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On the Existence of Solutions and Ulam-Type Stability for a Nonlinear ψ-Hilfer Fractional-Order Delay Integro-Differential Equation

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Abstract

In this work, we address a nonlinear ψ-Hilfer fractional-order Volterra integro-differential equation that incorporates n-multiple-variable time delays. Employing the ψ-Hilfer fractional derivative operator, we investigate the existence of a unique solution, as well as the Ulam–Hyers–Rassias stability, semi-Ulam–Hyers–Rassias stability, and Ulam–Hyers stability of the proposed ψ-Hilfer fractional-order Volterra integro-differential equation through the fixed-point approach. In this study, we enhance and generalize existing results in the literature on ψ-Hilfer fractional-order Volterra integro-differential equations, both including and excluding single delay, by establishing new findings for nonlinear ψ-Hilfer fractional-order Volterra integro-differential equations involving n-multiple-variable time delays. This study provides novel theoretical insights that deepen the qualitative understanding of fractional calculus.

Keywords: ψ-Hilfer FrOVI-DE; unique solution; UHR stability; semi-UHR stability; UH stability; fixed-point approach

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1. Introduction

Fractional calculus is widely recognized as a natural extension of classical integral and differential calculus, allowing the order of integration and differentiation to take non-integer values. In recent decades, the field of fractional calculus has emerged as an effective analytical framework across various disciplines, including mathematics, physics, engineering, and medicine, due to its broad applicability and effectiveness in modeling complex phenomena, as evidenced by a substantial and growing body of research. For a detailed analysis of fractional differential equations, the reader is referred to [1–3]. Further discussions on fractional calculus and its applications to fractional differential equations can be found in [4–8]. In line with this increasing interest, numerous definitions of fractional derivatives, such as the Riemann–Liouville (RL) derivative, Caputo (C) derivative, Grünwald–Letnikov (GL) derivative, and ψ-Hilfer fractional derivative, along with various associated fractional integrals and some related results, have been proposed, highlighting the field's ongoing theoretical development and expansion. In this regard, we refer the reader to [9] for the existence and Ulam stability of nonlinear implicit fractional differential equations with the Hadamard derivative; to [4] for applications of fractional calculus in

physics; to [10] for definitions of fractional integrals and derivatives; to [11] for the definition of the ψ-Hilfer fractional derivative; and to [12] for recent developments in the stability theory of fractional differential equations.

Depending on these definitions, the qualitative analysis of mathematical models involving these fractional derivatives has become a central focus for researchers. In particular, considerable, although still limited, attention has been given to investigating conditions under which solutions exist uniquely, as well as various types of stability, such as UH stability, UHR stability, etc., in the context of ψ-Hilfer fractional-order Volterra integro-differential equations (ψ-Hilfer-type FrOVI-DEs), both with and without delay, primarily using fixed-point techniques. Specifically, we refer the reader to [13,14] for the Ulam stability results of fractional differential equations of the Hilfer–Hadamard type; to [15] for the Ulam stability of Hilfer-type fractional differential inclusions; to [16–19] for the Ulam stability of Hilfer fractional differential equations; and to [20–22] for the Ulam stability of Hilfer fractional integro-differential equations. Although there are relatively few studies addressing the existence of solutions and the Ulam-type stability for ψ-Hilfer-type models, a substantial body of literature is available concerning similar qualitative properties in the context of other mathematical formulations.

In the following, we provide a brief summary of key studies addressing the existence, Ulam-type stability, and other qualitative characteristics of various mathematical models, including ψ-Hilfer FrOVI-DEs, with a focus on both delayed and non-delayed formulations.

Abbas et al. [23] conducted a study on the existence of solutions and established UH stability results for a specific class of Hadamard partial fractional integral inclusions by employing the theory of weakly Picard operators.

Abbas et al. [13] investigated the existence and UHR stability of a class of functional differential equations involving the Hilfer–Hadamard-FrD. The existence of solutions was established through the application of Schauder's fixed-point theorem.

Abbas et al. [15] established results concerning the existence, data dependence, and Ulam stability for a class of differential inclusions involving the Hilfer-FDE. These findings were derived using the framework of the multivalued weakly Picard operator theory.

Abbas et al. [14] constructed suitable conditions under which solutions exist uniquely for certain coupled systems involving Hilfer and Hilfer–Hadamard FrODEs. Their approach utilized generalized forms of classical fixed-point theorems within the setting of generalized Banach spaces.

Brzdęk et al. [24] introduced a novel fixed-point result within the context of function spaces, offering a powerful and versatile framework for exploring operator inequalities, particularly those tied to Ulam-type stability. This result represents a substantial extension of the classical BCMP and integrates several known fixed-point theorems into a unified formulation.

Moroşanu and Petruşel [25] investigated a delay integro-differential equation within a Banach space framework. Using a fixed-point method alongside the properties of linear operator semigroups, they established the global existence and uniqueness of solutions for the equation, subject to specific initial conditions.

Petruşel and Rus [26] investigated the UH stability and generalized UH stability of the equation $T(x) = 0$ by introducing alternative criteria, thereby ensuring its stability. Their analysis relies on a fixed-point approach, specifically employing a theorem concerning mappings that exhibit contractive behavior when restricted to their graphs.

Zhang et al. [27] addressed a mixed-type nonlinear FrOVI-DE posed on a semi-infinite interval in the setting of a Banach space. The analysis, based on the application of the BCMP, led to the establishment of conditions under which solutions exist uniquely for the problem under consideration.

In their study, Sousa and de Oliveira [11] proposed a novel approach to fractional differentiation involving a function-dependent operator, now referred to as the ψ-Hilfer-FrD, and explored its foundational characteristics within the framework of fractional calculus.

Luo et al. [17] conducted a comprehensive analysis of ψ-Hilfer-type fractional integro-differential equations with time-dependent delays. By applying a generalized Gronwall inequality, they derived refined criteria for UH stability and established several complementary results pertinent to the dynamic behavior of the Hilfer FrOVI-DE under investigation.

Sousa and de Oliveira [22] explored the stability behavior, specifically the UH and UHR types, of a nonlinear ψ-Hilfer FrOVI-DE, utilizing fixed-point theory as the main analytical framework.

de Oliveira and Sousa [20] conducted a comprehensive study on a certain ψ-Hilfer-type FrOVI-DE with a time-dependent delay. By employing the ψ-Hilfer-FrD alongside the BCMP, they investigated the system's dynamic behavior and established results concerning UH stability, UHR stability, and semi-UHR stability over both fine and general intervals.

In 2021, Zhou et al. [28] dealt with the ψ-Hilfer FrOVI-DE with the initial condition in a real Banach space Y:

$$\begin{cases} {}^{H}D_{a^+}^{\alpha,\beta;\psi}x(t) = f\left(t, x(t), {}^{H}D_{a^+}^{u,v;\psi}x(t)\right) + \int_a^t K(t,\tau,x(\tau),x(\delta(\tau)))d\tau,\ t \in [a,b], \\ I_{a^+}^{1-\gamma;\psi}x(a) = 0. \end{cases} \tag{1}$$

In [28], Zhou et al. first proved the existence of a unique solution to problem (1) using the BCMP. Then, the UH stability, UHR stability, and semi-UHR stability of the ψ-Hilfer FrOVI-DE in (1) were analyzed in a finite interval using the same fixed-point approach.

Zhou et al. [29] first inferred the conditions under which solutions exist uniquely for problem (1) using the BCMP under suitable growth conditions in an appropriate Banach space. Additionally, they established the UHR stability, UH stability, and semi-UHR stability for the initial value problem.

Recently, Graef et al. [30] and Tunç and Tunç [21] investigated a class of Hammerstein-type nonlinear ψ-Hilfer FrOVI-DEs characterized by multiple-variable time delays. Employing the ψ-Hilfer fractional derivative operator (ψ-Hilfer-FrDO) in conjunction with the BCMP, the authors analyzed the Ulam-type stability properties of the proposed equations.

