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A Result Regarding the Existence and Attractivity for a Class of Nonlinear Fractional Difference Equations with Time-Varying Delays

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Abstract: In this paper, we are studying a class of nonlinear fractional difference equations with time-varying delays in Banach space. By means of mathematical induction and the Picard iteration method, we first obtain the existence result of this fractional difference system. Under some new criteria along with the Schauder's fixed point theorem, we then derive the attractivity conclusions. Subsequently, with the aid of Grönwall's inequality, we prove that the system is globally attractive. Finally, we give two examples to prove the validity of our theorems.

Keywords: fractional difference equations; existence; attractivity; time-varying delays

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1. Introduction

Over the past few decades, fractional calculus has gained increasing attention from researchers worldwide. Fractional difference equations (FDEs) have found extensive applications across various fields, including physics, dynamics, medicine, and communication engineering. Current research trends indicate a growing interest in fractional difference equations, with significant advancements being made in this area. For a comprehensive discussion on this subject, refer to this monograph [1] and these related studies [2–6].

Analyzing solution existence in FDEs remains a highly relevant and compelling subject in mathematical research. Recently, many researchers have obtained the existence and unique results [7,8] by using the Banach contraction principle and fixed point theorem [9–11], and other technologies.

In recent years, fractional calculus has received a lot of attention because of its unique advantages in describing systems with memory and heredity. Compared with integral calculus, fractional calculus can more accurately describe the dynamic behavior of complex systems, so it has been widely used in physics, biology, economics, and engineering. As an important branch of fractional calculus, FDEs play an important role in the modeling of difference systems, especially in dealing with non-locality and long-range dependence problems.

The existence and attraction of the solution of an FDE are the core problems in theoretical research. The existence of the solution guarantees the rationality of the model, while the attraction reveals the stability of the long-term behavior of the system. In recent years, many scholars have conducted in-depth research on the solution of FDEs and achieved a series of important results. For example, studies have researched the existence of solutions to linear fractional difference equations, and explored the attractability conditions

for nonlinear FDEs. However, for more general FDEs, especially those with nonlinear or delay terms, there are still many unanswered questions about the existence and attraction of their solutions.

This paper focuses on establishing the existence and attraction of solutions for a specific class of generalized FDEs. By establishing new analytical tools and techniques, we will give the existence conditions of solutions and study their attractive behavior. The research in this paper not only enriches the theoretical system of FDEs but also provides theoretical support for the application in related fields.

In the context of fractional systems, attractivity analysis constitutes another essential research theme, with several studies [10,12–15] addressing this aspect. In Paper [10], the authors studied the fractional difference equations

$$\begin{cases} \Delta_{\alpha-1}^{\alpha} u(\rho) = g(\rho + \alpha, u(\rho + \alpha)), \rho \in \mathbb{N}_0, \alpha \in (0, 1], \\ \Delta_{\alpha-1}^{\alpha-1} u(\rho)|_{\rho=0} = u_0, \end{cases} \quad (1)$$

where Δ^{α} is the Riemann–Liouville fractional difference, $g : \mathbb{N}_{\alpha} \times \mathbb{R} \rightarrow \mathbb{R}$, $g(\rho, u)$ is continuous with respect to ρ and u , and $\mathbb{N}_{\alpha} = \{\alpha, \alpha + 1, \alpha + 2, \dots\}$. The existence and attractivity properties were established through the application of Picard iteration techniques and Schauder’s fixed point theorem. Moreover, some other attractivity results for fractional systems have also been discussed by many researchers (see [12–15] and the references therein). However, we are particularly interested in fractional systems involving time-varying delays, which have a huge impact on the whole system.

Motivated by the above papers, in this article, we are concerned with the existence and attractivity of solutions for the following FDEs with time-varying delays

$$\begin{cases} {}^C\Delta_a^{\vartheta} x(\rho) = Ax(\rho) + Bx(\rho - h(\rho)) + g(\rho, x(\rho), x(\rho - h(\rho))), \rho \in J_1, \\ x(\rho) = \phi(\rho, x(\rho)), \rho \in J_2, \end{cases} \quad (2)$$

where ${}^C\Delta_a^{\vartheta}$ denotes the Caputo fractional difference operator with $\vartheta \in (0, 1]$, A, B are n –dimensional constant matrices, and $h(t)$ is a delay function and its upper bound is $\bar{h} \in \mathbb{R}^+$, and $J_1 = \{\rho \in \mathbb{Z} | a \leq \rho \leq +\infty\}$ and $J_2 = \{\rho \in \mathbb{Z} | a - \bar{h} \leq \rho \leq a\}$ with $a \in \mathbb{Z}$, $J = J_1 \cup J_2$. $x(\rho) \in \mathbb{R}^n$ is the state vector, $\phi : J_2 \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, and $g : J_1 \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$.

