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On Extended Numerical Discretization Technique of Fractional Models with Caputo-Type Derivatives

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Abstract: In this work, we investigate the extended numerical discretization technique for the solution of fractional Bernoulli equations and SIRD epidemic models under the Caputo fractional, which is accurate and versatile. We have demonstrated the method's strength in examining complex systems; it is found that the method produces solutions that are identical to the exact solution and approximate series solutions. The ENDT is its ability to proficiently handle complex systems governed by fractional differential equations while preserving memory and hereditary characteristics. Its simplicity, accuracy, and flexibility render it an effective instrument for replicating real-world phenomena in physics and biology. The ENDT method offers accuracy, stability, and efficiency compared to traditional methods. It effectively handles challenges in complex systems, supports any fractional order, is simple to implement, improves computing efficiency with sophisticated methodologies, and applies it to epidemic predictions and biological simulations.

Keywords: fractional ordinary differential equations; Caputo derivatives; Bernoulli equation; SIRD models; fractional-order systems; accurate numerical methods



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1. Introduction

Fractional calculus, involving non-integer orders of derivatives and integrals, has developed rapidly in recent decades in science and engineering [1,2]. They extend the classical calculus so that one can model problems that the standard approach cannot [3,4]. Fractional derivatives have turned out to be crucial because they can help solve some real-world problems. It has opened up new areas of theoretical opportunities and practical applications in physics, control theory, signal processing, materials science, and biological modeling [5–11].

In contrast to traditional calculus, fractional calculus can describe systems with memory and hereditary features, making it popular in recent decades. Currently, it is widely applied in physics, engineering, biology, finance, etc. Fractional calculus improves long-term memory modeling and offers improvements in control theory. Fractional-time derivatives explain the relationship between stresses and strains in viscoelastic materials [12–14]. In biomedical sciences, fractional models are more flexible and accurate for disease propagation than integer-order models. The applications of FC are expanding and crucial for scientific and technical research [15–17], emerging as a versatile tool [18,19].

Epidemiological models are the basis for infectious disease dynamics and allow researchers and policymakers to predict how a disease spreads, perform impact assessments, and evaluate intervention strategies. These integrate mathematical frameworks into real data to simulate the behavior of diseases in alternative scenarios [20,21]. Given the complexity of infectious diseases, which depend on parameters such as transmission rates,

population mobility, and environmental conditions, formulation and simulation have become an integrated part of effective public health planning [22,23]. With the use of advanced numerical techniques and computational simulations, studies on the impact of all control measures, vaccination campaigns, quarantine protocols, and treatment strategies under various conditions could be made. The ability to model these scenarios provides great insight into how to reduce the impact of outbreaks and maintain health resources effectively. The role of simulations in epidemiology has grown dramatically, with rapid improvements in technology and mathematical modeling techniques that provide strong tools to solve global health problems [24,25].

Research on the derivation of efficient and accurate numerical algorithms for solving problems composed of ODEs has recently shown increasingly rapid development [26–29]. Several discretization schemes were introduced, according to [30–33], for special types of time-fractional PDEs. In most cases, these discretization techniques, for both integer-order and time-fractional ODEs, form a large linear or nonlinear system where unknowns approximate solution values at interior mesh points.

This article shows an innovative way to solve a fractional-order Bernoulli equation and a fractional epidemic model that includes susceptible, infected, recovered, and dead (SIRD) cases. It carries this out by using a scheme of fractional-order time derivatives. We demonstrate the efficiency and accuracy of the developed method by comparing approximate solutions with exact solutions, resulting in a perfect match. The proposed approach represents a new avenue to extend fractional-order methods to a wide class of systems with high-accuracy solutions. In the near future, this may be a very promising approach to developing and realizing complex fractional models in different fields, such as physics and engineering.

2. Basic Principles

This section defines the Riemann–Liouville and Caputo operators, which form the foundation for fractional differential equations used.

Definition 1. The Riemann–Liouville (R-L) fractional integral operator of order $\alpha > 0$ is defined as follows [34]:

$$I^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds, \quad t > 0. \quad (1)$$

Definition 2. Applying the Riemann–Liouville (R-L) operators, $n-1 < \alpha < n$ and $n \in \mathbb{N}$ are given by [34]

$${}^{RL}D_t^\alpha g(t) = \frac{d^n}{dt^n} I^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} g(s) ds, \quad t > 0. \quad (2)$$

Definition 3. For α ($n-1 < \alpha < n$), Caputo operators are used [35].

$${}^CD_t^\alpha g(t) = I^{n-\alpha} \frac{d^n}{dt^n} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} g^{(n)}(s) ds, \quad t > 0. \quad (3)$$

3. The Extended Numerical Discretization Technique (ENDT)

These methods assume the initial value problem for $0 < t \leq T$.

$$\begin{cases} {}^CD_t^\alpha u(t) = g(t, u(t)), & t \in [0, T], \\ u^{(i)}(0) = u_0^{(i)}, & i = 0, 1, \dots, [\alpha], \end{cases} \quad (4)$$

is equivalent to the Volterra-type integral equation [36]:

$$u(t) = \sum_{i=0}^{[\alpha]-1} u_0^{(i)} \frac{t^i}{i!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, u(s)) ds, \quad (5)$$

A continuous function serves as a solution to (4) if and only if it also satisfies (5).