Additionally, several recent contributions have addressed topics such as Ulam-type stability (see [31–33]), Lyapunov stability, and related qualitative behavior in both fractional and classical mathematical models (see [34–36]).

The motivation for this study stems from the aforementioned papers, the references cited within this work, and in particular from the paper by Zhou et al. [28]. In this context, we examine the ψ-Hilfer FrOVI-DE with n-time-varying delays in a real Banach space Y, described as follows:

$$\begin{cases} {}^{H}D_{0^+}^{\alpha,\beta;\psi}\nu(t) = \sum_{i=1}^{n} F_i\left(t, \nu(t), \nu(t-\tau_i(t)), {}^{H}D_{0^+}^{u,\vartheta;\psi}\nu(t)\right) \\ + \int_0^t \left(\sum_{i=1}^{n} K_i(t,s,\nu(s),\nu(s-\tau_i(s)))\right) ds, t \in J = (0,b], \\ I_{0^+}^{1-\gamma;\psi}\nu(0^+) = 0,\ \nu(t) = \theta(t),\ t \in [-\tau, 0], \end{cases} \tag{2}$$

where ${}^{H}D_{0^+}^{\alpha,\beta;\psi}(.)$ and ${}^{H}D_{0^+}^{u,\vartheta;\psi}(.)$ are the ψ-Hilfer fractional derivatives of order $0 < \alpha$, $u < 1$, $(u < \alpha)$ and type $0 \leq \beta$, $\vartheta \leq 1$, $I_{0^+}^{1-\gamma;\psi}(.)$ is ψ-Riemann-Liouville fractional integral of order $1-\gamma$, $\gamma = \alpha + \beta(1-\alpha)$ with respect to ψ, $\tau_i \in C(J,J)$, $\tau = \max\{\tau_i(t)\}$, $\theta \in C[-\tau, 0]$, $F_i \in C(J \times Y \times Y \times Y, Y)$ and $K_i \in C(J \times J \times Y \times Y, Y)$ are given functions.

The existing literature indicates a growing scholarly interest in the qualitative behavior of various types of fractional differential equations, particularly those involving multiple-variable delays, due to their broad applicability in modeling complex real-world phenomena. The principal objective of this ongoing study is to examine the existence of a unique solution, the UHR stability, the semi-UHR stability, and the UH stability of the ψ-Hilfer FrOVI-DE in (2), which involves multiple-variable delays on a finite interval, employing the fixed-point approach. While the ψ-Hilfer FrOVI-DE in (1) incorporates a single-variable delay, the formulation ψ-Hilfer FrOVI-DE in (2) represents a more general framework involving n-multiple-variable delays, thereby providing a broader perspective and enhancing and improving the existence of a unique solution and the Ulam-type stability results established for the ψ-Hilfer FrOVI-DE in (1). This constitutes the key novelty and original contribution of the present study. Furthermore, the existing literature predominantly focuses on the qualitative properties of ψ-Hilfer FrODEs, ψ-Hilfer FrODDEs, and ψ-Hilfer FrOVI-DEs, particularly concerning the UHR stability and UH stability for such classes of equations. However, there is a noticeable gap in research addressing the conditions under which solutions exist, as well as the concept of semi-UHR stability for these equations. In addition to addressing UHR stability and UH stability, the present study also aims to bridge this gap by examining both the existence of unique solutions and the semi-UHR stability for the general ψ-Hilfer FrOVI-DE in (2), thereby offering a novel and meaningful contribution to the field. Finally, it is evident that the ψ-Hilfer FrOVI-DE in (2), which incorporates multiple-variable delays, represents a significantly more general and distinct framework than existing models in the literature (see the aforementioned references and the relevant literature). This generalized formulation constitutes a novel contribution to the qualitative analysis of ψ-Hilfer FrOVI-DEs and enhances the current theoretical understanding of such types of equations.

The present article is structured in the following manner to facilitate a coherent exposition of the main results. In Section 2, we introduce key definitions, some lemmas with respect to the ψ-Hilfer-FrDO and integral, and a core result. In Sections 3 and 4, we establish three original theorems concerning the existence and uniqueness of solutions, as well as UHR stability, semi-UHR stability, and UH stability. These results are supplemented with the necessary definitions and analytical remarks related to the ψ-Hilfer FrOVI-DE incorporating multiple-variable time delays. Section 5 presents an example that illustrates the application of the main results of this article. Finally, Section 6 provides the concluding remarks, summarizing the main outcomes and presenting potential directions for prospective studies.

2. Background

In this section, we summarize the essential properties of the ψ-Riemann–Liouville fractional integral $I_{a^+}^{\alpha;\psi}(.)$ and the ψ-Hilfer fractional derivative ${}^{H}D_{a^+}^{\alpha,\beta;\psi}(.)$, and provide some definitions and auxiliary results pertinent to the subsequent analysis in this work.

Definition 1 ([11]). *Let (a,b), $(-\infty \leq a < b \leq \infty)$, be a finite interval (or infinite interval) of R, and let $\alpha > 0$. Also, let $\psi(x)$ be an increasing and positive monotone function on $(a,b]$ with the property that $\psi \in C^1(a,b)$ holds. The left-sided and right-sided Riemann–Liouville fractional integral of a function f in terms of a function ψ on the interval $[a,b]$ is described by*

$$I_{a^+}^{\alpha;\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (\psi(x) - \psi(s))^{\alpha-1} \psi'(s) f(s) ds$$

and

$$I_{b^-}^{\alpha;\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\psi(s) - \psi(x))^{\alpha-1} \psi'(s) f(s) ds,$$

respectively.

Definition 2 ([11]). *Let $\alpha \in (n-1, n)$, $n \in N$, $I = [a, b]$ be an interval with the property that $-\infty \leq a < b \leq \infty$ and f, $\psi \in C^n[a, b]$ be two functions, which ensure that ψ is an increasing function with the property that $\psi'(t) \neq 0$ for each $t \in I$, where $C^n[a, b]$ is the space of n-times continuously differentiable functions on $[a, b]$. Then, the left-sided ψ-Hilfer-FrD of the function f, i.e., ${}^H D_{a^+}^{\alpha,\beta;\psi} f(x)$, of order α and type $0 \leq \beta \leq 1$, is described by*

$${}^H D_{a^+}^{\alpha,\beta;\psi} f(x) = I_{a^+}^{\beta(n-\alpha);\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n I_{a^+}^{(1-\beta)(n-\alpha);\psi} f(x).$$

The ψ-fractional integral ${}^H D_{b^-}^{\alpha,\beta;\psi} f(x)$ is defined in a similar manner.

Lemma 1 ([11]). *Let $\alpha > 0$ and $\beta > 0$. Then, we have the semigroup property described by*

$$I_{a^+}^{\alpha;\psi} I_{a^+}^{\beta;\psi} f(x) = I_{a^+}^{\alpha+\beta;\psi} f(x).$$

Lemma 2 ([11]). *If $f \in C^n[a, b]$, $n - 1 < \alpha < n$, and $0 \leq \beta \leq 1$, then*

$$I_{a^+}^{\alpha;\psi} {}^H D_{a^+}^{\alpha,\beta;\psi} f(x) = f(x) - \sum_{k=1}^{n} \frac{(\psi(x) - \psi(a))^{\gamma-k}}{\Gamma(\gamma - k + 1)} f_{\psi}^{[n-k]} I_{a^+}^{(1-\beta)(n-\alpha);\psi} f(a),$$

with $\gamma = \alpha + \beta(n - \alpha)$.

Notably, if $f \in C^1[a, b]$, $0 < \alpha < 1$, and $0 \leq \beta \leq 1$, then

$$I_{a^+}^{\alpha;\psi} {}^H D_{a^+}^{\alpha,\beta;\psi} f(x) = f(x) - \frac{(\psi(x) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} I_{a^+}^{(1-\beta)(1-\alpha);\psi} f(a)$$

with $(1 - \alpha)(1 - \beta) = (1 - \gamma)$.

