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A Study of a Nonlocal Coupled Integral Boundary Value Problem for Nonlinear Hilfer–Hadamard-Type Fractional Langevin Equations

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Abstract: We discuss the existence criteria and Ulam–Hyers stability for solutions to a nonlocal integral boundary value problem of nonlinear coupled Hilfer–Hadamard-type fractional Langevin equations. Our results rely on the Leray–Schauder alternative and Banach's fixed point theorem. Examples are included to illustrate the results obtained.

Keywords: Hilfer–Hadamard fractional derivative; Langevin equations; system; nonlocal integral boundary conditions; existence; fixed point

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1. Introduction

Fractional differential equations find extensive applications in the mathematical modeling of several phenomena occurring in natural, technical and social sciences. Examples include immune systems [1], chaotic synchronization [2], diffusion processes [3,4], ecological systems [5], neural networks [6], financial economics, etc. In contrast to the classical derivative, different types of fractional derivatives exist, like Riemann–Liouville, Liouville–Caputo, Hadamard, Hilfer, etc. The Hilfer fractional derivative introduced in [7] is reduced into the Riemann–Liouville and Caputo fractional derivatives, respectively, for the smallest and largest values of its parameter. For further details on the Hilfer derivative, we refer the reader to [8,9], while some results on boundary value problems involving the Hilfer fractional derivate can be found in the review article [10]. The Hadamard fractional derivative [11] contains a logarithmic function in its definition, and Caputo–Hadamard and Hilfer–Hadamard are regarded as variants of this derivative. One notices that the Hilfer–Hadamard fractional derivative is specialized into the Hadamard and Caputo–Hadamard fractional derivatives for the smallest and largest values of its parameter, respectively. For a detailed account of the work on Hadamard fractional differential equations, inclusions and inequalities, we refer the reader to the text [12]. One can find interesting results on Caputo–Hadamard fractional boundary value problems in [13,14]. The authors studied Hilfer–Hadamard fractional differential equations with multi-point and integral boundary conditions in [15,16].

Langevin [17] presented Newton's second law of motion for stochastic physics during his study of Brownian motion, which is known as the Langevin equation. He was of the view that his approach to Brownian motion was "infinitely more simple" than that offered by Einstein. Later, it was discovered that the Langevin equation failed to represent complex systems. This was the pretext for offering certain generalizations of the Langevin equation to describe physical phenomena in disordered regions [18], statistical physics [19], fluctuation–dissipation configurations [20], etc. A fractional counterpart of the Langevin equation was proposed in [21] by replacing the ordinary derivative it contains with a fractional counterpart, while a Langevin equation of two different fractional orders was presented in [22]. One can find other variants of the Langevin equation in [23,24]. For results on boundary value problems involving a fractional-order Langevin equation, we refer the reader to [25–27]. Considerable interest has also been shown in studying nonlinear systems of fractional-order Langevin equations involving different fractional derivatives and boundary conditions; for instance, see [28,29] and the references cited therein.

The notion of Ulam–Hyers stability [30,31] appears in several areas of research, such as functional equations [32], the Black–Scholes equation [33], etc. The Ulam's stability of an integral boundary value problem of coupled fractional differential equations was studied in [34]. The authors of [35] discussed the Hyers–Ulam stability for fractional differential equations using the Mittag–Leffler kernel. Some results on the Ulam–Hyers stability of Langevin fractional differential equations can be found in [36,37].

In this paper, influenced by the work described in the preceding paragraphs, we introduce a new set of nonlocal integral boundary conditions and study a coupled system of Hilfer–Hadamard fractional Langevin equations supplemented with these conditions. Precisely, we develop the existence theory and Ulam–Hyers stability for the system

$$\begin{cases} {}^{HH}D_{1^+}^{\kappa_1,\xi_1}\left({}^{HH}D_{1^+}^{\kappa_2,\xi_2}+\chi_1\right)x(t)=f_1(t,x(t),y(t)), & t\in\mathcal{J}, \\ {}^{HH}D_{1^+}^{\kappa_3,\xi_3}\left({}^{HH}D_{1^+}^{\kappa_4,\xi_4}+\chi_2\right)y(t)=f_2(t,x(t),y(t)), & t\in\mathcal{J}, \end{cases} \tag{1}$$

equipped with the nonlocal integral boundary conditions

$$\begin{cases} x(1)=0, \quad x(\mu_1)=\sum\limits_{i=1}^{m_1}\lambda_i {}^{H}I_{1^+}^{\alpha_i}y(\eta_i), \quad x(T)=\sum\limits_{j=1}^{n}\rho_j\int_{\zeta_j}^{\omega_j}\frac{1}{s}y(s)ds, \\ y(1)=0, \quad y(\mu_2)=\sum\limits_{l=1}^{m_2}\sigma_l {}^{H}I_{1^+}^{\beta_l}x(\nu_l), \quad y(T)=\sum\limits_{j=1}^{n}\tau_j\int_{\zeta_j}^{\omega_j}\frac{1}{s}x(s)ds, \end{cases} \tag{2}$$

where ${}^{HH}D_{1^+}^{\kappa_p,\xi_p}$ denotes the Hilfer–Hadamard fractional derivative operator of order κ_p and type $\xi_p(0\le\xi_p\le1)$; $p=1,2,3,4$, and ${}^{H}I_{1^+}^{\alpha_i}$ $(i=1,\dots,m_1)$ and ${}^{H}I_{1^+}^{\beta_l}$ $(l=1,\dots,m_2)$ are the Hadamard fractional integral operators of orders $\alpha_i,\ \beta_l>0$, respectively, with $1<\kappa_1,\kappa_3\le2$ and $0<\kappa_2,\kappa_4\le1$, $\chi_1,\chi_2>0$, $\lambda_i,\sigma_l,\rho_j,\tau_j\in\mathbb{R}$, $\mathcal{J}=[1,T]$, $\mu_1,\mu_2\in(1,T)$, $\eta_i,\nu_l\in(1,T)$, $1<\zeta_j<\omega_j<T$, $2<\gamma_1+\kappa_2\le3$, $2<\gamma_2+\kappa_4\le3$, and $f_1,f_2\in C(\mathcal{J}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$.

We make use of the Leray–Schauder alternative and Banach's fixed point theorem to derive results on the existence and uniqueness of the given problem, respectively. It is well known that the fixed point technique is an effective and fruitful method for developing a variety of results on the existence of boundary value problems under different criteria.

We have arranged the remaining content of this paper as follows. Some related concepts from fractional calculus and auxiliary results are presented in Section 2. Section 3 contains the main results, while examples illustrating these results are given in Section 4. We discuss the Ulam–Hyers stability for problem (1) and (2) in Section 5. New results for special cases are indicated in the final section.

2. Subsidiary Results

We begin this section with some definitions.

Definition 1 ([38]). *For a continuous function $g : [a, \infty) \to \mathbb{R}$, we define the Hadamard fractional integral of order $q > 0$ as*

$$ {}^{H}I^{q}_{a+}g(t) = \frac{1}{\Gamma(q)} \int_{a}^{t} \left(\log \frac{t}{s} \right)^{q-1} \frac{g(s)}{s} ds, \quad t > a, $$

where $\log(.) = \log_e(.)$.

Definition 2 ([38]). *For a continuous function $g : [a, \infty) \to \mathbb{R}$, the Hadamard fractional derivative of order $q > 0$ is defined by*

$$ {}^{H}D^{q}_{a+}g(t) = \delta^n \left({}^{H}I^{n-q}_{a+} g \right)(t), \quad n = [q] + 1, $$

where $\delta^n = t^n \frac{d^n}{dt^n}$, and $[q]$ denotes the integer part of the real number q.

Lemma 1 ([38]). *For $0 < a < \infty$, and $p, q > 0$, we have*

1. $\left({}^{H}I^{q}_{a+} (\log \frac{t}{a})^{p-1} \right)(x) = \frac{\Gamma(p)}{\Gamma(p+q)} (\log \frac{x}{a})^{p+q-1}$;
2. $\left({}^{H}D^{q}_{a+} (\log \frac{t}{a})^{p-1} \right)(x) = \frac{\Gamma(p)}{\Gamma(p-q)} (\log \frac{x}{a})^{p-q-1}$.

   In particular, for $p = 1$, we have

$$ \left({}^{H}D^{q}_{a+} 1 \right)(x) = \frac{1}{\Gamma(1-q)} \left(\log \frac{x}{a} \right)^{-q} \neq 0, \quad 0 < q < 1. $$

Definition 3 ([7]). *We define the Hilfer–Hadamard fractional derivative of order $q \in (n-1, n]$ and type $p \in [0, 1]$ for $g \in L^1(a, b), 0 < a < b < \infty$, as*

$$ \begin{aligned} \left({}^{HH}D^{q,p}_{a+} g \right)(t) &= \left({}^{H}I^{p(n-q)}_{a+} \delta^n {}^{H}I^{(n-q)(1-p)}_{a+} g \right)(t) \\ &= \left({}^{H}I^{p(n-q)}_{a+} \delta^n {}^{H}I^{(n-\gamma)}_{a+} g \right)(t) \\ &= \left({}^{H}I^{p(n-q)}_{a+} {}^{H}D^{\gamma}_{a+} g \right)(t), \quad \gamma = q + np - qp, \end{aligned} $$

where ${}^{H}I^{(.)}_{a^+}$ and ${}^{H}D^{(.)}_{a^+}$ are given in Definitions 1 and 2, respectively.

Theorem 1 ([39]). *For $g \in L^1(a, b)$ and $\left({}^{H}I^{n-\gamma}_{a+} g \right)(t) \in AC^n_\delta[a, b]$, we have*

$$ \begin{aligned} {}^{H}I^{q}_{a^+} \left({}^{HH}D^{q,p}_{a+} g \right)(t) &= {}^{H}I^{\gamma}_{a^+} \left({}^{HH}D^{\gamma}_{a+} g \right)(t) \\ &= g(t) - \sum_{j=0}^{n-1} \frac{\left(\delta^{(n-j-1)} \left({}^{H}I^{n-\gamma}_{a+} g \right) \right)(a)}{\Gamma(\gamma - j)} \left(\log \frac{t}{a} \right)^{\gamma - j - 1}, \end{aligned} $$

where $0 < a < b < \infty$, and $\gamma = q + np - qp$, $n = [q] + 1, q > 0, p \in [0, 1]$ and $AC^n_\delta[a, b] = \{ f : [a, b] \to \mathbb{R} : \delta^{n-1}[f(t)] \in AC[a, b] \}$. Notice that $\Gamma(\gamma - j)$ exists for all $j = 1, 2, \ldots, n-1$ and $\gamma \in [q, n]$.

In the current work, we write the Hadamard fractional integral operator ${}^{H}I^{q}_{a+}$ and the Hilfer–Hadamard fractional derivative operator ${}^{HH}D^{q}_{a+}$ as ${}^{H}I^{q}$ and ${}^{HH}D^{q}$, respectively.

In the following lemma, we solve the linear version of system (1) complemented with boundary data (2). This lemma plays a key role in transforming the given nonlinear problem into a fixed point problem.