3.1. Numerical Approximation Framework

To solve different kinds of ordinary differential equations (ODEs), many numerical methods with discretization schemes have been created [37–39]. Researchers have been working on making numerical schemes that are both efficient and reliable for solving ODEs for a long time because they are useful in both engineering and science. Typically, most discretization schemes that are used to numerically solve ODEs end up in algebraic systems where the unknowns are approximations at discrete time steps. This article describes an ODE's discretization scheme by reducing the computational complexity through minimizing the solution of large-scale equations of systems and, therefore, an effective applied strategy to solve initial value problems.

$${}^c D_t^\alpha u(t) = G(t, u, u', u''), \quad 0 < t < T, \quad (6)$$

where G depends on $t, u, u', u'', T > 0$, and ${}^c D_t^\alpha$ is the Caputo operator ($0 < \alpha \leq 1$) for variable t , with $u(0) = f(0)$, and boundary conditions (BCs) $u(0) = \phi(0), u(T) = \psi(T), 0 < t < T$.

We use finite differences to reduce (6) to ODEs, then solve them using the fractional Adams method, analyze the stability (see [39]), and finally, validate via simulations. Assuming $u \in C^1([0, T])$, we discretize $[0, T]$ with grid $\{t_j = jh_t : j = 0, \dots, N\}$, $h_t = T/N$, and approximate $u'(t_j)$ and $u''(t_j)$ using finite differences.

$${}^c D_t^\alpha v_i(t) = G(t, v_i(t), \frac{v_{i+1}(t) - v_i(t)}{h_t}, \frac{v_{i+1}(t) - 2v_i(t) + v_{i-1}(t)}{h_t^2}), \quad i = 1, 2, \dots, M-1. \quad (7)$$

with $v_0(t) = \phi(t), v_M(t) = \psi(t), v_i(0) = f_i (i = 1, \dots, M-1)$, yielding $M-1$ ODEs:

$${}^c D_t^\alpha v_i(t) = \begin{cases} G\left(t_i, v_1(t), \frac{v_2(t) - v_1(t)}{h_t}, \frac{v_2(t) - 2v_1(t) + v_0(t)}{h_t^2}\right), & i = 1, \\ \vdots \\ G\left(t_i, v_i(t), \frac{v_{i+1}(t) - v_i(t)}{h_t}, \frac{v_{i+1}(t) - 2v_i(t) + v_{i-1}(t)}{h_t^2}\right), & 2 \leq i \leq M-2, \\ \vdots \\ G\left(t_i, v_{M-1}(t), \frac{v_M(t) - v_{M-1}(t)}{h_t}, \frac{v_M(t) - 2v_{M-1}(t) + v_{M-2}(t)}{h_t^2}\right), & i = M-1. \end{cases} \quad (8)$$

To solve Equation (8), we extend the fractional Adams method to handle the $M-1$ equations. Specifically, for each $i = 1, 2, \dots, M-1$, we can compute $v_i(t_{k+1})$ based on the equivalence of the initial value problem to the integral equation, expressed as follows for $k = 0, 1, \dots, N-1$:

$$v_i(t_{k+1}) = g(t_i) + \frac{1}{\Gamma(\alpha)} \int_0^{t_{k+1}} (t_{k+1} - s)^{\alpha-1} G\left(t_i, s, v_i(s), \frac{v_{i+1}(s) - v_i(s)}{h_t}, \frac{v_{i+1}(s) - 2v_i(s) + v_{i-1}(s)}{h_t^2}\right) ds. \quad (9)$$

The corrector step is given by

$$v_{i,k+1} = g(t_i) + \frac{h_t^\alpha}{\Gamma(\alpha+2)} F_i(t_{k+1}) + \frac{h_t^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^k a_{j,k+1} F_i(t_j), \quad (10)$$

where

$$F_i(t_j) = G(t_j, v_{i,j}, \frac{v_{i+1,j} - v_{i,j}}{h_t}, \frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{h_t^2}), \quad (11)$$

$$v_0(t) = \phi(t), \quad v_M(t) = \psi(t), \quad (12)$$

and

$$a_{j,k+1} = \begin{cases} k^{\alpha+1} + (\alpha - k)(k + 1)^\alpha, & \text{for } j = 0, \\ (k - j)^{\alpha+1} + (k - j + 2)^{\alpha+1} - 2(k - j + 1)^{\alpha+1}, & \text{for } 1 \leq j \leq k. \end{cases} \quad (13)$$

The predictor step, which is crucial in evaluating $v_{i,k+1}$, involves applying the fractional Adams–Bashforth method to Equation (11), yielding the predictor value:

$$v_{i,k+1}^p \approx g(t_i) + \frac{h_t^\alpha}{\Gamma(\alpha + 1)} \sum_{j=0}^k [(1 - k + j)^\alpha - (j - k)^\alpha] F_j(t_j). \quad (14)$$