According to Lemma 2, we have the following notations ([28]):

$$f_{\psi^+}^{[n-k]} f(x) = \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^{n-k} f(x)$$

and

$$f_{\psi^-}^{[n-k]} f(x) = \left(-\frac{1}{\psi'(x)} \frac{d}{dx} \right)^{n-k} f(x).$$

Lemma 3 ([11]). *If $f \in C^1[a, b]$, $0 < \alpha < 1$, and $0 \leq \beta \leq 1$, then*

$${}^H D_{a^+}^{\alpha,\beta;\psi} I_{a^+}^{\alpha;\psi} f(x) = f(x).$$

Theorem 1 ([37]). *Let X be a nonempty set, (X, d) be a complete generalized metric space, and $T : X \to X$ be a strictly contractive operator with the Lipschitz constant $L_C < 1$. If there exists an $n \in Z$, $n \geq 0$, such that $d(T^{n+1}x, T^n x) < \infty$ for each $x \in X$, then the following properties hold:*

(C1) The sequence $\{T^n\}$ converges to a fixed point $x*$ of T;

(C2) $x*$ is the unique fixed point of T in $X^* = \{y \in X : d(T^n x, y) < \infty\}$;

(C3) If $y \in X^*$, then $d(y, x*) \leq (1 - L_C)^{-1} d(Ty, y)$.

3. Uniqueness of Solutions

In this section, we rigorously establish the uniqueness of solution to the ψ-Hilfer FrOVI-DE in (2) with n-time-varying delays, employing the BCMP.

Lemma 4. *Let X be a nonempty set and $\nu : J \to X$ be a function with continuous derivatives. Then, problem (2) is equivalent to the corresponding integral equation as follows:*

$$\begin{aligned}\nu(t) =& I_{0^+}^{\alpha;\psi}\left[\sum_{i=1}^{n} F_i\Big(t,\nu(t),\nu(t-\tau_i(t)),{}^{H}D_{0^+}^{u,\vartheta;\psi}\nu(t)\Big)\right]\\ &+ I_{0^+}^{\alpha;\psi}\left[\int_0^t\left(\sum_{i=1}^{n} K_i(t,s,\nu(s),\nu(s-\tau_i(s)))\right)ds\right],\ t\in(0,b].\end{aligned} \tag{3}$$

Proof. Upon applying the ψ-fractional integral operator $I_{0^+}^{\alpha;\psi}(.)$ to both sides of the ψ-Hilfer FrOVI-DE in (2) and making use of Lemma 2, we derive the relation as follows:

$$\begin{aligned}\nu(t) =& \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} I_{0^+}^{(1-\gamma);\psi}\nu(0)\\ &+ I_{0^+}^{\alpha;\psi}\left[\sum_{i=1}^{n} F_i\Big(t,\nu(t),\nu(t-\tau_i(t)),{}^{H}D_{0^+}^{u,\vartheta;\psi}\nu(t)\Big)\right]\\ &+ I_{0^+}^{\alpha;\psi}\left[\int_0^t\left(\sum_{i=1}^{n} K_i(t,s,\nu(s),\nu(s-\tau_i(s)))\right)ds\right],\ t\in(0,b],\end{aligned}$$

where $(1-\alpha)(1-\beta)=1-\gamma$. Using the initial value condition $I_{0^+}^{1-\gamma;\psi}\nu(0^+)$ from the above formula, we get

$$\begin{aligned}\nu(t) =& I_{0^+}^{\alpha;\psi}\left[\sum_{i=1}^{n} F_i\Big(t,\nu(t),\nu(t-\tau_i(t)),{}^{H}D_{0^+}^{u,\vartheta;\psi}\nu(t)\Big)\right]\\ &+ I_{0^+}^{\alpha;\psi}\left[\int_0^t\left(\sum_{i=1}^{n} K_i(t,s,\nu(s),\nu(s-\tau_i(s)))\right)ds\right],\ \text{for each } t\in(0,b].\end{aligned}$$

Conversely, if $\nu(t)$ is a solution of Equation (3), by applying the ψ-Hilfer fractional derivative operator ${}^{H}D_{0^+}^{\alpha,\beta;\psi}(.)$ to both sides of (3) and invoking Lemma 3, we obtain the following expression:

$$\begin{aligned}{}^{H}D_{0^+}^{\alpha,\beta;\psi}\nu(t) =& \sum_{i=1}^{n} F_i\Big(t,\nu(t),\nu(t-\tau_i(t)),{}^{H}D_{0^+}^{u,\vartheta;\psi}\nu(t)\Big)\\ &+ \int_0^t\left(\sum_{i=1}^{n} K_i(t,s,\nu(s),\nu(s-\tau_i(s)))\right)ds,\ \text{for each } t\in(0,b].\end{aligned}$$

Thus, in light of the preceding analysis, a function with continuous derivatives satisfies the initial value problem (2) if and only if it satisfies the corresponding integral equation (3). This concludes the proof of Lemma 4. □

Remark 1. *Let Y be a real Banach space. Throughout this study, inspired by the work of Zhou et al. [28] and the methodological distinctions among the approaches in [11,24,27,29,32,37], we construct a specialized functional framework denoted by*

$$X=\left\{\nu\,\middle|\,\nu(t)\in C^1(J,Y),\ {}^{H}D_{0^+}^{u,\vartheta;\psi}\nu(t)\in C^1(J,Y)\right\}$$

with the associated norm

$$\|\nu\|_X = \max\left\{\sup_{t\in J}\|\nu(t)\|, \sup_{t\in J}\ \|\nu(t-\tau_i(t))\|,\ i=1,...,n,\ \sup_{t\in J}\left\|{}^{H}D_{0^+}^{u,\vartheta;\psi}\nu(t)\right\|\right\},$$

where $(X,\ \|.\|_X)$ *is a Banach space.*

Theorem 2 below establishes the first principal result of this work concerning the uniqueness of the solution of the ψ-Hilfer FrOVI-DE in (2) with the present initial condition.

Theorem 2. *Let the conditions below be satisfied:*
(Hp-1) Let $L_{F_i}(.)$, $M_{F_i}(.)$, $N_{F_i}(.)$, $\bar{L}_{K_i}(.)$, *and* $\bar{M}_{K_i}(.)$ *be nonnegative continuous functions and* $F_i \in C^1(J\times X\times X\times X, X)$, $K_i \in C(J\times J\times X\times X, X)$, $i=1,2,\ldots,n$, *such that*

$$\left\|F_i(t,\nu,\ell,\rho)-F_i(t,\nu',\ell',\rho')\right\| \le L_{F_i}(t)\left\|\nu(t)-v'(t)\right\| + M_{F_i}(t)\left\|\ell(t)-\ell'(t)\right\|$$

$$+N_{F_i}(t)\left\|\rho(t)-\rho'(t)\right\|,\ \textit{for each } t\in J$$

and

$$\left\|K_i(t,s,\nu,\ell)-K_i(t,s,\nu',\ell')\right\| \le \bar{L}_{K_i}(t)\left\|\nu(t)-v'(t)\right\| + \bar{M}_{K_i}(t)\left\|\ell(t)-\ell'(t)\right\|,$$

for each $t,\ s\in J$.
(Hp-2) Let $L>0$, $L\in R$, *and the functions* $L_{F_i}(t)$, $M_{F_i}(t)$, $N_{F_i}(t)$, $\bar{L}_{K_i}(t)$, *and* $\bar{M}_{K_i}(t)$ *satisfy*

$$\frac{(\psi(b)-\psi(0))^{\eta}}{\Gamma(\eta+1)}\max_{t\in(0,b]}\sum_{i=1}^{n}\left[L_{F_i}(t)+M_{F_i}(t)+N_{F_i}(t)+b\left(\bar{L}_{K_i}(t)+\bar{M}_{K_i}(t)\right)\right] \le L,$$

for each $t\in J$, $L\in(0,1)$, *with* $\eta=\alpha$ *or* $\eta=\alpha-u$. *Then, the ψ-Hilfer FrOVI-DE in (2) with the present initial condition has a unique solution.*