In comparison with existing results in the literature [10,12–15], this work makes three contributions:

- (1) In [10], the study examined the existence and attractivity of FDEs with constant delays. However, our work considers a more general case where the delay term in System (2) is expressed as a function of the variable ρ . This represents a substantial advancement in the analysis of fractional systems with time-varying delays.
- (2) The model we investigate represents a more generalized formulation, encompassing several existing cases in the literature as special instances. Previous studies [10,12–15] examined fractional systems with one-dimensional real coefficients, which may be overly simplistic for accurate mathematical modeling. In contrast, our work employs constant matrix coefficients in System (2), establishing a novel framework for analyzing generalized fractional systems.
- (3) This study use an innovative analytical approach combining the Lagrange mean-value theorem with generalized Grönwall’s inequality to investigate the attractivity and global attractivity of solutions for FDEs with time-varying delays. The derived theoretical results represent fundamentally novel contributions to the field.

The remainder of this paper is structured as follows. Section 2 presents preliminary results, including essential definitions and lemmas that form the theoretical foundation for our analysis. In Section 3, we establish existence criteria for solutions to the proposed fractional difference system. Section 4 extends this analysis to investigate solution stability properties. Finally, Section 5 illustrates the theoretical findings through two representative numerical examples.

2. Preliminaries

We define $t^\nu := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$, for any t and ν for which the right-hand side is defined. We also appeal to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^\nu = 0$. We now present the necessary preparations, including relevant definitions and lemmas, etc.

Definition 1 ([8]). The ϑ -th fractional sum of a function f , for $\vartheta > 0$, is defined to be

$$\Delta_{a_1}^{-\vartheta} g(\rho) = \Delta_{a_1}^{-\vartheta} g(\rho; a_1) := \frac{1}{\Gamma(\vartheta)} \sum_{s=a_1}^{\rho-\vartheta} (\rho-s-1)^{\vartheta-1} g(s),$$

where $\rho \in \{a_1 + \vartheta, a_1 + \vartheta + 1, \dots\} =: \mathbb{N}_{a_1+\vartheta}$. We also define the ϑ -th fractional difference, where $\vartheta > 0$ and $0 \leq K-1 < \vartheta \leq K$ with $K \in \mathbb{N}$, to be $\Delta^\vartheta g(\rho) := \Delta^K \Delta^{\vartheta-K} g(\rho)$, where $\rho \in \mathbb{N}_{a_1+\vartheta}$.

The following delta discrete Leibniz integral law is Proposition 15 in [16].

Lemma 1 ([16,17]). Assume that $\mu > 0$ and g is define on $\mathbb{N}_{a_1}$. Then

$$\Delta^{-\mu} \Delta^\mu g(\rho) = g(\rho) - \sum_{k=0}^{n-1} \frac{(\rho-a_1)^{\underline{k}}}{k!} \Delta^k g(a_1) = g(\rho) + \lambda_0 + \lambda_1 \rho + \dots + \lambda_{n-1} \rho^{n-1},$$

$$\lambda_i \in \mathbb{R}, i = 0, 1, 2, \dots, n-1.$$

Lemma 2 ([18,19]). Let $\vartheta \in \mathbb{R}$ and $\rho, s \in \mathbb{R}$ such that $(\rho-s)^\vartheta$ is well defined, then $\Delta_s(\rho-s)^\vartheta = -\vartheta(\rho-s-1)^{\vartheta-1}$.

Definition 2 ([10,20]). The discrete Mittag-Leffler function is defined by

$$F_\alpha(c, \rho) = \sum_{n=0}^{\infty} c^n \frac{\rho^{n\alpha}}{\Gamma(n\alpha+1)},$$

where $\alpha \in \mathbb{R}^+$ and $c \in \mathbb{C}$; then the series is absolutely convergent for $|c| < 1$.

Some other representations for the discrete delta Mittag-Leffler function were given in [16,19] and the nabla version was presented in [19,21].

Lemma 3 ([22]). Let $\alpha > 0$, and $u(\rho), v(\rho)$ be nonnegative functions and $w(\rho)$ be a nonnegative, nondecreasing function for $\rho \in \mathbb{N}_{a_1}$. If

$$u(\rho) \leq v(\rho) + w(\rho) \sum_{s=a_1}^{\rho-\alpha} (\rho-s-1)^{\alpha-1} u(s), \quad (3)$$

then

$$u(\rho) \leq v(\rho) + \sum_{k=1}^{\infty} \frac{[w(\rho)\Gamma(\alpha)]^k}{\Gamma(k\alpha)} \sum_{s=a_1}^{\rho-k\alpha} (\rho-s-1)^{k\alpha-1} v(s).$$

A nabla discrete version of Grönwall's inequality can be found in Theorem 1 of [11].

Definition 3 ([13]). The solution $x(\rho)$ of System (2) is attractive if there exists a constant K such that $|\phi(\rho, x(\rho))| \leq K$ (for all $\rho \in J_2$) implies that $x(\rho) \rightarrow 0$ as $\rho \rightarrow \infty$.

