2,\delta}(y))|z-y|^{-(N+qs)}\, dy\, dz \\
&\quad = \lambda \int_\Omega g(z, \varphi_2) v_{2,\delta}(z)\, dz. \qquad (14)
\end{aligned}$$

Adding the above two Equations (13) and (14) and using the fact that

$$\phi_m(\varphi_i(z) - \varphi_i(y)) = \phi_m((\varphi_i + \delta)(z) - (\varphi_i + \delta)(y)) \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad m \in \{r, q\},$$

we deduce

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\phi_r((\varphi_1+\delta)(z)-(\varphi_1+\delta)(y))}{|z-y|^{N+rs}}\left(\frac{\varphi_{2,\delta}^r}{(\varphi_1+\delta)^{r-1}}(z)-\frac{\varphi_{2,\delta}^r}{(\varphi_1+\delta)^{r-1}}(y)\right)dy\,dz\\
&-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\phi_r(\varphi_1(z)-\varphi_1(y))(\varphi_{1,\delta}(z)-\varphi_{1,\delta}(y))|z-y|^{-(N+rs)}\,dy\,dz\\
&+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\phi_q((\varphi_1+\delta)(z)-(\varphi_1+\delta)(y))}{|z-y|^{N+qs}}\left(\frac{\varphi_{2,\delta}^r}{(\varphi_1+\delta)^{r-1}}(z)-\frac{\varphi_{2,\delta}^r}{(\varphi_1+\delta)^{r-1}}(y)\right)dy\,dz\\
&-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\phi_q(\varphi_1(z)-\varphi_1(y))(\varphi_{1,\delta}(z)-\varphi_{1,\delta}(y))|z-y|^{-(N+qs)}\,dy\,dz\\
&+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\phi_r((\varphi_2+\delta)(z)-(\varphi_2+\delta)(y))}{|z-y|^{N+rs}}\left(\frac{\varphi_{1,\delta}^r}{(\varphi_2+\delta)^{r-1}}(z)-\frac{\varphi_{1,\delta}^r}{(\varphi_2+\delta)^{r-1}}(y)\right)dy\,dz\\
&-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\phi_r(\varphi_2(z)-\varphi_2(y))(\varphi_{2,\delta}(z)-\varphi_{2,\delta}(y))|z-y|^{-(N+rs)}\,dy\,dz\\
&+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\phi_q((\varphi_2+\delta)(z)-(\varphi_2+\delta)(y))}{|z-y|^{N+qs}}\left(\frac{\varphi_{1,\delta}^r}{(\varphi_2+\delta)^{r-1}}(z)-\frac{\varphi_{1,\delta}^r}{(\varphi_2+\delta)^{r-1}}(y)\right)dy\,dz\\
&-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\phi_q(\varphi_2(z)-\varphi_2(y))(\varphi_{2,\delta}(z)-\varphi_{2,\delta}(y))|z-y|^{-(N+qs)}\,dy\,dz\\
&=\lambda\left(\int_\Omega\left[g(z,\varphi_1)\left(\frac{\varphi_{2,\delta}^r}{(\varphi_1+\delta)^{r-1}}-\varphi_{1,\delta}\right)+g(z,\varphi_2)\left(\frac{\varphi_{1,\delta}^r}{(\varphi_2+\delta)^{r-1}}-\varphi_{2,\delta}\right)\right]dz\right).
\end{aligned}\tag{15}$$

We denote that $u_{1,\delta}:=\varphi_1+\delta$ and $u_{2,\delta}:=\varphi_2+\delta$. Now, exploiting Lemma 5 and the fact that $\xi\to\min\{|\xi|,\delta^{-1}\}$ is 1-Lipschitz, we deduce

$$\phi_r(u_{1,\delta}(z)-u_{1,\delta}(y))\left(\frac{\varphi_{2,\delta}^r}{u_{1,\delta}^{r-1}}(z)-\frac{\varphi_{2,\delta}^r}{u_{1,\delta}^{r-1}}(y)\right)\le|\varphi_2(z)-\varphi_2(y)|^r\tag{16}$$

and

$$\phi_r(u_{2,\delta}(z)-u_{2,\delta}(y))\left(\frac{\varphi_{1,\delta}^r}{u_{2,\delta}^{r-1}}(z)-\frac{\varphi_{1,\delta}^r}{u_{2,\delta}^{r-1}}(y)\right)\le|\varphi_1(z)-\varphi_1(y)|^r.\tag{17}$$