Proof. According to Lemma 4, the initial value problem (2) is equivalent to the integral Equation (3). Therefore, we transform the problem of solving the ψ-Hilfer FrOVI-DE in (2) into establishing a fixed-point problem for the operator $T: X\to X$, defined as follows:

$$T\nu(t)=\begin{cases}\varphi(t),\ t\in[-\tau,0]\\ I_{0^+}^{\alpha;\psi}\left[\sum_{i=1}^{n}F_i\left(t,\nu(t),\nu(t-\tau_i(t)),{}^{H}D_{0^+}^{u,\vartheta;\psi}\nu(t)\right)\right]\\ +I_{0^+}^{\alpha;\psi}\left[\int_0^t\left(\sum_{i=1}^{n}K_i(t,s,\nu(s),\nu(s-\tau_i(s)))\right)ds\right],\ t\in(0,b].\end{cases}\tag{4}$$

The uniqueness of the solution to the ψ-Hilfer FrOVI-DE in (2) is established through the application of the BCMP. To establish our main result, we define an operator $T: X\to X$ such that $T(\nu)\in X$ for each $\nu\in X$. We also have $\nu(t)\in C^1(J,Y)$ and ${}^{H}D_{0^+}^{u,\vartheta;\psi}\nu(t)\in C^1(J,Y)$. According to the definitions of the ψ-Hilfer-FrDO and ψ-Riemann–Liouville fractional integral, it can be obtained that $T\nu(t)\in C^1(J,Y)$ and ${}^{H}D_{0^+}^{u,\vartheta;\psi}T\nu(t)\in C^1(J,Y)$. Hence, the operator $T: X\to X$. Subsequently, we prove that the operator T is strictly contractive for each $\nu,\ w\in T$. Using (Hp-1) and (Hp-2), and in view of (4), we deduce that

$$\|T\nu(t)-Tw(t)\|=0, t\in[-\tau,0]$$

and

$$\begin{aligned}\|T\nu(t)-Tw(t)\| \leq & \frac{1}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(\xi))^{\alpha-1}\psi'(\xi)\\ & \times\sum_{i=1}^{n}\Big\|F_i\Big(\xi,\nu(\xi),\nu_i(\xi-\tau(\xi)),{}^{H}D_{0^+}^{u,\vartheta;\psi}\nu(\xi)\Big)\\ & -F_i\Big(t,w(\xi),w_i(\xi-\tau(\xi)),{}^{H}D_{0^+}^{u,\vartheta;\psi}w(\xi)\Big)\Big\|d\xi\\ & +\frac{1}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(\xi))^{\alpha-1}\psi'(\xi)\end{aligned}$$

$$\begin{aligned}&\times\left[\int_0^t\left(\sum_{i=1}^{n}\|K_i(t,s,\nu(s),\nu(s-\tau_i(s)))-K_i(t,s,w(s),w(s-\tau_i(s)))\|\right)ds\right]d\xi\\ \leq&\frac{1}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(\xi))^{\alpha-1}\psi'(\xi)\\ &\times\sum_{i=1}^{n}\big[L_{F_i}(\xi)\|\nu(\xi)-w(\xi)\|\\ &+\sum_{i=1}^{n}M_{F_i}(\xi)\|\nu(\xi-\tau_i(\xi))-\omega(s-\tau_i(\xi))\|+N_{F_i}(\xi)\Big\|{}^{H}D_{0^+}^{u,\vartheta;\psi}\nu(\xi)-{}^{H}D_{0^+}^{u,\vartheta;\psi}w(\xi)\Big\|\big]d\xi\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(\xi))^{\alpha-1}\psi'(\xi)\\ &\times\left[\int_0^\xi\left(\sum_{i=1}^{n}\bar{L}_{K_i}(\xi)\|\nu(s)-w(s)\|+\bar{M}_{K_i}(s)\|\nu(s-\tau_i(s))-w(s-\tau_i(s))\|\right)ds\right]d\xi\\ \leq&\frac{\|\nu-w\|_X}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(\xi))^{\alpha-1}\psi'(\xi)\\ &\times\sum_{i=1}^{n}\left[L_{F_i}(\xi)+M_{F_i}(\xi)+N_{F_i}(\xi)+\int_0^\xi\big(\bar{L}_{K_i}(s)+\bar{M}_{K_i}(s)\big)ds\right]d\xi\\ \leq&\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\max_{t\in[a,b]}\sum_{i=1}^{n}\big[L_{F_i}(t)+M_{F_i}(t)+N_{F_i}(t)+b\big(\bar{L}_{K_i}(t)+\bar{M}_{K_i}(t)\big)\big]\|\nu-w\|_X\\ \leq&L\|\nu-w\|_X, L\in(0,1).\end{aligned}$$

It follows additionally that

$$\begin{aligned}\|{}^{H}D_{0^+}^{u,\vartheta;\psi}T\nu(t)-{}^{H}D_{0^+}^{u,\vartheta;\psi}Tw(t)\| \leq & \frac{\|\nu-w\|_X}{\Gamma(\alpha-u)}\int_0^t(\psi(t)-\psi(\xi))^{\alpha-u-1}\psi'(\xi)\\ & \times\sum_{i=1}^{n}\left[L_{F_i}(\xi)+M_{F_i}(\xi)+N_{F_i}(\xi)+\int_0^\xi\big(\bar{L}_{K_i}(s)+\bar{M}_{K_i}(s)\big)ds\right]d\xi\\ \leq&L\|\nu-w\|_X, L\in(0,1).\end{aligned}$$

Thus, for each $\nu,\ w \in X$, we conclude that

$$\|T\nu(t) - Tw(t)\|_X \leq L\|\nu - w\|_X, \text{ for each } t \in J, L \in (0,1).$$

This result implies that T is a contraction mapping. Thus, T has a fixed point ν in the Banach space X, which leads to $T\nu = \nu$. Therefore, problem (2) has a unique solution. Accordingly, this concludes the proof of Theorem 2.

□

4. Ulam-Type Stability

In this subsection, we present the concepts related to UHR stability, UH stability, and semi-UHR stability on a finite interval. Additionally, we establish the relevant stability theorems for the ψ-Hilfer FrOVI-DE in (2).

A rigorous formulation of the stability results requires establishing adequate metrics on the Banach space X. The metric $d_1(.)$ on the Banach space X is defined by

$$d_1(x,y) = \inf\{ C \in [0,\infty) |\ \|\nu(t) - w(t)\| \leq C\Phi(t),\ \|\nu(t-\tau_i(t)) - w(t-\tau_i(t))\| \leq C\Phi(t),$$

$$\left\| {}^{H}D_{0^+}^{u,\vartheta;\psi} T\nu(t) - {}^{H}D_{0^+}^{u,\vartheta;\psi} Tw(t) \right\| \leq C\Phi(t),\ i = 1,...,n,\ t \in [0,b] \Big\},$$

where C is a constant and $\Phi(t)$ is a positive, non-decreasing continuous function.

The metric $d_2(.)$ on the Banach space X is defined by

$$d_2(x,y) = \sup\Bigg\{ C \in [0,\infty) |\ \frac{\|\nu(t) - w(t)\|}{\Phi(t)} \leq C,\ \frac{\|\nu(t-\tau_i(t)) - w(t-\tau_i(t))\|}{\Phi(t)} \leq C,$$

$$\frac{\left\| {}^{H}D_{0^+}^{u,\vartheta;\psi} T\nu(t) - {}^{H}D_{0^+}^{u,\vartheta;\psi} Tw(t) \right\|}{\Phi(t)} \leq C,\ i = 1,...,n,\ t \in [0,b] \Bigg\},$$

where $\Phi(t)$ is a positive, nonincreasing continuous function on $[0,b]$. As discussed in [38] and related works, we can prove that $d_1(.)$ and $d_2(.)$ are metrics on the Banach space X.