For $k = 0, \dots, N - 1$, $u_{i,k+1} \approx u(x_i, t_{k+1})$ ($i = 1, \dots, M - 1$) is computed via (8), (10), and (13):

$$v_{i,k+1} = \begin{cases} g(t_1) + \frac{h_t^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^k a_{j,k+1} G\left(t_1, v_1(t), \frac{v_2(t) - v_1(t)}{h_t}, \frac{v_2(t) - 2v_1(t) + v_0(t)}{h_t^2}\right) \\ \quad + \frac{h_t^\alpha}{\Gamma(\alpha + 2)} G\left(t_{k+1}, v_{1,k+1}^p, \frac{v_{2,k+1}^p - v_{1,k+1}^p}{h_t}, \frac{v_{2,k+1}^p - 2v_{1,k+1}^p + \phi(t_{k+1})}{h_t^2}\right), & i = 1, \\ g(t_i) + \frac{h_t^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^k a_{j,k+1} G\left(t_i, v_i(t), \frac{v_{i+1}(t) - v_i(t)}{h_t}, \frac{v_{i+1}(t) - 2v_i(t) + v_{i-1}(t)}{h_t^2}\right) \\ \quad + \frac{h_t^\alpha}{\Gamma(\alpha + 2)} G\left(t_{k+1}, v_{i,k+1}^p, \frac{v_{i+1,k+1}^p - v_{i,k+1}^p}{h_t}, \frac{v_{i+1,k+1}^p - 2v_{i,k+1}^p + v_{i-1,k+1}^p}{h_t^2}\right), & 2 \leq i \leq M - 2, \\ \vdots \\ g(t_{M-1}) + \frac{h_t^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^k a_{j,k+1} G\left(t_{M-1}, v_{M-1}(t), \frac{v_M(t) - v_{M-1}(t)}{h_t}, \frac{v_M(t) - 2v_{M-1}(t) + v_{M-2}(t)}{h_t^2}\right) \\ \quad + \frac{h_t^\alpha}{\Gamma(\alpha + 2)} G\left(t_{k+1}, v_{M-1,k+1}^p, \frac{\phi(t_{k+1}) - v_{M-1,k+1}^p}{h_t}, \frac{\phi(t_{k+1}) - 2v_{M-1,k+1}^p + v_{M-2,k+1}^p}{h_t^2}\right), & i = M. \end{cases} \quad (15)$$

with the weights defined in Equation (14).

$$v_{i,k+1}^p = \begin{cases} g(t_1) + \frac{h_t^\alpha}{\Gamma(\alpha + 1)} \sum_{j=0}^k ((k + 1 - j)^\alpha - (k - j)^\alpha) G\left(t_1, v_1(t), \frac{v_2(t) - v_1(t)}{h_t}, \frac{v_2(t) - 2v_1(t) + v_0(t)}{h_t^2}\right), & i = 1, \\ g(t_i) + \frac{h_t^\alpha}{\Gamma(\alpha + 1)} \sum_{j=0}^k ((k + 1 - j)^\alpha - (k - j)^\alpha) G\left(t_i, v_i(t), \frac{v_{i+1}(t) - v_i(t)}{h_t}, \frac{v_{i+1}(t) - 2v_i(t) + v_{i-1}(t)}{h_t^2}\right), & 2 \leq i \leq M - 2, \\ \vdots \\ g(t_{M-1}) + \frac{h_t^\alpha}{\Gamma(\alpha + 1)} \sum_{j=0}^k ((k + 1 - j)^\alpha - (k - j)^\alpha) G\left(t_i, v_{M-1}(t), \frac{v_M(t) - v_{M-1}(t)}{h_t}, \frac{v_M(t) - 2v_{M-1}(t) + v_{M-2}(t)}{h_t^2}\right), & i = M. \end{cases} \quad (16)$$

We outline the steps to approximate $v_{i,j}$: $j = 0, 1, \dots, k + 1, i = 0, 1, \dots, M$. The scheme is simple for all values of α, h_t , with $v_{i,k+1}$ computed using previously known values without solving equations. This method can also be applied to other integer- and fractional-order IBVPs using divided differences for derivatives of u .

3.2. Stability Analysis

If the aim is for an initial value problem (IVP) solver to be numerically stable, then it should be ensured that small changes in the initial conditions do not cause big changes in the solutions obtained [40,41]. In the following analysis, we consider the scheme's stability.

Theorem 1. Assume $G = G(t, u, u', u'')$ satisfies Lipschitz conditions with respect to its variables as

$$|G(t, u, u', u'') - G(t, v, v', v'')| \leq A|u - v|, \quad (17)$$

$$|G(t, u, u', u'') - G(t, u, v', v'')| \leq B|u' - v'|, \quad (18)$$

$$|G(t, u, u', u'') - G(t, u, u', v'')| \leq C|u'' - v''|. \quad (19)$$

For each $i = 1, 2, \dots, M - 1$, let $\mu_{i,k+1}$ and $\nu_{i,k+1}$ be two numerical solutions obtained using Formulas (15) and (16) with the initial approximations $\mu_{i,0}$ and $\nu_{i,0}$, respectively. Then, there exists a positive constant K such that, for $k = 1, 2, \dots, N$,

$$|\mu_{i,k+1} - \nu_{i,k+1}| \leq K \max_{1 \leq j \leq M-1} |\mu_{i,0} - \nu_{i,0}|. \quad (20)$$

Proof. From (15),

$$\begin{aligned} |\mu_{i,k+1} - \nu_{i,k+1}| &= |\mu_{i,0} - \nu_{i,0}| + A \frac{h_t^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^k a_{j,k+1} |\mu_{i,j} - \nu_{i,j}| \\ &\quad + B \frac{h_t^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^k b_{j,k+1} |\mu_{i,j} - \nu_{i,j}| \\ &\quad + C \frac{h_t^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^k |\mu_{i,k+1} - \nu_{i,k+1}|. \end{aligned} \quad (21)$$

Since G satisfies Lipschitz conditions (17)–(19), using (16), we obtain

$$\begin{aligned} |\mu_{i,k+1} - \nu_{i,k+1}| &= |\mu_{i,0} - \nu_{i,0}| + A \frac{h_t^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^k b_{j,k+1} |\mu_{i,j} - \nu_{i,j}| \\ &\quad + B \frac{h_t^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^k |\mu_{i,k+1} - \nu_{i,k+1}|. \end{aligned} \quad (22)$$

