fractal and fractional

Article

Sharp Bounds on Hankel Determinant of *q*-Starlike and *q*-Convex Functions Subordinate to Secant Hyperbolic Functions

Lifen Zhang [1], Zhigang Wang [2,*] and Lei Shi [1,*]

[1] School of Mathematics and Statistics, Anyang Normal University, Anyang 455002, China; zhanglifen@aynu.edu.cn
[2] School of Mathematics and Statistics, Hunan First Normal University, Changsha 410205, China
* Correspondence: wangmath@163.com (Z.W.); shimath@aynu.edu.cn (L.S.)

Abstract: In the present paper, using the q-difference operator, we introduce two classes of q-starlike functions and q-convex functions subordinate to secant hyperbolic functions. As functions in these classes have unique characteristic of missing coefficients on the second term in their analytic expansions, we define a new functional to unify the Hankel determinants with entries of the original coefficients, inverse coefficients, logarithmic coefficients, and inverse logarithmic coefficients for these functions. We obtain the sharp bounds on the new functional for functions in the two classes, and as a consequence, the best results on Hankel determinant for the starlike and convex functions subordinate to secant hyperbolic functions are given. The outcomes include some existing findings as corollaries and may help to deepen the understanding the properties of q-analogue analytic functions.

Keywords: Hankel determinant; q-starlike functions; q-convex functions; secant hyperbolic functions

MSC: 05A30; 30C45; 11B65; 47B38

Academic Editors: Carlo Cattani, Georgia Irina Oros and Gheorghe Oros

Received: 26 April 2025
Revised: 21 May 2025
Accepted: 24 May 2025
Published: 26 May 2025

Citation: Zhang, L.; Wang, Z.; Shi, L. Sharp Bounds on Hankel Determinant of q-Starlike and q-Convex Functions Subordinate to Secant Hyperbolic Functions. *Fractal Fract.* **2025**, *9*, 346. https://doi.org/10.3390/fractalfract9060346

1. Introduction and Definitions

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc and $\mathcal{A}$ be the group of analytic functions f in $\mathbb{D}$ with the normalization $f(0) = f'(0) - 1 = 0$. For $f \in \mathcal{A}$, it can be written as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1}$$

If a function never takes a value twice, it is called univalent. Traditionally, $\mathcal{S}$ is used to represent the set of such functions in geometric function theory. For an analytic function ω with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{D}$, we call it a Schwarz function. Let $\mathcal{P}$ denote the class of functions $f \in \mathcal{A}$ with $\Re(p(z)) > 0 (z \in \mathbb{D})$ and normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad z \in \mathbb{D}.$$

For $p \in \mathcal{P}$, it is often called a Carathéodory function [1].

For two given analytic functions f and g, $f \prec g$ means that f is subordinate to g, i.e., there exists a Schwarz function ω in the manner of

$$f(z) = g(\omega(z)), \quad z \in \mathbb{D}.$$

To illustrate our main idea, the notions of q-calculus need to be addressed. Throughout this paper, q is fixed to be $(0,1)$. The q-number $[\zeta]_q$ is introduced as

$$[\zeta]_q = \begin{cases} \frac{1-q^\zeta}{1-q}, & if\ \zeta \in \mathbb{C} \setminus \mathbb{N}, \\ 1+q+\cdots+q^{m-1}, & if\ \zeta = m \in \mathbb{N}. \end{cases} \tag{2}$$

The q-factorial $[m]_q!$ is used to denote

$$[m]_q! = \begin{cases} 1, & if\ m = 0, \\ [m]_q \cdot [m-1]_q \cdots [2]_q \cdot [1]_q, & if\ m \in \mathbb{N}. \end{cases} \tag{3}$$

In particular, $\lim_{q\to 1^-}[m]_q = m$.

The q-difference operator of a function ϕ is defined as

$$D_q\phi(z) = \frac{\phi(qz)-\phi(z)}{(q-1)z}, \quad z \in \mathbb{D} \setminus \{0\}, \tag{4}$$

see [2]. Clearly, $\lim_{q\to 1^-} D_q\phi(z) = \phi'(z)$, and $D_q z^m = [m]_q z^{m-1}$. This operator is widely used in the theory of hypergeometric series and quantum physics and is also known as the Jackson q-difference operator; we refer to [3–5] for more details.

Using the q-difference operator, Ismail et al. [6] first proposed the concept of q-starlike functions. In [7], it is proved that the conditions of q-starlike functions can be equivalently characterized by $f \in \mathcal{A}$ and

$$\frac{zD_qf(z)}{f(z)} \prec \frac{1+z}{1-qz}, \quad z \in \mathbb{D}. \tag{5}$$

For $0 \leq \alpha < 1$, Seoudy and Aouf [8] introduced the subclasses $\mathcal{S}_q^*(\alpha)$ and $\mathcal{K}_q(\alpha)$ defined, respectively, by

$$\mathcal{S}_q^* := \left\{ f \in \mathcal{A} : \Re\left(\frac{zD_qf(z)}{f(z)}\right) > \alpha, z \in \mathbb{D} \right\} \tag{6}$$

and

$$\mathcal{K}_q := \left\{ f \in \mathcal{A} : \Re\left(\frac{D_q(zD_qf(z))}{D_qf(z)}\right) > \alpha, z \in \mathbb{D} \right\}. \tag{7}$$

When $q \to 1^-$, $\mathcal{S}_q^*(\alpha)$ and $\mathcal{K}_q(\alpha)$ reduce to the class of starlike functions of order α and convex of order α in $\mathbb{D}$ (see Duren [9]). Afterwards, the research on q-starlike functions and q-convex functions continued to enrich, including the works on q-starlike functions associated with the Janowski functions [10], the q-exponential function [11], the q-Bernoulli numbers [12] and some others like [13–15].

In [16], Bano et al. introduced a novel class of starlike functions $\mathcal{S}^*(\text{sech})$ defined by

$$\mathcal{S}^*(\text{sech}) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \text{sech}(z), \quad z \in \mathbb{D} \right\}.$$

We remark that the function $\text{sech}(z)$ is not univalent in $\mathbb{D}$. By virtue of $\text{sech}(z) = \frac{2}{e^z+e^{-z}}$, it is clear that $\text{sech}\left(\frac{1}{2}\right) = \text{sech}\left(-\frac{1}{2}\right)$. As $\Re(\text{sech}(z)) > 0$ in $\mathbb{D}$, functions in the class $\mathcal{S}^*(\text{sech})$ are starlike and thus univalent. Recently, the coefficient problems for this class were studied in [17,18] and an interesting observation is that $a_2 \equiv 0$ when $f \in \mathcal{S}^*(\text{sech})$, with $a_2 = \frac{f''(0)}{2!}$.

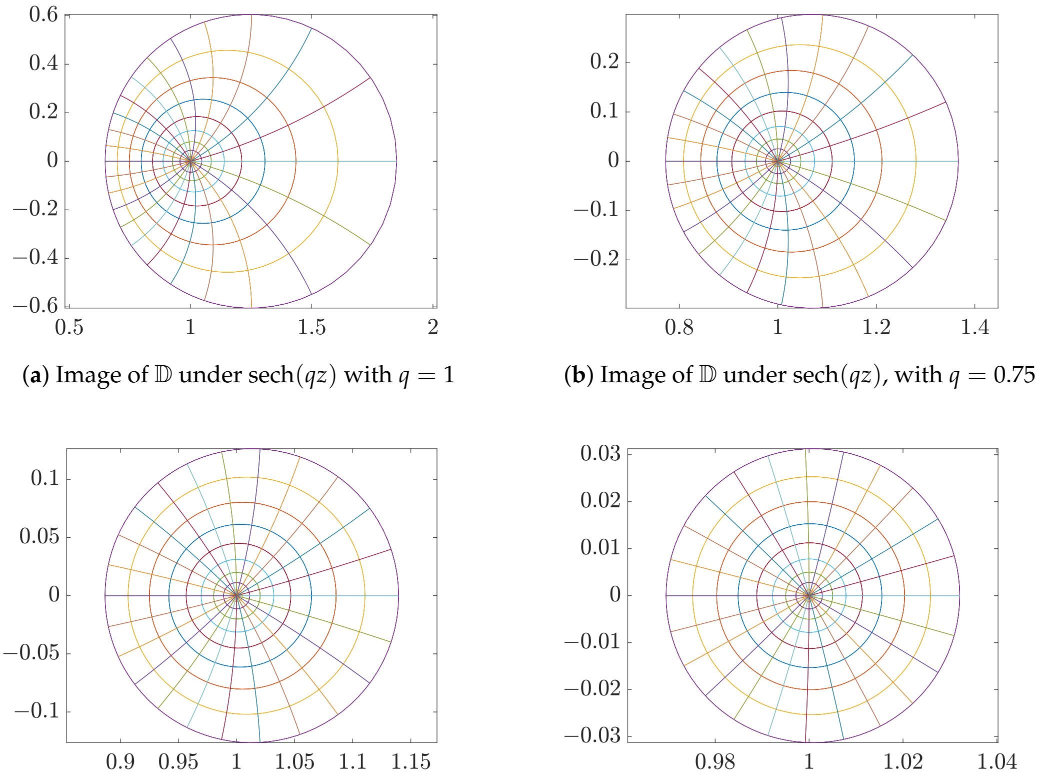
Inspired by the mentioned works, we consider the classes $\mathcal{S}_q^*(\text{sech})$ and $\mathcal{K}_q(\text{sech})$ defined, respectively, by

$$\mathcal{S}_q^*(\text{sech}) := \left\{ f \in \mathcal{A} : \frac{zD_q f(z)}{f(z)} \prec \text{sech}(qz), \quad z \in \mathbb{D} \right\} \tag{8}$$

and

$$\mathcal{K}_q(\text{sech}) := \left\{ f \in \mathcal{A} : \frac{D_q(zD_q f(z))}{D_q f(z)} \prec \text{sech}(qz), \quad z \in \mathbb{D} \right\}. \tag{9}$$

For different choices of q, the images domains of $\text{sech}(qz)$ are presented in Figure 1a–d. Clearly, $\lim_{q\to 1^-} \mathcal{S}_q^*(\text{sech}) = \mathcal{S}^*(\text{sech})$. Denote $\lim_{q\to 1^-} \mathcal{K}_q(\text{sech}) = \mathcal{K}(\text{sech})$. We remark that $\mathcal{K}(\text{sech})$ is a subclass of convex functions.



(**a**) Image of $\mathbb{D}$ under $\text{sech}(qz)$ with $q = 1$

(**b**) Image of $\mathbb{D}$ under $\text{sech}(qz)$, with $q = 0.75$

(**c**) Image of $\mathbb{D}$ under $\text{sech}(qz)$ with $q = 0.5$

(**d**) Image of $\mathbb{D}$ under $\text{sech}(qz)$ with $q = 0.25$

Figure 1. Images of $\mathbb{D}$ under $\text{sech}(qz)$ with various values of q.

Hankel determinant is an important tool in the study of analytic functions. In [19,20], Pommerenke introduced the Hankel determinant $\mathcal{H}_{q,n}(f)$ with $a_1 = 1$ and $q, n \in \mathbb{N}$ for $f \in \mathcal{A}$. It is defined by the coefficients a_n of f arranged in the form

$$\mathcal{H}_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

Utilizing the initial coefficients a_2, a_3, a_4, and a_5 of f, we may write

$$\mathcal{H}_{2,3}(f) = a_3 a_5 - a_4^2, \tag{10}$$

$$\mathcal{H}_{3,1}(f) = 2a_2 a_3 a_4 - a_2^2 a_5 - a_4^2 + a_3 a_5 - a_3^3. \tag{11}$$

In recent years, the upper bounds of Hankel determinants for f belonging to various subfamilies of analytic functions were obtained. For example, the study on bounded turning functions [21,22], close-to-convex functions [23], bi-univalent functions [24], convex functions [25] and starlike functions [26,27]. The results are abundant enough, and those interested can also refer to [28–30].