Definition 4 ([13]). The solution $x(\rho)$ of System (2) is said to be globally attractive, if there are

$$\lim_{\rho \rightarrow \infty} (x(\rho) - \omega(\rho)) = 0,$$

for any solution $\omega = \omega(\rho)$ of System (2).

Definition 5 ([10]). A subset $\Lambda \subseteq E$ of bounded real sequences is called uniformly Cauchy if for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that for all sequences $x = \{x(n)\} \in \Lambda$ and all indices $p, q > N$, the oscillation satisfies $|x(p) - x(q)| < \varepsilon$.

Theorem 1 ([23]). Let Λ be a closed, convex, and nonempty subset of a Banach space X . If $T : \Lambda \rightarrow \Lambda$ is a continuous operator with $T\Lambda$ being relatively compact in X , then T possesses at least one fixed point in Λ .

According to Definition 1 and Lemma 1, a function $x(\rho)$ is called the solution of (2) if $x(\rho)$ satisfies

$$x(\rho) = \begin{cases} \frac{1}{\Gamma(\vartheta)} \sum_{s=a}^{\rho-\vartheta} (\rho-s-1)^{\vartheta-1} [Ax(s) + Bx(s-h(s)) + g(s, x(s), x(s-h(s)))] + \phi(a_1, x(a_1)), & \rho \in J_1, \\ \phi(\rho, x(\rho)), & \rho \in J_2. \end{cases} \quad (4)$$

3. Existence Results

Assume that $\|A\|$ denotes the spectral norm defined by $\sqrt{\lambda_{\max}(A^T A)}$ and $\lambda_{\max}(A^T A)$ represents the maximum eigenvalue of the matrix $A^T A$, and let $\|x\|$ be the norm of $x(\rho) = (x_1(\rho), x_2(\rho), \dots, x_n(\rho))^T \in \mathbb{R}^n$ defined by $\|x\| = \max_{\rho \in J} \left(\sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}}$. Suppose that $\mathcal{B}^+(J)$ denotes the set of all nonnegative bounded functions on $J = J_1 \cup J_2 = \{\rho \in \mathbb{Z} | a \leq \rho \leq +\infty\} \cup \{\rho \in \mathbb{Z} | a - \bar{h} \leq \rho \leq a\}$. Assume that the nonlinear function $g : J \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies the condition as:

(H_1). there exists a function $l(\rho) \in \mathcal{B}^+(J)$ such that

$$\|g(\rho, x_1, \omega_1) - g(\rho, x_2, \omega_2)\| \leq l(\rho)(\|x_1 - x_2\| + \|\omega_1 - \omega_2\|),$$

where $g(\rho, 0, 0) = 0$. We assume that:

$$\|A\| = \bar{a}_1, \quad \|B\| = \bar{a}_2, \quad \sup_{s \in [a_1, \rho-v]} l(s) = \bar{L}, \quad \rho \in J.$$

Now, we use mathematical induction and the Picard iteration method to obtain the existence result of FDE (2).

Theorem 2. Assume that $0 < \bar{a}_1 + \bar{a}_2 + 2\bar{L} < 1$, then the FDE (2) has at least one solution if the condition (H_1) holds.

Proof. For all $\rho \in J$, we define the sequence $\{\varphi_n(\cdot) : n \in \mathbb{N}_0\}$ as follows:

$$\begin{aligned}\varphi_0(\rho) &= \phi(a_1, x(a_1)), \\ \varphi_n(\rho) &= \varphi_0(\rho) + \frac{1}{\Gamma(\vartheta)} \sum_{s=a_1}^{\rho-\vartheta} (\rho-s-1)^{\vartheta-1} \\ &\quad \times [A\varphi_{n-1}(s) + B\varphi_{n-1}(s-h(s)) + g(s, \varphi_{n-1}(s), \varphi_{n-1}(s-h(s)))],\end{aligned}$$

and $\sup_{\rho \in J} \|\varphi_0(\rho)\| = \bar{\varphi}$, $\bar{\varphi} \in \mathbb{R}^+$. By induction we have

$$\|\varphi_n - \varphi_{n-1}\| \leq \frac{(\bar{a}_1 + \bar{a}_2 + 2\bar{L})^n \cdot \bar{\varphi}}{\Gamma(n\vartheta + 1)} (\rho - a)^{n\vartheta}, \quad \rho \in J. \quad (5)$$

In fact, for $n = 1$, one can obtain that

$$\begin{aligned}\|\varphi_1 - \varphi_0\| &\leq \frac{1}{\Gamma(\vartheta)} \sum_{s=a_1}^{\rho-\vartheta} (\rho-s-1)^{\vartheta-1} \\ &\quad \times [\bar{a}_1 \|\varphi_0(s)\| + \bar{a}_2 \|\varphi_0(s-h(s))\| + l(s)(\|\varphi_0(s)\| + \|\varphi_0(s-h(s))\|)] \\ &\leq \frac{(\bar{a}_1 + \bar{a}_2 + 2\bar{L}) \cdot \bar{\varphi}}{\Gamma(\vartheta)} \cdot \sum_{s=a_1}^{\rho-\vartheta} (\rho-s-1)^{\vartheta-1} = \frac{(\bar{a}_1 + \bar{a}_2 + 2\bar{L}) \cdot (\rho - a_1)^{\vartheta} \cdot \bar{\varphi}}{\Gamma(\vartheta + 1)}.\end{aligned} \quad (6)$$