For each $j = 0, 1, \dots, k + 1$, set $\omega_j = \max_{1 \leq i \leq M} |\mu_j - \nu_j|$, and then we get

$$\begin{aligned} |\mu_{i,k+1} - \nu_{i,k+1}| &\leq |w_0 - A \sum_{j=0}^k a_{j,k+1} \omega_j| \\ &\quad + B \frac{h_t^\alpha}{\Gamma(\alpha + 1)} \sum_{j=0}^k b_{j,k+1} \omega_j \\ &\quad + C \frac{h_t^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^k a_{j,k+1} \frac{2u_j}{h_t} \\ &\quad + C \frac{h_t^\alpha}{\Gamma(\alpha + 1)} \sum_{j=0}^k b_{j,k+1} \frac{4u_j}{h_t} \end{aligned} \quad (23)$$

and so,

$$|\mu_{i,k+1} - \nu_{i,k+1}| \leq \left[1 + \frac{h_t^\alpha}{\Gamma(\alpha + 2)} K_1 \sum_{j=0}^k a_{j,k+1} \omega_j \right] + \frac{h_t^\alpha}{\Gamma(\alpha + 1)} K_2 \sum_{j=0}^k b_{j,k+1} \omega_j, \quad (24)$$

where $K_1 = A + B \frac{2}{h_t} + C \frac{4}{h_t^2}$, $K_2 = B^2 + AB \frac{4}{h_t^2} + C^2 + BC \frac{16}{h_t^4}$.

Using the discrete Grönwall's inequality, we obtain

$$|\mu_{i,k+1} - \nu_{i,k+1}| \leq \left[1 + \frac{h_t^\alpha}{\Gamma(\alpha + 2)} K_1 \right] \exp \left(\frac{h_t^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^k K_1 a_{j,k+1} + \frac{h_t^\alpha}{\Gamma(\alpha + 1)} K_2 b_{j,k+1} \right) w_0. \quad (25)$$

For weights $b_{j,k+1}$ and $a_{j,k+1}$ ($j = 0, \dots, k$), the relations are

$$h_t^\alpha \sum_{j=0}^k b_{j,k+1} = h_t^\alpha (k+1) \leq T^\alpha, \quad (26)$$

$$h_t^\alpha \sum_{j=0}^k a_{j,k+1} = h_t^\alpha ((a+1)(k+1) - 1) \leq (a+1)T^\alpha. \quad (27)$$

Using (25) and (26), Equation (27) simplifies to

$$|\mu_{i,k+1} - v_{i,k+1}| \leq \left[1 + \frac{h_t^\alpha}{\Gamma(\alpha+2)} K_1 \right] \exp \left(\frac{1}{\Gamma(\alpha+1)} K_1 T^\alpha + \frac{h_t^\alpha}{\Gamma(\alpha+2)} K_2 T^\alpha \right) w_0. \quad (28)$$

Finally, the constant K in Equation (20) is set as

$$K = \left[1 + \frac{h_t^\alpha}{\Gamma(\alpha+2)} K_1 \right] \exp \left(\frac{1}{\Gamma(\alpha+1)} K_1 T^\alpha + \frac{h_t^\alpha}{\Gamma(\alpha+2)} K_2 T^\alpha \right).$$

Thus, we obtain the desired result. Since G satisfies the Lipschitz condition, there exists a positive constant k such that, following a similar approach, we can prove $\mu_{P_i,k+1}^P$. $\square$

4. Numerical Results from Simulations

The work at hand proposes a new schema to solve two completely different problems: the Bernoulli equation and epidemic disease models. Given its widespread application in fluid dynamics, thermodynamics, and various other scientific fields, we begin with the Bernoulli equation. Here, we propose an innovative approach to the search for solutions by taking advantage of advanced techniques and algorithms and offering new insight into this classical problem. We next consider epidemic disease models, which have been gaining increasing importance in formulating the dynamics of infectious diseases. With our new schema applied to such models, we would like to make predictions of disease spread more accurately and efficiently while determining the best strategies for its control. Mathematical modeling will be combined with computational methods in order to obtain more reliable predictions in the context of real-world epidemics. We use our schema to solve the Bernoulli equation and models of epidemic diseases to show how useful and powerful this approach is for solving a wide range of difficult problems.

Problem 1. We start with the Fractional Bernoulli equation [42]:

$${}^c D_t^\alpha u(t) = 2u(t) - 4u^2(t), \quad (29)$$

where $0 < \alpha \leq 1$ and $u(0) = 1$. The exact solution of the Bernoulli equation, when $\alpha = 1$ is

$$u(t) = \frac{1}{2 - e^{-2t}}, \quad (30)$$

under the initial condition $u(0) = 1$, where ${}^c D_t^\alpha$ is defined by Equation (4) with the parameter α . When $\alpha = 1$, the Bernoulli Equation (29) has an exact solution corresponding to the proposed numerical approach.

Table 1 shows the numerical solutions of the Bernoulli equation for various values of N and T at $\alpha = 1$. The results show the behavior of the proposed numerical method for the approximating solution to the exact solution u_{exact} . It can be seen that as the number of grid points N is increased, the numerical solution starts converging toward the exact solution for all the tested values of $T = 10, 13, 16$. This demonstrates the accuracy and stability of

the method at large resolutions. For example, the solution with $N = 600$ is very accurate and has a negligible difference from the exact solution at $T = 10$. At $T = 16$, for all N continuously, the numerical method agrees with the exact solution, reflecting its reliability for large time intervals.