Lemma 2. *For $h_1, h_2 \in C(\mathcal{J}, \mathbb{R})$, the unique solution of the linear system*

$$\begin{cases} {}^{HH}D^{\kappa_1,\xi_1}\big({}^{HH}D^{\kappa_2,\xi_2} + \chi_1\big)x(t) = h_1(t), \quad t \in \mathcal{J}, \\ {}^{HH}D^{\kappa_3,\xi_3}\big({}^{HH}D^{\kappa_4,\xi_4} + \chi_2\big)y(t) = h_2(t), \quad t \in \mathcal{J}, \end{cases} \tag{3}$$

subject to the boundary conditions in (2) is given by

$$\begin{aligned} x(t) \;=\; & \int_1^t \left[\frac{\big(\log\frac{t}{s}\big)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{h_1(s)}{s} - \chi_1 \frac{\big(\log\frac{t}{s}\big)^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{x(s)}{s} \right] ds \\ & + \mathbb{M}_1(t) \Bigg\{ \sum_{i=1}^{m_1} \lambda_i \int_1^{\eta_i} \left[\frac{\big(\log\frac{\eta_i}{s}\big)^{\alpha_i+\kappa_3+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_3+\kappa_4)} \frac{h_2(s)}{s} - \chi_2 \frac{\big(\log\frac{\eta_i}{s}\big)^{\alpha_i+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_4)} \frac{y(s)}{s} \right] ds \\ & - \int_1^{\mu_1} \left[\frac{\big(\log\frac{\mu_1}{s}\big)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{h_1(s)}{s} - \chi_1 \frac{\big(\log\frac{\mu_1}{s}\big)^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{x(s)}{s} \right] ds \Bigg\} \\ & + \mathbb{M}_2(t) \Bigg\{ \sum_{j=1}^{n} \rho_j \int_{\zeta_j}^{\omega_j} \frac{1}{s} \int_1^s \left[\frac{\big(\log\frac{s}{u}\big)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{h_2(u)}{u} - \chi_2 \frac{\big(\log\frac{s}{u}\big)^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{y(u)}{u} \right] du ds \\ & - \int_1^{T} \left[\frac{\big(\log\frac{T}{s}\big)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{h_1(s)}{s} - \chi_1 \frac{\big(\log\frac{T}{s}\big)^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{x(s)}{s} \right] ds \Bigg\} \\ & + \mathbb{M}_3(t) \Bigg\{ \sum_{l=1}^{m_2} \sigma_l \int_1^{\nu_l} \left[\frac{\big(\log\frac{\nu_l}{s}\big)^{\beta_l+\kappa_1+\kappa_2-1}}{\Gamma(\beta_l+\kappa_1+\kappa_2)} \frac{h_1(s)}{s} - \chi_1 \frac{\big(\log\frac{\nu_l}{s}\big)^{\beta_l+\kappa_2-1}}{\Gamma(\beta_l+\kappa_2)} \frac{x(s)}{s} \right] ds \\ & - \int_1^{\mu_2} \left[\frac{\big(\log\frac{\mu_2}{s}\big)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{h_2(s)}{s} - \chi_2 \frac{\big(\log\frac{\mu_2}{s}\big)^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{y(s)}{s} \right] ds \Bigg\} \\ & + \mathbb{M}_4(t) \Bigg\{ \sum_{j=1}^{n} \tau_j \int_{\zeta_j}^{\omega_j} \frac{1}{s} \int_1^s \left[\frac{\big(\log\frac{s}{u}\big)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{h_1(u)}{u} - \chi_1 \frac{\big(\log\frac{s}{u}\big)^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{x(u)}{u} \right] du ds \\ & - \int_1^{T} \left[\frac{\big(\log\frac{T}{s}\big)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{h_2(s)}{s} - \chi_2 \frac{\big(\log\frac{T}{s}\big)^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{y(s)}{s} \right] ds \Bigg\}, \quad t \in \mathcal{J}, \end{aligned} \tag{4}$$

and

$$\begin{aligned} y(t) \;=\; & \int_1^t \left[\frac{\big(\log\frac{t}{s}\big)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{h_2(s)}{s} - \chi_2 \frac{\big(\log\frac{t}{s}\big)^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{y(s)}{s} \right] ds \\ & + \mathbb{N}_1(t) \Bigg\{ \sum_{i=1}^{m_1} \lambda_i \int_1^{\eta_i} \left[\frac{\big(\log\frac{\eta_i}{s}\big)^{\alpha_i+\kappa_3+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_3+\kappa_4)} \frac{h_2(s)}{s} - \chi_2 \frac{\big(\log\frac{\eta_i}{s}\big)^{\alpha_i+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_4)} \frac{y(s)}{s} \right] ds \\ & - \int_1^{\mu_1} \left[\frac{\big(\log\frac{\mu_1}{s}\big)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{h_1(s)}{s} - \chi_1 \frac{\big(\log\frac{\mu_1}{s}\big)^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{x(s)}{s} \right] ds \Bigg\} \\ & + \mathbb{N}_2(t) \Bigg\{ \sum_{j=1}^{n} \rho_j \int_{\zeta_j}^{\omega_j} \frac{1}{s} \int_1^s \left[\frac{\big(\log\frac{s}{u}\big)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{h_2(u)}{u} - \chi_2 \frac{\big(\log\frac{s}{u}\big)^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{y(u)}{u} \right] du ds \\ & - \int_1^{T} \left[\frac{\big(\log\frac{T}{s}\big)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{h_1(s)}{s} - \chi_1 \frac{\big(\log\frac{T}{s}\big)^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{x(s)}{s} \right] ds \Bigg\} \\ & + \mathbb{N}_3(t) \Bigg\{ \sum_{l=1}^{m_2} \sigma_l \int_1^{\nu_l} \left[\frac{\big(\log\frac{\nu_l}{s}\big)^{\beta_l+\kappa_1+\kappa_2-1}}{\Gamma(\beta_l+\kappa_1+\kappa_2)} \frac{h_1(s)}{s} - \chi_1 \frac{\big(\log\frac{\nu_l}{s}\big)^{\beta_l+\kappa_2-1}}{\Gamma(\beta_l+\kappa_2)} \frac{x(s)}{s} \right] ds \\ & - \int_1^{\mu_2} \left[\frac{\big(\log\frac{\mu_2}{s}\big)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{h_2(s)}{s} - \chi_2 \frac{\big(\log\frac{\mu_2}{s}\big)^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{y(s)}{s} \right] ds \Bigg\} \\ & + \mathbb{N}_4(t) \Bigg\{ \sum_{j=1}^{n} \tau_j \int_{\zeta_j}^{\omega_j} \frac{1}{s} \int_1^s \left[\frac{\big(\log\frac{s}{u}\big)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{h_1(u)}{u} - \chi_1 \frac{\big(\log\frac{s}{u}\big)^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{x(u)}{u} \right] du ds \\ & - \int_1^{T} \left[\frac{\big(\log\frac{T}{s}\big)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{h_2(s)}{s} - \chi_2 \frac{\big(\log\frac{T}{s}\big)^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{y(s)}{s} \right] ds \Bigg\}, \quad t \in \mathcal{J}, \end{aligned} \tag{5}$$

where

$$
\begin{aligned}
&\mathbb{M}_1(t) = r_4(\log t)^{\gamma_1+\kappa_2-1} + r_3(\log t)^{\gamma_1+\kappa_2-2}, \quad \mathbb{M}_2(t) = z_4(\log t)^{\gamma_1+\kappa_2-1} + z_3(\log t)^{\gamma_1+\kappa_2-2},\\
&\mathbb{M}_3(t) = u_4(\log t)^{\gamma_1+\kappa_2-1} + u_3(\log t)^{\gamma_1+\kappa_2-2}, \quad \mathbb{M}_4(t) = s_4(\log t)^{\gamma_1+\kappa_2-1} + s_3(\log t)^{\gamma_1+\kappa_2-2},\\
&\mathbb{N}_1(t) = r_2(\log t)^{\gamma_2+\kappa_4-1} + r_1(\log t)^{\gamma_2+\kappa_4-2}, \quad \mathbb{N}_2(t) = z_2(\log t)^{\gamma_2+\kappa_4-1} + z_1(\log t)^{\gamma_2+\kappa_4-2},\\
&\mathbb{N}_3(t) = u_2(\log t)^{\gamma_2+\kappa_4-1} + u_1(\log t)^{\gamma_2+\kappa_4-2}, \quad \mathbb{N}_4(t) = s_2(\log t)^{\gamma_2+\kappa_4-1} + s_1(\log t)^{\gamma_2+\kappa_4-2},\\
&r_1 = \tfrac{1}{A_1\Delta}[-k_3(E_1+B_1\Delta_1) + k_1(L_1+B_1\Delta_2)], \quad r_2 = \tfrac{1}{A_1k_1}[E_1 - A_1k_2r_1 + B_1\Delta_1],\\
&r_3 = -\tfrac{1}{\Delta_3}[B_1 + (A_4B_1 - B_4A_1)r_1 + (A_3B_1 - B_3A_1)r_2], \quad r_4 = \tfrac{1}{A_1}(-A_2r_3 + A_4r_1 + A_3r_2 + 1),\\
&z_1 = \tfrac{1}{\Delta}(k_3\Delta_1 - k_1\Delta_2), \quad z_2 = \tfrac{-1}{k_1}(\Delta_1 + k_2z_1), \quad z_3 = \tfrac{1}{\Delta_3}[A_1 - (A_4B_1 - B_4A_1)z_1 - (A_3B_1 - B_3A_1)z_2],\\
&z_4 = \tfrac{-1}{A_1}(A_2z_3 - A_3z_2 - A_4z_1), \quad u_1 = \tfrac{-k_3}{\Delta}, \quad u_2 = \tfrac{1}{k_1}(1 - k_2u_1),\\
&u_3 = -\tfrac{1}{\Delta_3}[(A_3B_1 - B_3A_1)u_2 + (A_4B_1 - B_4A_1)u_1], \quad u_4 = \tfrac{1}{A_1}[-A_2u_3 + A_3u_2 + A_4u_1], \quad s_1 = \tfrac{k_1}{\Delta},\\
&s_2 = \tfrac{-k_2}{\Delta}, \quad s_3 = -\tfrac{1}{\Delta\Delta_3}[k_1(A_4B_1 - B_4A_1) - k_2(A_3B_1 - B_3A_1)], \quad s_4 = \tfrac{1}{A_1}(A_4s_1 - A_2s_3 + A_3s_2),\\
&\Delta_1 = \tfrac{1}{\Delta_3}(A_2E_1 - E_2A_1), \quad \Delta_2 = \tfrac{1}{\Delta_3}(A_2L_1 - L_2A_1), \quad \Delta_3 = A_1B_2 - B_1A_2 \neq 0, \quad \Delta = k_1k_4 - k_2k_3 \neq 0,\\
&k_1 = \tfrac{1}{A_1}[(A_1E_3 - E_1A_3) - \Delta_1(A_3B_1 - B_3A_1)], \quad k_2 = \tfrac{1}{A_1}[(A_1E_4 - E_1A_4) - \Delta_1(A_4B_1 - B_4A_1)],\\
&k_3 = \tfrac{1}{A_1}[(A_1L_3 - L_1A_3) - \Delta_2(A_3B_1 - B_3A_1)], \quad k_4 = \tfrac{1}{A_1}[(A_1L_4 - L_1A_4) - \Delta_2(A_4B_1 - B_4A_1)],\\
&A_1 = (\log\mu_1)^{\gamma_1+\kappa_2-1}, \quad A_2 = (\log\mu_1)^{\gamma_1+\kappa_2-2}, \quad A_3 = \sum_{i=1}^{m_1}\tfrac{\lambda_i\Gamma(\gamma_2+\kappa_4)}{\Gamma(\alpha_i+\gamma_2+\kappa_4)}(\log\eta_i)^{\alpha_i+\gamma_2+\kappa_4-1},\\
&A_4 = \sum_{i=1}^{m_1}\tfrac{\lambda_i\Gamma(\gamma_2+\kappa_4-1)}{\Gamma(\alpha_i+\gamma_2+\kappa_4-1)}(\log\eta_i)^{\alpha_i+\gamma_2+\kappa_4-2}, \quad B_1 = (\log T)^{\gamma_1+\kappa_2-1}, \quad B_2 = (\log T)^{\gamma_1+\kappa_2-2},\\
&B_3 = \tfrac{1}{(\gamma_2+\kappa_4)}\sum_{j=1}^{n}\rho_j\Big[(\log\omega_j)^{\gamma_2+\kappa_4} - (\log\zeta_j)^{\gamma_2+\kappa_4}\Big],\\
&B_4 = \tfrac{1}{(\gamma_2+\kappa_4-1)}\sum_{j=1}^{n}\rho_j\Big[(\log\omega_j)^{\gamma_2+\kappa_4-1} - (\log\zeta_j)^{\gamma_2+\kappa_4-1}\Big],\\
&E_1 = \sum_{l=1}^{m_2}\tfrac{\sigma_l\Gamma(\gamma_1+\kappa_2)}{\Gamma(\beta_l+\gamma_1+\kappa_2)}(\log\nu_l)^{\beta_l+\gamma_1+\kappa_2-1}, \quad E_2 = \sum_{l=1}^{m_2}\tfrac{\sigma_l\Gamma(\gamma_1+\kappa_2-1)}{\Gamma(\beta_l+\gamma_1+\kappa_2-1)}(\log\nu_l)^{\beta_l+\gamma_1+\kappa_2-2},\\
&E_3 = (\log\mu_2)^{\gamma_2+\kappa_4-1}, \quad E_4 = (\log\mu_2)^{\gamma_2+\kappa_4-2}, \quad L_1 = \tfrac{1}{(\gamma_1+\kappa_2)}\sum_{j=1}^{n}\tau_j\Big[(\log\omega_j)^{\gamma_1+\kappa_2} - (\log\zeta_j)^{\gamma_1+\kappa_2}\Big],\\
&L_2 = \tfrac{1}{(\gamma_1+\kappa_2-1)}\sum_{j=1}^{n}\tau_j\Big[(\log\omega_j)^{\gamma_1+\kappa_2-1} - (\log\zeta_j)^{\gamma_1+\kappa_2-1}\Big],\\
&L_3 = (\log T)^{\gamma_2+\kappa_4-1}, \; L_4 = (\log T)^{\gamma_2+\kappa_4-2}.
\end{aligned}
\tag{6}
$$