In the subsequent result, we give the second result regarding the UHR stability of the ψ-Hilfer FrOVI-DE in (2) that incorporates n-multiple-variable time delays.

Definition 3 (UHR stability). *For each continuously differentiable function $\nu : J \to X$ satisfying*

$$\Bigg\| \nu(t) - I_{0^+}^{\alpha;\psi} \left[\sum_{i=1}^{n} F_i\Big(t, \nu(t), \nu_i(t-\tau(t)), {}^{H}D_{0^+}^{u,\vartheta;\psi}\nu(t)\Big) \right]$$

$$- I_{0^+}^{\alpha;\psi} \left[\int_0^t \left(\sum_{i=1}^{n} K_i(t,s,\nu(s),\nu(s-\tau_i(s))) \right) ds \right] \Bigg\| \leq I_{0^+}^{\alpha;\psi}\Phi(t)$$

and

$$\Bigg\| {}^{H}D_{0^+}^{u,\vartheta;\psi}\nu(t) - I_{0^+}^{\alpha-u;\psi} \left[\sum_{i=1}^{n} F_i\Big(t, \nu(t), \nu_i(t-\tau(t)), {}^{H}D_{0^+}^{u,\vartheta;\psi}\nu(t)\Big) \right]$$

$$- I_{0^+}^{\alpha-u;\psi} \left[\int_0^t \left(\sum_{i=1}^{n} K_i(t,s,\nu(s),\nu(s-\tau_i(s))) \right) ds \right] \Bigg\| \leq I_{0^+}^{\alpha-u;\psi}\Phi(t),\ \forall t \in J,$$

where $\Phi(t)$ is a positive, nonincreasing continuous function on a finite interval $[0, b]$, there exists a solution v_0 of the ψ-Hilfer FrOVI-DE in (2) and a constant $C > 0$, which is independent of v and v_0, such that

$$\|v(t) - v_0(t)\| \leq C\Phi(t)$$

and

$$\left\| {}^{H}D_{0^+}^{u,\vartheta;\psi} v(t) - {}^{H}D_{0^+}^{u,\vartheta;\psi} v_0(t) \right\| \leq C\Phi(t), \forall t \in J.$$

Remark 2. *Under these circumstances, we can say that the ψ-Hilfer FrOVI-DE in (2) possesses UHR stability on $(0, b]$.*

Remark 3 (UH stability)**.** *By replacing the nonnegative continuous function Φ with $\theta \geq 0$, $\theta \in R$ in Definition 3, the ψ-Hilfer FrOVI-DE possesses UH stability on $(0, b]$.*

Definition 4 (Semi-UHR stability)**.** *For each continuously differentiable function $v : J \to X$ satisfying*

$$\left\| v(t) - I_{0^+}^{\alpha;\psi} \left[\sum_{i=1}^{n} F_i \left(t, v(t), v_i(t - \tau(t)), {}^{H}D_{0^+}^{u,\vartheta;\psi} v(t) \right) \right] \right.$$
$$\left. - I_{0^+}^{\alpha;\psi} \left[\int_0^t \left(\sum_{i=1}^{n} K_i(t, s, v(s), v(s - \tau_i(s))) \right) ds \right] \right\| \leq I_{0^+}^{\alpha;\psi} \theta$$

and

$$\left\| {}^{H}D_{0^+}^{u,\vartheta;\psi} v(t) - I_{0^+}^{\alpha-u;\psi} \left[\sum_{i=1}^{n} F_i \left(t, v(t), v_i(t - \tau(t)), {}^{H}D_{0^+}^{u,\vartheta;\psi} v(t) \right) \right] \right.$$
$$\left. - I_{0^+}^{\alpha-u;\psi} \left[\int_0^t \left(\sum_{i=1}^{n} K_i(t, s, v(s), v(s - \tau_i(s))) \right) ds \right] \right\| \leq I_{0^+}^{\alpha-u;\psi} \theta, \ \forall t \in J,$$

where $\theta \geq 0$, $\theta \in R$, there exists a solution v_0 of the ψ-Hilfer FrOVI-DE in (2) and a constant $C > 0$, which is independent of v, v_0, such that

$$\|v(t) - v_0(t)\| \leq C\Phi(t)$$

and

$$\left\| {}^{H}D_{0^+}^{u,\vartheta;\psi} v(t) - {}^{H}D_{0^+}^{u,\vartheta;\psi} v_0(t) \right\| \leq C\Phi(t), \forall t \in J.$$

Remark 4. *Under these circumstances, we can say that the ψ-Hilfer FrOVI-DE in (2) possesses semi-UHR stability on $(0, b]$.*

We now introduce the theorems and associated proofs related to the UHR stability, UH stability, and semi-UHR stability of the ψ-Hilfer FrOVI-DE in (2) on the finite interval $[0, b]$.

Theorem 3. *Let (Hp-1) and (Hp-2) of Theorem 2 hold and $\Phi(t)$ be a positive, non-decreasing continuous function. Additionally, for each continuously differentiable function $v : J \to Y$, the following conditions hold:*

$$\left\| v(t) - I_{0^+}^{\alpha;\psi} \left[\sum_{i=1}^{n} F_i \left(t, v(t), v_i(t - \tau(t)), {}^{H}D_{0^+}^{u,\vartheta;\psi} v(t) \right) \right] \right.$$
$$\left. - I_{0^+}^{\alpha;\psi} \left[\int_0^t \left(\sum_{i=1}^{n} K_i(t, s, v(s), v(s - \tau_i(s))) \right) ds \right] \right\| \leq I_{0^+}^{\alpha;\psi} \Phi(t) \tag{5}$$

and

$$\left\| {}^{H}D_{0^+}^{u,\vartheta;\psi}\nu(t) - I_{0^+}^{\alpha-u;\psi}\left[\sum_{i=1}^{n} F_i\Big(t,\nu(t),\nu_i(t-\tau(t)),{}^{H}D_{0^+}^{u,\vartheta;\psi}\nu(t)\Big)\right] - I_{0^+}^{\alpha-u;\psi}\left[\int_0^t\left(\sum_{i=1}^{n} K_i(t,s,\nu(s),\nu(s-\tau_i(s)))\right)ds\right]\right\| \leq I_{0^+}^{\alpha-u;\psi}\Phi(t),\ \forall t\in J. \tag{6}$$

Consequently, a unique solution $\nu_0 \in X$ exists with the properties that

$$\|\nu(t)-\nu_0(t)\| \leq \frac{1}{1-L}\min\left\{\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)},\frac{(\psi(b)-\psi(a))^{\alpha-u}}{\Gamma(\alpha-u+1)}\right\}\Phi(t)$$

and

$$\left\|{}^{H}D_{0^+}^{u,\vartheta;\psi}\nu(t) - {}^{H}D_{0^+}^{u,\vartheta;\psi}\nu_0(t)\right\| \leq \frac{1}{1-L}\min\left\{\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)},\ \frac{(\psi(b)-\psi(a))^{\alpha-u}}{\Gamma(\alpha-u+1)}\right\}\Phi(t), \forall t\in J, L\in(0,1).$$

Remark 5. *The result of this theorem means that the ψ-Hilfer FrOVI-DE in (2) possesses UHR stability on $(0,b]$.*