The numbers in Table 2 are the answers to the Bernoulli equation for the fractional order $\alpha = 0.99$ at different times $T = 10, 13, 16$, when the grid sizes are N . The results show that the method converges as N increases, and they also show that it works well for solving problems with fractional orders close to 1. The following examples demonstrate that the accuracy of numerical solutions tends to increase as N increases. For $T = 10$, the solution for $N = 600$ is closer to the exact solution u_{exact} than coarser grids like $N = 50$. A similar trend is seen for $T = 13$ and $T = 16$, where the solution successively approaches u_{exact} for increasing N .

The numbers in Table 3 below show the answers to the Bernoulli equation when $\alpha = 0.95$, with the corresponding values of T and N . For $N = 50$ to 600, the solutions converge, and the successive values almost overlap, indicating that the precision is stable and effective. The solutions tend to a stable number for each T , and the divergence becomes small for higher N , particularly for $T = 0.5$ and $T = 1$. Additionally, the solutions improve as T decreases, with values slightly higher at $T = 0.1$ compared to those at $T = 1$. This shows that T changes how the equations behave. In general, the findings show that the numerical technique is reliable and stable, with the equation stabilizing for higher values of N and the solutions becoming more accurate as N grows.

However, it is also clear that the fractional order $\alpha = 0.99$ introduces a small but consistent deviation from the exact solution observed at the integer order $\alpha = 1$. Thus, even for slight variations in the order, that difference reflects the expected sensitivity for fractional-order systems. The method retains robustness in capturing those subtle differences while keeping accuracy high.

Table 1. Numerical solutions of the Bernoulli equation at $\alpha = 1$.

N	$T = 10$	$T = 13$	$T = 16$
50	0.500000000001739	0.500000000000000	0.500000000000000
80	0.500000000023599	0.500000000000005	0.500000000000000
100	0.500000000048409	0.500000000000020	0.500000000000000
110	0.500000000061942	0.500000000000031	0.500000000000000
150	0.500000000115477	0.500000000000095	0.500000000000000
200	0.500000000172562	0.500000000000195	0.500000000000000
600	0.500000000363980	0.500000000000709	0.500000000000001
u_{exact}	0.500000000515288	0.500000000001277	0.500000000000003

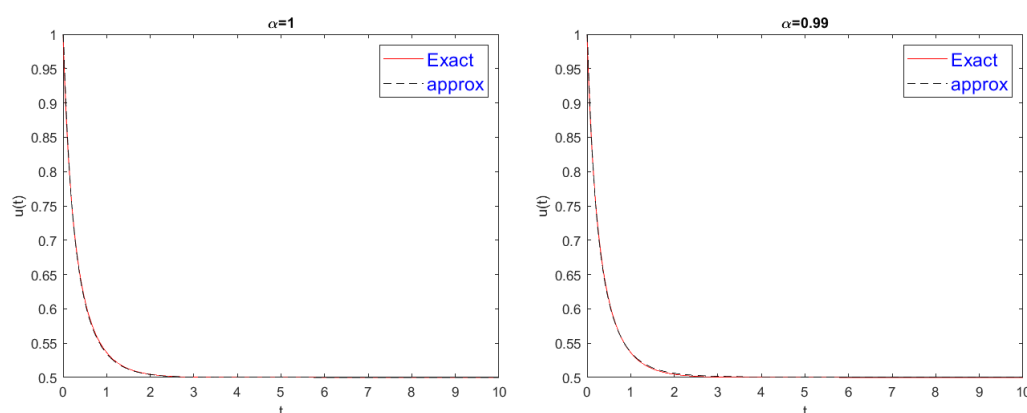
Table 2. Numerical solutions of the Bernoulli equation at $\alpha = 0.99$.

N	$T = 10$	$T = 13$	$T = 16$
50	0.500273426320075	0.500205941921804	0.500165206791134
80	0.500277208471413	0.500208868588402	0.500167656847179
100	0.500278406426705	0.500209779043973	0.500168406791393
110	0.500278833892965	0.500210101428939	0.500168670803320
150	0.500279952395590	0.500210937842773	0.500169351132783
200	0.500280703422183	0.500211493040043	0.500169798362421
600	0.500282163921836	0.500212555870457	0.500170642622861

Table 3. Numerical solutions of the Bernoulli equation at $\alpha = 0.95$.

N	$T = 0.1$	$T = 0.5$	$T = 1$
50	0.554439473935439	0.577254687841527	0.554423787473707
80	0.577197107971268	0.554352743115358	0.546285614347391
100	0.577220141522140	0.554385897300327	0.546313839430827
110	0.577233960625649	0.554402474287876	0.546327952638142
150	0.577243172934483	0.554412420451679	0.546327952638142
200	0.577249752946447	0.554419051217137	0.546342066300731
600	0.577254687841527	0.554423787473707	0.546346098861132

These two graphs in Figure 1 compare exact (solid line) and approximate (dashed line) solutions of the system in Equation (27) for α values of 1 (left) and 0.99 (right), with time t . Both cases show nearly identical behavior. The remarkable overlap between exact and approximate solutions in both scenarios validates the novel method's accuracy.