Proof. Using the Hadamard integral operators ${}^{H}I^{\kappa_1}$ and ${}^{H}I^{\kappa_3}$ on the first and second equations in (3), respectively, we obtain

$$
\begin{cases}
\left({}^{HH}D^{\kappa_2,\xi_2} + \chi_1\right)x(t) = {}^{H}I^{\kappa_1}h_1(t) + c_1(\log t)^{\gamma_1-1} + c_2(\log t)^{\gamma_1-2}, & t \in \mathcal{J},\\
\left({}^{HH}D^{\kappa_4,\xi_4} + \chi_2\right)y(t) = {}^{H}I^{\kappa_3}h_2(t) + d_1(\log t)^{\gamma_2-1} + d_2(\log t)^{\gamma_2-2}, & t \in \mathcal{J},
\end{cases}
\tag{7}
$$

where $\gamma_1 = \kappa_1 + (2-\kappa_1)\xi_1, \gamma_2 = \kappa_3 + (2-\kappa_3)\xi_3$, and $c_1, c_2, d_1, d_2 \in \mathbb{R}$ are unknown arbitrary constants.

Now, applying the Hadamard integral operators ${}^{H}I^{\kappa_2}$ and ${}^{H}I^{\kappa_4}$ to the first and second equations in (7), respectively, we obtain

$$
\begin{cases}
x(t) &= {}^{H}I^{\kappa_1+\kappa_2}h_1(t) - \chi_1{}^{H}I^{\kappa_2}x(t) + c_1\frac{\Gamma(\gamma_1)}{\Gamma(\gamma_1+\kappa_2)}(\log t)^{\gamma_1+\kappa_2-1}\\
&\quad + c_2\frac{\Gamma(\gamma_1-1)}{\Gamma(\gamma_1+\kappa_2-1)}(\log t)^{\gamma_1+\kappa_2-2} + c_3(\log t)^{\gamma_3-1}, \quad t \in \mathcal{J},\\
y(t) &= {}^{H}I^{\kappa_3+\kappa_4}h_2(t) - \chi_2{}^{H}I^{\kappa_4}y(t) + d_1\frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2+\kappa_4)}(\log t)^{\gamma_2+\kappa_4-1}\\
&\quad + d_2\frac{\Gamma(\gamma_2-1)}{\Gamma(\gamma_2+\kappa_4-1)}(\log t)^{\gamma_2+\kappa_4-2} + d_3(\log t)^{\gamma_4-1}, \quad t \in \mathcal{J},
\end{cases}
\tag{8}
$$

where $\gamma_3 = \kappa_2 + (1-\kappa_2)\xi_2, \ \gamma_4 = \kappa_4 + (1-\kappa_4)\xi_4$, and $c_3, d_3 \in \mathbb{R}$ are unknown arbitrary constants.

Combining the conditions $x(1) = 0 = y(1)$ with (8), we find that $c_3 = d_3 = 0$ as $\gamma_3 \in [\kappa_2, 1]$ and $\gamma_4 \in [\kappa_4, 1]$. Letting $\widetilde{c}_1 = c_1 \frac{\Gamma(\gamma_1)}{\Gamma(\gamma_1+\kappa_2)}, \widetilde{c}_2 = c_2 \frac{\Gamma(\gamma_1-1)}{\Gamma(\gamma_1+\kappa_2-1)}, \widetilde{d}_1 = d_1 \frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2+\kappa_4)}$, and $\widetilde{d}_2 = d_2 \frac{\Gamma(\gamma_2-1)}{\Gamma(\gamma_2+\kappa_4-1)}$, system (8) together with $c_3 = d_3 = 0$ takes the form

$$\begin{cases} x(t) &= {}^H I^{\kappa_1+\kappa_2} h_1(t) - \chi_1 {}^H I^{\kappa_2} x(t) + \widetilde{c}_1 (\log t)^{\gamma_1+\kappa_2-1} + \widetilde{c}_2 (\log t)^{\gamma_1+\kappa_2-2}, \quad t \in \mathcal{J}, \\ y(t) &= {}^H I^{\kappa_3+\kappa_4} h_2(t) - \chi_2 {}^H I^{\kappa_4} y(t) + \widetilde{d}_1 (\log t)^{\gamma_2+\kappa_4-1} + \widetilde{d}_2 (\log t)^{\gamma_2+\kappa_4-2}, \quad t \in \mathcal{J}. \end{cases} \tag{9}$$

Now, using (9) for the remaining conditions given by (2), we obtain

$$\begin{aligned} A_1\widetilde{c}_1 + A_2\widetilde{c}_2 - A_3\widetilde{d}_1 - A_4\widetilde{d}_2 &= \mathbb{J}_1, \\ B_1\widetilde{c}_1 + B_2\widetilde{c}_2 - B_3\widetilde{d}_1 - B_4\widetilde{d}_2 &= \mathbb{J}_2, \\ -E_1\widetilde{c}_1 - E_2\widetilde{c}_2 + E_3\widetilde{d}_1 + E_4\widetilde{d}_2 &= \mathbb{J}_3, \\ -L_1\widetilde{c}_1 - L_2\widetilde{c}_2 + L_3\widetilde{d}_1 + L_4\widetilde{d}_2 &= \mathbb{J}_4, \end{aligned} \tag{10}$$

where $A_p, B_p, E_p, L_p, p = 1,2,3,4$, are given in (6), and

$$\begin{aligned} \mathbb{J}_1 &= \sum_{i=1}^{m_1} \lambda_i \big[{}^H I^{\alpha_i+\kappa_3+\kappa_4} h_2(\eta_i) - \chi_2 {}^H I^{\alpha_i+\kappa_4} y(\eta_i) \big] - \big[{}^H I^{\kappa_1+\kappa_2} h_1(\mu_1) - \chi_1 {}^H I^{\kappa 2} x(\mu_1) \big], \\ \mathbb{J}_2 &= \sum_{j=1}^{n} \rho_j \int_{\zeta_j}^{\omega_j} \frac{1}{s} \big[{}^H I^{\kappa_3+\kappa_4} h_2(s) - \chi_2 {}^H I^{\kappa_4} y(s) \big] ds - \big[{}^H I^{\kappa_1+\kappa_2} h_1(T) - \chi_1 {}^H I^{\kappa_2} x(T) \big], \\ \mathbb{J}_3 &= \sum_{l=1}^{m_2} \sigma_l \big[{}^H I^{\beta_l+\kappa_1+\kappa_2} h_1(\nu_l) - \chi_1 {}^H I^{\beta_l+\kappa_2} x(\nu_l) \big] - \big[{}^H I^{\kappa_3+\kappa_4} h_2(\mu_2) - \chi_2 {}^H I^{\kappa 4} y(\mu_2) \big], \\ \mathbb{J}_4 &= \sum_{j=1}^{n} \tau_j \int_{\zeta_j}^{\omega_j} \frac{1}{s} \big[{}^H I^{\kappa_1+\kappa_2} h_1(s) - \chi_1 {}^H I^{\kappa_2} x(s) \big] ds - \big[{}^H I^{\kappa_3+\kappa_4} h_2(T) - \chi_2 {}^H I^{\kappa_4} y(T) \big]. \end{aligned} \tag{11}$$

Solving system (10) for $\widetilde{c}_1, \widetilde{c}_2, \widetilde{d}_1$, and $\widetilde{d}_2$, we find that

$$\begin{aligned} \widetilde{c}_1 &= r_4\mathbb{J}_1 + z_4\mathbb{J}_2 + u_4\mathbb{J}_3 + s_4\mathbb{J}_4, \\ \widetilde{c}_2 &= r_3\mathbb{J}_1 + z_3\mathbb{J}_2 + u_3\mathbb{J}_3 + s_3\mathbb{J}_4, \\ \widetilde{d}_1 &= r_2\mathbb{J}_1 + z_2\mathbb{J}_2 + u_2\mathbb{J}_3 + s_2\mathbb{J}_4, \\ \widetilde{d}_2 &= r_1\mathbb{J}_1 + z_1\mathbb{J}_2 + u_1\mathbb{J}_3 + s_1\mathbb{J}_4. \end{aligned} \tag{12}$$

Using (12) together with (6) and (11) in (9), we obtain the solution (4) and (5). The converse of the lemma follows through direct computation. □

3. The Main Results

According to Lemma 2, problem(1) and (2) can be transformed into a fixed point problem as $(x, y) = \mathbb{A}(x, y)$, where $\mathbb{A} : \mathbb{X} \times \mathbb{X} \to \mathbb{X} \times \mathbb{X}$ is an operator defined by

$$\mathbb{A}(x,y)(t) = \begin{pmatrix} \mathbb{A}_1(x,y)(t) \\ \mathbb{A}_2(x,y)(t) \end{pmatrix}, \tag{13}$$