Without loss of generality, we set

$$\|\varphi_{n-1}(\rho) - \varphi_{n-2}(\rho)\| \leq \frac{(\bar{a}_1 + \bar{a}_2 + 2\bar{L})^{n-1} \cdot \bar{\varphi}}{\Gamma((n-1)\vartheta + 1)} (\rho - a_1)^{(n-1)\vartheta}, \quad \rho \in J. \quad (7)$$

Then

$$\begin{aligned}\|\varphi_n(\rho) - \varphi_{n-1}(\rho)\| &\leq \frac{1}{\Gamma(\vartheta)} \sum_{s=a_1}^{\rho-\vartheta} (\rho-s-1)^{\vartheta-1} \\ &\quad \times [(\bar{a}_1 + l(s)) \|\varphi_{n-1}(s) - \varphi_{n-2}(s)\| + (\bar{a}_2 + l(s)) \|\varphi_{n-1}(s-h(s)) - \varphi_{n-2}(s-h(s))\|] \\ &\leq \frac{(\bar{a}_1 + \bar{L})(\bar{a}_1 + \bar{a}_2 + 2\bar{L})^{n-1} \cdot \bar{\varphi}}{\Gamma(\vartheta) \cdot \Gamma((n-1)\vartheta + 1)} \cdot \sum_{s=a_1}^{\rho-\vartheta} (\rho-s-1)^{\vartheta-1} \cdot (s-a)^{(n-1)\vartheta} \\ &\quad + \frac{(\bar{a}_2 + \bar{L})(\bar{a}_1 + \bar{a}_2 + 2\bar{L})^{n-1} \cdot \bar{\varphi}}{\Gamma(\vartheta) \cdot \Gamma((n-1)\vartheta + 1)} \cdot \sum_{s=a_1}^{\rho-\vartheta} (\rho-s-1)^{\vartheta-1} \cdot (s-h(s)-a_1)^{(n-1)\vartheta} \\ &\leq \frac{(\bar{a}_1 + \bar{a}_2 + 2\bar{L})^n \cdot \bar{\varphi}}{\Gamma(\vartheta) \cdot \Gamma((n-1)\vartheta + 1)} \cdot \sum_{s=a_1}^{\rho-\vartheta} (\rho-s-1)^{\vartheta-1} \cdot (s-a_1)^{(n-1)\vartheta} \\ &= \frac{(\bar{a}_1 + \bar{a}_2 + 2\bar{L})^n \cdot \bar{\varphi}}{\Gamma((n-1)\vartheta + 1)} \cdot \Delta_{a_1}^{-\vartheta} (\rho - a_1)^{(n-1)\vartheta} \\ &= \frac{(\bar{a}_1 + \bar{a}_2 + 2\bar{L})^n \cdot \bar{\varphi}}{\Gamma((n-1)\vartheta + 1)} \cdot \frac{\Gamma((n-1)\vartheta + 1)}{\Gamma((n-1)\vartheta + \vartheta + 1)} (\rho - a_1)^{n\vartheta} \\ &= \frac{(\bar{a}_1 + \bar{a}_2 + 2\bar{L})^n \cdot \bar{\varphi}}{\Gamma(n\vartheta + 1)} (\rho - a_1)^{n\vartheta}.\end{aligned} \quad (8)$$

Let

$$\begin{aligned}\varphi(\rho) &= \lim_{n \rightarrow \infty} \varphi_n(\rho) = \lim_{n \rightarrow \infty} (\varphi_n(\rho) - \varphi_0(\rho)) + \varphi_0(\rho) \\ &= \sum_{k=1}^{\infty} (\varphi_k(\rho) - \varphi_{k-1}(\rho)) + \varphi_0(\rho) \\ &\leq \|\phi\| \cdot \sum_{k=1}^{\infty} (\bar{a}_1 + \bar{a}_2 + 2\bar{L})^k \frac{(\rho - a_1)^{k\vartheta}}{\Gamma(k\vartheta + 1)} + \varphi_0(\rho).\end{aligned}\quad (9)$$

By Definition 2, the series is absolutely convergent if $0 < \bar{a}_1 + \bar{a}_2 + 2\bar{L} < 1$, and then the existence of solutions for the System (2) is proved. The proof is completed. $\square$

4. Attractivity Results

In this section, we will derive the attractivity conclusions with the help of Schauder's fixed point theorem. Subsequently, with the aid of Grönwall's inequality, we will prove that the system is globally attractive.