If $f \in \mathcal{S}$ defined in $\mathbb{D}$, the inverse of f exists and is univalent at least in a disk of radius $1/4$. Denote

$$F(w) := f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + \cdots .$$

As $f(F(w)) = w$, the coefficients of F are closely related with f. Researchers endeavor to study the inverse functions from different perspectives; in particular, the Hankel determinant using the inverse coefficients is smoothly introduced [31–34]. We note that $\mathcal{H}_{2,3}(f^{-1})$ and $\mathcal{H}_{3,1}(f^{-1})$ are given by

$$\mathcal{H}_{2,3}(f^{-1}) = A_3A_5 - A_4^2, \tag{12}$$

$$\mathcal{H}_{3,1}(f^{-1}) = 2A_2A_3A_4 - A_2^2A_5 - A_4^2 + A_3A_5 - A_3^3. \tag{13}$$

The logarithmic coefficients γ_n of $f \in \mathcal{S}$ are well discussed for the reason of their connection with the Bieberbach conjecture. They are presented by

$$F_f := \log\left(\frac{f(z)}{z}\right) = 2\sum_{n=1}^{\infty} \gamma_n z^n, \quad \log 1 = 0. \tag{14}$$

The idea of taking γ_n as the entries of the Hankel determinant was first proposed in [35] and later widely accepted by researchers [36–39]. Based on existing representation methods, the second Hankel determinant of logarithmic coefficients is denoted by

$$\mathcal{H}_{2,1}\left(F_f/2\right) = \gamma_1\gamma_3 - \gamma_2^2, \tag{15}$$

$$\mathcal{H}_{2,2}\left(F_f/2\right) = \gamma_2\gamma_4 - \gamma_3^2. \tag{16}$$

In [40], Ponnusamy et al. first introduced the concept of logarithmic coefficients of inverse functions. It is defined by

$$\log\left(\frac{F(w)}{w}\right) = 2\sum_{n=1}^{\infty} \varpi_n w^n, \quad |w| < \frac{1}{4}. \tag{17}$$

To broaden the fields on coefficient problems for univalent functions, it is bound to consider the Hankel determinant, with a_n replaced by ϖ_n; see [41–43]. Using this idea, we have

$$\mathcal{H}_{2,1}\left(F_{f^{-1}}/2\right) = \varpi_1\varpi_3 - \varpi_3^2, \tag{18}$$

$$\mathcal{H}_{2,2}\left(F_{f^{-1}}/2\right) = \varpi_2\varpi_4 - \varpi_3^2. \tag{19}$$

Taking a_n to express the Hankel determinant $\mathcal{H}_{2,3}(f^{-1})$ and $\mathcal{H}_{3,1}(f^{-1})$, it is calculated that

$$\begin{aligned}\mathcal{H}_{2,3}(f^{-1}) = {} & 3a_2^6 - 6a_2^4a_3 + 2a_2^3a_4 - 2a_2^2a_5 + 2a_2^2a_3^2 + 4a_2a_3a_4 \\ & + a_3a_5 - a_4^2 - 3a_3^3,\end{aligned} \tag{20}$$

$$\mathcal{H}_{3,1}(f^{-1}) = a_2^6 - 3a_2^4a_3 + 3a_2^2a_3^2 - a_2^2a_5 + 2a_2a_3a_4 + a_3a_5 - a_4^2 - 2a_3^3. \tag{21}$$

Differentiating (14) and using (1), we may obtain the correspondence between a_n and γ_n of f. Substituting γ_n with a_n leads to

$$\mathcal{H}_{2,2}\left(F_f/2\right) = \frac{1}{288}a_2^6 - \frac{1}{48}a_2^4a_3 - \frac{1}{24}a_2^3a_4 + \frac{1}{16}a_2^2a_3^2 - \frac{1}{8}a_2^2a_5 + \frac{1}{4}a_2a_3a_4 + \frac{1}{4}a_3a_5 - \frac{1}{4}a_4^2 - \frac{1}{8}a_3^3. \tag{22}$$

In [44], it is shown that

$$\mathcal{H}_{2,2}\left(F_{f^{-1}}/2\right) = \frac{145}{288}a_2^6 - \frac{55}{48}a_2^4a_3 + \frac{5}{24}a_2^3a_4 + \frac{11}{16}a_2^2a_3^2 - \frac{5}{8}a_2^2a_5 + \frac{3}{4}a_2a_3a_4 + \frac{1}{4}a_3a_5 - \frac{1}{4}a_4^2 - \frac{5}{8}a_3^3. \tag{23}$$

When $a_2 = 0$, it is noted that

$$\mathcal{H}_{2,3}(f) = a_3a_5 - a_4^2, \tag{24}$$

$$\mathcal{H}_{3,1}(f) = a_3a_5 - a_4^2 - a_3^3, \tag{25}$$

$$\mathcal{H}_{2,3}\left(f^{-1}\right) = a_3a_5 - a_4^2 - 3a_3^3, \tag{26}$$

$$\mathcal{H}_{3,1}\left(f^{-1}\right) = a_3a_5 - a_4^2 - 2a_3^3, \tag{27}$$

$$\mathcal{H}_{2,2}\left(F_f/2\right) = \frac{1}{4}\left(a_3a_5 - a_4^2 - \frac{1}{2}a_3^3\right), \tag{28}$$

$$\mathcal{H}_{2,2}\left(F_{f^{-1}}/2\right) = \frac{1}{4}\left(a_3a_5 - a_4^2 - \frac{5}{2}a_3^3\right). \tag{29}$$

It is interesting that they are all connected with $a_3a_5 - a_4^2 - \mu a_3^3$, where μ is a real number, and $\mu \geq 0$. Thus, we may expect this expression as a new functional of analytic functions.

Let $\mu \in [0, +\infty)$ and $f \in \mathcal{A}$ be of the form (1). Define

$$\mathcal{H}_\mu(f) = a_3a_5 - a_4^2 - \mu a_3^3, \tag{30}$$

where $a_n := \frac{f^{(n)}(0)}{n!}$ for $n \geq 2$. For different choices of the parameter μ, this functional may be used as a unified tool to give the upper bound of a certain Hankel determinant.

In this article, we aim to study the sharp bounds on the new functional $\mathcal{H}_\mu(f)$ for the functions in the classes $\mathcal{S}_q^*(\text{sech})$ and $\mathcal{K}_q(\text{sech})$. As a consequence, some useful results on the bounds of the second and third Hankel determinants with different entries are obtained.

2. Lemmas

In this section, we list two crucial lemmas that will be applied to investigate the main results of this work. As we know, an efficient way to solve coefficient problems for various classes of analytic functions is to associate them with the coefficients of Carathéodory functions. The first lemma is frequently used, as it provides a parametric representation of some initial coefficients for Carathéodory functions.

Lemma 1 (see [45])**.** *Let $p \in \mathcal{P}$ be of the form (1), and let $c_1 \geq 0$. Then, for some $x, \kappa \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$,*

$$2c_2 = c_1^2 + \left(4 - c_1^2\right)x, \tag{31}$$

$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)\kappa. \tag{32}$$

Lemma 2 (see [46])**.** *Let τ_1, τ_2 , $\tau_3 \in \mathbb{R}$, and be defined as*

$$U(\tau_1,\tau_2,\tau_3)=\max_{z\in\overline{\mathbb{D}}}\left\{\left|\tau_1+\tau_2 z+\tau_3 z^2\right|+1-|z|^2\right\}. \tag{33}$$

If $\tau_1 \le 0$ and $\tau_3 < 0$, then

$$U(\tau_1,\tau_2,\tau_3)=\begin{cases} -\tau_1+|\tau_2|-\tau_3, & if\ |\tau_2|\ge 2(1+\tau_3),\\ 1-\tau_1+\frac{\tau_2^2}{4(1+\tau_3)}, & if\ |\tau_2|<2(1+\tau_3).\end{cases}$$

If $\tau_1 > 0$ and $\tau_3 < 0$, then

$$U(\tau_1,\tau_2,\tau_3)=\begin{cases} 1-\tau_1+\frac{\tau_2^2}{4(1+\tau_3)}, & if\ \tau_2^2\ge -\frac{4\tau_1\tau_3^3}{1-\tau_3^2},\ |\tau_2|<2(1+\tau_3),\\ 1+\tau_1+\frac{\tau_2^2}{4(1-\tau_3)}, & if\ \tau_2^2<\min\left\{4(1-\tau_3)^2,-\frac{4\tau_1\tau_3^3}{1-\tau_3^2}\right\},\\ V(\tau_1,\tau_2,\tau_3), & otherwise,\end{cases}$$

where

$$V(\tau_1,\tau_2,\tau_3)=\begin{cases} \tau_1+|\tau_2|+\tau_3, & if\ -\tau_3(4\tau_1+|\tau_2|)\le\tau_1|\tau_2|,\\ -\tau_1+|\tau_2|-\tau_3, & if\ -\tau_3(-4\tau_1+|\tau_2|)\ge\tau_1|\tau_2|,\\ (\tau_1-\tau_3)\sqrt{1-\frac{\tau_2^2}{4\tau_1\tau_3}}, & otherwise.\end{cases} \tag{34}$$

3. Main Results

At first, we will discuss the upper bound of $\mathcal{H}_\mu(f)$ for functions in the class $\mathcal{S}_q^*(\text{sech})$.

Theorem 1. *If $f \in \mathcal{S}_q^*(\text{sech})$, then*

$$\left|\mathcal{H}_\mu(f)\right|\le\begin{cases} \frac{q^2}{3072[2]_q^3[3]_q^2[4]_q}\max_{t\in[0,2]}\left\{\Lambda_0 t^6+\Lambda_2 t^4+\Lambda_3 t^2\right\}, & if\ \mu\in\left[0,\frac{6[2]_q+5q[2]_q^2}{6[4]_q}\right],\\ \frac{q^2}{3072[2]_q^3[3]_q^2[4]_q}\max_{t\in[0,2]}\left\{\Lambda_1 t^6+\Lambda_2 t^4+\Lambda_3 t^2\right\}, & if\ \mu\in\left(\frac{6[2]_q+5q[2]_q^2}{6[4]_q},+\infty\right),\end{cases}$$

where

$$\begin{aligned}\Lambda_0&=48[2]_q^3[4]_q-36[2]_q^2[3]_q^2+6q[2]_q[3]_q^2+5q^2[2]_q^2[3]_q^2-6\mu q[3]_q^2[4]_q,\\ \Lambda_1&=48[2]_q^3[4]_q-36[2]_q^2[3]_q^2-6q[2]_q[3]_q^2-5q^2[2]_q^2[3]_q^2+6\mu q[3]_q^2[4]_q,\\ \Lambda_2&=192[2]_q^2[3]_q^2-384[2]_q^3[4]_q,\\ \Lambda_3&=768[2]_q^3[4]_q-192[2]_q^2[3]_q^2.\end{aligned}$$

Proof. Let $f \in \mathcal{S}_q^*(\text{sech})$. According to the subordination principle, there is a Schwarz function ω such that

$$\frac{zD_q f(z)}{f(z)}=\text{sech}(q\omega(z)),\quad z\in\mathbb{D}.$$

Taking

$$p(z)=\frac{1+\omega(z)}{1-\omega(z)}=1+c_1z+c_2z^2+c_3z^3+c_4z^4+\cdots,\quad z\in\mathbb{D},$$

it is seen that $p \in \mathcal{P}$, and

$$\omega(z) = \frac{c_1}{2}z + \frac{2c_2 - c_1^2}{4}z^2 + \frac{4c_3 - 4c_2c_1 + c_1^3}{8}z^3 + \frac{8c_4 - 8c_3c_1 - 4c_2^2 + 6c_1^2c_2 - c_1^4}{16}z^4 + \cdots, \quad z \in \mathbb{D}.$$