with

$$\begin{aligned}
\mathbb{A}_1(x,y)(t) \quad = \quad & \int_1^t \left[\frac{\left(\log\frac{t}{s}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{f_1(s,x(s),y(s))}{s} - \chi_1 \frac{\left(\log\frac{t}{s}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{x(s)}{s} \right] ds \\
& + \mathbb{M}_1(t) \left\{ \sum_{i=1}^{m_1} \lambda_i \int_1^{\eta_i} \left[\frac{\left(\log\frac{\eta_i}{s}\right)^{\alpha_i+\kappa_3+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_3+\kappa_4)} \frac{f_2(s,x(s),y(s))}{s} - \chi_2 \frac{\left(\log\frac{\eta_i}{s}\right)^{\alpha_i+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_4)} \frac{y(s)}{s} \right] ds \right. \\
& \left. - \int_1^{\mu_1} \left[\frac{\left(\log\frac{\mu_1}{s}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{f_1(s,x(s),y(s))}{s} - \chi_1 \frac{\left(\log\frac{\mu_1}{s}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{x(s)}{s} \right] ds \right\} \\
& + \mathbb{M}_2(t) \left\{ \sum_{j=1}^{n} \rho_j \int_{\zeta_j}^{\omega_j} \frac{1}{s} \int_1^s \left[\frac{\left(\log\frac{s}{u}\right)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{f_2(u,x(u),y(u))}{u} - \chi_2 \frac{\left(\log\frac{s}{u}\right)^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{y(u)}{u} \right] du ds \right. \\
& \left. - \int_1^{T} \left[\frac{\left(\log\frac{T}{s}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{f_1(s,x(s),y(s))}{s} - \chi_1 \frac{\left(\log\frac{T}{s}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{x(s)}{s} \right] ds \right\} \\
& + \mathbb{M}_3(t) \left\{ \sum_{l=1}^{m_2} \sigma_l \int_1^{\nu_l} \left[\frac{\left(\log\frac{\nu_l}{s}\right)^{\beta_l+\kappa_1+\kappa_2-1}}{\Gamma(\beta_l+\kappa_1+\kappa_2)} \frac{f_1(s,x(s),y(s))}{s} - \chi_1 \frac{\left(\log\frac{\nu_l}{s}\right)^{\beta_l+\kappa_2-1}}{\Gamma(\beta_l+\kappa_2)} \frac{x(s)}{s} \right] ds \right. \\
& \left. - \int_1^{\mu_2} \left[\frac{\left(\log\frac{\mu_2}{s}\right)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{f_2(s,x(s),y(s))}{s} - \chi_2 \frac{\left(\log\frac{\mu_2}{s}\right)^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{y(s)}{s} \right] ds \right\} \\
& + \mathbb{M}_4(t) \left\{ \sum_{j=1}^{n} \tau_j \int_{\zeta_j}^{\omega_j} \frac{1}{s} \int_1^s \left[\frac{\left(\log\frac{s}{u}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{f_1(u,x(u),y(u))}{u} - \chi_1 \frac{\left(\log\frac{s}{u}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{x(u)}{u} \right] du ds \right. \\
& \left. - \int_1^{T} \left[\frac{\left(\log\frac{T}{s}\right)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{f_2(s,x(s),y(s))}{s} - \chi_2 \frac{\left(\log\frac{T}{s}\right)^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{y(s)}{s} \right] ds \right\},
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
\mathbb{A}_2(x,y)(t) \quad = \quad & \int_1^t \left[\frac{\left(\log\frac{t}{s}\right)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{f_2(s,x(s),y(s))}{s} - \chi_2 \frac{\left(\log\frac{t}{s}\right)^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{y(s)}{s} \right] ds \\
& + \mathbb{N}_1(t) \left\{ \sum_{i=1}^{m_1} \lambda_i \int_1^{\eta_i} \left[\frac{\left(\log\frac{\eta_i}{s}\right)^{\alpha_i+\kappa_3+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_3+\kappa_4)} \frac{f_2(s,x(s),y(s))}{s} - \chi_2 \frac{\left(\log\frac{\eta_i}{s}\right)^{\alpha_i+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_4)} \frac{y(s)}{s} \right] ds \right. \\
& \left. - \int_1^{\mu_1} \left[\frac{\left(\log\frac{\mu_1}{s}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{f_1(s,x(s),y(s))}{s} - \chi_1 \frac{\left(\log\frac{\mu_1}{s}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{x(s)}{s} \right] ds \right\} \\
& + \mathbb{N}_2(t) \left\{ \sum_{j=1}^{n} \rho_j \int_{\zeta_j}^{\omega_j} \frac{1}{s} \int_1^s \left[\frac{\left(\log\frac{s}{u}\right)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{f_2(u,x(u),y(u))}{u} - \chi_2 \frac{\left(\log\frac{s}{u}\right)^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{y(u)}{u} \right] du ds \right. \\
& \left. - \int_1^{T} \left[\frac{\left(\log\frac{T}{s}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{f_1(s,x(s),y(s))}{s} - \chi_1 \frac{\left(\log\frac{T}{s}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{x(s)}{s} \right] ds \right\} \\
& + \mathbb{N}_3(t) \left\{ \sum_{l=1}^{m_2} \sigma_l \int_1^{\nu_l} \left[\frac{\left(\log\frac{\nu_l}{s}\right)^{\beta_l+\kappa_1+\kappa_2-1}}{\Gamma(\beta_l+\kappa_1+\kappa_2)} \frac{f_1(s,x(s),y(s))}{s} - \chi_1 \frac{\left(\log\frac{\nu_l}{s}\right)^{\beta_l+\kappa_2-1}}{\Gamma(\beta_l+\kappa_2)} \frac{x(s)}{s} \right] ds \right. \\
& \left. - \int_1^{\mu_2} \left[\frac{\left(\log\frac{\mu_2}{s}\right)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{f_2(s,x(s),y(s))}{s} - \chi_2 \frac{\left(\log\frac{\mu_2}{s}\right)^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{y(s)}{s} \right] ds \right\} \\
& + \mathbb{N}_4(t) \left\{ \sum_{j=1}^{n} \tau_j \int_{\zeta_j}^{\omega_j} \frac{1}{s} \int_1^s \left[\frac{\left(\log\frac{s}{u}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{f_1(u,x(u),y(u))}{u} - \chi_1 \frac{\left(\log\frac{s}{u}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{x(u)}{u} \right] du ds \right. \\
& \left. - \int_1^{T} \left[\frac{\left(\log\frac{T}{s}\right)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{f_2(s,x(s),y(s))}{s} - \chi_2 \frac{\left(\log\frac{T}{s}\right)^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{y(s)}{s} \right] ds \right\}.
\end{aligned} \tag{15}$$

Here, the product space $\mathbb{X} \times \mathbb{X}$ is a Banach space equipped with the norm $\| (x,y) \| = \| x \| + \| y \|, (x,y) \in \mathbb{X} \times \mathbb{X}$, where $\mathbb{X}$ is the Banach space of all of the continuous functions from $\mathcal{J} \to \mathbb{R}$ endowed with the supremum norm $\| x \| = \sup_{t \in \mathcal{J}} |x(t)|$.

Note that the fixed points of the operator $\mathbb{A}$ correspond to solutions to problem (1) and (2).

In the sequel, we need the following assumptions:

(H_1) Real constants $\hat{m}_p, \hat{n}_p \geq 0,\ p = 1,2$, and $\hat{m}_0, \hat{n}_0 > 0$ exist such that $\forall x, y \in \mathbb{R}$,

$$|f_1(t,x,y)| \leq \hat{m}_0 + \hat{m}_1|x| + \hat{m}_2|y|,$$
$$|f_2(t,x,y)| \leq \hat{n}_0 + \hat{n}_1|x| + \hat{n}_2|y|;$$

(H_2) For all $t \in \mathcal{J}, x_p, y_p \in \mathbb{R},\ p = 1,2$, positive constants exist $\mathbb{L}_1$ and $\mathbb{L}_2$ such that

$$|f_1(t,x_2,y_2) - f_1(t,x_1,y_1| \leq \mathbb{L}_1(|x_2 - x_1| + |y_2 - y_1|),$$
$$|f_2(t,x_2,y_2) - f_2(t,x_1,y_1| \leq \mathbb{L}_2(|x_2 - x_1| + |y_2 - y_1|).$$

Next, we introduce the notation

$$\begin{aligned}
\Psi_1 &= \frac{(\log T)^{\kappa_1+\kappa_2}}{\Gamma(\kappa_1+\kappa_2+1)} + \overline{\mathbb{M}}_1 \frac{(\log \mu_1)^{\kappa_1+\kappa_2}}{\Gamma(\kappa_1+\kappa_2+1)} + \overline{\mathbb{M}}_2 \frac{(\log T)^{\kappa_1+\kappa_2}}{\Gamma(\kappa_1+\kappa_2+1)} + \overline{\mathbb{M}}_3 \sum_{l=1}^{m_2} \frac{|\sigma_l|(\log \nu_l)^{\beta_l+\kappa_1+\kappa_2}}{\Gamma(\beta_l+\kappa_1+\kappa_2+1)} \\
&\quad + \overline{\mathbb{M}}_4 \sum_{j=1}^{n} \frac{|\tau_j|}{\Gamma(\kappa_1+\kappa_2+2)} \left((\log \omega_j)^{\kappa_1+\kappa_2+1} - (\log \zeta_j)^{\kappa_1+\kappa_2+1} \right), \\
\Psi_2 &= \overline{\mathbb{M}}_1 \sum_{i=1}^{m_1} \frac{|\lambda_i|(\log \eta_i)^{\alpha_i+\kappa_3+\kappa_4}}{\Gamma(\alpha_i+\kappa_3+\kappa_4+1)} + \overline{\mathbb{M}}_2 \sum_{j=1}^{n} \frac{|\rho_j|}{\Gamma(\kappa_3+\kappa_4+2)} \left((\log \omega_j)^{\kappa_3+\kappa_4+1} - (\log \zeta_j)^{\kappa_3+\kappa_4+1} \right) \\
&\quad + \overline{\mathbb{M}}_3 \frac{(\log \mu_2)^{\kappa_3+\kappa_4}}{\Gamma(\kappa_3+\kappa_4+1)} + \overline{\mathbb{M}}_4 \frac{(\log T)^{\kappa_3+\kappa_4}}{\Gamma(\kappa_3+\kappa_4+1)}, \\
\Psi_3 &= \chi_1 \Bigg[\frac{(\log T)^{\kappa_2}}{\Gamma(\kappa_2+1)} + \overline{\mathbb{M}}_1 \frac{(\log \mu_1)^{\kappa_2}}{\Gamma(\kappa_2+1)} + \overline{\mathbb{M}}_2 \frac{(\log T)^{\kappa_2}}{\Gamma(\kappa_2+1)} + \overline{\mathbb{M}}_3 \sum_{l=1}^{m_2} \frac{|\sigma_l|(\log \nu_l)^{\beta_l+\kappa_2}}{\Gamma(\beta_l+\kappa_2+1)} \\
&\quad + \overline{\mathbb{M}}_4 \sum_{j=1}^{n} \frac{|\tau_j|}{\Gamma(\kappa_2+2)} \left((\log \omega_j)^{\kappa_2+1} - (\log \zeta_j)^{\kappa_2+1} \right) \Bigg], \\
\Psi_4 &= \chi_2 \Bigg[\overline{\mathbb{M}}_1 \sum_{i=1}^{m_1} \frac{|\lambda_i|(\log \eta_i)^{\alpha_i+\kappa_4}}{\Gamma(\alpha_i+\kappa_4+1)} + \overline{\mathbb{M}}_2 \sum_{j=1}^{n} \frac{|\rho_j|}{\Gamma(\kappa_4+2)} \left((\log \omega_j)^{\kappa_4+1} - (\log \zeta_j)^{\kappa_4+1} \right) \\
&\quad + \overline{\mathbb{M}}_3 \frac{(\log \mu_2)^{\kappa_4}}{\Gamma(\kappa_4+1)} + \overline{\mathbb{M}}_4 \frac{(\log T)^{\kappa_4}}{\Gamma(\kappa_4+1)} \Bigg], \\
\Phi_1 &= \overline{\mathbb{N}}_1 \frac{(\log \mu_1)^{\kappa_1+\kappa_2}}{\Gamma(\kappa_1+\kappa_2+1)} + \overline{\mathbb{N}}_2 \frac{(\log T)^{\kappa_1+\kappa_2}}{\Gamma(\kappa_1+\kappa_2+1)} + \overline{\mathbb{N}}_3 \sum_{l=1}^{m_2} \frac{|\sigma_l|(\log \nu_l)^{\beta_l+\kappa_1+\kappa_2}}{\Gamma(\beta_l+\kappa_1+\kappa_2+1)} \qquad (16) \\
&\quad + \overline{\mathbb{N}}_4 \sum_{j=1}^{n} \frac{|\tau_j|}{\Gamma(\kappa_1+\kappa_2+2)} \left((\log \omega_j)^{\kappa_1+\kappa_2+1} - (\log \zeta_j)^{\kappa_1+\kappa_2+1} \right), \\
\Phi_2 &= \frac{(\log T)^{\kappa_3+\kappa_4}}{\Gamma(\kappa_3+\kappa_4+1)} + \overline{\mathbb{N}}_1 \sum_{i=1}^{m_1} \frac{|\lambda_i|(\log \eta_i)^{\alpha_i+\kappa_3+\kappa_4}}{\Gamma(\alpha_i+\kappa_3+\kappa_4+1)} \\
&\quad + \overline{\mathbb{N}}_2 \sum_{j=1}^{n} \frac{|\rho_j|}{\Gamma(\kappa_3+\kappa_4+2)} \left((\log \omega_j)^{\kappa_3+\kappa_4+1} - (\log \zeta_j)^{\kappa_3+\kappa_4+1} \right) \\
&\quad + \overline{\mathbb{N}}_3 \frac{(\log \mu_2)^{\kappa_3+\kappa_4}}{\Gamma(\kappa_3+\kappa_4+1)} + \overline{\mathbb{N}}_4 \frac{(\log T)^{\kappa_3+\kappa_4}}{\Gamma(\kappa_3+\kappa_4+1)}, \\
\Phi_3 &= \chi_1 \Bigg[\overline{\mathbb{N}}_1 \frac{(\log \mu_1)^{\kappa_2}}{\Gamma(\kappa_2+1)} + \overline{\mathbb{N}}_2 \frac{(\log T)^{\kappa_2}}{\Gamma(\kappa_2+1)} + \overline{\mathbb{N}}_3 \sum_{l=1}^{m_2} \frac{|\sigma_l|(\log \nu_l)^{\beta_l+\kappa_2}}{\Gamma(\beta_l+\kappa_2+1)} \\
&\quad + \overline{\mathbb{N}}_4 \sum_{j=1}^{n} \frac{|\tau_j|}{\Gamma(\kappa_2+2)} \left((\log \omega_j)^{\kappa_2+1} - (\log \zeta_j)^{\kappa_2+1} \right) \Bigg], \\
\Phi_4 &= \chi_2 \Bigg[\frac{(\log T)^{\kappa_4}}{\Gamma(\kappa_4+1)} + \overline{\mathbb{N}}_1 \sum_{i=1}^{m_1} \frac{|\lambda_i|(\log \eta_i)^{\alpha_i+\kappa_4}}{\Gamma(\alpha_i+\kappa_4+1)} + \overline{\mathbb{N}}_2 \sum_{j=1}^{n} \frac{|\rho_j|}{\Gamma(\kappa_4+2)} \left((\log \omega_j)^{\kappa_4+1} - (\log \zeta_j)^{\kappa_4+1} \right) \\
&\quad + \overline{\mathbb{N}}_3 \frac{(\log \mu_2)^{\kappa_4}}{\Gamma(\kappa_4+1)} + \overline{\mathbb{N}}_4 \frac{(\log T)^{\kappa_4}}{\Gamma(\kappa_4+1)} \Bigg],
\end{aligned}$$

where

$$\overline{\mathbb{M}}_p = \sup_{t \in \mathcal{J}} |\mathbb{M}_p(t)|, \quad \overline{\mathbb{N}}_p = \sup_{t \in \mathcal{J}} |\mathbb{N}_p(t)|, \quad p = 1,2,3,4.$$

Now, we proceed to presenting our main results. In our first result, we establish the existence of solutions to the problem(1) and (2), which relies on the Leray–Schauder alternative [40].