In compliance with Lemma 6 with $f(\xi)=\xi^{r-1}$, we know that

$$\begin{aligned}
&\phi_q(u_{1,\delta}(z)-u_{1,\delta}(y))\left(\frac{\varphi_{2,\delta}^r}{u_{1,\delta}^{r-1}}(z)-\frac{\varphi_{2,\delta}^r}{u_{1,\delta}^{r-1}}(y)\right)\\
&\qquad\le|\varphi_{2,\delta}(z)-\varphi_{2,\delta}(y)|^r|\varphi_1(z)-\varphi_1(y)|^{q-r}
\end{aligned}\tag{18}$$

and

$$\begin{aligned}
&\phi_q(u_{2,\delta}(z)-u_{2,\delta}(y))\left(\frac{\varphi_{1,\delta}^r}{u_{2,\delta}^{r-1}}(z)-\frac{\varphi_{1,\delta}^r}{u_{2,\delta}^{r-1}}(y)\right)\\
&\qquad\le|\varphi_{1,\delta}(z)-\varphi_{1,\delta}(y)|^r|\varphi_2(z)-\varphi_2(y)|^{q-r}.
\end{aligned}\tag{19}$$

Based on Young's inequality, inequalities (18) and (19) imply

$$\begin{aligned}
&\phi_q(u_{1,\delta}(z)-u_{1,\delta}(y))\left(\frac{\varphi_{2,\delta}^r}{u_{1,\delta}^{r-1}}(z)-\frac{\varphi_{2,\delta}^{pr}}{u_{1,\delta}^{r-1}}(y)\right)\\
&\qquad\le\frac{r}{q}|\varphi_{2,\delta}(z)-\varphi_{2,\delta}(y)|^q+\frac{q-r}{q}|\varphi_1(z)-\varphi_1(y)|^q
\end{aligned}\tag{20}$$

and

$$\phi_q(u_{2,\delta}(z)-u_{2,\delta}(y))\left(\frac{\varphi_{1,\delta}^r}{u_{2,\delta}^{r-1}}(z)-\frac{\varphi_{1,\delta}^r}{u_{2,\delta}^{r-1}}(y)\right)\\ \leq \frac{r}{q}|\varphi_{1,\delta}(z)-\varphi_{1,\delta}(y)|^q+\frac{q-r}{q}|\varphi_2(z)-\varphi_2(y)|^q. \tag{21}$$

Because $\varphi_{i,\delta}\to\varphi_i$ and $u_{i,\delta}\to\varphi_i$, as $\delta\to 0$ for $i=1,2$, by setting the limit as $\delta\to 0$ in (16), (17) and (20)–(21), we infer that

$$\phi_r(\varphi_1(z)-\varphi_1(y))\left(\frac{\varphi_2^r}{\varphi_1^{r-1}}(z)-\frac{\varphi_2^r}{\varphi_1^{r-1}}(y)\right)\leq|\varphi_2(z)-\varphi_2(y)|^r,\\ \phi_r(\varphi_2(z)-\varphi_2(y))\left(\frac{\varphi_1^r}{\varphi_2^{r-1}}(z)-\frac{\varphi_1^r}{\varphi_2^{r-1}}(y)\right)\leq|\varphi_1(z)-\varphi_1(y)|^r \tag{22}$$

and

$$\phi_q(\varphi_1(z)-\varphi_1(y))\left(\frac{\varphi_2^r}{\varphi_1^{r-1}}(z)-\frac{\varphi_2^r}{\varphi_1^{r-1}}(y)\right)\\ \leq\frac{r}{q}|\varphi_2(z)-\varphi_2(y)|^q+\frac{q-r}{q}|\varphi_1(z)-\varphi_1(y)|^q,\\ \phi_q(\varphi_2(z)-\varphi_2(y))\left(\frac{\varphi_1^r}{\varphi_2^{r-1}}(z)-\frac{\varphi_1^r}{\varphi_2^{rs-1}}(y)\right)\\ \leq\frac{r}{q}|\varphi_1(z)-\varphi_1(y)|^q+\frac{q-r}{q}|\varphi_2(z)-\varphi_2(y)|^q. \tag{23}$$