Then,

$$\operatorname{sech}(q\omega(z)) = 1 - \frac{1}{8}q^2c_1^2z^2 + \frac{1}{8}q^2\left(-2c_1c_2 + c_1^3\right)z^3 + \frac{1}{384}q^2\left[144c_1^2c_2 - 96c_1c_3 - 48c_2^2 - \left(36 - 5q^2\right)c_1^4\right]z^4 + \cdots, \quad z \in \mathbb{D}. \tag{35}$$

Using the form of (1), we obtain

$$\frac{zD_qf(z)}{f(z)} = 1 + qa_2z + q\left([2]_qa_3 - a_2^2\right)z^2 + q\left[[3]_qa_4 - (1+[2]_q)a_2a_3 + a_2^3\right]z^3 + q\left[[4]_qa_5 - (1+[3]_q)a_2a_4 - [2]_qa_3^2 + (2+[2]_q)a_2^2a_3 - a_2^4\right]z^4 + \cdots, \quad z \in \mathbb{D}. \tag{36}$$

By comparing the coefficients on the right side of (35) and (36), we have

$$a_2 = 0, \tag{37}$$

$$a_3 = -\frac{q}{8[2]_q}c_1^2, \tag{38}$$

$$a_4 = -\frac{q}{8[3]_q}\left(2c_1c_2 - c_1^3\right), \tag{39}$$

$$a_5 = \frac{q}{384[4]_q}\left[-\left(36 - 5q^2 - \frac{6q}{[2]_q}\right)c_1^4 + 144c_1^2c_2 - 96c_1c_3 - 48c_2^2\right]. \tag{40}$$

Let $f \in \mathcal{S}_q^*(\operatorname{sech})$ and $f_\theta(z) = e^{-i\theta}f(e^{i\theta}z)$, with $\theta \in \mathbb{R}$. Noting that

$$\frac{zD_qf_\theta(z)}{f_\theta(z)} = \frac{z \cdot \frac{e^{-i\theta}f\left(e^{i\theta}qz\right) - e^{-i\theta}f\left(e^{i\theta}z\right)}{(q-1)z}}{e^{-i\theta}f\left(e^{i\theta}z\right)} = \frac{e^{i\theta}z \cdot \frac{f\left(e^{i\theta}qz\right) - f\left(e^{i\theta}z\right)}{e^{i\theta}(q-1)z}}{f\left(e^{i\theta}z\right)} = \frac{e^{i\theta}zD_qf\left(e^{i\theta}z\right)}{f\left(e^{i\theta}z\right)} \in \mathcal{S}_q^*(\operatorname{sech})$$

and

$$H_\mu(f_\theta) = e^{4i\theta}H_\mu(f),$$

the functional $\left|\mathcal{H}_\mu(f_\theta)\right| = \left|\mathcal{H}_\mu(f)\right|$ for all $\theta \in \mathbb{R}$ and $f \in \mathcal{S}_q^*(\operatorname{sech})$. Hence, we are able to assume $c_1 = c \in [0,2]$ in estimating the upper bound of $\left|\mathcal{H}_\mu(f)\right|$ for $f \in \mathcal{S}_q^*(\operatorname{sech})$. Using (38)–(40) and (30), we obtain

$$\mathcal{H}_\mu(f) = \frac{q^2c^2}{3072[2]_q^3[3]_q^2[4]_q}\left(\alpha_1c^4 + \alpha_2c^2c_2 + \alpha_3cc_3 + \alpha_4c_2^2\right), \tag{41}$$

where

$$\begin{aligned}
\alpha_1 &= 6\mu q[3]_q^2[4]_q - 48[2]_q^3[4]_q + 36[2]_q^2[3]_q^2 - 6q[2]_q[3]_q^2 - 5q^2[2]_q^2[3]_q^2, \\
\alpha_2 &= 48[2]_q^2\left(4[2]_q[4]_q - 3[3]_q^2\right), \\
\alpha_3 &= 96[2]_q^2[3]_q^2, \\
\alpha_4 &= 48[2]_q^2\left([3]_q^2 - 4[2]_q[4]_q\right).
\end{aligned}$$

Using Lemma 1 and substituting c_2 and c_3 into (41) gives

$$H_\mu(f) = \frac{q^2c^2}{3072[2]_q^3[3]_q^2[4]_q}\left[\beta_1 + \beta_2 x + \beta_3 x^2 + 48[2]_q^2[3]_q^2 c(4-c^2)\left(1-|x|^2\right)\kappa\right],$$

where $x, \kappa \in \overline{\mathbb{D}}$, and

$$\begin{aligned}
\beta_1 &= q[3]_q^2\left(6\mu[4]_q - 6[2]_q - 5q[2]_q^2\right)c^4,\\
\beta_2 &= 0,\\
\beta_3 &= -12[2]_q^2\left(4-c^2\right)\left[4\left(4[2]_q[4]_q - [3]_q^2\right) + \left(3[3]_q^2 - 4[2]_q[4]_q\right)c^2\right].
\end{aligned}$$

When $c = 0$, it is clear that $\mathcal{H}_\mu(f) = 0$. When $c = 2$,

$$\mathcal{H}_\mu(f) = \frac{q^3\left(6\mu[4]_q - 6[2]_q - 5q[2]_q^2\right)}{48[2]_q^3[4]_q}, \quad c = 2. \tag{42}$$

Now we assume that $c \in (0,2)$. By taking $|\kappa| \le 1$, it is achieved that

$$\begin{aligned}
\left|\mathcal{H}_\mu(f)\right| &\le \frac{q^2c^3(4-c^2)}{64[2]_q[4]_q}\left(\left|\sigma_1 + \sigma_2 x + \sigma_3 x^2\right| + 1 - |x|^2\right)\\
&:= \frac{q^2c^3(4-c^2)}{64[2]_q[4]_q}U(\sigma_1,\sigma_2,\sigma_3),
\end{aligned} \tag{43}$$

where U is defined in (33), and

$$\begin{aligned}
\sigma_1 &= \frac{\left(6\mu[4]_q - 6[2]_q - 5q[2]_q^2\right)qc^3}{48[2]_q^2(4-c^2)},\\
\sigma_2 &= 0,\\
\sigma_3 &= -\frac{16[2]_q[4]_q - 4[3]_q^2 + \left(3[3]_q^2 - 4[2]_q[4]_q\right)c^2}{4[3]_q^2 c}.
\end{aligned}$$

Since

$$3[3]_q^2 - 4[2]_q[4]_q = -q^4 - 2q^3 + q^2 - 2q - 1 < 0 \tag{44}$$

for all $q \in (0,1)$, we have

$$16[2]_q[4]_q - 4[3]_q^2 + \left(3[3]_q^2 - 4[2]_q[4]_q\right)c^2 > 16[2]_q[4]_q - 4[3]_q^2 + 4\left(3[3]_q^2 - 4[2]_q[4]_q\right) = 8[3]_q^2 > 0$$

for $c \in (0,2)$. Thus, $\sigma_3 < 0$. From (44), it is obvious that $[3]_q^2 - 4[2]_q[4]_q < 0$. Hence,

$$1 + \sigma_3 = \frac{(2-c)\left[2\left([3]_q^2 - 4[2]_q[4]_q\right) + \left(3[3]_q^2 - 4[2]_q[4]_q\right)c\right]}{4[3]_q^2 c} < 0,$$

which means that $\sigma_3 < -1$.

When $\mu \le \frac{6[2]_q + 5q[2]_q^2}{6[4]_q}$, $\sigma_1 \le 0$. It is observed that $\sigma_1 \le 0$, $\sigma_3 < 0$, and $|\sigma_2| \ge 2(1+\sigma_3)$; thus, an application of Lemma 2 yields

$$U(\sigma_1,\sigma_2,\sigma_3) \le -\sigma_1 + |\sigma_2| - \sigma_3 = \frac{\psi_1 c^4 + \psi_2 c^2 + \psi_3}{48[2]_q^2[3]_q^2 c(4-c^2)}, \tag{45}$$

where

$$\psi_1 = 48[2]_q^3[4]_q - 36[2]_q^2[3]_q^2 + 6q[2]_q[3]_q^2 + 5q^2[2]_q^2[3]_q^2 - 6\mu q[3]_q^2[4]_q, \tag{46}$$
$$\psi_2 = 192[2]_q^2[3]_q^2 - 384[2]_q^3[4]_q, \tag{47}$$
$$\psi_3 = 768[2]_q^3[4]_q - 192[2]_q^2[3]_q^2. \tag{48}$$

Combining (43) and (45), we conclude that

$$|\mathcal{H}_\mu(f)| \leq \frac{q^2}{3072[2]_q^3[3]_q^2[4]_q}\left(\psi_1 c^6 + \psi_2 c^4 + \psi_3 c^2\right), \quad c \in (0,2). \tag{49}$$

Let

$$\Phi(t) = \frac{q^2}{3072[2]_q^3[3]_q^2[4]_q}\left(\psi_1 t^6 + \psi_2 t^4 + \psi_3 t^2\right), \quad t \in [0,2], \tag{50}$$

where ψ_1, ψ_2, and ψ_3 are given, respectively, by (46)–(48). Based on (42), (49), and

$$\Phi(2) = -\frac{q^3\left(6\mu[4]_q - 6[2]_q - 5q[2]_q^2\right)}{48[2]_q^3[4]_q}, \tag{51}$$

we obtain

$$|\mathcal{H}_\mu(f)| \leq \max_{t\in[0,2]}\{\Phi(t)\}, \quad \mu \in \left[0, \frac{6[2]_q + 5q[2]_q^2}{6[4]_q}\right], \tag{52}$$

where Φ is defined in (50).

In the following, we consider $\mu > \frac{6[2]_q+5q[2]_q^2}{6[4]_q}$. In this case, $\sigma_1 > 0$, and $\sigma_3 < 0$. Furthermore, as $\sigma_3 < -1$, it is known that $-\frac{4\sigma_1\sigma_3^3}{1-\sigma_3^2} < 0$. Then, we have $|\sigma_2| \geq 2(1+\sigma_3)$ and $\sigma_2^2 \geq -\frac{4\sigma_1\sigma_3^3}{1-\sigma_3^2}$. Applying Lemma 2, it is seen that

$$U(\sigma_1, \sigma_2, \sigma_3) = V(\sigma_1, \sigma_2, \sigma_3), \tag{53}$$

where V is defined in (34). Noting that $-\sigma_3(4\sigma_1 + |\sigma_2|) = -4\sigma_1\sigma_3 > 0$ and $-\sigma_3(-4\sigma_1 + |\sigma_2|) = 4\sigma_1\sigma_3 < 0$, we have

$$V(\sigma_1, \sigma_2, \sigma_3) \leq (\sigma_1 - \sigma_3)\sqrt{1 - \frac{\sigma_2^2}{4\sigma_1\sigma_3}} = \frac{\xi_1 c^4 + \xi_2 c^2 + \xi_3}{48[2]_q^2[3]_q^2 c(4-c^2)}, \tag{54}$$

where

$$\xi_1 = 48[2]_q^3[4]_q - 36[2]_q^2[3]_q^2 - 6q[2]_q[3]_q^2 - 5q^2[2]_q^2[3]_q^2 + 6\mu q[3]_q^2[4]_q, \tag{55}$$
$$\xi_2 = 192[2]_q^2[3]_q^2 - 384[2]_q^3[4]_q, \tag{56}$$
$$\xi_3 = 768[2]_q^3[4]_q - 192[2]_q^2[3]_q^2. \tag{57}$$

From (43), (53), and (54), we obtain

$$|\mathcal{H}_\mu(f)| \leq \frac{q^2}{3072[2]_q^3[3]_q^2[4]_q}\left(\xi_1 c^6 + \xi_2 c^4 + \xi_3 c^2\right), \quad c \in (0,2). \tag{58}$$

Define

$$\Psi(t) = \frac{q^2}{3072[2]_q^3[3]_q^2[4]_q}\left(\xi_1 t^6 + \xi_2 t^4 + \xi_3 t^2\right), \quad t \in [0,2], \tag{59}$$

where ξ_1, ξ_2, and ξ_3 are given by (55)–(57). As

$$\Psi(2) = \frac{q^3\left(6\mu[4]_q - 6[2]_q - 5q[2]_q^2\right)}{48[2]_q^3[4]_q},$$

using (42) and (58), we have

$$\left|\mathcal{H}_\mu(f)\right| \le \max_{t\in[0,2]}\{\Psi(t)\}, \quad \mu \in \left(\frac{6[2]_q + 5q[2]_q^2}{6[4]_q}, +\infty\right), \tag{60}$$

where Ψ is defined in (59). Combining (52) and (60), we obtain the inequalities in Theorem 1. The proof is completed. □

Taking $q \to 1^-$ in Theorem 1, we obtain the upper bound of $\mathcal{H}_\mu(f)$ for $f \in \mathcal{S}^*(\text{sech})$.