Theorem 2. *Let $f_1, f_2 \in C(\mathcal{J} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and the condition (H_1) is satisfied. Then, at least one solution to the problem (1) and (2) on $\mathcal{J}$ existsif $0 < \mathbb{S}_1, \mathbb{S}_2 < 1$, where*

$$\begin{aligned}
\mathbb{S}_1 &= \hat{m}_1(\Psi_1 + \Phi_1) + \hat{n}_1(\Psi_2 + \Phi_2) + (\Psi_3 + \Phi_3), \\
\mathbb{S}_2 &= \hat{m}_2(\Psi_1 + \Phi_1) + \hat{n}_2(\Psi_2 + \Phi_2) + (\Psi_4 + \Phi_4),
\end{aligned} \qquad (17)$$

and Ψ_p, Φ_p, $p = 1,2,3,4$, are given in (16).

Proof. Let us first establish that the operator $\mathbb{A} : \mathbb{X} \times \mathbb{X} \to \mathbb{X} \times \mathbb{X}$, given by (13), is completely continuous. Obviously, the operator $\mathbb{A}$ is continuous in view of the continuity of f_1 and f_2. For a fixed number $\hat{r}$, we consider a bounded set $\mathbb{O} = \{(x,y) \in \mathbb{X} \times \mathbb{X} : \| (x,y) \| \leq \hat{r}\} \subset \mathbb{X} \times \mathbb{X}$. According to (H_1), we have

$$\begin{aligned} |f_1(t,x,y)| &\leq \hat{m}_0 + \hat{m}_1|x| + \hat{m}_2|y| \\ &\leq \hat{m}_0 + \hat{m}_1(\| x \| + \| y \|) + \hat{m}_2(\| x \| + \| y \|) \\ &\leq \hat{m}_0 + (\hat{m}_1 + \hat{m}_2)\hat{r} =: \mathbb{K}_1. \end{aligned}$$

Likewise, we have $|f_2(t,x,y)| \leq \hat{n}_0 + (\hat{n}_1 + \hat{n}_2)\hat{r} =: \mathbb{K}_2$. Then, for any $(x,y) \in \mathbb{O}$, we obtain

$$\begin{aligned} &\| \mathbb{A}_1(x,y) \| \\ \leq\; & \sup_{t\in\mathcal{J}} \Bigg\{ \int_1^t \left[\frac{(\log\frac{t}{s})^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{|f_1(s,x(s),y(s))|}{s} + \chi_1 \frac{(\log\frac{t}{s})^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{|x(s)|}{s} \right] ds \\ & + |\mathbb{M}_1(t)| \Bigg\{ \sum_{i=1}^{m_1} |\lambda_i| \int_1^{\eta_i} \left[\frac{(\log\frac{\eta_i}{s})^{\alpha_i+\kappa_3+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_3+\kappa_4)} \frac{|f_2(s,x(s),y(s))|}{s} + \chi_2 \frac{(\log\frac{\eta_i}{s})^{\alpha_i+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_4)} \frac{|y(s)|}{s} \right] ds \\ & + \int_1^{\mu_1} \left[\frac{(\log\frac{\mu_1}{s})^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{|f_1(s,x(s),y(s))|}{s} + \chi_1 \frac{(\log\frac{\mu_1}{s})^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{|x(s)|}{s} \right] ds \Bigg\} \\ & + |\mathbb{M}_2(t)| \Bigg\{ \sum_{j=1}^{n} |\rho_j| \int_{\zeta_j}^{\omega_j} \frac{1}{s} \int_1^s \left[\frac{(\log\frac{s}{u})^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{|f_2(u,x(u),y(u))|}{u} + \chi_2 \frac{(\log\frac{s}{u})^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{|y(u)|}{u} \right] du ds \\ & + \int_1^{T} \left[\frac{(\log\frac{T}{s})^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{|f_1(s,x(s),y(s))|}{s} + \chi_1 \frac{(\log\frac{T}{s})^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{|x(s)|}{s} \right] ds \Bigg\} \\ & + |\mathbb{M}_3(t)| \Bigg\{ \sum_{l=1}^{m_2} |\sigma_l| \int_1^{\nu_l} \left[\frac{(\log\frac{\nu_l}{s})^{\beta_l+\kappa_1+\kappa_2-1}}{\Gamma(\beta_l+\kappa_1+\kappa_2)} \frac{|f_1(s,x(s),y(s))|}{s} + \chi_1 \frac{(\log\frac{\nu_l}{s})^{\beta_l+\kappa_2-1}}{\Gamma(\beta_l+\kappa_2)} \frac{|x(s)|}{s} \right] ds \\ & + \int_1^{\mu_2} \left[\frac{(\log\frac{\mu_2}{s})^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{|f_2(s,x(s),y(s))|}{s} + \chi_2 \frac{(\log\frac{\mu_2}{s})^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{|y(s)|}{s} \right] ds \Bigg\} \\ & + |\mathbb{M}_4(t)| \Bigg\{ \sum_{j=1}^{n} |\tau_j| \int_{\zeta_j}^{\omega_j} \frac{1}{s} \int_1^s \left[\frac{(\log\frac{s}{u})^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{|f_1(u,x(u),y(u))|}{u} + \chi_1 \frac{(\log\frac{s}{u})^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{|x(u)|}{u} \right] du ds \\ & + \int_1^{T} \left[\frac{(\log\frac{T}{s})^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{|f_2(s,x(s),y(s))|}{s} + \chi_2 \frac{(\log\frac{T}{s})^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{|y(s)|}{s} \right] ds \Bigg\} \Bigg\} \\ \leq\; & \mathbb{K}_1 \Bigg[\frac{(\log T)^{\kappa_1+\kappa_2}}{\Gamma(\kappa_1+\kappa_2+1)} + \overline{\mathbb{M}}_1 \frac{(\log \mu_1)^{\kappa_1+\kappa_2}}{\Gamma(\kappa_1+\kappa_2+1)} + \overline{\mathbb{M}}_2 \frac{(\log T)^{\kappa_1+\kappa_2}}{\Gamma(\kappa_1+\kappa_2+1)} + \overline{\mathbb{M}}_3 \sum_{l=1}^{m_2} \frac{|\sigma_l|(\log \nu_l)^{\beta_l+\kappa_1+\kappa_2}}{\Gamma(\beta_l+\kappa_1+\kappa_2+1)} \\ & + \overline{\mathbb{M}}_4 \sum_{j=1}^{n} \frac{|\tau_j|}{\Gamma(\kappa_1+\kappa_2+2)} \Big((\log \omega_j)^{\kappa_1+\kappa_2+1} - (\log \zeta_j)^{\kappa_1+\kappa_2+1} \Big) \Bigg] \\ & + \mathbb{K}_2 \Bigg[\overline{\mathbb{M}}_1 \sum_{i=1}^{m_1} \frac{|\lambda_i|(\log \eta_i)^{\alpha_i+\kappa_3+\kappa_4}}{\Gamma(\alpha_i+\kappa_3+\kappa_4+1)} + \overline{\mathbb{M}}_2 \sum_{j=1}^{n} \frac{|\rho_j|}{\Gamma(\kappa_3+\kappa_4+2)} \Big((\log \omega_j)^{\kappa_3+\kappa_4+1} - (\log \zeta_j)^{\kappa_3+\kappa_4+1} \Big) \\ & + \overline{\mathbb{M}}_3 \frac{(\log \mu_2)^{\kappa_3+\kappa_4}}{\Gamma(\kappa_3+\kappa_4+1)} + \overline{\mathbb{M}}_4 \frac{(\log T)^{\kappa_3+\kappa_4}}{\Gamma(\kappa_3+\kappa_4+1)} \Bigg] \\ & + \chi_1 \Bigg[\frac{(\log T)^{\kappa_2}}{\Gamma(\kappa_2+1)} + \overline{\mathbb{M}}_1 \frac{(\log \mu_1)^{\kappa_2}}{\Gamma(\kappa_2+1)} + \overline{\mathbb{M}}_2 \frac{(\log T)^{\kappa_2}}{\Gamma(\kappa_2+1)} + \overline{\mathbb{M}}_3 \sum_{l=1}^{m_2} \frac{|\sigma_l|(\log \nu_l)^{\beta_l+\kappa_2}}{\Gamma(\beta_l+\kappa_2+1)} \\ & + \overline{\mathbb{M}}_4 \sum_{j=1}^{n} \frac{|\tau_j|}{\Gamma(\kappa_2+2)} \Big((\log \omega_j)^{\kappa_2+1} - (\log \zeta_j)^{\kappa_2+1} \Big) \Bigg] \| x \| \\ & + \chi_2 \Bigg[\overline{\mathbb{M}}_1 \sum_{i=1}^{m_1} \frac{|\lambda_i|(\log \eta_i)^{\alpha_i+\kappa_4}}{\Gamma(\alpha_i+\kappa_4+1)} + \overline{\mathbb{M}}_2 \sum_{j=1}^{n} \frac{|\rho_j|}{\Gamma(\kappa_4+2)} \Big((\log \omega_j)^{\kappa_4+1} - (\log \zeta_j)^{\kappa_4+1} \Big) \\ & + \overline{\mathbb{M}}_3 \frac{(\log \mu_2)^{\kappa_4}}{\Gamma(\kappa_4+1)} + \overline{\mathbb{M}}_4 \frac{(\log T)^{\kappa_4}}{\Gamma(\kappa_4+1)} \Bigg] \| y \| \\ \leq\; & \mathbb{K}_1\Psi_1 + \mathbb{K}_2\Psi_2 + \Psi_3\| x \| + \Psi_4\| y \|, \end{aligned} \tag{18}$$

where $\Psi_p,\ p = 1,2,3,4$, are given in (16).

Similarly, one can find that

$$\| \mathbb{A}_2(x,y) \| \leq \mathbb{K}_1\Phi_1 + \mathbb{K}_2\Phi_2 + \Phi_3\| x \| + \Phi_4\| y \|, \tag{19}$$

where $\Phi_p, \ p = 1,2,3,4$, are given in (16).

From (18) and (19), we have

$$\begin{aligned} \| \mathbb{A}(x,y) \| &= \| \mathbb{A}_1(x,y) \| + \|\mathbb{A}_2(x,y) \| \\ &\leq \mathbb{K}_1(\Psi_1 + \Phi_1) + \mathbb{K}_2(\Psi_2 + \Phi_2) + \max(\Psi_3 + \Phi_3, \Psi_4 + \Phi_4)\| (x,y) \| \\ &\leq \mathbb{K}_1(\Psi_1 + \Phi_1) + \mathbb{K}_2(\Psi_2 + \Phi_2) + \max(\Psi_3 + \Phi_3, \Psi_4 + \Phi_4)\hat{r}, \end{aligned}$$

which implies that $\mathbb{A}(\mathbb{O})$ is uniformly bounded.