Before the proving, we define the operator T as follows:

$$Tx(\rho) = \begin{cases} \frac{1}{\Gamma(\vartheta)} \sum_{s=a}^{\rho-\vartheta} (\rho - s - 1)^{\vartheta-1} [Ax(s) + Bx(s - h(s)) + g(s, x(s), x(s - h(s)))] + \phi(a_1, x(a_1)), & \rho \in J_1, \\ \phi(\rho, x(\rho)), & \rho \in J_2, \end{cases} \quad (10)$$

and clearly, $x(\rho)$ serves as a solution to System (2) if and only if it constitutes a fixed point of the operator T .

Theorem 3. Suppose the validity of condition (H_1) , then following assumptions $(H_2) : \phi(\rho, x(\rho))$ satisfies the Lipschitz condition, namely, there exists a nonnegative function $L_\phi(\rho)$ such that $\|\phi(\rho, x) - \phi(\rho, \omega)\| \leq L_\phi(\rho)\|x - \omega\|$, and $\phi(\rho, 0) = 0$, and (H_3) : for $\forall \rho \in J_1$, then the inequality

$$\sup_{\rho \in J_1} \left\{ L_\phi(\rho) + \frac{(\bar{a}_1 + \bar{a}_2 + 2\bar{L}) \cdot (\rho - a_1)^{\vartheta}}{\Gamma(\vartheta + 1)} \right\} \leq 1$$

holds. Then System (2) has at least one solution in space

$$S = \{x \in C(J, \mathbb{R}^n) : \|x(\rho)\| \leq r, r \in \mathbb{R}^+ \text{ for all } \rho \in J\}.$$

Furthermore, the solutions of (2) are attractive.

Proof. Define the set $S \subset C(J, \mathbb{R}^n)$ by

$$S = \{x \in C(J, \mathbb{R}^n) : \|x(\rho)\| \leq r, r \in \mathbb{R}^+ \text{ for all } \rho \in J\}.$$

Obviously, we can obtain that the set S is a nonempty, closed, bounded, and convex subset of $C(J, \mathbb{R}^n)$. The existence of solutions to System (2) can be established by demonstrating that the operator T admits at least one fixed point in S .

Now we show that T is continuous in S . Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence of functions such that $u_n \in S$ for all $n \in \mathbb{N}$ and $u_n \rightarrow u$ as $n \rightarrow \infty$, for any $\rho \in J$,

$$\begin{aligned}
 & \|Tu_n(\rho) - Tu(\rho)\| \\
 & \leq L_\phi(\rho) \|u_n(a) - u(a)\| + \frac{1}{\Gamma(\vartheta)} \sum_{s=a}^{\rho-\vartheta} (\rho-s-1)^{\vartheta-1} \\
 & \quad \times [A \|u_n(s) - u(s)\| + B \|u_n(s-h(s)) - u(s-h(s))\| \\
 & \quad + \|f(s, u_n(s), u_n(s-h(s))) - f(s, u(s), u(s-h(s)))\|] \\
 & \leq L_\phi(\rho) \|u_n(a_1) - u(a_1)\| + \frac{(\bar{a}_1 + \bar{a}_2 + 2\bar{L}) \|u_n(\rho) - u(\rho)\|}{\Gamma(\vartheta)} \sum_{s=a}^{\rho-\vartheta} (\rho-s-1)^{\vartheta-1} \\
 & \leq L_\phi(\rho) \|u_n(a_1) - u(a_1)\| + \frac{(\bar{a}_1 + \bar{a}_2 + 2\bar{L})(\rho-a_1)^\vartheta \|u_n(\rho) - u(\rho)\|}{\Gamma(\vartheta+1)} \\
 & \rightarrow 0 \text{ as } n \rightarrow \infty,
 \end{aligned} \tag{11}$$

which implies that operator T is continuous.

Next, we show that $T(S)$ is equicontinuous. For all $\rho_2, \rho_1 \in J$ and $\rho_2 > \rho_1$,

$$\begin{aligned}
 & \|Tx(\rho_2) - Tx(\rho_1)\| \\
 & \leq \frac{1}{\Gamma(\vartheta)} \sum_{s=a}^{\rho_1-\vartheta} [(\rho_2-s-1)^{\vartheta-1} - (\rho_1-s-1)^{\vartheta-1}] \\
 & \quad \times [(\bar{a} + l(s)) \|x(s)\| + (\bar{b} + l(s)) \|x(s-h(s))\|] \\
 & \quad + \frac{1}{\Gamma(\vartheta)} \sum_{s=\rho_1-\vartheta}^{\rho_2-\vartheta} (\rho_2-s-1)^{\vartheta-1} [(\bar{a}_1 + l(s)) \|x(s)\| + (\bar{a}_2 + l(s)) \|x(s-h(s))\|].
 \end{aligned} \tag{12}$$

Let $h(\rho, s) = (\rho-s-1)^{\vartheta-1}$, then by the Lagrange mean-value theorem, there exists $\rho_{12} \in [t_1, t_2]$, and we have

$$(\rho_2-s-1)^{\vartheta-1} - (\rho_1-s-1)^{\vartheta-1} = h(\rho_2, s) - h(\rho_1, s) = h'(\rho_{12}, s)(\rho_2 - \rho_1). \tag{13}$$