Passing to the limit $\delta\to 0$ in (15) and applying the Fatou Lemma in the first, third, fifth, and seventh terms, as well as utilizing the dominated convergence theorem for all the other terms, it follows from (22) and (23) that

$$\begin{aligned}0\geq&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\phi_r(\varphi_1(z)-\varphi_1(y))\left(\frac{\varphi_2^r}{\varphi_1^{r-1}}(z)-\frac{\varphi_2^r}{\varphi_1^{r-1}}(y)\right)|z-y|^{-(N+rs)}\,dy\,dz\\ &-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\varphi_1(z)-\varphi_1(y)|^r|z-y|^{-(N+rs)}\,dy\,dz\\ &+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\phi_q(\varphi_1(z)-\varphi_1(y))\left(\frac{\varphi_2^r}{\varphi_1^{r-1}}(z)-\frac{\varphi_2^r}{\varphi_1^{r-1}}(y)\right)|z-y|^{-(N+qs)}\,dy\,dz\\ &-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\varphi_1(z)-\varphi_1(y)|^q|z-y|^{-(N+qs)}\,dy\,dz\\ &+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\phi_r(\varphi_2(z)-\varphi_2(y))\left(\frac{\varphi_1^r}{\varphi_2^{r-1}}(z)-\frac{\varphi_1^r}{\varphi_2^{r-1}}(y)\right)|z-y|^{-(N+rs)}\,dy\,dz\\ &-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\varphi_2(z)-\varphi_2(y)|^r|z-y|^{-(N+rs)}\,dy\,dz\\ &+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\phi_q(\varphi_2(z)-\varphi_2(y))\left(\frac{\varphi_1^r}{\varphi_2^{r-1}}(z)-\frac{\varphi_1^r}{\varphi_2^{r-1}}(y)\right)|z-y|^{-(N+qs)}\,dy\,dz\\ &-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\varphi_2(z)-\varphi_2(y)|^q|z-y|^{-(N+qs)}\,dy\,dz\\ &\geq\lambda\left(\int_\Omega g(z,\varphi_1)\left(\frac{\varphi_2^r}{\varphi_1^{r-1}}-\varphi_1\right)+g(z,\varphi_2)\left(\frac{\varphi_1^r}{\varphi_2^{r-1}}-\varphi_2\right)dz\right)\\ &=-\lambda\int_\Omega\left(\frac{g(z,\varphi_1)}{\varphi_1^{r-1}}-\frac{g(z,\varphi_2)}{\varphi_2^{r-1}}\right)(\varphi_1^r-\varphi_2^r)\,dz.\end{aligned} \tag{24}$$

Due to Lemma 6 on the left-hand side of (24), we find that

$$\int_\Omega \left(\frac{g(z,\varphi_1)}{\varphi_1^{r-1}} - \frac{g(z,\varphi_2)}{\varphi_2^{r-1}} \right) (\varphi_1^r - \varphi_2^r)\, dz \geq 0.$$

Hence, since the function $\xi \mapsto \frac{g(z,\xi)}{\xi^{r-1}}$ is decreasing in $(0,+\infty)$, we derive that $\varphi_1 = \varphi_2$. Consequently, we conclude that problem (2) has a unique positive solution. □