To show that $\mathbb{A}(\mathbb{O})$ is equicontinuous, we take $t_1, t_2 \in \mathcal{J}$ with $t_1 < t_2$. Then, we obtain

$$\begin{aligned} & |\mathbb{A}_1(x,y)(t_2) - \mathbb{A}_1(x,y)(t_1)| \\ \leq & \left| \int_1^{t_2} \left[\frac{\left(\log \frac{t_2}{s}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{f_1(s,x(s),y(s))}{s} - \chi_1 \frac{\left(\log \frac{t_2}{s}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{x(s)}{s} \right] ds \right. \\ & \left. - \int_1^{t_1} \left[\frac{\left(\log \frac{t_1}{s}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{f_1(s,x(s),y(s))}{s} - \chi_1 \frac{\left(\log \frac{t_1}{s}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{x(s)}{s} \right] ds \right| \\ & + |\mathbb{M}_1(t_2) - \mathbb{M}_1(t_2)||\mathbb{J}_1| + |\mathbb{M}_2(t_2) - \mathbb{M}_2(t_2)||\mathbb{J}_2| \\ & + |\mathbb{M}_3(t_2) - \mathbb{M}_3(t_2)||\mathbb{J}_3| + |\mathbb{M}_4(t_2) - \mathbb{M}_4(t_2)||\mathbb{J}_4| \\ \leq & \frac{\mathbb{K}_1}{\Gamma(\kappa_1+\kappa_2+1)} \left[2(\log t_2 - \log t_1)^{\kappa_1+\kappa_2} + \left| (\log t_2)^{\kappa_1+\kappa_2} - (\log t_1)^{\kappa_1+\kappa_2} \right| \right] \\ & + \frac{\chi_1}{\Gamma(\kappa_2+1)} \left[2(\log t_2 - \log t_1)^{\kappa_2} + \left| (\log t_2)^{\kappa_2} - (\log t_1)^{\kappa_2} \right| \right] \hat{r} \\ & + \left[|r_4| \left| (\log t_2)^{\gamma_1+\kappa_2-1} - (\log t_1)^{\gamma_1+\kappa_2-1} \right| \right. \\ & \left. + |r_3| \left| (\log t_2)^{\gamma_1+\kappa_2-2} - (\log t_1)^{\gamma_1+\kappa_2-2} \right| \right] |\mathbb{J}_1| \\ & + \left[|z_4| \left| (\log t_2)^{\gamma_1+\kappa_2-1} - (\log t_1)^{\gamma_1+\kappa_2-1} \right| \right. \\ & \left. + |z_3| \left| (\log t_2)^{\gamma_1+\kappa_2-2} - (\log t_1)^{\gamma_1+\kappa_2-2} \right| \right] |\mathbb{J}_2| \\ & + \left[|u_4| \left| (\log t_2)^{\gamma_1+\kappa_2-1} - (\log t_1)^{\gamma_1+\kappa_2-1} \right| \right. \\ & \left. + |u_3| \left| (\log t_2)^{\gamma_1+\kappa_2-2} - (\log t_1)^{\gamma_1+\kappa_2-2} \right| \right] |\mathbb{J}_3| \\ & + \left[|s_4| \left| (\log t_2)^{\gamma_1+\kappa_2-1} - (\log t_1)^{\gamma_1+\kappa_2-1} \right| \right. \\ & \left. + |s_3| \left| (\log t_2)^{\gamma_1+\kappa_2-2} - (\log t_1)^{\gamma_1+\kappa_2-2} \right| \right] |\mathbb{J}_4| \\ \longrightarrow & \ 0 \text{ as } (t_2 - t_1) \to 0 \text{ independently of } (x,y) \in \mathbb{O}, \end{aligned}$$

where $\mathbb{J}_i, \ \ i = 1,2,3,4$, are given in (11) with f_1, f_2 instead of h_1, h_2 here. Analogously, it can be shown that

$$|\mathbb{A}_2(x,y)(t_2) - \mathbb{A}_2(x,y)(t_1)| \longrightarrow 0 \text{ as } (t_2 - t_1) \to 0 \text{ independently of } (x,y) \in \mathbb{O}.$$

Thus, $\mathbb{A}_1(\mathbb{O})$ and $\mathbb{A}_2(\mathbb{O})$ are equicontinuous, and hence $\mathbb{A}(\mathbb{O})$ is equicontinuous. Hence, we deduce using the Arzelá–Ascoli theorem that $\mathbb{A}(\mathbb{O})$ is completely continuous.

Finally, we consider a set $\mathbb{H} = \{(x,y) \in \mathbb{X} \times \mathbb{X} : (x,y) = \tau\mathbb{A}(x,y), \ 0 < \tau \leq 1\}$ and verify its boundedness. Let $(x,y) \in \mathbb{H}$. Then, $(x,y) = \tau\mathbb{A}(x,y)$ implies that $x(t) = \tau\mathbb{A}_1(x,y)(t)$ and $y(t) = \tau\mathbb{A}_2(x,y)(t)$ for $t \in \mathcal{J}$. Then, according to assumption (H_1), we have

$$
\begin{aligned}
\| x \| &= \sup_{t\in\mathcal{J}}|x(t)| \le \sup_{t\in\mathcal{J}}|\mathbb{A}_1(x,y)(t)| \\
&\le \sup_{t\in\mathcal{J}}\Bigg\{ \int_1^t \left[\frac{(\log\frac{t}{s})^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{(\hat{m}_0+\hat{m}_1|x|+\hat{m}_2|y|)}{s} + \chi_1 \frac{(\log\frac{t}{s})^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{|x(s)|}{s} \right] ds \\
&\quad +|\mathbb{M}_1(t)|\Bigg\{ \sum_{i=1}^{m_1} |\lambda_i| \int_1^{\eta_i} \left[\frac{(\log\frac{\eta_i}{s})^{\alpha_i+\kappa_3+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_3+\kappa_4)} \frac{(\hat{n}_0+\hat{n}_1|x|+\hat{n}_2|y|)}{s} + \chi_2 \frac{(\log\frac{\eta_i}{s})^{\alpha_i+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_4)} \frac{|y(s)|}{s} \right] ds \\
&\quad + \int_1^{\mu_1} \left[\frac{(\log\frac{\mu_1}{s})^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{(\hat{m}_0+\hat{m}_1|x|+\hat{m}_2|y|)}{s} + \chi_1 \frac{(\log\frac{\mu_1}{s})^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{|x(s)|}{s} \right] ds \Bigg\} \\
&\quad +|\mathbb{M}_2(t)|\Bigg\{ \sum_{j=1}^{n} |\rho_j| \int_{\zeta_j}^{\omega_j} \frac{1}{s} \int_1^s \left[\frac{(\log\frac{s}{u})^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{(\hat{n}_0+\hat{n}_1|x|+\hat{n}_2|y|)}{u} + \chi_2 \frac{(\log\frac{s}{u})^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{|y(u)|}{u} \right] du ds \\
&\quad + \int_1^{T} \left[\frac{(\log\frac{T}{s})^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{(\hat{m}_0+\hat{m}_1|x|+\hat{m}_2|y|)}{s} + \chi_1 \frac{(\log\frac{T}{s})^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{|x(s)|}{s} \right] ds \Bigg\} \\
&\quad +|\mathbb{M}_3(t)|\Bigg\{ \sum_{l=1}^{m_2} |\sigma_l| \int_1^{\nu_l} \left[\frac{(\log\frac{\nu_l}{s})^{\beta_l+\kappa_1+\kappa_2-1}}{\Gamma(\beta_l+\kappa_1+\kappa_2)} \frac{(\hat{m}_0+\hat{m}_1|x|+\hat{m}_2|y|)}{s} + \chi_1 \frac{(\log\frac{\nu_l}{s})^{\beta_l+\kappa_2-1}}{\Gamma(\beta_l+\kappa_2)} \frac{|x(s)|}{s} \right] ds \\
&\quad + \int_1^{\mu_2} \left[\frac{(\log\frac{\mu_2}{s})^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{(\hat{n}_0+\hat{n}_1|x|+\hat{n}_2|y|)}{s} + \chi_2 \frac{(\log\frac{\mu_2}{s})^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{|y(s)|}{s} \right] ds \Bigg\} \\
&\quad +|\mathbb{M}_4(t)|\Bigg\{ \sum_{j=1}^{n} |\tau_j| \int_{\zeta_j}^{\omega_j} \frac{1}{s} \int_1^s \left[\frac{(\log\frac{s}{u})^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{(\hat{m}_0+\hat{m}_1|x|+\hat{m}_2|y|)}{u} + \chi_1 \frac{(\log\frac{s}{u})^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{|x(u)|}{u} \right] du ds \\
&\quad + \int_1^{T} \left[\frac{(\log\frac{T}{s})^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{(\hat{n}_0+\hat{n}_1|x|+\hat{n}_2|y|)}{s} + \chi_2 \frac{(\log\frac{T}{s})^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{|y(s)|}{s} \right] ds \Bigg\} \Bigg\},
\end{aligned}
$$

which implies that

$$\| x \| \le (\hat{m}_0\Psi_1 + \hat{n}_0\Psi_2) + (\hat{m}_1\Psi_1 + \hat{n}_1\Psi_2 + \Psi_3)\| x \| + (\hat{m}_2\Psi_1 + \hat{n}_2\Psi_2 + \Psi_4)\| y \|. \tag{20}$$

In a similar manner, we can find that

$$\| y \| \le (\hat{m}_0\Phi_1 + \hat{n}_0\Phi_2) + (\hat{m}_1\Phi_1 + \hat{n}_1\Phi_2 + \Phi_3)\| x \| + (\hat{m}_2\Phi_1 + \hat{n}_2\Phi_2 + \Phi_4)\| y \|. \tag{21}$$

From (20) and (21), it follows that

$$\| x \| + \| y \| \le \hat{m}_0(\Psi_1 + \Phi_1) + \hat{n}_0(\Psi_2 + \Phi_2) + \max(\mathbb{S}_1, \mathbb{S}_2)\| (x, y) \|,$$

where $\mathbb{S}_1$ and $\mathbb{S}_2$ are given in (17). The above inequality can alternatively be written as

$$\| (x, y) \| \le \frac{1}{\mathbb{G}_0}[\hat{m}_0(\Psi_1 + \Phi_1) + \hat{n}_0(\Psi_2 + \Phi_2)],$$

where

$$\mathbb{G}_0 = 1 - \max(\mathbb{S}_1, \mathbb{S}_2).$$

Thus, the set $\mathbb{H}$ is bounded. Consequently, the operator $\mathbb{A}$ has at least one fixed point through application of the Leray–Schauder alternative [40]. Therefore, problem (1) and (2) admit at least one solution on $\mathcal{J}$. □

Next, we obtain a result on the uniqueness of the problem (1) and (2) with the aid of Banach's fixed point theorem [40].

Theorem 3. *Assume that $f_1, f_2 \in C(\mathcal{J} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. In addition, we suppose that the condition (H_2) is satisfied and*

$$\mathbb{L}_1(\Psi_1 + \Phi_1) + \mathbb{L}_2(\Psi_2 + \Phi_2) + \max(\Psi_3 + \Phi_3, \Psi_4 + \Phi_4) < 1, \tag{22}$$

where Ψ_p and Φ_p, $p = 1,2,3,4$, are given in (16). Then, the problem (1) and (2) have a unique solution on $\mathcal{J}$.