**Figure 1.** Comparison of exact and approximate solutions for Equation (29).

Problem 2. We discuss the model of epidemic disease as follows [43]:

$$\begin{aligned}
 {}^c D_t^\alpha S(t) &= -\beta S(t)I(t), \\
 {}^c D_t^\alpha I(t) &= \beta S(t)I(t) - (\delta + \psi)I(t), \\
 {}^c D_t^\alpha R(t) &= (\delta - \psi)I(t) - \psi R(t), \\
 {}^c D_t^\alpha D(t) &= (\psi - \delta)I(t) - \psi D(t).
 \end{aligned} \tag{31}$$

We define the parameters β, δ , and ψ , each representing a specific rate related to disease dynamics: β indicates the infection rate of susceptible individuals, valued at 0.0007; δ represents the recovery rate of infected individuals, set at 0.09; and ψ denotes the mortality rate of infected individuals, established at 0.08.

When setting the initial conditions, it is presumed that the total population, represented by N , does not change over time. This assumption implies that the population at the initial time is the combined sum of individuals in each compartment: susceptible, infected, recovered, and deceased. Mathematically, this relationship is described by the equation $S(0) + I(0) + R(0) + D(0) = N$, where $S(0)$, $I(0)$, $R(0)$, and $D(0)$ represent the initial numbers in each respective group. We assume initial conditions $S_0 = 350$, $I_0 = 400$, $R_0 = 400$, and $D_0 = 350$ (see [43]). The choice of the fractional order α is based on the fact that it is able to capture the memory effect and anomalous dynamics that occur in the propagation of infectious diseases. For this study, α is selected in the range $(0, 1]$ to represent subdiffusive behavior, with smaller values of α producing stronger memory effects. The specific value of α is chosen to yield the best fit for the epidemic data observed or simulated, such that the model describes real dynamics.

Table 4 gives the susceptible population in time t for different values of α . At first ($t = 0$), the population is uniformly distributed as 350 and decreases with increasing t according to the value of α . When $\alpha = 1$, the decrease occurs quickly, whereas the smaller the α , the slower the decrease; this reflects a moderating effect from fractional dynamics. This reveals the flexibility of the fractional-order model to capture the behaviors of epidemics, particularly in systems that have memory or hereditary effects.

Table 4. Susceptible population for different fractional orders α .

t	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$
0	350.000000	350.000000	350.000000	350.000000	350.000000
0.1	345.451913	343.457788	343.220710	343.961546	345.100000
0.2	330.378279	325.304428	326.897803	330.954892	335.400964
0.3	323.710728	315.270471	315.704199	320.157276	325.907055
0.4	319.043339	307.721697	306.527476	310.475368	316.622121
0.5	315.397991	301.555873	298.600671	301.582377	307.549270
0.6	312.386988	296.300557	291.563650	293.310671	298.690896
0.7	309.812743	291.700752	285.208219	285.556438	290.048713
0.8	307.559486	287.600194	279.399399	278.248225	281.623778
0.9	305.553024	283.894989	274.042992	271.333348	273.416537
1	303.742724	280.511986	269.069767	264.771013	265.426854

Table 5 shows the infected population as a function of time, t , for different values of α . For $t = 0$, all the α values have an infected population of 400, providing a consistent initial condition. As t increases, the infected population grows consistently for all α , with higher α values, for example, $\alpha = 1$ exhibits a faster growth rate compared to lower α values. This trend reflects the impact of fractional-order dynamics, where smaller values of α lead to a dampened development of infections, likely reflecting memory effects or delays in the spread. These results confirm the flexibility of fractional-order models to represent different infection dynamics.

Table 5. Infected population for different fractional orders α .

t	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$
0	400.000000	400.000000	400.000000	400.000000	400.000000
0.1	401.392272	402.002718	402.075293	401.848506	401.500000
0.2	405.855637	407.298252	406.832167	405.664986	404.373536
0.3	407.316677	409.476375	409.500036	408.469286	406.993095
0.4	408.224022	410.816939	411.336135	410.693566	409.359146
0.5	408.858639	411.708342	412.650031	412.478773	411.472892
0.6	409.331919	412.315557	413.590977	413.902776	413.336226
0.7	409.698933	412.726028	414.246981	415.017061	414.951694
0.8	409.990996	412.992359	414.676024	415.859223	416.322450
0.9	410.227618	413.148882	414.919110	416.458506	417.452209
1	410.421761	413.219326	415.006701	416.838666	418.345205

Table 6 presents variations in the recovered population with respect to time, t , for different values of α . When $t = 0$, the recovered population is always 400 for all α values. As time t increases, the recovered population in all cases decreases; however, the smaller the α value, the slower the decline in the recovered population compared to higher α values. This would confirm that the underlying dynamics are of fractional order, and a smaller value of α indicates a more delayed or moderated recovery process. These results highlight the flexibility of the fractional-order models in modeling the recovery patterns driven by memory or hereditary effects.