Remark 1. *Our hypothesis* (F3) *can be regarded as a special case of that of [14,25] where g satisfies assumptions*

$$\alpha_0(z) = \lim_{\xi \to 0^+} \frac{g(z,\xi)}{\xi^{r-1}} \qquad \text{and} \qquad \alpha_\infty(z) = \lim_{\xi \to +\infty} \frac{g(z,\xi)}{\xi^{r-1}}$$

for almost all $z \in \Omega$. Let us define $\Lambda_1(\alpha_0)$ and $\Lambda_1(\alpha_\infty)$ as

$$\Lambda_1(\alpha_0) = \inf_{\varrho \in \mathbb{W}} \left\{ [\varrho]_{s,r,q} - \int_\Omega \alpha_0 |\varrho(z)|^r \, dz \ : \ \|\varrho\|_{L^r(\Omega)} = 1 \right\}$$

and

$$\Lambda_1(\alpha_\infty) = \inf_{\varrho \in \mathbb{W}} \left\{ [\varrho]_{s,r,q} - \int_\Omega \alpha_\infty |\varrho(z)|^r \, dz \ : \ \|\varrho\|_{L^r(\Omega)} = 1 \right\}.$$

If $\Lambda_1(\alpha_0) < 0 < \Lambda_1(\alpha_\infty)$ instead of (F3) *holds, then obvious modifications of the proof of Theorem 2 and analogous arguments as in [14] imply that problem* (2) *has a unique positive solution.*

4. Application

In this section, as an application of Theorems 1 and 2, we derive the existence result of a unique nontrivial positive solution to Brézis–Oswald-type problems involving the fractional (r,q)-Laplacian operators and discontinuous Kirchhoff-type coefficients. To do this, we introduce the abstract theorem suggested by B. Ricceri [17].

Definition 2. *Let $\mathbb{W}$ be a topological space; a function $h : \mathbb{W} \to \mathbb{R}$ is inf-compact if the set $h^{-1}((-\infty,\xi])$ is compact for each $\xi \in \mathbb{R}$.*

The following abstract global minimum principle in [17] plays a decisive role in finding our second main consequence.

Theorem 3. *Let $\mathbb{W}$ be a topological space, and let $\Upsilon : \mathbb{W} \to \mathbb{R}$, with $\Upsilon^{-1}(0) \neq \emptyset$ and $\Phi : \mathbb{W} \to \mathbb{R}$ being two functions such that, for each $\gamma > 0$, the function $\gamma\Upsilon - \Phi$ is lower semicontinuous, is inf-compact, and has a unique global minimum. Furthermore, assume that Φ has no global maxima in $\mathbb{W}$. Additionally, assume that $\Lambda \subseteq (0,+\infty)$ is an open interval and $M : \Lambda \to \mathbb{R}$ is an increasing function satisfying $M(\Lambda) = (0,+\infty)$. Then, there is a unique $\tilde{u} \in \mathbb{W}$ such that $\Upsilon(\tilde{u}) \in \Lambda$ and*

$$M(\Upsilon(\tilde{u}))\Upsilon(\tilde{u}) - \Phi(\tilde{u}) = \inf_{u \in \mathbb{W}} (M(\Upsilon(\tilde{u}))\Upsilon(u) - \Phi(u)).$$

As each assumption of Theorem 3 is assured, we obtain the following theorem. The fundamental idea of the proof of the uniqueness of positive solutions to problem (1) follows from [9,14]; see [29] for more details.

Theorem 4. *Assume that there is an open interval $\Lambda \subseteq (0,+\infty)$ such that $M(J) = (0,+\infty)$ and the restriction of M to J is increasing. Let $g : \Omega \times [0,+\infty) \to (0,+\infty)$ be a function satisfying assumptions* (F1)–(F4) *and $g(z,0) = 0$ for almost all $z \in \Omega$. Then, problem* (1) *has a unique positive weak solution $\widetilde{v}$, which is the unique global minimum in $\mathbb{W}$ of the functional*

$$\varrho \mapsto M\big([\widetilde{v}]_{s,r,q}\big)\left[\frac{1}{r}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varrho(z)-\varrho(y)|^r}{|z-y|^{N+rs}}\,dy\,dz+\frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varrho(z)-\varrho(y)|^q}{|z-y|^{N+qs}}\,dy\,dz\right]$$
$$-\int_\Omega\Big(\int_0^{\varrho^+(z)} g(z,t)\,dt\Big)\,dz,$$

where $\varrho^+ := \max\{\varrho, 0\}$.