Proof. To verify the hypotheses of Banach's fixed point theorem, we consider a closed ball $\mathbb{B}_r = \{(x,y) \in \mathbb{X} \times \mathbb{X} : \| (x,y) \| \leq r\}$ with

$$r \geq \frac{\mathcal{M}_1(\Psi_1+\Phi_1)+\mathcal{M}_2(\Psi_2+\Phi_2)}{1-[\mathbb{L}_1(\Psi_1+\Phi_1)+\mathbb{L}_2(\Psi_2+\Phi_2)]-\max(\Psi_3+\Phi_3,\Psi_4+\Phi_4)}, \tag{23}$$

where $\sup_{t\in\mathcal{J}}|f_1(t,0,0)| = \mathcal{M}_1 < \infty$ and $\sup_{t\in\mathcal{J}}|f_2(t,0,0)| = \mathcal{M}_2 < \infty$. Now, we establish that $\mathbb{A}\mathbb{B}_r \subset \mathbb{B}_r$, where $\mathbb{A} : \mathbb{B}_r \longrightarrow \mathbb{X} \times \mathbb{X}$ is given by (13). According to (H_2), we have

$$\begin{aligned} |f_1(t,x(t),y(t))| &= |f_1(t,x(t),y(t)) - f_1(t,0,0) + f_1(t,0,0)| \leq \mathbb{L}_1(\| x \| + \| y \|) + \mathcal{M}_1 \\ &\leq \mathbb{L}_1 r + \mathcal{M}_1, \\ |f_2(t,x(t),y(t)| &= |f_2(t,x(t),y(t)) - f_2(t,0,0) + f_2(t,0,0)| \leq \mathbb{L}_2(\| x \| + \| y \|) + \mathcal{M}_2 \\ &\leq \mathbb{L}_2 r + \mathcal{M}_2. \end{aligned} \tag{24}$$

For $(x,y) \in \mathbb{B}_r$, it follows by using (24) that

$$\begin{aligned} \| \mathbb{A}_1(x,y) \| \leq\ & \sup_{t\in\mathcal{J}}\Bigg\{ \int_1^t \Bigg[\frac{(\log\frac{t}{s})^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)}\frac{(\mathbb{L}_1 r+\mathcal{M}_1)}{s} + \chi_1\frac{(\log\frac{t}{s})^{\kappa_2-1}}{\Gamma(\kappa_2)}\frac{|x(s)|}{s}\Bigg]ds \\ & +|\mathbb{M}_1(t)|\Bigg\{ \sum_{i=1}^{m_1}|\lambda_i|\int_1^{\eta_i}\Bigg[\frac{(\log\frac{\eta_i}{s})^{\alpha_i+\kappa_3+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_3+\kappa_4)}\frac{(\mathbb{L}_2 r+\mathcal{M}_2)}{s} + \chi_2\frac{(\log\frac{\eta_i}{s})^{\alpha_i+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_4)}\frac{|y(s)|}{s}\Bigg]ds \\ & +\int_1^{\mu_1}\Bigg[\frac{(\log\frac{\mu_1}{s})^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)}\frac{(\mathbb{L}_1 r+\mathcal{M}_1)}{s} + \chi_1\frac{(\log\frac{\mu_1}{s})^{\kappa_2-1}}{\Gamma(\kappa_2)}\frac{|x(s)|}{s}\Bigg]ds\Bigg\} \\ & +|\mathbb{M}_2(t)|\Bigg\{ \sum_{j=1}^{n}|\rho_j|\int_{\zeta_j}^{\omega_j}\frac{1}{s}\int_1^s\Bigg[\frac{(\log\frac{s}{u})^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)}\frac{(\mathbb{L}_2 r+\mathcal{M}_2)}{u} + \chi_2\frac{(\log\frac{s}{u})^{\kappa_4-1}}{\Gamma(\kappa_4)}\frac{|y(u)|}{u}\Bigg]duds \\ & +\int_1^{T}\Bigg[\frac{(\log\frac{T}{s})^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)}\frac{(\mathbb{L}_1 r+\mathcal{M}_1)}{s} + \chi_1\frac{(\log\frac{T}{s})^{\kappa_2-1}}{\Gamma(\kappa_2)}\frac{|x(s)|}{s}\Bigg]ds\Bigg\} \\ & +|\mathbb{M}_3(t)|\Bigg\{ \sum_{l=1}^{m_2}|\sigma_l|\int_1^{\nu_l}\Bigg[\frac{(\log\frac{\nu_l}{s})^{\beta_l+\kappa_1+\kappa_2-1}}{\Gamma(\beta_l+\kappa_1+\kappa_2)}\frac{(\mathbb{L}_1 r+\mathcal{M}_1)}{s} + \chi_1\frac{(\log\frac{\nu_l}{s})^{\beta_l+\kappa_2-1}}{\Gamma(\beta_l+\kappa_2)}\frac{|x(s)|}{s}\Bigg]ds \\ & +\int_1^{\mu_2}\Bigg[\frac{(\log\frac{\mu_2}{s})^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)}\frac{(\mathbb{L}_2 r+\mathcal{M}_2)}{s} + \chi_2\frac{(\log\frac{\mu_2}{s})^{\kappa_4-1}}{\Gamma(\kappa_4)}\frac{|y(s)|}{s}\Bigg]ds\Bigg\} \\ & +|\mathbb{M}_4(t)|\Bigg\{ \sum_{j=1}^{n}|\tau_j|\int_{\zeta_j}^{\omega_j}\frac{1}{s}\int_1^s\Bigg[\frac{(\log\frac{s}{u})^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)}\frac{(\mathbb{L}_1 r+\mathcal{M}_1)}{u} + \chi_1\frac{(\log\frac{s}{u})^{\kappa_2-1}}{\Gamma(\kappa_2)}\frac{|x(u)|}{u}\Bigg]duds \\ & +\int_1^{T}\Bigg[\frac{(\log\frac{T}{s})^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)}\frac{(\mathbb{L}_2 r+\mathcal{M}_2)}{s} + \chi_2\frac{(\log\frac{T}{s})^{\kappa_4-1}}{\Gamma(\kappa_4)}\frac{|y(s|)}{s}\Bigg]ds\Bigg\}\Bigg\} \\ & \leq (\mathbb{L}_1 r+\mathcal{M}_1)\Psi_1 + \Psi_3\| x \| + (\mathbb{L}_2 r+\mathcal{M}_2)\Psi_2 + \Psi_4\| y \|. \end{aligned} \tag{25}$$

Likewise, we can find that

$$\| \mathbb{A}_2(x,y) \| \leq (\mathbb{L}_1 r+\mathcal{M}_1)\Phi_1 + \Phi_3\| x \| + (\mathbb{L}_2 r+\mathcal{M}_2)\Phi_2 + \Phi_4\| y \|. \tag{26}$$

From (25) and (26), together with (23),

$$\begin{aligned} \| \mathbb{A}(x,y) \| &= \| \mathbb{A}_1(x,y) \| + \|\mathbb{A}_2(x,y) \| \\ &\leq [\mathbb{L}_1(\Psi_1+\Phi_1)+\mathbb{L}_2(\Psi_2+\Phi_2)]r + \mathcal{M}_1(\Psi_1+\Phi_1)+\mathcal{M}_2(\Psi_2+\Phi_2) \\ &\quad + \max(\Psi_3+\Phi_3,\Psi_4+\Phi_4)r \leq r, \end{aligned}$$

which shows that $\mathbb{A}(x,y) \in \mathbb{B}_r$. Hence, $\mathbb{A}\mathbb{B}_r \subset \mathbb{B}_r$ since $(x,y) \in \mathbb{B}_r$ is an arbitrary element.

Next, we verify that the operator $\mathbb{A}$ is a contraction. For this, let $(x_1,y_1),(x_2,y_2) \in \mathbb{X} \times \mathbb{X}$. Then, for any $t \in \mathcal{J}$, we obtain

$$
\begin{aligned}
& \| \mathbb{A}_1(x_2,y_2) - \mathbb{A}_1(x_1,y_1) \| \\
\leq\ & \sup_{t\in\mathcal{J}} \Bigg\{ \int_1^t \Bigg[\frac{(\log\frac{t}{s})^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{|f_1(s,x_2(s),y_2(s))-f_1(s,x_1(s),y_1(s))|}{s} \\
& +\chi_1 \frac{(\log\frac{t}{s})^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{|x_2(s)-x_1(s)|}{s} \Bigg] ds \\
& +|\mathbb{M}_1(t)| \Bigg\{ \sum_{i=1}^{m_1} |\lambda_i| \int_1^{\eta_i} \Bigg[\frac{(\log\frac{\eta_i}{s})^{\alpha_i+\kappa_3+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_3+\kappa_4)} \frac{|f_2(s,x_2(s),y_2(s))-f_2(s,x_1(s),y_1(s))|}{s} \\
& +\chi_2 \frac{(\log\frac{\eta_i}{s})^{\alpha_i+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_4)} \frac{|y_2(s)-y_1(s)|}{s} \Bigg] ds \\
& +\int_1^{\mu_1} \Bigg[\frac{(\log\frac{\mu_1}{s})^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{|f_1(s,x_2(s),y_2(s))-f_1(s,x_1(s),y_1(s))|}{s} \\
& +\chi_1 \frac{(\log\frac{\mu_1}{s})^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{|x_2(s)-x_1(s)|}{s} \Bigg] ds \Bigg\} \\
& +|\mathbb{M}_2(t)| \Bigg\{ \sum_{j=1}^{n} |\rho_j| \int_{\zeta_j}^{\omega_j} \frac{1}{s} \int_1^s \Bigg[\frac{(\log\frac{s}{u})^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{|f_2(u,x_2(u),y_2(u))-f_2(u,x_1(u),y_1(u))|}{u} \\
& +\chi_2 \frac{(\log\frac{s}{u})^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{|y_2(u)-y_1(u)|}{u} \Bigg] du ds \\
& +\int_1^{T} \Bigg[\frac{(\log\frac{T}{s})^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{|f_1(s,x_2(s),y_2(s))-f_1(s,x_1(s),y_1(s))|}{s} \\
& +\chi_1 \frac{(\log\frac{T}{s})^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{|x_2(s)-x_1(s)|}{s} \Bigg] ds \Bigg\} \\
& +|\mathbb{M}_3(t)| \Bigg\{ \sum_{l=1}^{m_2} |\sigma_l| \int_1^{\nu_l} \Bigg[\frac{(\log\frac{\nu_l}{s})^{\beta_l+\kappa_1+\kappa_2-1}}{\Gamma(\beta_l+\kappa_1+\kappa_2)} \frac{|f_1(s,x_2(s),y_2(s))-f_1(s,x_1(s),y_1(s))|}{s} \\
& +\chi_1 \frac{(\log\frac{\nu_l}{s})^{\beta_l+\kappa_2-1}}{\Gamma(\beta_l+\kappa_2)} \frac{|x_2(s)-x_1(s)|}{s} \Bigg] ds \\
& +\int_1^{\mu_2} \Bigg[\frac{(\log\frac{\mu_2}{s})^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{|f_2(s,x_2(s),y_2(s))-f_2(s,x_1(s),y_1(s))|}{s} \\
& +\chi_2 \frac{(\log\frac{\mu_2}{s})^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{|y_2(s)-y_1(s)|}{s} \Bigg] ds \Bigg\} \\
& +|\mathbb{M}_4(t)| \Bigg\{ \sum_{j=1}^{n} |\tau_j| \int_{\zeta_j}^{\omega_j} \frac{1}{s} \int_1^s \Bigg[\frac{(\log\frac{s}{u})^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{|f_1(u,x_2(u),y_2(u))-f_1(u,x_1(u),y_1(u))|}{u} \\
& +\chi_1 \frac{(\log\frac{s}{u})^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{|x_2(u)-x_1(u)|}{u} \Bigg] du ds \\
& +\int_1^{T} \Bigg[\frac{(\log\frac{T}{s})^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{|f_2(s,x_2(s),y_2(s))-f_2(s,x_1(s),y_1(s))|}{s} \\
& +\chi_2 \frac{(\log\frac{T}{s})^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{|y_2(s)-y_1(s)|}{s} \Bigg] ds \Bigg\} \Bigg\} \\
\leq\ & (\mathbb{L}_1\Psi_1 + \mathbb{L}_2\Psi_2)(\| x_2 - x_1\| + \|y_2 - y_1\|) + \Psi_3 \| x_2 - x_1\| + \Psi_4 \| y_2 - y_1 \|.
\end{aligned} \tag{27}
$$

Similarly, we can find that

$$
\| \mathbb{A}_2(x_2,y_2) - \mathbb{A}_2(x_1,y_1)\| \leq (\mathbb{L}_1\Phi_1 + \mathbb{L}_2\Phi_2)(\| x_2 - x_1\| + \|y_2 - y_1\|) + \Phi_3 \| x_2 - x_1\| + \Phi_4 \| y_2 - y_1 \|. \tag{28}
$$

From (27) and (28), we have

$$
\begin{aligned}
& \| \mathbb{A}(x_2,y_2) - \mathbb{A}(x_1,y_1) \| \\
\leq\ & [\mathbb{L}_1(\Psi_1+\Phi_1) + \mathbb{L}_2(\Psi_2+\Phi_2) + \max(\Psi_3+\Phi_3, Z_4+N_4)](\| x_2 - x_1\| + \|y_2 - y_1\|),
\end{aligned}
$$

which, according to (22), implies that $\mathbb{A}$ is a contraction. Therefore, a unique fixed point exists for the operator $\mathbb{A}$ through the application of Banach's fixed point theorem. Thus, a unique solution to problem (1) and (2) exists on $\mathcal{J}$. □

4. Examples

Here, we present examples illustrating the results obtained in the previous section.