Theorem 2. *If $f \in \mathcal{S}^*(\text{sech})$, then*

$$\left|\mathcal{H}_\mu(f)\right| \le \begin{cases} \frac{1}{36864}\max_{t\in[0,2]}\{\Gamma_1(t)\}, & \text{if } \mu \in \left[0, \frac{4}{3}\right], \\ \frac{1}{36864}\max_{t\in[0,2]}\{\Gamma_2(t)\}, & \text{if } \mu \in \left(\frac{4}{3}, +\infty\right), \end{cases}$$

where Γ_1 and Γ_2 are defined, respectively, by

$$\Gamma_1(t) = (22 - 9\mu)t^6 - 224t^4 + 736t^2, \quad t \in [0,2],$$
$$\Gamma_2(t) = (9\mu - 2)t^6 - 224t^4 + 736t^2, \quad t \in [0,2].$$

Proof. Setting $q \to 1^-$ in Theorem 1, it is seen that $\Lambda_0 \to 528 - 216\mu$, $\Lambda_1 \to 216\mu - 48$, $\Lambda_2 \to -5376$, and $\Lambda_3 \to 17664$. Also, $\frac{q^2}{3072[2]_q^3[3]_q^2[4]_q} \to \frac{1}{884736}$, and $\frac{6[2]_q+5q[2]_q^2}{6[4]_q} \to \frac{4}{3}$. Substituting these results, the assertion in Theorem 2 follows. □

By $\mathcal{H}_{2,3}(f) = a_3a_5 - a_4^2$, taking $\mu = 0$ in Theorem 2 yields the upper bound of the second Hankel determinant for $f \in \mathcal{S}^*(\text{sech})$.

Corollary 1. *Let $f \in \mathcal{S}^*(\text{sech})$. Then,*

$$|\mathcal{H}_{2,3}(f)| \le \frac{10051}{470448} = 0.021364\ldots. \tag{61}$$

The bound is sharp, with the extremal functions g_1 given by

$$g_1(z) = z\exp\left(\int_0^z \frac{\text{sech}\left(\frac{p_1(s)-1}{p_1(s)+1}\right) - 1}{s}ds\right), \quad z \in \mathbb{D}, \tag{62}$$

where

$$p_1(z) = \frac{1 + \varrho_1 z + z^2}{1 - z^2}, \quad z \in \mathbb{D} \tag{63}$$

and $\varrho_1 = \frac{2\sqrt{759}}{33} \approx 1.669694$.

Proof. From the definition, we know that $\mathcal{H}_{2,3}(f) = \mathcal{H}_0(f)$. Applying Theorem 2 yields

$$|\mathcal{H}_{2,3}(f)| \le \max_{t\in[0,2]} \frac{1}{36864}\left(22t^6 - 224t^4 + 736t^2\right).$$

Let $r_0(t) = \frac{1}{36864}(22t^6 - 224t^4 + 736t^2)$, with $t \in [0,2]$. Since r_0 has a maximum value $\frac{10051}{470448}$ attained at $\varrho_1 = \frac{2\sqrt{759}}{33}$, the inequality (61) in Corollary 1 is thus obtained. Now, we consider the sharpness. Taking the logarithmic derivative on both sides in (62), we obtain

$$\frac{zg_1'(z)}{g_1(z)} = \operatorname{sech}\left(\frac{p_1(z)-1}{p_1(z)+1}\right),$$

where p_1 is defined by (63). As $\varrho_1 \in [0,2]$, it is known that $p_1 \in \mathcal{P}$, and $g_1 \in \mathcal{S}^*(\operatorname{sech})$. In view of

$$g_1(z) = z - \frac{23}{132}z^3 - \frac{10\sqrt{759}}{3267}z^4 + \frac{1069}{13068}z^5 + \cdots, \quad z \in \mathbb{D},$$

we conclude that

$$|\mathcal{H}_{2,3}(g_1)| = \frac{10051}{470448}.$$

The proof of Corollary 1 is completed. □

Remark 1 (In [18], Theorem 2.7)**.** *It is asserted that the sharp bound of $\mathcal{H}_{2,3}(f)$ for $f \in \mathcal{S}^*(\operatorname{sech})$ is $\frac{1}{48}$. Indeed, a minor mistake occurs in their proof.*

Since $a_2 \equiv 0$ for $f \in \mathcal{S}^*(\operatorname{sech})$, we have $\mathcal{H}_{3,1}(f) = a_3a_5 - a_4^2 - a_3^3$. Choosing $\mu = 1$ in Theorem 2 gives the known result on the third Hankel determinant for $f \in \mathcal{S}^*(\operatorname{sech})$.

Corollary 2 ([18], Theorem 2.6)**.** *Let $f \in \mathcal{S}^*(\operatorname{sech})$. Then,*

$$|H_{3,1}(f)| \leq \frac{671\sqrt{1342} - 12460}{657072} = 0.018446\ldots.$$

The inequality is sharp, with the extremal function g_2 presented by

$$g_2(z) = z\exp\left(\int_0^z \frac{\operatorname{sech}\left(\frac{p_2(z)-1}{p_2(z)+1}\right) - 1}{s}ds\right), \quad z \in \mathbb{D},$$

where

$$p_2(z) = \frac{1 + \varrho_2 z + z^2}{1 - z^2}, \quad z \in \mathbb{D}$$

and $\varrho_2 = \frac{2}{39}\sqrt{2184 - 39\sqrt{1342}} \approx 1.409371$.

Regarding the Hankel determinant with entry of the inverse coefficients, it is noted that $\mathcal{H}_{2,3}(f^{-1}) = \mathcal{H}_3(f)$, and $\mathcal{H}_{3,1}(f^{-1}) = \mathcal{H}_2(f)$ for $f \in \mathcal{S}^*(\operatorname{sech})$. Hence, we may obtain the two existing outcomes by assigning $\mu = 3$ and $\mu = 2$ in Theorem 2, respectively.

Corollary 3 ([18], Theorem 3.4)**.** *Suppose that $f \in \mathcal{S}^*(\operatorname{sech})$. Then,*

$$\left|\mathcal{H}_{2,3}\left(f^{-1}\right)\right| \leq \frac{5}{192}.$$

The extremal function is given by

$$g_3(z) = z\exp\left(\int_0^z \frac{\operatorname{sech}(s) - 1}{s}ds\right), \quad z \in \mathbb{D}. \tag{64}$$

Corollary 4 ([18], Theorem 3.3)**.** *Suppose that $f \in \mathcal{S}^*(\operatorname{sech})$. Then,*

$$\left|\mathcal{H}_{3,1}\left(f^{-1}\right)\right| \leq \frac{77 + 29\sqrt{58}}{15552} = 0.019152\ldots.$$

Remark 2 (In [18], Theorem 3.3)**.** *The authors gave the upper bound of $\mathcal{H}_{2,3}(f^{-1})$ for $f \in \mathcal{S}^*(\mathrm{sech})$ while the extremal function is missing. In fact, the bound is sharp, with the function g_4 defined by*

$$g_4(z) = z \exp\left(\int_0^z \frac{\mathrm{sech}\left(i\frac{p_3(z)-1}{p_3(z)+1}\right) - 1}{s} ds \right), \quad z \in \mathbb{D}, \tag{65}$$

where

$$p_3(z) = \frac{1-z^2}{1+\varrho_3 z + z^2}, \quad z \in \mathbb{D}$$

and $\varrho_3 = \sqrt{\frac{14-\sqrt{58}}{3}} \approx 1.458793.$

Regarding the Hankel determinant with elements of logarithmic coefficients for $f \in \mathcal{S}^*(\mathrm{sech})$, we have $\mathcal{H}_{2,2}\left(F_f/2\right) = \frac{1}{4}\mathcal{H}_{\frac{1}{2}}(f)$. Thus, an application of Theorem 2 leads to the new finding on the upper bound of the second Hankel determinant for logarithmic functions.

Corollary 5. *Suppose that $f \in \mathcal{S}^*(\mathrm{sech})$. Then,*

$$\left|\mathcal{H}_{2,2}\left(F_f/2\right)\right| \leq \frac{3892+103\sqrt{721}}{1360800} = 0.0048924\ldots.$$

The result is sharp, with the extremal function g_5 presented by

$$g_5(z) = z \exp\left(\int_0^z \frac{\mathrm{sech}\left(\frac{p_4(z)-1}{p_4(z)+1}\right) - 1}{s} ds \right), \quad z \in \mathbb{D}, \tag{66}$$

where

$$p_4(z) = \frac{1+\varrho_4 z + z^2}{1-z^2}, \quad z \in \mathbb{D}$$

and

$$\varrho_4 = \sqrt{\frac{448-8\sqrt{721}}{105}} \approx 1.490249. \tag{67}$$

Proof. Let $f \in \mathcal{S}^*(\mathrm{sech})$. Taking $\mu = \frac{1}{2}$ in Theorem 2, we obtain

$$\left|\mathcal{H}_{2,2}\left(F_f/2\right)\right| = \left|\frac{1}{4}\mathcal{H}_{\frac{1}{2}}(f)\right| \leq \max_{t\in[0,2]} \frac{1}{147456}\left(\frac{35}{2}t^6 - 224t^4 + 736t^2\right).$$

Let $r_1(t) = \frac{1}{147456}\left(\frac{35}{2}t^6 - 224t^4 + 736t^2\right)$, with $t \in [0,2]$. The only critical point of r_1 in $(0,2)$ is ϱ_4 given in (67) at which r_1 attains its maximum value $\frac{3892+103\sqrt{721}}{1360800}$.

For the sharpness, it is seen that g_5 defined in (66) has the form

$$g_5(z) = z - \frac{\varrho_4^2}{16}z^3 - \frac{\varrho_4(4-\varrho_4^2)}{24}z^4 - \frac{7\varrho_4^4 - 48\varrho_4 + 48}{384}z^5 + \cdots, \quad z \in \mathbb{D}$$

and

$$\left|\mathcal{H}_{2,2}(F_{g_5}/2)\right| = \left|-\frac{1}{147456}\left(\frac{35}{2}\varrho_4^6 - 224\varrho_4^4 + 736\varrho_4^2\right)\right| = \frac{3892+103\sqrt{721}}{1360800}.$$

The proof of Corollary 5 is completed. □

In view of $\mathcal{H}_{2,2}\left(F_{f^{-1}}/2\right) = \frac{1}{4}\mathcal{H}_{\frac{5}{2}}(f)$ for $f \in \mathcal{S}^*(\text{sech})$, we are able to obtain the exact bound of the Hankel determinant with inverse logarithmic coefficients as input for functions in this group.