Therefore, through (12) and (13), one can obtain

$$\begin{aligned}
 & \|Tx(\rho_2) - Tx(\rho_1)\| \\
 & \leq \frac{1}{\Gamma(\vartheta)} \sum_{s=a}^{\rho_1-\vartheta} h'(\rho_{12}, s)(\rho_2 - \rho_1) [(\bar{a}_1 + l(s)) \|x(s)\| + (\bar{a}_2 + l(s)) \|x(s-h(s))\|] \\
 & \quad + \frac{1}{\Gamma(\vartheta)} \sum_{s=\rho_1-\vartheta}^{\rho_2-\vartheta} (\rho_2-s-1)^{\vartheta-1} [(\bar{a}_1 + l(s)) \|x(s)\| + (\bar{a}_2 + l(s)) \|x(s-h(s))\|],
 \end{aligned} \tag{14}$$

which tends to zero when $\rho_1 \rightarrow \rho_2$, and we conclude that $T(S)$ is equicontinuous on J .

In the following, we prove that T maps S into S . For all $x \in S$, $\rho \in J_1$, we have

$$\begin{aligned}
& \|Tx(\rho)\| \\
& \leq \frac{1}{\Gamma(\vartheta)} \sum_{s=a_1}^{\rho-\vartheta} (\rho-s-1)^{\vartheta-1} \|Ax(s) + Bx(s-h(s)) + g(s, x(s), x(s-h(s)))\| + \|\phi(a_1, x(a_1))\| \\
& \leq \frac{1}{\Gamma(\vartheta)} \sum_{s=a_1}^{\rho-\vartheta} (\rho-s-1)^{\vartheta-1} \|Ax(s) + Bx(s-h(s)) + l(s)(x(s) + x(s-h(s)))\| + L_\phi(\rho) \|x(a_1)\| \\
& \leq \frac{(\bar{a}_1 + \bar{a}_2 + 2\bar{L}) \cdot r}{\Gamma(\vartheta)} \sum_{s=a_1}^{\rho-\vartheta} (\rho-s-1)^{\vartheta-1} + L_\phi(\rho) \cdot r \\
& \leq \left(L_\phi(\rho) + \frac{(\bar{a}_1 + \bar{a}_2 + 2\bar{L}) \cdot (\rho - a_1)^\vartheta}{\Gamma(\vartheta + 1)} \right) \leq r.
\end{aligned} \tag{15}$$

If $\rho \in J_2$, it is easy for us to verify

$$\|Tx(\rho)\| \leq r,$$

which implies that $T(S) \subset S$, $\rho \in J$. Therefore, $T(S)$ is a relatively compact set in $C(J, \mathbb{R}^n)$ and all the conditions of Schauder's fixed point theorem are fulfilled. Hence, the operator T has a fixed point which is the solution of System (2). All functions in S tend to 0 as $\rho \rightarrow +\infty$, and we can obtain that the solution of (2) is attractive. The proof is completed. $\square$

Remark 1. It should be noted that the stability results established in Theorem 3 do not necessarily guarantee global attractivity as defined in Definition 4.

Theorem 4. If $L_\phi(\rho) + \sum_{k=1}^{\infty} \frac{(\bar{a}_1 + \bar{a}_2 + 2\bar{L})^k}{\Gamma(k\vartheta)} \sum_{s=a_1}^{\rho-k\vartheta} (\rho-s-1)^{k\vartheta-1} L_\phi(s)$ vanishes at infinity, then the solution $x(\rho)$ of System (2) is said to be globally attractive in view of assumptions (H_1) and (H_2) .

Proof. For any solution $y(\rho)$ of System (2), we get

$$\begin{aligned}
& \|\omega(\rho) - x(\rho)\| \\
& \leq L_\phi(\rho) \|\omega(a_1) - x(a_1)\| + \frac{1}{\Gamma(\vartheta)} \sum_{s=a_1}^{\rho-\vartheta} (\rho-s-1)^{\vartheta-1} \\
& \quad \times [(A + l(s)) \|\omega(s) - x(s)\| + (B + l(s)) \|\omega(s-h(s)) - x(s-h(s))\|].
\end{aligned} \tag{16}$$

Let $z(\rho) = \|\omega(\rho) - x(\rho)\|$, then

$$\begin{aligned}
z(\rho) & \leq L_\phi(\rho) z(a_1) + \frac{1}{\Gamma(\vartheta)} \sum_{s=a_1}^{\rho-\vartheta} (\rho-s-1)^{\vartheta-1} [(A + l(s)) z(s) + (B + l(s)) z(s-h(s))] \\
& \leq L_\phi(\rho) z(a_1) + \frac{\bar{a}_1 + \bar{L}}{\Gamma(\vartheta)} \sum_{s=a_1}^{\rho-\vartheta} (\rho-s-1)^{\vartheta-1} z(s) + \frac{\bar{a}_2 + \bar{L}}{\Gamma(\vartheta)} \sum_{s=a_1}^{\rho-\vartheta} (\rho-s-1)^{\vartheta-1} z(s-h(s)).
\end{aligned} \tag{17}$$