Proof. Define an operator $T: X\to X$ by

$$T\nu(t) = \begin{cases} \phi(t),\ t\in[-\tau,0] \\ I_{0^+}^{\alpha;\psi}\left[\sum\limits_{i=1}^{n} F_i\Big(t,\nu(t),\nu(t-\tau_i(t)),{}^{H}D_{0^+}^{u,\vartheta;\psi}\nu(t)\Big)\right] \\ +I_{0^+}^{\alpha;\psi}\left[\int\limits_0^t\left(\sum\limits_{i=1}^{n} K_i(t,s,\nu(s),\nu(s-\tau_i(s)))\right)ds\right],\ \forall t\in J. \end{cases} \tag{7}$$

We can derive from the metric $d_1(.)$, the conditions (Hp-1) and (Hp-2) on the Banach space X, and (7) that

$$\|T\nu(t)-Tw(t)\| = 0, -\tau\leq t\leq 0,$$

and

$$\begin{aligned}
\|T\nu(t)-Tw(t)\| \leq{}& \frac{1}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(\xi))^{\alpha-1}\psi'(\xi)\times\sum_{i=1}^{n}\Big[L_{F_i}(\xi)\|\nu(\xi)-w(\xi)\| \\
&+\sum_{i=1}^{n}M_{F_i}(\xi)\|\nu(\xi-\tau_i(\xi))-\omega(s-\tau_i(\xi))\|+N_{F_i}(\xi)\left\|{}^{H}D_{0^+}^{u,\vartheta;\psi}\nu(\xi)-{}^{H}D_{0^+}^{u,\vartheta;\psi}w(\xi)\right\|\Big]d\xi \\
&+\frac{C}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(\xi))^{\alpha-1}\psi'(\xi) \\
&\times\left[\int_0^{\xi}\left(\sum_{i=1}^{n}\bar{L}_{K_i}(s)\|\nu(s)-w(s)\|+\bar{M}_{K_i}(s)\|\nu(s-\tau_i(s))-w(s-\tau_i(s))\|\right)ds\right]d\xi \\
\leq{}& \frac{C}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(\xi))^{\alpha-1}\psi'(\xi)\sum_{i=1}^{n}\big[L_{F_i}(\xi)\Phi(\xi)+M_{F_i}(\xi)\Phi(\xi-\tau_i(\xi))+N_{F_i}(\xi)\Phi(\xi)\big]d\xi \\
&+\frac{C}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(\xi))^{\alpha-1}\psi'(\xi)\sum_{i=1}^{n}\left[\int_0^{\xi}\big(\bar{L}_{K_i}(s)\Phi(s)+\bar{M}_{K_i}(s)\Phi(s-\tau_i(s))\big)ds\right]d\xi
\end{aligned}$$

$$\leq \frac{C\Phi(t)}{\Gamma(\alpha)} \int_0^t (\psi(t)-\psi(\xi))^{\alpha-1}\psi'(\xi) \sum_{i=1}^{n} \left[L_{F_i}(\xi)+M_{F_i}(\xi)+N_{F_i}(\xi)\right] d\xi$$

$$+ \frac{C\Phi(t)}{\Gamma(\alpha)} \int_0^t (\psi(t)-\psi(\xi))^{\alpha-1}\psi'(\xi) \sum_{i=1}^{n} \left[\int_0^{\xi} \left(\bar{L}_{K_i}(s)+\bar{M}_{K_i}(s)\right) ds\right] d\xi$$

$$\leq CL\Phi(t), L \in (0,1), \text{ for each } \nu,\ w \in X.$$

Likewise, we derive

$$\left\| {}^{H}D_{0^+}^{u,\vartheta;\psi} T\nu(t) - {}^{H}D_{0^+}^{u,\vartheta;\psi} Tw(t) \right\|$$

$$\leq \frac{C\Phi(t)}{\Gamma(\alpha-u)} \int_0^t (\psi(t)-\psi(\xi))^{\alpha-u-1}\psi'(\xi)$$

$$\times \sum_{i=1}^{n} \left[L_{F_i}(\xi)+M_{F_i}(\xi)+N_{F_i}(\xi)+\int_0^{\xi} \left(\bar{L}_{K_i}(s)+\bar{M}_{K_i}(s)\right) ds\right] d\xi$$

$$\leq CL\Phi(t), L \in (0,1).$$

Accordingly, for each $\nu,\ w \in X$, we conclude that

$$d_1(T\nu, Tw) \leq CL = Ld_1(\nu, w), L \in (0,1).$$

Consequently, using (5) and (6), we deduce that

$$\|\nu(t) - T\nu(t)\| \leq I_{0^+}^{\alpha;\psi}\Phi(t) \leq \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\Phi(t)$$

and

$$\|{}^{H}D_{0^+}^{u,\vartheta;\psi}\nu(t) - {}^{H}D_{0^+}^{u,\vartheta;\psi} T\nu(t)\| \leq I_{0^+}^{\alpha-u;\psi}\Phi(t)$$

$$\leq \frac{(\psi(b)-\psi(a))^{\alpha-u}}{\Gamma(\alpha-u+1)}\Phi(t), \text{ for each } t \in J.$$

Accordingly, we determine that

$$d_1(\nu, T\nu) \leq \min\left\{\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}, \frac{(\psi(b)-\psi(a))^{\alpha-u}}{\Gamma(\alpha-u+1)}\right\} < \infty.$$

Using (C1) and (C2) of Theorem 1, there exists a unique fixed point ν_0 with the property that $T\nu_0 = \nu_0$. Subsequently, from (C3) of Theorem 1, we conclude that

$$d_1(\nu, \nu_0) \leq \frac{1}{1-L} d_1(T\nu, \nu) \leq \frac{1}{1-L} \min\left\{\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}, \frac{(\psi(b)-\psi(a))^{\alpha-u}}{\Gamma(\alpha-u+1)}\right\},\ L \in (0,1).$$

From the above analysis, we deduce that the ψ-Hilfer FrOVI-DE in (2) exhibits UHR stability, thereby completing the proof of the theorem. □

Remark 6. *Let $\Phi(t) = 1$ in Theorem 3. Then, the ψ-Hilfer FrOVI-DE in (2) possesses UH stability on $(0, b]$.*

Theorem 4. *Let (Hp-1) and (Hp-2) of Theorem 2 hold and $\Phi(t)$ be a positive, nonincreasing continuous function. Furthermore, let $v : J \to X$ be a continuously differentiable function satisfying*

$$\left\| v(t) - I_{0^+}^{\alpha;\psi}\left[\sum_{i=1}^{n} F_i\Big(t, v(t), v_i(t-\tau(t)), {}^{H}D_{0^+}^{u,\vartheta;\psi} v(t)\Big)\right] - I_{0^+}^{\alpha;\psi}\left[\int_0^t \left(\sum_{i=1}^{n} K_i(t,s,v(s),v(s-\tau_i(s)))\right) ds\right] \right\| \leq I_{0^+}^{\alpha;\psi}\theta \tag{8}$$

and

$$\left\| {}^{H}D_{0^+}^{u,\vartheta;\psi} v(t) - I_{0^+}^{\alpha-u;\psi}\left[\sum_{i=1}^{n} F_i\Big(t, v(t), v_i(t-\tau(t)), {}^{H}D_{0^+}^{u,\vartheta;\psi} v(t)\Big)\right] - I_{0^+}^{\alpha-u;\psi}\left[\int_0^t \left(\sum_{i=1}^{n} K_i(t,s,v(s),v(s-\tau_i(s)))\right) ds\right] \right\| \leq I_{0^+}^{\alpha-u;\psi}\theta,\ \forall t \in J, \tag{9}$$

where $\theta > 0$, $\theta \in R$. Then, there exists a unique solution $v_0 \in X$ and an $M \in R$, $M > 0$, with the property that

$$\|v(t) - v_0(t)\| \leq \max\left\{\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}, \frac{(\psi(b)-\psi(a))^{\alpha-u}}{\Gamma(\alpha-u+1)}\right\}\frac{\theta M}{1-L}\Phi(t)$$

and

$$\left\| {}^{H}D_{0^+}^{u,\vartheta;\psi} v(t) - {}^{H}D_{0^+}^{u,\vartheta;\psi} v_0(t)\right\| \leq \max\left\{\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}, \frac{(\psi(b)-\psi(a))^{\alpha-u}}{\Gamma(\alpha-u+1)}\right\}\frac{\theta M}{1-L}\Phi(t), \forall t \in J, L \in (0,1).$$