Corollary 6. *Suppose that* $f \in \mathcal{S}^*(\text{sech})$. *Then,*

$$\left|\mathcal{H}_{2,2}\left(F_{f^{-1}}/2\right)\right| \leq \frac{62020 + 307\sqrt{307}}{13071456} = 0.005156\ldots.$$

The equality is attained by the function g_6 *given by*

$$g_6(z) = z\exp\left(\int_0^z \frac{\text{sech}\left(i\frac{p(z)-1}{p(z)+1}\right) - 1}{s} ds\right), \quad z \in \mathbb{D}, \tag{68}$$

where

$$p(z) = \frac{1 - z^2}{1 + \varrho_5 z + z^2}, \quad z \in \mathbb{D}$$

and

$$\varrho_5 = \sqrt{\frac{448 - 8\sqrt{307}}{123}} \approx 1.581984. \tag{69}$$

Proof. Let $f \in \mathcal{S}^*(\text{sech})$. Using Theorem 2, we obtain

$$\left|\mathcal{H}_{2,2}\left(F_{f^{-1}}/2\right)\right| = \left|\frac{1}{4}\mathcal{H}_{\frac{5}{2}}(f)\right| \leq \max_{t\in[0,2]} \frac{1}{147456}\left(\frac{41}{2}t^6 - 224t^4 + 736t^2\right).$$

Let $r_2(t) = \frac{1}{147456}\left(\frac{41}{2}t^6 - 224t^4 + 736t^2\right)$, with $t \in [0,2]$. The unique critical point of r_2 in $(0,2)$ is ϱ_5 given in (69) at which r_2 achieves its maximum value $\frac{62020+307\sqrt{307}}{13071456}$.

For the sharpness, we note that g_6 defined in (68) has the expansion of

$$g_6(z) = z + \frac{\varrho_5^2}{16}z^3 + \frac{\varrho_5(4 - \varrho_5^2)}{24}z^4 + \frac{11\varrho_5^4 - 48\varrho_5^2 + 48}{384}, \quad z \in \mathbb{D}$$

and

$$\left|\mathcal{H}_{2,2}\left(F_{g_6^{-1}}/2\right)\right| = \left|-\frac{1}{147456}\left(\frac{41}{2}\varrho_5^6 - 224\varrho_5^4 + 736\varrho_5^2\right)\right| = \frac{62020 + 307\sqrt{307}}{13071456}.$$

The proof of Corollary 6 is then completed. □

Now, we aim to determine the bound of $\mathcal{H}_\mu(f)$ for $f \in \mathcal{K}_q(\text{sech})$.

Theorem 3. *Let* $f \in \mathcal{K}_q(\text{sech})$. *Then,*

$$|\mathcal{H}_\mu(f)| \leq \begin{cases} \frac{q^2}{3072[2]_q^3[3]_q^3[4]_q^2[5]_q}\max_{t\in[0,2]}\{\Pi_0 t^6 + \Pi_2 t^4 + \Pi_3 t^2\}, & if\ \mu \in \left[0, \frac{6[2]_q[3]_q^2 + 5q[2]_q^2[3]_q^2}{6[4]_q[5]_q}\right], \\ \frac{q^2}{3072[2]_q^3[3]_q^3[4]_q^2[5]_q}\max_{t\in[0,2]}\{\Pi_1 t^6 + \Pi_2 t^4 + \Pi_3 t^2\}, & if\ \mu \in \left(\frac{6[2]_q[3]_q^2 + 5q[2]_q^2[3]_q^2}{6[4]_q[5]_q}, +\infty\right), \end{cases}$$

where

$$\Pi_0(t) = 48[2]_q^3[3]_q[5]_q - 36[2]_q^2[3]_q^2[4]_q + 6q[2]_q[3]_q^2[4]_q + 5q^2[2]_q^2[3]_q^2[4]_q - 6\mu q[4]_q^2[5]_q,$$
$$\Pi_1(t) = 48[2]_q^3[3]_q[5]_q - 36[2]_q^2[3]_q^2[4]_q - 6q[2]_q[3]_q^2[4]_q - 5q^2[2]_q^2[3]_q^2[4]_q + 6\mu q[4]_q^2[5]_q,$$
$$\Pi_2(t) = 192[2]_q^2[3]_q^2[4]_q - 384[2]_q^3[3]_q[5]_q,$$
$$\Pi_3(t) = 768[2]_q^3[3]_q[5]_q - 192[2]_q^2[3]_q^2[4]_q.$$

Proof. Suppose that $f \in \mathcal{K}_q(\text{sech})$ and

$$f(z) = z + b_2z^2 + b_3z^3 + b_4z^4 + b_5z^5 + \cdots, \quad z \in \mathbb{D}.$$

Based on the relationship between the class $\mathcal{S}_q^*(\text{sech})$ and $\mathcal{K}_q(\text{sech})$, we know that $g(z) = zD_qf(z) \in \mathcal{S}_q^*(\text{sech})$. Thus, $b_n = \frac{a_n}{[n]_q} (n \geq 2)$, where a_n are the corresponding coefficients of $g \in \mathcal{S}_q^*(\text{sech})$. From the proof of Theorem 1, we can write

$$b_2 = 0, \tag{70}$$
$$b_3 = -\frac{q}{8[2]_q[3]_q}d_1^2, \tag{71}$$
$$b_4 = -\frac{q}{8[3]_q[4]_q}\left(2d_1d_2 - d_1^3\right), \tag{72}$$
$$b_5 = \frac{q}{384[4]_q[5]_q}\left[-\left(36 - 5q^2 - \frac{6q}{[2]_q}\right)d_1^4 + 144d_1^2d_2 - 96d_1d_3 - 48d_2^2\right] \tag{73}$$

for some $p \in \mathcal{P}$, with

$$p(z) = 1 + d_1z + d_2z^2 + d_3z^3 + \cdots, \quad z \in \mathbb{D}.$$

Let $f \in \mathcal{K}_q(\text{sech})$ and $f_\theta(z) = e^{-i\theta}f\left(e^{i\theta}z\right)$, with $\theta \in \mathbb{R}$. From the definition,

$$D_qf_\theta(z) = \frac{f_\theta(qz) - f_\theta(z)}{(q-1)z} = \frac{e^{-i\theta}f\left(e^{i\theta}qz\right) - e^{-i\theta}f\left(e^{i\theta}z\right)}{(q-1)z} = D_qf\left(e^{i\theta}z\right).$$

Setting $u = e^{i\theta}z$, it is noted that $D_qf_\theta(z) = D_qf(u)$, and thus, $D_q^2f_\theta(z) = e^{i\theta}D_q^2f(u)$. Using the basic property of the q-difference operator, we have

$$\frac{zD_q\left(zD_qf_\theta(z)\right)}{zD_qf_\theta(z)} = z \cdot \frac{qD_qf_\theta(z) + zD_q^2f_\theta(qz)}{zD_qf_\theta(z)} = z \cdot \frac{qD_qf(u) + uD_q^2f(u)}{zD_qf(u)} = \frac{uD_q\left(uD_qf(u)\right)}{uD_qf(u)}.$$

Thus, $f_\theta \in \mathcal{K}_q(\text{sech})$. As the class $\mathcal{K}_q(\text{sech})$ and the functional $\mathcal{H}_\mu(f)$ are rotation-invariant, we may assume that $d_1 = d \in [0,2]$. Substituting (71)–(73) into $\mathcal{H}_\mu(f)$ defined in (30), we obtain

$$\mathcal{H}_\mu(f) = \frac{q^2d^2}{3072[2]_q^3[3]_q^3[4]_q^2[5]_q}\left(\nu_1d^4 + \nu_2d^2d_2 + \nu_3dd_3 + \nu_4d_2^2\right),$$

where

$$\nu_1 = 36[2]_q^2[3]_q^2[4]_q - 48[2]_q^3[3]_q[5]_q - 6q[2]_q[3]_q^2[4]_q - 5q^2[2]_q^2[3]_q^2[4]_q + 6\mu q[4]_q^2[5]_q,$$
$$\nu_2 = -48[2]_q^2[3]_q\left(3[3]_q[4]_q - 4[2]_q[5]_q\right),$$
$$\nu_3 = 96[2]_q^2[3]_q^2[4]_q,$$
$$\nu_4 = 48[2]_q^2[3]_q\left([3]_q[4]_q - 4[2]_q[5]_q\right).$$

Using (58) and (60) in Lemma 1, we obtain

$$\mathcal{H}_\mu(f) = \frac{q^2 d^2}{3072[2]_q^3[3]_q^3[4]_q^2[5]_q}\left[\lambda_1 + \lambda_2 x + \lambda_3 x^2 + 48[2]_q^2[3]_q^2[4]_q d\left(4 - d^2\right)\left(1 - |x|^2\right)\kappa\right]$$

for some $x, \kappa \in \overline{\mathbb{D}}$, with

$$\lambda_1 = q[4]_q\left(6\mu[4]_q[5]_q - 6[2]_q[3]_q^2 - 5q[2]_q^2[3]_q^2\right)d^4,$$
$$\lambda_2 = 0,$$
$$\lambda_3 = -12[2]_q^2[3]_q\left(4 - d^2\right)\left[16[2]_q[5]_q - 4[3]_q[4]_q - \left(4[2]_q[5]_q - 3[3]_q[4]_q\right)d^2\right].$$

When $d = 0$, $\mathcal{H}_\mu(f) = 0$. When $d = 2$,

$$\mathcal{H}_\mu(f) = \frac{\left(6\mu[4]_q[5]_q - 6[2]_q[3]_q^2 - 5q[2]_q^2[3]_q^2\right)q^3}{48[2]_q^3[3]_q^3[4]_q[5]_q}, \quad d = 2. \tag{74}$$

Consider $d \in (0, 2)$. From $|\kappa| \leq 1$, we obtain

$$\left|\mathcal{H}_\mu(f)\right| \leq \frac{q^2 d^3 (4 - d^2)}{64[2]_q[3]_q[4]_q[5]_q}\left(\left|\varsigma_1 + \varsigma_2 x + \varsigma_3 x^2\right| + 1 - |x|^2\right)$$
$$:= \frac{q^2 d^3 (4 - d^2)}{64[2]_q[3]_q[4]_q[5]_q} U(\varsigma_1, \varsigma_2, \varsigma_3),$$

where U is defined in (33), and

$$\varsigma_1 = \frac{q\left(6\mu[4]_q[5]_q - 6[2]_q[3]_q^2 - 5q[2]_q^2[3]_q^2\right)d^3}{48[2]_q^2[3]_q^2(4 - d^2)}, \tag{75}$$

$$\varsigma_2 = 0, \tag{76}$$

$$\varsigma_3 = \frac{4[3]_q[4]_q - 16[2]_q[5]_q + \left(4[2]_q[5]_q - 3[3]_q[4]_q\right)d^2}{4[3]_q[4]_q d}. \tag{77}$$

It is easily seen that

$$4[2]_q[5]_q - 3[3]_q[4]_q = 1 + 2q - q^2 - q^3 + 2q^4 + q^5 > 0, \quad q \in (0, 1), \tag{78}$$

which leads to

$$4[3]_q[4]_q - 16[2]_q[5]_q + \left(4[2]_q[5]_q - 3[3]_q[4]_q\right)d^2 < 4[3]_q[4]_q - 16[2]_q[5]_q + 4\left(4[2]_q[5]_q - 3[3]_q[4]_q\right) < 0.$$

From (77), we have $\varsigma_3 < 0$. Indeed, using (78), it yields

$$2[3]_q[4]_q - 8[2]_q[5]_q + \left(3[3]_q[4]_q - 4[2]_q[5]_q\right)d < 2[3]_q[4]_q - 8[2]_q[5]_q < 0,$$

which implies that

$$1 + \varsigma_3 = \frac{(2 - d)\left[2[3]_q[4]_q - 8[2]_q[5]_q + \left(3[3]_q[4]_q - 4[2]_q[5]_q\right)d\right]}{4[3]_q[4]_q d} < 0,$$

i.e., $\varsigma_3 < -1$.