Assume that $z^* = \sup_{\theta \in [-h, 0]} z(\rho + \theta)$, then we have

$$z^*(\rho) \leq L_\phi(\rho) z(a_1) + \frac{\bar{a}_1 + \bar{a}_2 + 2\bar{L}}{\Gamma(\vartheta)} \sum_{s=a_1}^{\rho-\vartheta} (\rho-s-1)^{\vartheta-1} z^*(s). \tag{18}$$

Through Lemma 3, we have

$$z^*(\rho) \leq z(a_1) \left[L_\phi(\rho) + \sum_{k=1}^{\infty} \frac{(\bar{a}_1 + \bar{a}_2 + 2\bar{L})^k}{\Gamma(k\vartheta)} \sum_{s=a_1}^{\rho-k\vartheta} (\rho-s-1)^{k\vartheta-1} L_\phi(s) \right] \rightarrow 0 \text{ as } \rho \rightarrow +\infty, \quad (19)$$

which implies that $\lim_{\rho \rightarrow \infty} (x(\rho) - \omega(\rho)) = 0$; then the solution $x(\rho)$ of System (2) is said to be globally attractive. $\square$

5. Illustrative Examples

Two numerical examples are presented in this section to validate the proposed theory.

Example 1. Consider the Fractional System (2) with $\vartheta \in (0, 1]$, and $g(\cdot) = 0.08 \sin x(\rho) + 0.08 \cos x(\rho - h(\rho))$ with $x = (x_1(\rho), x_2(\rho))^T$, where

$$A = \begin{pmatrix} 0.11 & 0 \\ 0 & 0.08 \end{pmatrix}, \quad (20)$$

$$B = \begin{pmatrix} 0.1 & 0.4 \\ 0.2 & 0.1 \end{pmatrix}. \quad (21)$$

We have $\|A\| = \bar{a} = 0.11$, $\|B\| = \bar{b} = 0.44$, and $\bar{L} = 0.08$, which imply that the condition $0 < \bar{a}_1 + \bar{a}_2 + 2\bar{L} = 0.71 < 1$ referred in Theorem 2 holds. Consequently, there exists at least one solution to the specified System (2).

Remark 2. The existing literature on nonlinear FDEs with time-varying delays remains limited, as evidenced by previous studies [8,10,12,13,15,24]. Notably, none of these established results can be directly applied to Example 1, which consequently highlights the novel contributions of our current work.

Example 2. Assuming that $v = 0.01$, $a = 0$, all other data are the same as in the above Example 1, and let $\phi(\rho, x) = \frac{\sin x(\rho)}{20e^\rho}$, then e is a natural constant. We can get that $L_\phi(\rho) = \frac{1}{20}e^{-\rho}$ and

$$\sup_{\rho \in J_1} \left\{ L_\phi(\rho) + \frac{(\bar{a}_1 + \bar{a}_2 + 2\bar{L}) \cdot (\rho - a_1)^{\vartheta}}{\Gamma(\vartheta + 1)} \right\} = 0.89 \leq 1,$$

and

$$L_\phi(\rho) + \sum_{k=1}^{\infty} \frac{(\bar{a}_1 + \bar{a}_2 + 2\bar{L})^k}{\Gamma(k\vartheta)} \sum_{s=a_1}^{\rho-k\vartheta} (\rho-s-1)^{k\vartheta-1} L_\phi(s) \rightarrow 0,$$

as $\rho \rightarrow +\infty$. Hence, Theorems 3 and 4 are satisfied, proving that the solutions of the Fractional System (2) are attractive and globally attractive.

6. Conclusions

This paper investigates a novel class of nonlinear FDEs characterized by time-varying delays. The inclusion of functional delay terms represents a significant departure from conventional formulations with constant delays examined in prior studies [10,12–15]. By means of mathematical induction and the Picard iteration method, we first obtain the existence result of this fractional difference system. Under some new criteria along with Schauder's fixed point theorem, we then derive the attractivity conclusions. Subsequently, with the aid of Grönwall's inequality, we prove that the system is globally attractive.

Finally, several representative examples are presented at the end of this paper to illustrate the validity and practicality of the proposed criteria.

In our subsequent work, we will consider obtaining the existence of the solution by using the discrete Laplace transform [25] and take into account the influence of the fuzzy environment on attractivity.