Table 6. Recovered population for different fractional orders α .

t	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$
0	400.000000	400.000000	400.000000	400.000000	400.000000
0.1	398.700546	398.130796	402.075293	398.274727	398.600000
0.2	394.368286	392.899835	398.063060	394.530599	395.812700
0.3	392.361825	389.875023	393.358745	391.358049	393.050572
0.4	390.930655	387.534855	390.026864	388.457637	390.313160
0.5	389.795978	385.576924	387.224283	385.741098	387.600014
0.6	388.846799	383.871667	384.744596	383.163848	384.910687
0.7	388.026258	382.349131	380.411583	380.698749	382.244738
0.8	387.300873	380.966419	378.468265	378.327487	379.601732
0.9	386.649086	379.694982	376.637708	376.036847	376.981240
1	386.056120	378.514692	374.902217	373.816838	374.382842

Table 7 presents the dead population over time (t) for various values of α . For $t = 0$, all the α values have a population of 350, which acts as a uniform starting point. For larger t , the dead population decreases with increasing rates across different values of α . A higher value of α , for example, $\alpha = 1$, gives a faster decline in the dead population, whereas a smaller value of α exhibits a slower decay, which may reveal the contribution of fractional-order dynamics in controlling mortality rates. These findings highlight the utility of fractional-order models in capturing nuanced death dynamics driven by memory effects or varied characteristics of the system.

Table 7. Dead population for different fractional orders α .

t	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$
0	350.0000	350.0000	350.0000	350.0000	350.0000
0.1	395.5492	395.3753	395.3920	395.5774	395.8915
0.2	395.6400	395.5641	395.6749	395.6774	395.9915
0.3	395.7400	395.6641	395.9714	396.3089	396.7254
0.4	395.8463	395.9857	396.2838	396.3089	396.8254
0.5	395.8463	396.2253	396.5838	397.0777	397.5550
0.6	396.1116	396.3253	396.9697	397.0877	397.7550
0.7	396.2785	396.7918	397.3540	397.8975	397.9677
0.8	396.4833	397.5746	397.3540	397.9975	398.3787
0.9	396.4833	398.1681	398.2643	398.8007	398.7876
1	397.1815	399.2681	398.8607	398.9007	399.5985

Figure 2 shows the epidemic model simulations, demonstrating population dynamics under varying fractional-order derivatives ($\alpha = 1.0, 0.95, 0.75, 0.55$) over a 100-day period. The system shows well-separated behavioral patterns: for higher α values, such as 1.0 and 0.95, the populations have rapid initial decline with a stabilization around day 40 and are characterized by lower final population values and more complete resolution of the disease. Lower values of α , such as 0.75 and 0.55, show a more gradual decline in curves, with higher sustained populations and incomplete resolution at day 100. All four cases start with population peaks around 400–450 individuals, where susceptible (S, blue) and infected (I, red) populations immediately show steep declines, while recovery (R, green) and death (D, cyan) are proportional to α . The model serves to illustrate the impact of different intervention intensities of the α value on the course of disease progression and resolution. Higher α values correspond to more aggressive intervention strategies that yield faster containment of the disease but perhaps higher initial mortality rates, while lower α values suggest more moderate interventions, leading to a prolonged presence of the disease but perhaps more manageable healthcare demands over time.

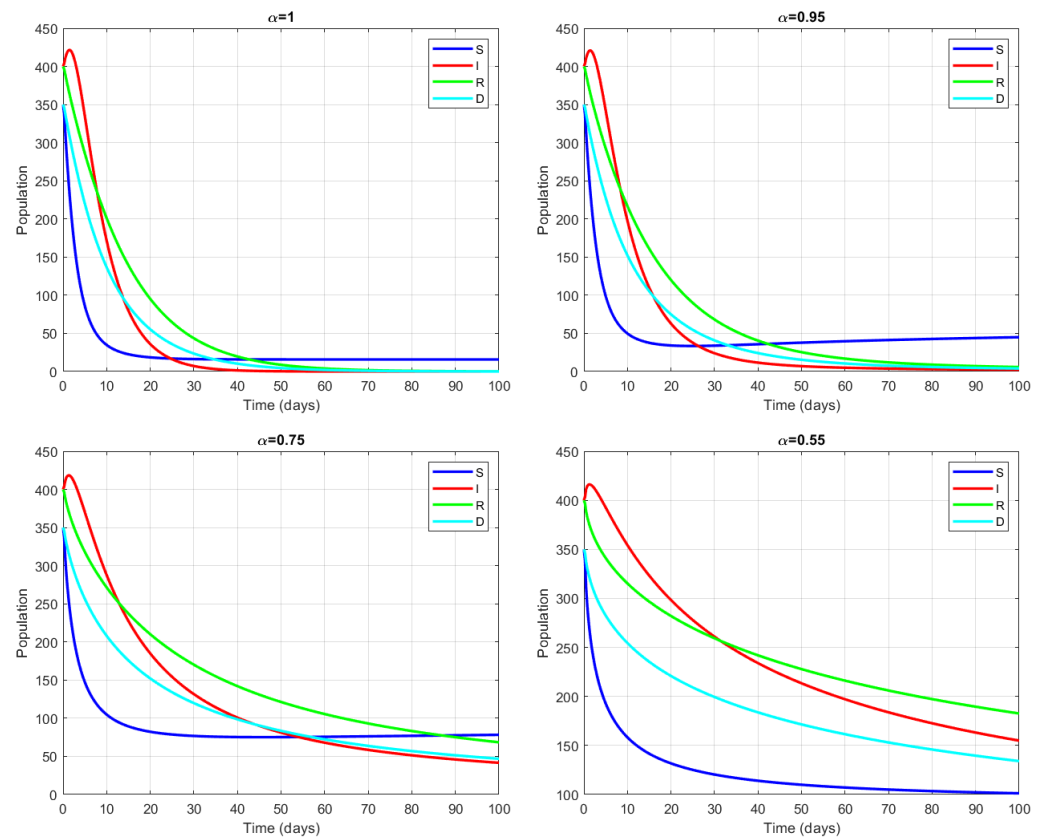


Figure 2. Four views of the system for different values of α .