Proof. First, extend g to $\mathbb{R}$, setting $g(z,\xi) = 0$ for all $\xi < 0$. To utilize Theorem 3, consider Y given in (4) and define Φ by

$$\Phi(\varrho) := \int_\Omega G(z, \varrho^+(z))\,dz$$

for all $\varrho \in \mathbb{W}$. Let $\gamma > 0$ be fixed and choose

$$\delta \in \left(0, \frac{\gamma(\mathfrak{C}_0+1)}{2\mathfrak{C}_0}\right),$$

where $\mathfrak{C}_0$ is given in Lemma 3. In accordance with condition (F3), Lemma 3, and the relation (9), we know that

$$\begin{aligned}\gamma\mathrm{Y}(\varrho)-\Phi(\varrho) &\geq \frac{\gamma}{r}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varrho(z)-\varrho(y)|^r}{|z-y|^{N+rs}}\,dy\,dz+\frac{\gamma}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varrho(z)-\varrho(y)|^q}{|z-y|^{N+qs}}\,dy\,dz\\ &\quad-\delta\int_\Omega|\varrho(z)|^r dz - C(\delta)|\Omega|\\ &\geq \gamma\left(\frac{1}{2}+\frac{1}{2\mathfrak{C}_0}\right)\|\varrho\|_{\mathbb{W}}^r-\delta\int_\Omega|\varrho(z)|^r dz - C(\delta)|\Omega|\\ &\geq \left(\frac{\gamma(\mathfrak{C}_0+1)}{2\mathfrak{C}_0}-\delta\right)\|\varrho\|_{\mathbb{W}}^r - C(\delta)|\Omega|\end{aligned}$$

for any $\varrho \in \mathbb{W}$. Hence, it follows from the choice of δ that

$$\lim_{\|\varrho\|_{\mathbb{W}}\to+\infty}(\gamma\mathrm{Y}(\varrho)-\Phi(\varrho)) = +\infty.$$

This fact, jointly with Lemma 4, the Eberlein–Smulyan theorem, and the reflexivity of $\mathbb{W}$, implies that the functional $\gamma\mathrm{Y}-\Phi$ is weakly inf-compact. Now, we claim that it has a unique global minimum in $\mathbb{W}$. Its critical points are exactly the weak solutions to the problem

$$\begin{cases}(-\Delta)_r^s\varrho+(-\Delta)_q^s\varrho=\frac{1}{\gamma}g(z,\varrho) & \text{in } \Omega,\\ \varrho=0 & \text{on } \partial\Omega,\end{cases} \tag{25}$$

where $\varrho \in \mathbb{W}$ is said to be a weak solution to problem (25) if

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varrho(z)-\varrho(y)|^{r-2}}{|z-y|^{N+rs}}(\varrho(z)-\varrho(y))(\phi(z)-\phi(y))\,dy\,dz\\ &+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varrho(z)-\varrho(y)|^{q-2}}{|z-y|^{N+qs}}(\varrho(z)-\varrho(y))(\phi(z)-\phi(y))\,dy\,dz=\frac{1}{\gamma}\int_\Omega g(z,\varrho)\phi\,dz\end{aligned} \tag{26}$$

for any $\phi \in \mathbb{W}$. From the same arguments made in Theorems 1 and 2, problem (25) has at most one positive solution for any $\gamma > 0$. As a result, we derive that the functional $\gamma\mathrm{Y}-\Phi$ has a unique global minimum in $\mathbb{W}$, since, otherwise, in consideration of [33] Corollary 1, it would have at least three critical points. In view of the estimate (12), we know that 0 is

not a global minimum for the functional $\gamma Y - \Phi$ and, thus, the global minimum of this functional is consistent with its only nonzero critical point.