When $\mu \leq \frac{6[2]_q[3]_q^2+5q[2]_q^2[3]_q^2}{6[4]_q[5]_q}$, clearly, $\varsigma_1 \leq 0$. Since $|\varsigma_2| \geq 2(1+\varsigma_3)$, using Lemma 2, we obtain

$$U(\varsigma_1,\varsigma_2,\varsigma_3) \leq -\varsigma_1 + |\varsigma_2| - \varsigma_3 = \frac{\vartheta_1 d^4 + \vartheta_2 d^2 + \vartheta_3}{48[2]_q^2[3]_q^2[4]_q d(4-d^2)},$$

where

$$\begin{aligned}
\vartheta_1 &= 48[2]_q^3[3]_q[5]_q - 36[2]_q^2[3]_q^2[4]_q + 6q[2]_q[3]_q^2[4]_q + 5q^2[2]_q^2[3]_q^2[4]_q - 6\mu q[4]_q^2[5]_q,\\
\vartheta_2 &= 192[2]_q^2[3]_q^2[4]_q - 384[2]_q^3[3]_q[5]_q,\\
\vartheta_3 &= 768[2]_q^3[3]_q[5]_q - 192[2]_q^2[3]_q^2[4]_q.
\end{aligned}$$

Therefore,

$$\left|\mathcal{H}_\mu(f)\right| \leq \frac{q^2}{3072[2]_q^3[3]_q^3[4]_q^2[5]_q}\left(\vartheta_1 d^6 + \vartheta_2 d^4 + \vartheta_3 d^2\right), \quad d \in (0,2). \tag{79}$$

Define

$$\Upsilon(t) = \frac{q^2}{3072[2]_q^3[3]_q^3[4]_q^2[5]_q}\left(\vartheta_1 t^6 + \vartheta_2 t^4 + \vartheta_3 t^2\right), \quad t \in [0,2].$$

Clearly,

$$\Upsilon(2) = -\frac{\left(6\mu[4]_q[5]_q - 6[2]_q[3]_q^2 - 5q[2]_q^2[3]_q^2\right)q^3}{48[2]_q^3[3]_q^3[4]_q[5]_q}.$$

Combining (74) and (79), we conclude that

$$\left|\mathcal{H}_\mu(f)\right| \leq \max_{t\in[0,2]}\{\Upsilon(t)\}, \quad \mu \in \left[0, \frac{6[2]_q[3]_q^2 + 5q[2]_q^2[3]_q^2}{6[4]_q[5]_q}\right]. \tag{80}$$

Now, we consider $\mu > \frac{6[2]_q[3]_q^2+5q[2]_q^2[3]_q^2}{6[4]_q[5]_q}$. Then, $\varsigma_1 > 0$, and $\varsigma_3 < 0$. As $\varsigma_3 < -1$, we see that $1-\varsigma_3^2 < 0$, and thus, $\frac{-4\varsigma_1\varsigma_3^3}{1-\varsigma_3^2} < 0$. Combining the fact that $|\varsigma_2| \geq 2(1+\varsigma_3)$ and $\varsigma_2^2 \geq \frac{-4\varsigma_1\varsigma_3^3}{1-\varsigma_3^2}$, an application of Lemma 2 leads to $U(\varsigma_1,\varsigma_2,\varsigma_3) \leq V(\varsigma_1,\varsigma_2,\varsigma_3)$, where V is defined in (34). Obviously, $-\varsigma_3(4\varsigma_1 + |\varsigma_2|) > \varsigma_1|\varsigma_2|$, and $-\varsigma_3(-4\varsigma_1 + |\varsigma_2|) < \varsigma_1|\varsigma_2|$ because $\varsigma_1 > 0$, $\varsigma_2 = 0$, and $\varsigma_3 < 0$. Therefore, we may find that

$$V(\varsigma_1,\varsigma_2,\varsigma_3) \leq (\varsigma_1 - \varsigma_3)\sqrt{1 - \frac{\varsigma_2^2}{4\varsigma_1\varsigma_3}} = \frac{\iota_1 d^4 + \iota_2 d^2 + \iota_3}{48[2]_q^2][3]_q^2[4]_q d(4-d^2)},$$

where

$$\begin{aligned}
\iota_1 &= 48[2]_q^3[3]_q[5]_q - 36[2]_q^2[3]_q^2[4]_q - 6q[2]_q[3]_q^2[4]_q - 5q^2[2]_q^2[3]_q^2[4]_q + 6\mu q[4]_q^2[5]_q,\\
\iota_2 &= 192[2]_q^2[3]_q^2[4]_q - 384[2]_q^3[3]_q[5]_q,\\
\iota_3 &= 768[2]_q^3[3]_q[5]_q - 192[2]_q^2[3]_q^2[4]_q.
\end{aligned}$$

It follows that

$$\left|\mathcal{H}_\mu(f)\right| \leq \frac{q^2}{3072[2]_q^3[3]_q^3[4]_q^2[5]_q}\left(\iota_1 d^6 + \iota_2 d^4 + \iota_3 d^2\right), \quad d \in (0,2). \tag{81}$$

Define

$$\Xi(t) = \frac{q^2}{3072[2]_q^3[3]_q^3[4]_q^2[5]_q}\left(\iota_1 d^6 + \iota_2 d^4 + \iota_3 d^2\right), \quad t \in [0,2].$$

By $\Xi(2) = \frac{(6\mu[4]_q[5]_q - 6[2]_q[3]_q^2 - 5q[2]_q^2[3]_q^2)q^3}{48[2]_q^3[3]_q^3[4]_q[5]_q}$, along with (74) and (81), we obtain

$$\left|\mathcal{H}_\mu(f)\right| \le \max_{t\in[0,2]}\{\Xi(t)\}, \quad \mu \in \left(\frac{6[2]_q[3]_q^2 + 5q[2]_q^2[3]_q^2}{6[4]_q[5]_q}, +\infty\right]. \tag{82}$$

Combining (80) and (82), the assertion in Theorem 3 follows. □

Let $q \to 1^-$; then, we obtain the estimation on the functional $\mathcal{H}_\mu(f)$ for functions in the class $\mathcal{K}(\text{sech})$, which is a subfamily of convex functions.

Theorem 4. *Let $f \in \mathcal{K}(\text{sech})$. Then,*

$$\left|\mathcal{H}_\mu(f)\right| \le \begin{cases} \frac{1}{552960}\max_{t\in[0,2]}\{\Gamma_3(t)\}, & \text{if } \mu \in \left[0, \frac{12}{5}\right], \\ \frac{1}{552960}\max_{t\in[0,2]}\{\Gamma_4(t)\}, & \text{if } \mu \in \left(\frac{12}{5}, +\infty\right), \end{cases}$$

where Γ_3 and Γ_4 are defined, respectively, by

$$\Gamma_3(t) = (18 - 5\mu)t^6 - 192t^4 + 672t^2, \quad t \in [0,2],$$
$$\Gamma_4(t) = (5\mu - 6)t^6 - 192t^4 + 672t^2, \quad t \in [0,2].$$

Proof. Let $q \to 1^-$ in Theorem 3. Then, $\Pi_0 \to 1728 - 480\mu$, $\Pi_1 \to -576 + 480\mu$, $\Pi_2 \to -18432$, and $\Pi_3 \to 64512$. Also, $\frac{q^2}{3072[2]_q^3[3]_q^3[4]_q^2[5]_q} \to \frac{1}{53084160}$, and $\frac{6[2]_q[3]_q^2+5q[2]_q^2[3]_q^2}{6[4]_q[5]_q} \to \frac{12}{5}$. Substituting these results, the assertion in Theorem 4 follows. □

Taking $\mu = 0,\ 1,\ 3,\ 2,\ \frac{1}{2},\ \frac{5}{2}$ in Theorem 4, we obtain the sharp bounds of the Hankel determinant with the original coefficients, inverse coefficients, logarithmic coefficients, and inverse logarithmic coefficients, respectively, for functions in the class $\mathcal{K}(\text{sech})$.

Corollary 7. *Let $f \in \mathcal{K}(\text{sech})$. Then,*

$$|\mathcal{H}_{2,3}(f)| \le \frac{49}{34992} = 0.001400\ldots. \tag{83}$$

The estimate is sharp with the extremal function h_1 expressed by

$$h_1(z) = \int_0^z \exp\left(\int_0^u \frac{\text{sech}\left(\frac{\widehat{p}_1(u)-1}{\widehat{p}_1(u)+1}\right) - 1}{u} du\right) ds, \quad z \in \mathbb{D}, \tag{84}$$

where

$$\widehat{p}_1(z) = \frac{1 + \frac{2\sqrt{7}}{3}z + z^2}{1 - z^2}, \quad z \in \mathbb{D}.$$

Proof. Suppose that $f \in \mathcal{K}(\text{sech})$. From Theorem 4, we know that

$$|\mathcal{H}_{2,3}(f)| = |\mathcal{H}_0(f)| \le \max_{t\in[0,2]} \frac{1}{552960}\left(18t^6 - 192t^4 + 672t^2\right).$$

Let $r_3(t) = \frac{1}{552960}\left(18t^6 - 192t^4 + 672t^2\right)$, with $t \in [0,2]$. It is seen that r_3 has a maximum value $\frac{49}{34992}$ achieved at $\widetilde{t_0} = \frac{2\sqrt{7}}{3}$. The inequality (83) in Corollary 7 is thus obtained.

For the sharpness, we observe that the function h_1 defined in (84), satisfying $zh_1'(z) \in \mathcal{S}^*(\mathrm{sech})$, which implies that $h_1 \in \mathcal{K}(\mathrm{sech})$ according to the Alexander relationship. We note that

$$h_1(z) = z - \frac{7}{108}z^3 - \frac{\sqrt{7}}{162}z^4 + \frac{17}{972}z^5 + \cdots, \quad z \in \mathbb{D}$$

and $|\mathcal{H}_{2,3}(h_1)| = \frac{49}{34992}$. This completes the proof of Corollary 7. □

Corollary 8. *Let $f \in \mathcal{K}(\mathrm{sech})$. Then,*

$$|H_{3,1}(f)| \leq \frac{136 + 37\sqrt{74}}{365040} = 0.001244\ldots. \tag{85}$$

The result is sharp, with the extremal function h_2 presented by

$$h_2(z) = \int_0^z \exp\left(\int_0^u \frac{\mathrm{sech}\left(\frac{\widehat{p}_2(u)-1}{\widehat{p}_2(u)+1}\right) - 1}{u} du\right) ds, \quad z \in \mathbb{D}, \tag{86}$$

where

$$\widehat{p}_2(z) = \frac{1 + \chi_1 z + z^2}{1 - z^2}, \quad z \in \mathbb{D}$$

and

$$\chi_1 = \sqrt{\frac{64 - 4\sqrt{74}}{13}} \approx 1.508710. \tag{87}$$

Proof. Assume that $f \in \mathcal{K}(\mathrm{sech})$. Utilizing Theorem 4, we obtain

$$|H_{3,1}(f)| = |H_1(f)| \leq \max_{t \in [0,2]} \frac{1}{552960}\left(13t^6 - 192t^4 + 62t^2\right).$$

Define $r_4(t) = \frac{1}{552960}\left(13t^6 - 192t^4 + 62t^2\right)$ with $t \in [0,2]$. It is found that r_4 has a maximum value $\frac{136+37\sqrt{74}}{365040}$ attained at χ_1, which is given in (87). The inequality (85) in Corollary 8 is thus obtained.