Remark 7. *The result of Theorem 4 implies that the ψ-Hilfer FrOVI-DE in (2) possesses semi-UH stability on $(0, b]$.*

Proof. Define an operator $T : X \to X$ by

$$Tv(t) = \begin{cases} \phi(t),\ t \in [-\tau, 0] \\ I_{0^+}^{\alpha;\psi}\left[\sum_{i=1}^{n} F_i\Big(t, v(t), v(t-\tau_i(t)), {}^{H}D_{0^+}^{\alpha,\beta;\psi} v(t)\Big)\right] \\ +I_{0^+}^{\alpha;\psi}\left[\int_0^t \left(\sum_{i=1}^{n} K_i(t,s,v(s),v(s-\tau_i(s)))\right) ds\right],\ \forall t \in J. \end{cases} \tag{10}$$

Similar to the proofs of Theorems 2 and 3, using the conditions of Theorem 4 and the metric $d_2(.)$ on the Banach space X, we deduce from (10) that

$$\|Tv(t) - T\omega(t)\| = 0, -\tau \leq t \leq 0$$

and

$$\frac{\|Tv(t) - Tw(t)\|}{\Phi(t)} \leq \frac{1}{\Gamma(\alpha)}\int_0^t (\psi(t)-\psi(\xi))^{\alpha-1}\psi'(\xi) \times \sum_{i=1}^{n}\left[L_{F_i}(\xi)\frac{\|v(\xi)-w(\xi)\|}{\Phi(\xi)} + \sum_{i=1}^{n} M_{F_i}(\xi)\frac{\|v(\xi-\tau_i(\xi)) - \omega(s-\tau_i(\xi))\|}{\Phi(\xi)} + N_{F_i}(\xi)\frac{\left\|{}^{H}D_{0^+}^{\alpha,\beta;\psi} v(\xi) - {}^{H}D_{0^+}^{\alpha,\beta;\psi} w(\xi)\right\|}{\Phi(\xi)}\right] d\xi$$

$$+\frac{C}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(\xi))^{\alpha-1}\psi'(\xi)$$

$$\times\left[\int_0^\xi\left(\sum_{i=1}^n\bar{L}_{K_i}(s)\frac{\|\nu(s)-w(s)\|}{\Phi(s)}+\bar{M}_{K_i}(s)\frac{\|\nu(s-\tau_i(s))-w(s-\tau_i(s))\|}{\Phi(s)}\right)ds\right]d\xi$$

$$\leq\frac{C}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(\xi))^{\alpha-1}\psi'(\xi)\sum_{i=1}^n\left[L_{F_i}(\xi)+M_{F_i}(\xi)+N_{F_i}(\xi)\right]d\xi$$

$$+\frac{C}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(\xi))^{\alpha-1}\psi'(\xi)\sum_{i=1}^n\left[\int_0^\xi\left(\bar{L}_{K_i}(s)+\bar{M}_{K_i}(s)\right)ds\right]d\xi$$

$$\leq CL, L\in(0,1).$$

Likewise, we obtain

$$\frac{\left\|{}^HD_{0^+}^{u,\vartheta;\psi}T\nu(t)-{}^HD_{0^+}^{u,\vartheta;\psi}Tw(t)\right\|}{\Phi(t)}$$

$$\leq\frac{C}{\Gamma(\alpha-u)}\int_0^t(\psi(t)-\psi(\xi))^{\alpha-u-1}\psi'(\xi)$$

$$\times\sum_{i=1}^n\left[L_{F_i}(\xi)+M_{F_i}(\xi)+N_{F_i}(\xi)+\int_0^\xi\left(\bar{L}_{K_i}(s)+\bar{M}_{K_i}(s)\right)ds\right]d\xi$$

$$\leq CL, L\in(0,1).$$

Since $\Phi(t)$ is a positive, nonincreasing continuous function, for each $t\in J$, we have

$$\frac{1}{\Phi(t)}\leq M,\ \text{for each } t\in[0,b],$$

where $M\in R,\ M>0$, is a positive constant. Subsequently, from (8) and (9), we infer that

$$\frac{\|\nu(t)-T\nu(t)\|}{\Phi(t)}\leq\frac{I_{0^+}^{\alpha;\psi}\theta}{\Phi(t)}\leq\frac{\theta(\psi(b)-\psi(a))^\alpha}{\Phi(t)\Gamma(\alpha+1)}\leq\frac{\theta(\psi(b)-\psi(a))^\alpha M}{\Gamma(\alpha+1)}\tag{11}$$

and

$$\frac{\left\|{}^HD_{0^+}^{u,\vartheta;\psi}\nu(t)-{}^HD_{0^+}^{u,\vartheta;\psi}T\nu(t)\right\|}{\Phi(t)}\leq\frac{I_{0^+}^{\alpha-u;\psi}\theta}{\Phi(t)}\leq\frac{\theta(\psi(b)-\psi(a))^{\alpha-u}}{\Phi(t)\Gamma(\alpha-u+1)}\leq\frac{\theta(\psi(b)-\psi(a))^{\alpha-u}M}{\Gamma(\alpha-u+1)}.\tag{12}$$

Considering (11) and (12), we also get

$$d_2(\nu,T\nu)\leq\max\left\{\frac{\theta M(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)},\frac{\theta M(\psi(b)-\psi(a))^{\alpha-u}}{\Gamma(\alpha-u+1)}\right\}.$$

Using (C1) and (C2) of Theorem 1, there exists a unique fixed point ν_0 with the property that $T\nu_0=\nu_0$. Then from (C1) of Theorem 1, we deduce that

$$d_2(\nu,\nu_0)\leq\frac{1}{1-L}d_2(\nu(t),T\nu(t))\leq\frac{1}{1-L}\max\left\{\frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)},\frac{(\psi(b)-\psi(a))^{\alpha-u}}{\Gamma(\alpha-u+1)}\right\}\frac{\theta M}{1-L},$$

$$L\in(0,1).$$

Thus, the proof of the theorem is complete. □

Remark 8. *From the above analysis, we deduce that the ψ-Hilfer FrOVI-DE in (2) possesses semi-UH stability on* $(0, b]$.

5. Application

In this section, we present a specific example of the ψ-Hilfer FrOVI-DE in (2) to substantiate the validity of the conditions stipulated in Theorems 2 and 3.

Example 1. *We consider the following ψ-Hilfer FrOVI-DE incorporating a variable delay:*

$$\begin{cases} {}^{H}D_{0^+}^{\frac{1}{4},\frac{1}{4};t^2}\nu(t) = \frac{1}{300(1+t^4)}\arctan(\nu(t)) + \frac{\nu(t)}{450(1+t^2)\left(1+\nu^2\left(2^{-1}t\right)\right)} \\ +\frac{1}{600}\int_0^t \frac{\sin(\nu(s))+\sin\left(\nu\left(2^{-1}s\right)\right)}{1+s^4+t^4}ds,\ t \in I = (0,1], \\ I_{0^+}^{1-\gamma}\nu(0) = \sigma, \\ \nu(t) = 0,\ t \in \left[-2^{-1}, 0\right]. \end{cases} \tag{13}$$