Finally, let us prove that Φ has no global maxima. Suppose, to the contrary, that $\widehat{\varrho} \in \mathbb{W}$ is a global maximum of Φ. Obviously, we know $\Phi(\widehat{\varrho}) > 0$. Since g is non-negative and $\xi \mapsto \frac{g(z,\xi)}{\xi^{r-1}}$ is strictly decreasing in $(0, +\infty)$ for almost all $z \in \Omega$, it follows that $g(z,\xi) > 0$ for all $\xi > 0$. In fact, if there is $\eta > 0$ such that $g(z,\eta) = 0$ for $\xi > \eta$, we would have

$$0 = \frac{g(z,\eta)}{\eta^{r-1}} > \frac{g(z,\xi)}{\xi^{r-1}},$$

and so $g(z,\xi) < 0$, contrary to our assumption. Now, let $\widehat{\varrho} \in X$ be such that $\Phi(\widehat{\varrho}) > 0$. We claim that the set

$$\Sigma := \{z \in \Omega : g(z, \widehat{\varrho}(z)) > 0\}$$

has a positive measure. Arguing by contradiction, we assume that $|\Sigma| = 0$. Then, one has

$$|\{z \in \Omega : g(z, \widehat{\varrho}(z)) > 0\}| = 0,$$

i.e., $g(z, \widehat{\varrho}(z)) \leq 0$ for almost all $z \in \Omega$. From this, since $g(z,\xi) > 0$ for all $\xi > 0$, we infer that $\widehat{\varrho}(z) \leq 0$ for almost all $z \in \Omega$. Consequently, $G(z, \widehat{\varrho}(z)) = 0$ for almost all $z \in \Omega$, which contradicts by $\Phi(\widehat{\varrho}) > 0$, as claimed. We fix a closed set $\mathcal{A} \subset \Sigma$ of positive measures. Let $\varphi \in \mathbb{W}$ be such that $\varphi \geq 0$ and $\varphi(z) = 1$ for almost all $z \in \mathcal{A}$. Then, we have

$$\int_\Omega g(z, \widehat{\varrho}(z))\varphi(z)\, dz \geq \int_{\mathcal{A}} g(z, \widehat{\varrho}(z))\, dz > 0,$$

and so $\Phi'(\widehat{\varrho}) \neq 0$, which is a contradiction.

Hence, each assumption of Theorem 3 is satisfied. Therefore, there is a unique $\widetilde{v} \in \mathbb{W}$, with $[\widetilde{v}]_{s,r,q} \in \Lambda$, such that

$$\begin{aligned} M\big([\widetilde{v}]_{s,r,q}\big)&\left[\frac{1}{r}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\widetilde{v}(z)-\widetilde{v}(y)|^r}{|z-y|^{N+rs}}\,dy\,dz+\frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\widetilde{v}(z)-\widetilde{v}(y)|^q}{|z-y|^{N+qs}}\,dy\,dz\right]\\ &-\int_\Omega G(z,\widetilde{v}^+(z))\,dz\\ =\inf_{\varrho\in\mathbb{W}}&\left\{M\big([\widetilde{v}]_{s,r,q}\big)\left[\frac{1}{r}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varrho(z)-\varrho(y)|^r}{|z-y|^{N+rs}}\,dy\,dz+\frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varrho(z)-\varrho(y)|^q}{|z-y|^{N+qs}}\,dy\,dz\right]\right.\\ &\left.-\int_\Omega G(z,\varrho^+(z))\,dz\right\}. \end{aligned}$$

Consequently, from what is seen above, the function $\widetilde{v}$ is the unique positive weak solution for problem (1). □

5. Conclusions

The first aim of this study was to examine the existence and uniqueness of positive solutions to the Brézis–Oswald-type problem involving the fractional (r,q)-Laplacian operators. To do this, we considered the eigenvalue problem associated with the fractional (r,q)-Laplacian and several versions of the discrete Picone inequalities in [8,12] as the key tools. To apply such existence and uniqueness results, we gave our second main result by exploiting the global minimum principle of Ricceri [17]. Thus far, the uniqueness results of positive solutions to the Brézis–Oswald-type fractional (r,q)-Laplacian problems with the discontinuous Kirchhoff coefficient have not been reported.