For the extremal function, clearly, h_2 defined in (86) belongs to the class $\mathcal{K}(\mathrm{sech})$ by $\widehat{p}_2 \in \mathcal{P}$. As

$$h_2(z) = z - \frac{\chi_1^2}{48}z^3 - \frac{\chi_1(4 - \chi_1^2)}{96}z^4 - \frac{7\chi_1^4 - 48\chi_1^2 - 48}{1920}z^5 + \cdots, \quad z \in \mathbb{D},$$

we have

$$|H_{3,1}(h_2)| = \left|-\frac{1}{552960}\left(13\chi_1^6 - 192\chi_1^4 + 672\chi_1^2\right)\right| = \frac{136 + 37\sqrt{74}}{365040}.$$

The proof of Corollary 8 is thus completed. □

Corollary 9. *Let $f \in \mathcal{K}(\mathrm{sech})$. Then,*

$$\left|\mathcal{H}_{2,3}\left(f^{-1}\right)\right| \leq \frac{65\sqrt{130} - 536}{174960} = 0.001172\ldots.$$

The equality is attained by the function h_3 given by

$$h_3(z) = \int_0^z \exp\left(\int_0^u \frac{\operatorname{sech}\left(i\frac{\widehat{p}_3(u)-1}{\widehat{p}_3(u)+1}\right) - 1}{u} du\right) ds, \quad z \in \mathbb{D}, \tag{88}$$

where

$$\widehat{p}_3(z) = \frac{1 - z^2}{1 + \chi_2 z + z^2}, \quad z \in \mathbb{D}$$

and

$$\chi_2 = \frac{2\sqrt{16 - \sqrt{130}}}{3} \approx 1.429568. \tag{89}$$

Proof. Let $f \in \mathcal{K}(\operatorname{sech})$. It is seen that

$$\left|\mathcal{H}_{2,3}\left(f^{-1}\right)\right| = |\mathcal{H}_3(f)| \leq \max_{t\in[0,2]} \frac{1}{552960}\left(9t^6 - 192t^4 + 672t^2\right).$$

Setting $r_5(t) = \frac{1}{552960}\left(9t^6 - 192t^4 + 672t^2\right)$, with $t \in [0,2]$, it is calculated that r_5 has a maximum value $\frac{65\sqrt{130}-536}{174960}$ achieved at χ_2, which is given in (89).

For the sharpness, we observe that h_3 presented in (88) belongs to the class $\mathcal{K}(\operatorname{sech})$. In view of

$$h_3(z) = z + \frac{\chi_2^2}{48}z^3 + \frac{\chi_2\left(4 - \chi_2^2\right)}{96}z^4 + \frac{11\chi_2^4 - 48\chi_2^2 + 48}{1920}z^5 + \cdots, \quad z \in \mathbb{D}$$

and

$$\left|H_{2,3}\left(h_3^{-1}\right)\right| = \left|-\frac{1}{184320}\left(3\chi_2^6 - 64\chi_1^4 + 224\chi_1^2\right)\right| = \frac{65\sqrt{130} - 536}{174960},$$

we complete the proof of Corollary 9. □

Corollary 10. *Suppose that $f \in \mathcal{K}(\operatorname{sech})$. Then,*

$$\left|\mathcal{H}_{3,1}\left(f^{-1}\right)\right| \leq \frac{1}{864} = 0.001157\ldots.$$

The equality holds for the function h_4 defined as

$$h_4(z) = \int_0^z \exp\left(\int_0^u \frac{\operatorname{sech}\left(\frac{\widehat{p}_4(u)-1}{\widehat{p}_4(u)+1}\right) - 1}{u} du\right) ds, \quad z \in \mathbb{D} \tag{90}$$

with

$$\widehat{p}_4(z) = \frac{1 + \sqrt{2}z + z^2}{1 - z^2}, \quad z \in \mathbb{D}.$$

Proof. Applying Theorem 4, we see that

$$\left|\mathcal{H}_{3,1}\left(f^{-1}\right)\right| = |\mathcal{H}_2(f)| \leq \max_{t\in[0,2]} \frac{1}{552960}\left(8t^6 - 192t^4 + 672t^2\right).$$

Let $r_6(t) = \frac{1}{552960}\left(8t^6 - 192t^4 + 672t^2\right)$ with $t \in [0,2]$. The only critical point of r_6 in $(0,2)$ is $\sqrt{2}$, and $r_6(t) \leq r_6\left(\sqrt{2}\right) = \frac{1}{864}$ for all $t \in [0,2]$.

For the equality, it is easy to know that h_4 given in (90) belongs to the class $\mathcal{K}(\operatorname{sech})$. Since

$$h_4(z) = z - \frac{1}{24}z^3 - \frac{\sqrt{2}}{48}z^4 + \frac{1}{96}z^5 + \cdots, \quad z \in \mathbb{D}$$

and

$$\left|H_{3,1}\left(h_4^{-1}\right)\right| = \frac{1}{864},$$

we obtain the desired result in Corollary 10. □

Corollary 11. *Suppose that $f \in \mathcal{K}(\text{sech})$. Then,*

$$\left|\mathcal{H}_{2,2}\left(F_f/2\right)\right| \leq \frac{1112 + 39\sqrt{39}}{4151520} = 0.000326\ldots. \tag{91}$$

The result is sharp, with the extremal function h_5 presented by

$$h_5(z) = \int_0^z \exp\left(\int_0^u \frac{\text{sech}\left(\frac{\widehat{p}_5(u)-1}{\widehat{p}_5(u)+1}\right) - 1}{u} du\right) ds, \quad z \in \mathbb{D}, \tag{92}$$

where

$$\widehat{p}_5(z) = \frac{1 + \chi_3 z + z^2}{1 - z^2}, \quad z \in \mathbb{D}$$

and

$$\chi_3 = \sqrt{\frac{128 - 8\sqrt{39}}{31}} \approx 1.586638. \tag{93}$$

Proof. Utilizing Theorem 4, we have

$$\left|\mathcal{H}_{2,2}\left(F_f/2\right)\right| = \left|\frac{1}{4}\mathcal{H}_{\frac{1}{2}}(f)\right| \leq \max_{t\in[0,2]} \frac{1}{2211840}\left(\frac{31}{2}t^6 - 192t^4 + 672t^2\right).$$

Define $r_7(t) = \frac{1}{2211840}\left(\frac{31}{2}t^6 - 192t^4 + 672t^2\right)$, with $t \in [0,2]$. We note that χ_3 given in (93) is the unique critical point of r_7 in $(0,2)$, and $r_7(t) \leq r_7(\chi_3) = \frac{1112+39\sqrt{39}}{4151520}$ for all $t \in [0,2]$. This yields the inequality (91) in Corollary 11.

To show the sharpness, we note that h_5 defined in (92) belongs to the class $\mathcal{K}(\text{sech})$. As

$$h_5(z) = z - \frac{\chi_3^2}{48}z^3 - \frac{\chi_3\left(4 - \chi_3^2\right)}{96}z^4 - \frac{7\chi_3^4 - 48\chi_3^2 + 48}{1920}z^5 + \cdots, \quad z \in \mathbb{D}$$

and

$$\left|\mathcal{H}_{2,2}\left(F_f/2\right)\right| = \frac{1}{2211840}\left(\frac{31}{2}\chi_3^6 - 192\chi_3^4 + 672\chi_3^2\right) = \frac{1112 + 39\sqrt{39}}{4151520},$$

we complete the proof of Corollary 11. □

Corollary 12. *Suppose that $f \in \mathcal{K}(\text{sech})$. Then,*

$$\left|\mathcal{H}_{2,2}\left(F_{f^{-1}}/2\right)\right| \leq \frac{165\sqrt{165} - 1912}{730080} \approx 0.000284\ldots. \tag{94}$$

The equality holds by function h_6 taking

$$h_6(z) = \int_0^z \exp\left(\int_0^u \frac{\text{sech}\left(i\frac{\widehat{p}_6(u)-1}{\widehat{p}_6(u)+1}\right) - 1}{u} du\right) ds, \quad z \in \mathbb{D}, \tag{95}$$

where

$$\widehat{p}_6(z) = \frac{1 - z^2}{1 + \chi_4 z + z^2}, \quad z \in \mathbb{D}$$

and

$$\chi_4 = \sqrt{\frac{128 - 8\sqrt{165}}{13}} \approx 1.393340. \tag{96}$$

Proof. From Theorem 4, we obtain

$$\left|\mathcal{H}_{2,2}\left(F_{f^{-1}}/2\right)\right| = \left|\frac{1}{4}\mathcal{H}_{\frac{5}{2}}(f)\right| \le \max_{t\in[0,2]} \frac{1}{2211840}\left(\frac{13}{2}t^6 - 192t^4 + 672t^2\right) = \frac{165\sqrt{165} - 1912}{730080}.$$

It is easy to check that h_6 defined in (95) belongs to the class $\mathcal{K}(\text{sech})$. Let χ_4 be given in (96). As

$$h_6(z) = z + \frac{\chi_4^2}{48}z^3 + \frac{\chi_4(4-\chi_4^2)}{96}z^4 + \frac{11\chi_4^4 - 48\chi_4^2 + 48}{1920}z^5 + \cdots, \quad z \in \mathbb{D}$$

and

$$\left|\mathcal{H}_{2,2}\left(F_{f^{-1}}/2\right)\right| = \frac{1}{2211840}\left(\frac{13}{2}\chi_4^6 - 192\chi_4^4 + 672\chi_4^2\right) = \frac{165\sqrt{165} - 1912}{730080},$$

we know the equality in (94) of Corollary 12 holds. □

4. Conclusions

In the study of q-starlike and q-convex functions, we recall that the Fekete–Szegö problem has attracted a great deal of attention; see [47–50]. Its analytic representation is $a_3 - \mu a_2^2$, where a_2 and a_3 are the initial coefficients of the considered functions, and μ is a constant. In the present paper, we define a new functional in the form of $a_3a_5 - a_4^2 - \mu a_3^3$, where $\mu \ge 0$. When $a_2 = 0$, it is found that some of the second and third Hankel determinants with different entries all take this form. Using this functional, we are able to give a unified expression of the desired coefficient problems. As an application, we introduce two classes q-starlike and q-convex functions subordinate to secant hyperbolic functions and calculate the sharp bounds on this functional. By $q \to 1^-$, we are able to obtain the bounds on this functional for functions in the families of starlike and convex functions. By taking $\mu = 0,\ 1,\ 3,\ 2,\ \frac{1}{2},\ \frac{5}{2}$, we obtain some known results and also new findings on the exact bounds of the Hankel determinant.

Although a significant amount of valuable work on q-analogue analytic functions has been done and the output is abundant, some important issues need to be addressed. An example includes what conditions guarantee functions in the q-analogue classes to be univalent. As $q \to 0$, the performances of the q-classes of analytic functions can be very complex.

Author Contributions: Conceptualization, L.Z. and L.S.; methodology, Z.W. and L.S.; writing—original draft preparation, L.Z.; writing—review and editing, Z.W. and L.S.; funding acquisition, L.S. All authors have read and agreed to the published version of the manuscript.

Funding: The present investigation was supported by the Natural Science Foundation of Changshaunder Grant no. kq2502003 and the Key Project of Natural Science Foundation of Educational Committee of Henan Province under Grant no. 24B110001 of China.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No data were used for this research.