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References

- Goodrich, C.; Peterson, A. *Discrete Fractional Calculus*; Springer International Publishing: Cham, Switzerland, 2015.
- Bagwan, A.; Pachpatte, D.; Kiseák, J.; Stehlík, M. On existence results of boundary value problems of Caputo fractional difference equations for weak-form efficient market hypothesis. *Stoch. Anal. Appl.* **2025**, *43*, 30–47. [\[CrossRef\]](#)
- Sharma, A.; Mishra, S.; Shukla, A. Schauder's fixed point theorem approach for stability analysis of nonlinear fractional difference equations. *Chaos Solitons Fractals* **2024**, *188*, 115586. [\[CrossRef\]](#)
- Dimitrov, N.; Jonnalagadda, J. Existence, uniqueness, and stability of solutions for Nabla fractional difference equations. *Fractal Fract.* **2024**, *8*, 591. [\[CrossRef\]](#)
- Dimitrov, N.; Jonnalagadda, J. Existence of positive solutions for a class of Nabla fractional boundary value problems. *Fractal Fract.* **2025**, *9*, 131. [\[CrossRef\]](#)
- El-Houari, H.; Krait, A.; Chadli, L.S.; Moussa, H. A SIR epidemic model involving fractional atangana derivative. In *International Conference on Partial Differential Equations and Applications, Modeling and Simulation*; Springer International Publishing: Cham, Switzerland, 2021; pp. 26–36.
- Abdeljawad, T.; Al-Mdallal, Q. Discrete Mittag-Leffler kernel type fractional difference initial value problems and Gronwall's inequality. *J. Comput. Appl. Math.* **2018**, *339*, 218–230. [\[CrossRef\]](#)
- Goodrich, C. Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions. *Comput. Math. Appl.* **2011**, *61*, 191–202. [\[CrossRef\]](#)
- Atici, F.; Eloe, P. Two-point boundary value problems for finite fractional difference equations. *J. Differ. Equ. Appl.* **2011**, *17*, 445–456. [\[CrossRef\]](#)
- Zhang, L.; Zhou, Y. Existence and attractivity of solutions for fractional difference equations. *Adv. Difference Equ.* **2018**, *2018*, 191. [\[CrossRef\]](#)
- Alzabut, J.; Abdeljawad, T. A generalized discrete fractional Gronwall inequality and its application on the uniqueness of solution and its application on the uniqueness of solutions for nonlinear delay fractional difference system. *Appl. Anal. Discrete Math.* **2018**, *12*, 036–048. [\[CrossRef\]](#)
- Wang, H.; Li, F. Existence and attractivity of global solutions for a class of fractional quadratic integral equations in Banach space. *Adv. Differ. Equ.* **2016**, *2016*, 186. [\[CrossRef\]](#)
- Losada, J.; Nieto, J.; Pourhadi, E. On the attractivity of solutions for a class of multi-term fractional functional differential equations. *J. Comput. Appl. Math.* **2017**, *312*, 2–12. [\[CrossRef\]](#)
- Zhou, Y. Attractivity for fractional differential equations in Banach space. *Appl. Math. Lett.* **2018**, *75*, 1–6. [\[CrossRef\]](#)
- Abbas, S.; Benchohra, M.; Diagana, T. Existence and attractivity results for some fractional order partial integro-differential equations with delay. *Afr. Diaspora J. Math.* **2013**, *15*, 87–100.
- Abdeljawad, T. On Riemann and Caputo fractional differences. *Comput. Math. Appl.* **2011**, *62*, 1602–1611. [\[CrossRef\]](#)
- Abdeljawad, T. Different type kernel h-fractional differences and their fractional h-sums. *Chaos Solitons Fractals* **2018**, *116*, 146–156. [\[CrossRef\]](#)
- Abdeljawad, T.; Abdalla, B. Monotonicity results for delta and nabla Caputo and Riemann fractional differences via dual identities. *Filomat* **2017**, *31*, 3671–3683. [\[CrossRef\]](#)

19. Abdeljawad, T. On Delta and Nabla Caputo fractional differences and Dual identities. *Discrete Dyn. Nat. Soc.* **2013**, *2013*, 406910. [[CrossRef](#)]
20. Wu, G.; Baleanu, D.; Zeng, S. Finite-time stability of discrete fractional delay systems: Gronwall inequality and stability criterion. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *57*, 299–308. [[CrossRef](#)]
21. Abdeljawad, T.; Jarad, F.; Baleanu, D. A semigroup-like property for discrete Mittag-Leffler functions. *Adv. Differ. Equ.* **2012**, *2012*, 72. [[CrossRef](#)]
22. Luo, D.; Luo, Z. Uniqueness and novel finite-time stability of solutions for a class of nonlinear fractional delay difference systems. *Discrete Dyn. Nat. Soc.* **2018**, *7*, 8476285. [[CrossRef](#)]
23. Yang, M.; Alsaedi, A.; Ahmad, B.; Zhou, Y. Attractivity for Hilfer fractional stochastic evolution equations. *Adv. Differ. Equ.* **2020**, *130*, 130. [[CrossRef](#)]
24. Khan, A.; Li, Y.; Shah, K.; Khan, T. On coupled p-Laplacian fractional differential equations with nonlinear boundary conditions. *Complexity* **2017**, *9*, 8197610.
25. Liu, X.; Ye, G.; Liu, W. Discrete Laplace transform for interval-valued functions and its applications to interval fractional difference equations. *Fuzzy Sets Syst.* **2025**, *509*, 109336. [[CrossRef](#)]

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