Furthermore, another goal was to consider the Brézis–Oswald-type fractional (r,q)-Laplacian problems involving the Kirchhoff coefficients and Hardy potentials

$$\begin{cases} M([\varrho]_{s,r,q})\left((-\Delta)_r^s\varrho+(-\Delta)_q^s\varrho\right)=\mu\left(\frac{|\varrho|^{r-2}\varrho}{|z|^r}+\frac{|\varrho|^{q-2}\varrho}{|z|^q}\right)+g(z,\varrho) & \text{in} \quad \Omega, \\ \varrho>0 & \text{in} \quad \Omega, \\ \varrho=0 & \text{on} \quad \mathbb{R}^N\setminus\Omega, \end{cases} \tag{27}$$

where $1<r<q<r_s^*$, M is a Kirchhoff-type function and $\mu\in(-\infty,\mu^*)$ for a positive constant μ^*. To the best of our knowledge, there are no studies on the existence and uniqueness results of a positive solution to problem (27). In a future study, we, therefore, aim to obtain similar results to those in Theorems 1, 2, and 3 for problem (27).

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References

1. Ambrosio, V.; Rădulescu, V.D. Fractional double-phase patterns: Concentration and multiplicity of solutions. *J. Math. Pures Appl.* **2020**, *142*, 101-145. [CrossRef]
2. Marano, S.A.; Mosconi, S. Some recent results on the Dirichlet problem for (p,q)-Laplace equations. *Discrete Contin. Dyn. Syst.-S* **2018**, *11*, 279–291. [CrossRef]
3. Alves, C.O.; Ambrosio, V.; Isernia, T. Existence, multiplicity and concentration for a class of fractional $p\&q$ Laplacian problems in $\mathbb{R}^N$. *Commun. Pure Appl. Anal.* **2019**, *18*, 2009–2045.
4. Ambrosio, V.; Isernia, T. Multiplicity of positive solutions for a fractional $p\&q$-Laplacian problem in $\mathbb{R}^N$. *J. Math. Anal. Appl.* **2021**, *501*, 124487.
5. Bhakta, M.; Mukherjee, D. Multiplicity results for (p,q) fractional elliptic equations involving critical nonlinearities. *Adv. Differ. Equ.* **2019**, *24*, 185–228. [CrossRef]
6. Brezis, H.; Oswald, L. Remarks on sublinear elliptic equations. *Nonlinear Anal.* **1986** , *10*, 55–64. [CrossRef]
7. Biagi, S.; Mugnai, D.; Vecchi, E. A Brézis-Oswald approach for mixed local and nonlocal operators. *Commun. Contemp. Math.* **2024**, *26*, 2250057. [CrossRef]
8. Brasco, L.; Franzina, G. Convexity properties of Dirichlet integrals and Picone-type inequalities. *Kodai Math. J.* **2014**, *37*, 769–799. [CrossRef]
9. Brasco, L.; Squassina, M. Optimal solvability for a nonlocal at critical growth. *J. Differ. Equ.* **2018**, *264*, 2242–2269. [CrossRef]
10. Carvalho, M.L.; Goncalves, J.V.; Silva, E.D.; Santos, C.A.P. A Type of Brézis–Oswald Problem to the Φ-Laplacian Operator with Very Singular Term. *Milan J. Math.* **2018**, *86*, 53–80. [CrossRef]
11. Díaz, J.I.; Saa, J.E. Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires. *Comptes Rendus Acad. Sci. Série I* **1987**, *305*, 521–524.
12. Giacomoni, J.; Gouasmia, A.; Mokrane, A. Discrete Picone inequalities and applications to non local and non homogenenous operators. *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat.* **2022**, *116*, 100. [CrossRef]
13. Kim, I.H.; Kim, Y.-H. Existence, uniqueness, and localization of positive solutions to nonlocal problems of the Kirchhoff type via the global minimum principle of Ricceri. *AIMS Math.* **2025**, *10*, 4540–4557. [CrossRef]
14. Mugnai, D.; Pinamonti, A.; Vecchi, E. Towards a Brezis-Oswald-type result for fractional problems with Robin boundary conditions. *Calc. Var. Partial. Differ. Equ.* **2020**, *59*, 1–25. [CrossRef]
15. Faria, L.F.O.; Miyagaki, O.H.; Motreanu, D. Comparison and positive solutions for problems with the (p,q)-Laplacian and a convection term. *Proc. Edinb. Math. Soc.* **2014**, *57*, 687–698. [CrossRef]
16. Frank, R.L.; Seiringer, R. Non-linear ground state representations and sharp Hardy inequalities. *J. Funct. Anal.* **2008**, *255*, 3407–3430. [CrossRef]