Acknowledgments: The authors would like to express their gratitude for the referees' valuable suggestions, which truly improved the present work.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

References

1. Zaprawa, P. On a coefficient inequality for Carathéodory functions. *Results Math.* **2024**, *79*, 30. [CrossRef]
2. Jackson, F.H. On q-functions and certain difference operator. *Trans. R. Soc. Edinb.* **1909**, *46*, 253–281. [CrossRef]
3. Srivastava, H.M. Univalent functions, fractional calculus, and associated generalized hypergeometric functions. In *Univalent Functions, Fractional Calculus, and Their Applications*; Srivastava, H.M., Owa, S., Eds.; Halsted Press: Chichester, UK; John Wiley and Sons: New York, NY, USA, 1989; pp. 329–354.
4. Kac, V.; Cheung, P. *Quantum Calculus*; Springer: Berlin/Heidelberg, Germany, 2001.
5. Qiu, J.-L.; Wang, Z.-G.; Li, M. Some characterizations for meromorphic ζ-starlike functions. *J. Contemp. Math. Anal.* **2025**, *60*, 36–47. [CrossRef]
6. Ismail, M.E.H.; Merkes, E.; Styer, D. A generalization of starlike functions. *Complex Var. Theory Appl.* **1990**, *14*, 77–84. [CrossRef]
7. Uçar, H.E.Ö. Coefficient inequality for q-starlike functions. *Appl. Math. Comput.* **2016**, *276*, 122–126.
8. Seoudy, T.M.; Aouf, M.K. Coefficient estimates of new classes of q-starlike and q-convex functions of complex order. *J. Math. Inequal.* **2016**, *10*, 135–145. [CrossRef]
9. Duren, P. *Univalent Functions*; Springer: Berlin/Heidelberg, Germany, 1983.
10. Srivastava, H.M.; Khan, B.; Khan, N.; Ahmad, Q.Z. Coefficient inequalities for q-starlike functions associated with the Janowski functions. *Hokkaido Math. J.* **2019**, *48*, 407–425. [CrossRef]
11. Srivastava, H.M.; Khan, B.; Khan, N.; Tahir, M.; Ahmad, S.; Khan, N. Upper bound of the third Hankel determinant for a subclass of q-starlike functions associated with the q-exponential function. *Bull. Sci. Math.* **2021**, *167*, 102942. [CrossRef]
12. Çağlar, M.; Orhan, H.; Srivastava, H.M. Coefficient bounds for q-starlike functions associated with q-Bernoulli numbers. *J. Appl. Anal. Comput.* **2023**, *15*, 2354–2364. [CrossRef]
13. Srivastava, H.M.; Hadi, S.H.; Darus, M. Some subclasses of p-valent γ-uniformly type q-starlike and q-convex functions defined by using a certain generalized q-Bernardi integral operator. *Rev. Real Acad. Cienc. Exactas Físicas Nat. Ser. A Matemáticas* **2023**, *117*, 50. [CrossRef]
14. Khan, M.F.; AbaOud, M. New applications of fractional q-calculus operator for a new subclass of q-starlike functions related with the cardioid domain. *Fractal Fract.* **2024**, *8*, 71. [CrossRef]
15. Gul, B.; Ritelli, D.; Alhefthi, R.K.; Arif, M. A novel family of starlike functions involving quantum calculus and a special function. *Fractal Fract.* **2025**, *9*, 179. [CrossRef]
16. Bano, K.; Raza, M.; Xin, Q.; Tchier, F.; Malik, S.N. Starlike functions associated with secant hyperbolic function. *Symmetry* **2023**, *15*, 737. [CrossRef]
17. Kumar, S.; Breaz, D.; Cotîrlă, L.I.; Çetinkaya, A. Hankel determinants of normalized analytic functions associated with hyperbolic secant function. *Symmetry* **2024**, *16*, 1303. [CrossRef]
18. Raza, M.; Bano, K.; Xin, Q.; Tchier, F.; Malik, S.N. Sharp coefficient inequalities of starlike functions connected with secant hyperbolic function. *J. Inequal. Appl.* **2024**, *1*, 56. [CrossRef]
19. Pommerenke, C. On the coefficients and Hankel determinants of univalent functions. *J. Lond. Math. Soc.* **1966**, *1*, 111–122. [CrossRef]
20. Pommerenke, C. On the Hankel determinants of univalent functions. *Mathematika* **1967**, *14*, 108–112. [CrossRef]
21. Krishna, D.V.; Venkateswarlua, B.; RamReddy, T. Third Hankel determinant for bounded turning functions of order alpha. *J. Niger. Math. Soc.* **2015**, *34*, 121–127. [CrossRef]
22. Wang, Z.-G.; Srivastava, H.M.; Arif, M.; Liu, Z.-H.; Ullah, K. Sharp bounds on Hankel determinants of bounded turning functions involving the hyperbolic tangent function. *Appl. Anal. Discrete Math.* **2024**, *18*, 551–571. [CrossRef]
23. Răducanu, D.; Zaprawa, P. Second Hankel determinant for close-to-convex functions. *C. R. Math. Acad. Sci. Paris* **2017**, *355*, 1063–1071. [CrossRef]
24. Srivastava, H.M.; Murugusundaramoorthy, G.; Bulboacă, T. The second Hankel determinant for subclasses of bi-univalent functions associated with a nephroid domain. *Rev. Real Acad. Cienc. Exactas Físicas Nat. Ser. A Matemáticas* **2022**, *116*, 145. [CrossRef]
25. Kowalczyk, B.; Lecko, A.; Sim, Y.J. The sharp bound of the Hankel determinant of the third kind for convex functions. *Bull. Aust. Math. Soc.* **2018**, *97*, 435–445. [CrossRef]
26. Kowalczyk, B.; Lecko, A.; Thomas, D.K. The sharp bound of the third Hankel determinant for starlike functions. *Forum Math.* **2022**, *34*, 1249–1254. [CrossRef]
27. Lecko, A.; Sim, Y.J.; Śmiarowska, B. The sharp bound of the Hankel determinant of the third kind for starlike functions of order 1/2. *Complex Anal. Oper. Theory* **2019**, *13*, 2231–2238. [CrossRef]
28. Shi, L.; Shutaywi, M.; Alreshidi, N.; Arif, M.; Ghufran, M.S. The sharp bounds of the third-order Hankel determinant for certain analytic functions associated with an eight-shaped domain. *Fractal Fract.* **2022**, *6*, 223. [CrossRef]
29. Wang, Z.-G.; Farooq, M.U.; Arif, M.; Malik, S.N.; Tawfiq, F.M.O. A class of meromorphic functions involving higher order derivative. *J. Contemp. Math. Anal.* **2024**, *59*, 419–429. [CrossRef]

30. Zaprawa, P. On Hankel determinant $\mathcal{H}_2(3)$ for univalent functions. *Results Math.* **2018**, *73*, 89. [CrossRef]
31. Sim, Y.J.; Lecko, A.; Thomas, D.K. The second Hankel determinant for strongly convex and Ozaki close-to-convex functions. *Ann. Mat. Pura Appl.* **2021**, *200*, 2515–2533. [CrossRef]
32. Obradović, M.; Tuneski, N. Hankel determinant of second order for inverse functions of certain classes of univalent functions. *Adv. Math. Sci. J.* **2023**, *12*, 519–528. [CrossRef]
33. Srivastava, H.M.; Cho, N.E.; Alderremy, A.A.; Lupas, A.A.; Mahmoud, E.E.; Khan, S. Sharp inequalities for a class of novel convex functions associated with Gregory polynomials. *J. Inequal. Appl.* **2024**, *1*, 140. [CrossRef]
34. Shi, L.; Arif, M. Sharp coefficient results on the inverse of silverman starlike functions. *J. Contemp. Math. Anal.* **2024**, *59*, 279–289. [CrossRef]
35. Kowalczyk, B.; Lecko, A. Second Hankel determinant of logarithmic coefficients of convex and starlike functions. *Bull. Aust. Math. Soc.* **2022**, *105*, 458–467. [CrossRef]
36. Srivastava, H.M.; Eker, S.S.; Şeker, B.; Çekiç, B. Second Hankel determinant of the logarithmic coefficients for a subclass of univalent functions. *Miskolc Math. Notes* **2024**, *25*, 479–488. [CrossRef]
37. Eker, S.S.; Lecko, A.; Çekiç, B.; Şeker, B. The second Hankel determinant of logarithmic coefficients for strongly Ozaki close-to-convex functions. *Bull. Malays. Math. Sci. Soc.* **2023**, *46*, 183. [CrossRef]
38. Allu, V.; Shaji, A. The Sharp bound of the second Hankel determinant of logarithmic coefficients for starlike and convex functions. *Bull. Aust. Math. Soc.* **2024**, 1–9. [CrossRef]
39. Palei, S.; Soren, M.M.; Cotîrlă, L.I. Coefficient bounds for alpha-convex functions involving the linear q-derivative operator connected with the cardioid domain. *Fractal Fract.* **2025**, *9*, 172. [CrossRef]
40. Ponnusamy, S.; Sharma, N.L.; Wirths, K.J. Logarithmic coefficients of the inverse of univalent functions. *Results Math.* **2018**, *73*, 160. [CrossRef]
41. Ali, M.F.; Nurezzaman, M. The second Hankel determinant for some classes of univalent functions. *Mediterr. J. Math.* **2024**, *21*, 220. [CrossRef]
42. Mandal, S.; Roy, P.P.; Ahamed, M.B. Hankel and toeplitz determinants of logarithmic coefficients of inverse functions for certain classes of univalent functions. *Iran. J. Sci.* **2025**, *49*, 243–252. [CrossRef]
43. Srivastava, H.M.; Shaba, T.G.; Ibrahim, M.; Tchier, F.; Khan, B. Coefficient bounds and second Hankel determinant for a subclass of symmetric bi-starlike functions involving Euler polynomials. *Bull. Sci. Math.* **2024**, *192*, 103405. [CrossRef]
44. Shi, L.; Abbas, M.; Raza, M.; Arif, M.; Kumam, P. Inverse logarithmic coefficient bounds for starlike functions subordinated to the exponential functions. *J. Inequal. Appl.* **2024**, *1*, 17. [CrossRef]
45. Kwon, O.S.; Lecko, A.; Sim, Y.J. On the fourth coefficient of functions in the Carathéodory class. *Comput. Methods Funct. Theory* **2018**, *18*, 307–314 [CrossRef]
46. Choi, J.H.; Kim, Y.C.; Sugawa, T. A general approach to the Fekete-Szegö problem. *J. Math. Soc. Jpn.* **2007**, *59*, 707–727. [CrossRef]
47. Cetinkaya, A.; Kahramaner, Y.; Polatoglu, Y. Fekete-Szegö inequalities for q-starlike and q-convex functions. *Acta Univ. Apulensis* **2018**, *53*, 55–64.
48. Srivastava, H.M.; Raza, N.; AbuJarad, E.S.; Srivastava, G.; AbuJarad, M.H. Fekete-Szegö inequality for classes of (p,q)-starlike and (p,q)-convex functions. *Rev. Real Acad. Cienc. Exactas Físicas Nat. Ser. A Matemáticas* **2019**, *113*, 3563–3584. [CrossRef]
49. Bulut, S. Fekete-Szegö problem for q-starlike functions in connected with k-Fibonacci numbers. *Hacet. J. Math. Stat.* **2022**, *51*, 1661–1673. [CrossRef]
50. Srivastava, H.M.; Khan, N.; Darus, M.; Khan, S.; Ahmad, Q.Z.; Hussain, S. Fekete-Szegö type problems and their applications for a subclass of q-starlike functions with respect to symmetrical points. *Mathematics* **2020**, *8*, 842. [CrossRef]