According to the ψ-Hilfer FrOVI-DE in (13), we have $\alpha = \beta = \frac{1}{4}$, $\psi(t) = t^2$, *and* $\tau_1(t) = \frac{t}{2}$. *Moreover,* $\gamma = \alpha + \beta(1-\alpha) = \frac{7}{16}$, $\tau = \max\left\{2^{-1}t\right\} = 2^{-1}$. *By comparing the ψ-Hilfer FrOVI-DE (13) and the ψ-Hilfer FrOVI-DE in (2), the corresponding relations can be formulated as follows:*

$$J = (0,1],\ a = 0,\ b = 1,$$

$$F_1\left(t, \nu, \nu(t-\tau_1(t)), {}^{H}D_{0^+}^{u,\vartheta;\psi}\nu(t)\right) = \frac{1}{300(1+t^4)}\arctan(\nu) + \frac{\nu}{450(1+t^2)(1+\nu^2(2^{-1}t))},$$

$$K_1(t,s,\nu(s),\nu(s-\tau_1(s))) = \frac{\sin(\nu(s))+\sin\left(\nu\left(2^{-1}s\right)\right)}{600(1+s^4+t^4)},$$

$$F_1 \in C((0,1]\times R\times R\times R, R),$$

$$K_1 \in C((0,1]\times(0,1]\times R\times R, R),$$

$$\begin{aligned} \left\|F_1(t,\nu,\ell,\rho) - F_1\left(t,\nu',\ell',\rho'\right)\right\| &\le \frac{1}{300(1+t^4)}\left|\arctan(\nu)-\arctan\left(\nu'\right)\right| \\ &+\frac{1}{450(1+t^2)}\left|\frac{\ell}{1+\ell^2(2^{-1}t)} - \frac{\ell'}{1+\ell'^2(2^{-1}t)}\right| \\ &\le \frac{1}{300(1+t^4)}\left|\nu-\nu'\right| + \frac{1}{450(1+t^2)}\left|\ell-\ell'\right|, \end{aligned}$$

where

$$L_{F_1}(t) = \frac{1}{300(1+t^4)}, M_{F_1}(t) = \frac{1}{450(1+t^2)}, N_{F_1}(t) = 0,$$

$$\begin{aligned} \left\|K_1(t,s,\nu,\ell) - K_1\left(t,s,\nu',\ell'\right)\right\| &\le \frac{1}{600(1+s^4+t^4)}\left|\sin(\nu(s)) - \sin\left(\nu'(s)\right)\right| \\ &+\frac{1}{600(1+s^4+t^4)}\left|\sin\left(\ell\left(2^{-1}s\right)\right) - \sin\left(\ell'\left(2^{-1}s\right)\right)\right| \\ &\le \frac{1}{600(1+s^4)}\left|\sin(\nu(s)) - \sin\left(\nu'(s)\right)\right| \\ &+\frac{1}{600(1+s^4)}\left|\sin\left(\ell\left(2^{-1}s\right)\right) - \sin\left(\ell'\left(2^{-1}s\right)\right)\right| \end{aligned}$$

$$\leq \frac{1}{600(1+s^4)}\left|\nu(s)-\nu'(s)\right| + \frac{1}{600(1+s^4)}\left|\ell\left(2^{-1}s\right)-\ell'\left(2^{-1}s\right)\right|,$$

where

$$\bar{L}_{K_1}(s) = \frac{1}{600(1+s^4)}, \bar{M}_{K_1}(s) = \frac{1}{600(1+s^4)}.$$

In light of the above computations, the conditions (Hp-1) and (Hp-2) stipulated in Theorem 2 are satisfied. Furthermore, the inequality stated in Theorem 2 is likewise satisfied, as shown below:

$$\frac{(\psi(b)-\psi(0))^{\eta}}{\Gamma(\eta+1)}\max_{t\in[a,b]}\sum_{i=1}^{n}\left[L_{F_i}(t)+M_{F_i}(t)+N_{F_i}(t)+b\left(\bar{L}_{K_i}(t)+\bar{M}_{K_i}(t)\right)\right]$$

$$= \frac{(\psi(1)-\psi(0))^{\frac{1}{4}}}{\Gamma\left(\frac{5}{4}\right)}\max_{t\in[0,1]}\left[L_{F_1}(t)+M_{F_1}(t)+N_{F_1}(t)+b\left(\bar{L}_{K_1}(t)+\bar{M}_{K_1}(t)\right)\right]$$

$$\cong \frac{10000}{9064}\max_{t\in[0,1]}\left[\frac{1}{300(1+t^4)}+\frac{1}{450(1+t^2)}+\frac{1}{300(1+t^4)}\right]$$

$$\leq \frac{10000}{9064}\times\frac{4}{450} = \frac{100}{10197} = L < 1.$$

Then, the ψ-Hilfer FrOVI-DE in (13) with the present initial condition has a unique solution. Moreover, let the continuously differentiable function $\nu : J \to X$ satisfy the following inequalities:

$$\left\|\nu(t) - I_{0^+}^{\frac{1}{4};t^2}\left[\frac{1}{300(1+t^4)}\arctan(\nu(t)) + \frac{\nu(t)}{450(1+t^2)(1+\nu^2(2^{-1}t))}\right]\right.$$

$$\left. - I_{0^+}^{\frac{1}{4};t^2}\left[\int_0^t\left(\frac{\sin(\nu(s))+\sin\left(\nu\left(2^{-1}s\right)\right)}{600(1+s^4+t^4)}\right)ds\right]\right\| \leq I_{0^+}^{\frac{1}{4};t^2}\Phi(t)$$

and

$$\left\|{}^{H}D_{0^+}^{u,\vartheta;\psi}\nu(t) - I_{0^+}^{\frac{1}{4}-u;t^2}\left[\frac{1}{300(1+t^4)}\arctan(\nu(t)) + \frac{\nu(t)}{450(1+t^2)(1+\nu^2(2^{-1}t))}\right]\right.$$

$$\left. - I_{0^+}^{\frac{1}{4}-u;\psi}\left[\int_0^t\left(\frac{\sin(\nu(s))+\sin\left(\nu\left(2^{-1}s\right)\right)}{600(1+s^4+t^4)}\right)ds\right]\right\| \leq I_{0^+}^{\frac{1}{4}-u;t^2}\Phi(t),\ \forall t\in(0,1],$$

where $\Phi(t)$ is a positive, nonincreasing continuous function on a finite interval $(0,1]$. Thus, it is concluded from the preceding results that the ψ-Hilfer FrOVI-DE in (13) possesses UHR stability. In a similar manner, one can verify that the conditions of Theorem 4 and the UH stability result stated in Remark 6 are also satisfied. For conciseness, the detailed verification is omitted.

6. Conclusions

In this work, we analyze the conditions under which unique solutions exist, as well as the UHR stability, UH stability, and semi-UH stability of a nonlinear ψ-Hilfer FrOVI-DE incorporating multiple-variable time delays with the initial condition. In order to establish the results presented in this study, we apply the BCMP with respect to a complete generalized metric, an appropriate norm, and other advanced techniques. In this study, we present three novel and original results pertaining to the qualitative concepts discussed. The results presented in this study are expected to make a substantial contribution to the advancement of future theoretical research and applications within the field of applied sciences. In future research endeavors, we plan to investigate the same qualitative properties of neutral ψ-Hilfer FrOVI-DEs and ψ-Hilfer FrOVI-DEs, including single- or multiple-variable advanced arguments, as open problems.

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Abbreviations

The abbreviations employed in this article are as follows:

ψ-Hilfer FrODE	ψ-Hilfer fractional-order differential equation
ψ-Hilfer FrODEs	ψ-Hilfer fractional-order differential equations
ψ-Hilfer FrOVI-DE	ψ-Hilfer fractional-order Volterra integro-differential equation
ψ-Hilfer FrOVI-DEs	ψ-Hilfer fractional-order Volterra integro-differential equations
ψ-Hilfer-FrDO	ψ-Hilfer fractional derivative operator
UH stability	Ulam–Hyers stability
UHR stability	Ulam–Hyers–Rassias stability
semi-UHR stability	semi-Ulam–Hyers–Rassias stability
BCMP	Banach contraction mapping principle

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