17. Ricceri, B. Existence, uniqueness, localization and minimization property of positive solutions for non-local problems involving discontinuous Kirchhoff functions. *Adv. Nonlinear Anal.* **2024**, *13*, 20230104. [CrossRef]
18. Kirchhoff, G.R. *Vorlesungen über Mathematische Physik, Mechanik*; Teubner: Leipzig, Germany, 1876.
19. Fiscella, A.; Valdinoci, E. A critical Kirchhoff type problem involving a nonlocal operator. *Nonlinear Anal.* **2014**, *94*, 156–170. [CrossRef]
20. Huang, T.; Deng, S. Existence of ground state solutions for Kirchhoff type problem without the Ambrosetti–Rabinowitz condition. *Appl. Math. Lett.* **2021**, *113*, 106866. [CrossRef]
21. Kim, I.H.; Kim, Y.-H.; Park, K. Multiple solutions to a non-local problem of Schrödinger-Kirchhoff type in $\mathbb{R}^N$. *Fractal Fract.* **2023**, *7*, 627. [CrossRef]
22. Nyamoradi, N. Existence of three solutions for Kirchhoff nonlocal operators of elliptic type. *Math. Commun.* **2013**, *18*, 489–502.
23. Pucci, P.; Xiang, M.Q.; Zhang, B.L. Multiple solutions for nonhomogeneous Schrödinger-Kirchhoff type equations involving the fractional p-Laplacian in $\mathbb{R}^N$. *Calc. Var. Partial Differ. Equ.* **2015**, *54*, 2785–2806. [CrossRef]
24. Xiang, M.Q.; Zhang, B.L.; Ferrara, M. Existence of solutions for Kirchhoff type problem involving the non-local fractional p-Laplacian. *J. Math. Anal. Appl.* **2015**, *424*, 1021–1041. [CrossRef]
25. Biagi, S.; Vecchi, E. On a Brezis-Oswald-type result for degenerate Kirchhoff problems. *Discrete Contin. Dyn. Syst.-S* **2024**, *44*, 702–717. [CrossRef]
26. Fiscella, A. Schrödinger-Kirchhoff-Hardy p-fractional equations without the Ambrosetti-Rabinowitz condition. *Discrete Contin. Dyn. Syst.-S* **2020**, *13*, 1993–2007.
27. Kim, Y.-H. Existence and Uniqueness of Solution to the p-Laplacian Equations Involving Discontinuous Kirchhoff Functions Via A Global Minimum Principle of Ricceri. *Minimax Theory Appl.* **2025**, *10*, 34–42.
28. Kim, Y.-H. Existence and uniqueness of a positive solution to double phase problems involving discontinuous Kirchhoff type function. *Bull. Korean Math. Soc.* 2024, *in press.*
29. Kim, Y.-H. Existence and Uniqueness of Solutions to Non-Local Problems of Brézis-Oswald Type and Its Application. *Fractal Fract.* **2024**, *8*, 622. [CrossRef]
30. Adams, R.A.; Fournier, J.J.F *Sobolev Spaces*, 2nd ed.; Academic Press: New York, NY, USA; London, UK, 2003.
31. Di Nezza, E.; Palatucci, G.; Valdinoci, E. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **2012**, *136*, 521–573. [CrossRef]
32. Franzina, G.; Palatucci, G. Fractional p-eigenvalues. *Riv. Mat. Univ. Parma* **2014**, *5*, 373–386.
33. Pucci, P.; Serrin, J. A mountain pass theorem. *J. Differ. Equ.* **1985**, *60*, 142–149. [CrossRef]

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