5. Numerical Method Accuracy

In this part, we find the numerical accuracy of the model from above by calculating the error in every compartment S , I , R , and D with the assistance of the following formula:

$$E_S = \|S_{h_t/2} - S_{h_t}\|, \quad E_I = \|I_{h_t/2} - I_{h_t}\|, \quad E_R = \|R_{h_t/2} - R_{h_t}\|, \quad E_D = \|D_{h_t/2} - D_{h_t}\|$$

where $\|\cdot\|$ is the L_∞ norm, or the maximum absolute error at any time step. This is found by comparing values at two time step sizes, $h_t/2$ and h_t , to evaluate the impact of halving the time step on accuracy. The respective errors for different step sizes (h_t) and fractional orders (α) are provided in Tables 8 and 9. The aim of this research is to verify the convergence of the numerical scheme and find out how modifications in the time step and fractional order affect the precision of the solution. Through a comparison of errors at different h_t , we can examine the behavior of convergence for the algorithm, while studying the influence of α helps one consider how the dynamics of the model are affected by fractional derivatives. This error analysis is crucial in order to make the optimal trade-off between efficiency of computation and accuracy, especially in real applications, such as epidemiological models, where the choice of the numerical parameters highly influences the reliability of the results.

Table 8. Errors of S , I , R , D at $\alpha = 1$ and $T = 0.03$.

h_t	E_S	E_I	E_R	E_D
1×10^{-5}	-	-	-	-
5×10^{-6}	0.0046549	0.0014249	0.00133	0.00152
2.5×10^{-6}	0.0023275	0.00071248	0.000665	0.00076
1.25×10^{-6}	0.0011637	0.00035625	0.0003325	0.00038

Table 9. Errors of S, I, R, D at $\alpha = 0.85$ and $T = 0.03$.

h_t	E_S	E_I	E_R	E_D
1×10^{-5}	-	-	-	-
5×10^{-6}	0.016954	0.0051889	0.0048441	0.0055362
2.5×10^{-6}	0.009406	0.0028791	0.0026875	0.0030714
1.25×10^{-6}	0.0052184	0.0015974	0.001491	0.001704

In Table 8, the errors for $\alpha = 1$ at $T = 0.03$ are presented. From these results, one can observe the tendency of convergence as the step size decreases. The error values for S, I, R , and D approximately halve each time h_t is halved, which aligns well with the expectations based on the numerical accuracy for this type of fractional model. Among the compartments, the error is highest for S , while I, R , and D exhibit comparable magnitudes.

In Table 9, at $T = 0.03$ for $\alpha = 0.85$, the errors are larger compared to the case for $\alpha = 1$, indicating greater complexity in the dynamics at fractional orders. The halving of h_t yields a significant reduction in the error across all compartments, confirming the model's reliability for fractional-order systems. The error for S remains dominant, while I, R , and D follow a similar trend.

6. Discussion

Fractional integral equations were successfully solved via the fractional calculus method by using various analytical and numerical methods such as Grünwald–Letnikov, finite element, and spectral ones. However, such techniques typically have problems with stability, convergence, and computing complexity.

The numerical discretization of the Grünwald–Letnikov difference (ENDT), a newly proposed method in this study, was found to be more accurate and efficient, particularly in allowing us to overcome the above concerns in chaotic and hyperchaotic systems. In contrast with traditional methods, ENDT is capable of ensuring the best rates of accuracy and stability when solving both Caputo IVP and boundary value problems (BVPs) through Caputo-type fractional derivatives. One of the applications illustrating the robustness and predictability of ENDT in all the series of models is through numerical simulations.

The method is meant to be global, with tools to be used for any fractional order, and this, of course, is given a limited domain. The positive numerical results show that ENDT is technically proficient and can be a useful tool for the precise solution of generalized fractional differential equations in several engineering and scientific applications.

The ENDT approach provides great precision, stability, and efficiency. By guaranteeing higher convergence, simplicity of implementation, and more general applicability in physics, biology, and engineering, it beats conventional approaches.

This research introduces an extended numerical discretization technique (ENDT) that solves fractional Bernoulli equations and SIRD epidemic models subject to Caputo fractional derivatives with the highest accuracy and stability. ENDT accurately matches series solutions and handles memory and heredity difficulties more efficiently than typical approaches. The approach may be used for stochastic and multidimensional systems in physics, biology, and engineering. The method's speed, variety, and reliability make it a powerful tool for solving difficult fractional differential equation problems, especially in daily settings like epidemic prediction and biological system modeling.

7. Conclusions

The extended numerical discretization technique for the solution of fractional Bernoulli equations and SIRD epidemic models under the Caputo fractional initial value problem, as discussed in the present study, is immensely accurate and versatile. The method presented has been shown to be a strong way to look at complicated systems that are controlled by FDEs. It provides solutions that are consistent, with exact results and very accurate approximate series solutions. Its memory capability and hereditary properties make it a useful tool for analyzing dynamics in physics, biology, and engineering. Furthermore, its adaptability suggests possible extensions to more intricate systems, including stochastic and multidimensional models. More research will be carried out on how to use this method for larger problems, how to combine it with more advanced computer methods, and how to use it in the real world for applications like predicting epidemics and simulating biological systems. This development opens new pathways for finding new approaches to solving complicated scientific problems. Future work will expand the suggested method to more complicated and stochastic systems, improve computing efficiency with sophisticated methodologies, and apply it to epidemic predictions and biological simulations.

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