Example 1. *Consider a coupled system of nonlinear Langevin equations*

$$\begin{cases} {}^{HH}D^{\kappa_1,\xi_1}\left({}^{HH}D^{\kappa_2,\xi_2}+\chi_1\right)x(t)=f_1(t,x(t),y(t)), & t\in\mathcal{J}, \\ {}^{HH}D^{\kappa_3,\xi_3}\left({}^{HH}D^{\kappa_4,\xi_4}+\chi_2\right)y(t)=f_2(t,x(t),y(t)), & t\in\mathcal{J}, \end{cases} \tag{29}$$

subject to the boundary data

$$\begin{cases} x(1)=0, \quad x(\mu_1)=\sum_{i=1}^{3}\lambda_i {}^{H}I^{\alpha_i}y(\eta_i), \quad x(T)=\sum_{j=1}^{3}\rho_j\int_{\zeta_j}^{\omega_j}\frac{1}{s}y(s)ds, \\ y(1)=0, \quad y(\mu_2)=\sum_{l=1}^{2}\sigma_l {}^{H}I^{\beta_l}x(\nu_l), \quad y(T)=\sum_{j=1}^{3}\tau_j\int_{\zeta_j}^{\omega_j}\frac{1}{s}x(s)ds, \end{cases} \tag{30}$$

with $\kappa_1=1.5$, $\kappa_2=0.5$, $\kappa_3=1.25$, $\kappa_4=0.6$, $0\le\xi_p\le1$ $(p=1,2,3,4)$, $\gamma_1=1.75$, $\gamma_2=1.5$, $\gamma_3=0.75$, $\gamma_4=0.7$, $\alpha_1=5/3$, $\alpha_2=1/3$, $\alpha_3=8/3$, $\beta_1=8/5$, $\beta_2=2/5$, $v_1=4/3$, $v_2=5/3$, $\zeta_1=6/3$, $\zeta_2=8/3$, $\zeta_3=10/3$, $\chi_1=1/20$, $\chi_2=1/50$, $\omega_1=7/3$, $\omega_2=9/3$, $\omega_3=11/3$, $\eta_1=12/3$, $\eta_2=13/3$, $\eta_3=14/3$, $\sigma_1=1/15$, $\sigma_2=1/10$, $\lambda_1=1/20$, $\lambda_2=1/4$, $\lambda_3=1/4$, $\rho_1=1/12$, $\rho_2=1/6$, $\rho_3=1/4$, $\tau_1=1/7$, $\tau_2=1/5$, $\tau_3=1/2$, $\mu_1=1.1$, $\mu_2=1.2$, $n=3$, $m_1=3$, $m_2=2$,, $\mathcal{J}=[1,5]$.

Using the given values, we find that $\Psi_1\approx2.74585874$, $\Psi_2\approx2.84924444$, $\Psi_3\approx0.23579603$, $\Psi_4\approx0.09318252$, $\Phi_1\approx0.13336712$, $\Phi_2\approx3.17256265$, $\Phi_3\approx0.02135806$, and $\Phi_4\approx0.09118365$ (Ψ_p and Φ_p, $p=1,2,3,4$, are given in (16)).

(a) To illustrate Theorem 2, we take

$$\begin{aligned} f_1(t,x(t),y(t)) &= e^{-5t}+\frac{\cos\ x(t)}{20t+4}+\frac{|y(t)|}{(t+20)(1+|y(t)|)}, \quad t\in\mathcal{J}, \\ f_2(t,x(t),y(t)) &= \frac{1}{\sqrt{3t+1}}+\frac{x(t)\sin y(t)}{10t+20}+\frac{|y(t)|}{(t+4)^3\left(\sqrt{1+|x(t)|^2}\right)}, \ t\in\mathcal{J}, \end{aligned} \tag{31}$$

and note that

$$|f_1(t,x(t),y(t))|\le\frac{1}{e^5}+\frac{1}{24}|x(t)|+\frac{1}{21}|y(t)|,$$

$$|f_2(t,x(t),y(t))|\le\frac{1}{2}+\frac{1}{50}|x(t)|+\frac{1}{125}|y(t)|,$$

that is, the assumption (H_1) holds with $\hat{m}_0=\frac{1}{e^5}$, $\hat{m}_1=\frac{1}{24}$, $\hat{m}_2=\frac{1}{21}$, $\hat{n}_0=\frac{1}{2}$, $\hat{n}_1=\frac{1}{30}$, $\hat{n}_2=\frac{1}{125}$.

Moreover, we find that $\mathbb{S}_1\approx0.57784874<1$ and $\mathbb{S}_2\approx0.36964662<1$. Thus, the hypotheses of Theorem 2 are verified. Therefore, the conclusion of Theorem 2 applies to the problem (29) and (30), with f_1 and f_2 given in (31).

(b) To illustrate Theorem 3, we take

$$\begin{aligned} f_1(t,x(t),y(t)) &= \frac{1}{5(t+8)}\tan^{-1}x(t)+\frac{\cos y(t)}{5\sqrt{(t+80)}}+e^t, \quad t\in\mathcal{J}, \\ f_2(t,x(t),y(t)) &= \frac{1}{10\sqrt{2t+7}}\left(\frac{|x(t)|}{1+|x(t)|}+\sin y(t)\right)+t^2, \quad t\in\mathcal{J}. \end{aligned} \tag{32}$$

For each $t\in\mathcal{J}$ and $x_1,y_1,x_2,y_2\in\mathbb{R}$, we notice that

$$|f(t,x_2,y_2)-f(t,x_1,y_1)|\le\frac{1}{45}(|x_2-x_1|+|y_2-y_1|),$$

$$|g(t, x_2, y_2) - g(t, x_1, y_1)| \leq \frac{1}{30}(|x_2 - x_1| + |y_2 - y_1|).$$

Thus, the condition (H_2) holds true with $\mathbb{L}_1 = \frac{1}{45}$ and $\mathbb{L}_2 = \frac{1}{30}$. Further, we have

$$\mathbb{L}_1(\Psi_1 + \Phi_1) + \mathbb{L}_2(\Psi_2 + \Phi_2) + \max(\Psi_3 + \Phi_3, \Psi_4 + \Phi_4) \approx 0.52186379 < 1.$$

Clearly, the hypothesis of Theorem 3 is satisfied. Hence, the problem (29) and (30) with the nonlinear functions f_1 and f_2 given by (32) have a unique solution on $\mathcal{J}$.

5. The Stability Analysis

Let us first develop the arguments for the Ulam–Hyers stability [41] of the problem (1) and (2).

For an arbitrary $\epsilon = (\epsilon_1, \epsilon_2) > 0$ and $t \in \mathcal{J}$, we consider a system of inequalities with the boundary conditions (2) given by

$$\begin{cases} \left|{}^{HH}D^{\kappa_1,\xi_1}\left({}^{HH}D^{\kappa_2,\xi_2} + \chi_1\right)\hat{x}(t) - f_1(t, \hat{x}(t), \hat{y}(t))\right| \leq \epsilon_1, \\ \left|{}^{HH}D^{\kappa_3,\xi_3}\left({}^{HH}D^{\kappa_4,\xi_4} + \chi_2\right)\hat{y}(t) - f_2(t, \hat{x}(t), \hat{y}(t))\right| \leq \epsilon_2. \end{cases} \tag{33}$$

If $(\hat{x}, \hat{y}) \in \mathbb{X} \times \mathbb{X}$ is a solution of the system of inequalities (33) with the boundary conditions (2), then functions $\lambda_1, \lambda_2 \in C(\mathcal{J}, \mathbb{R})$ exist such that $|\lambda_1(t)| \leq \epsilon_1$, $|\lambda_2(t)| \leq \epsilon_2$, $t \in \mathcal{J}$ and such that $(\hat{x}, \hat{y}) \in \mathbb{X} \times \mathbb{X}$ satisfies the system of Hilfer–Hadamard-type fractional Langevin equations

$$\begin{cases} {}^{HH}D^{\kappa_1,\xi_1}\left({}^{HH}D^{\kappa_2,\xi_2} + \chi_1\right)\hat{x}(t) = f_1(t, \hat{x}(t), \hat{y}(t)) + \lambda_1(t), \\ {}^{HH}D^{\kappa_3,\xi_3}\left({}^{HH}D^{\kappa_4,\xi_4} + \chi_2\right)\hat{y}(t) = f_2(t, \hat{x}(t), \hat{y}(t)) + \lambda_2(t). \end{cases}$$

Thus, for the forthcoming analysis, we consider the boundary value problem

$$\begin{cases} {}^{HH}D^{\kappa_1,\xi_1}\left({}^{HH}D^{\kappa_2,\xi_2} + \chi_1\right)\hat{x}(t) = f_1(t, \hat{x}(t), \hat{y}(t)) + \lambda_1(t), \quad t \in \mathcal{J}, \\ {}^{HH}D^{\kappa_3,\xi_3}\left({}^{HH}D^{\kappa_4,\xi_4} + \chi_2\right)\hat{y}(t) = f_2(t, \hat{x}(t), \hat{y}(t)) + \lambda_2(t), \quad t \in \mathcal{J}, \\ \hat{x}(1) = 0, \quad \hat{x}(\mu_1) = \sum_{i=1}^{m_1} \lambda_i {}^{H}I^{\alpha_i}\hat{y}(\eta_i), \quad \hat{x}(T) = \sum_{j=1}^{n} \rho_j \int_{\zeta_j}^{\omega_j} \frac{1}{s}\hat{y}(s)ds, \\ \hat{y}(1) = 0, \quad \hat{y}(\mu_2) = \sum_{l=1}^{m_2} \sigma_l {}^{H}I^{\beta_l}\hat{x}(\nu_l), \quad \hat{y}(T) = \sum_{j=1}^{n} \tau_j \int_{\zeta_j}^{\omega_j} \frac{1}{s}\hat{x}(s)ds. \end{cases} \tag{34}$$

Definition 4. *The system (1) and (2) is called Ulam–Hyers-stable if we can find* $c = \left(c_{f_1}, c_{f_2}\right) > 0$ *such that for each solution* $(\hat{x}, \hat{y}) \in \mathbb{X} \times \mathbb{X}$ *of (34), a unique solution* $(x, y) \in \mathbb{X} \times \mathbb{X}$ *to the system (1) and (2) exists satisfying*

$$\| (\hat{x}, \hat{y}) - (x, y) \| \leq c\epsilon^T, \quad t \in \mathcal{J}.$$

Definition 5. *The system (1) and (2) is said to be generalized Ulam–Hyers-stable if a unique solution* $(x, y) \in \mathbb{X} \times \mathbb{X}$ *to the system (1) and (2) exists satisfying*

$$\| (\hat{x}, \hat{y}) - (x, y) \| \leq \Omega(\epsilon), \quad t \in \mathcal{J},$$

for each solution $(\hat{x}, \hat{y}) \in \mathbb{X} \times \mathbb{X}$ *of (34), where* $\Omega = (\Omega_1, \Omega_2) \in C(\mathbb{R}^+, \mathbb{R}^+)$ *with* $\Omega(0) = 0$ $(\Omega_1(0) = \Omega_2(0) = 0)$.

Theorem 4. *Let* $f_1, f_2 \in C(\mathcal{J} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ *and the assumption* (H_2) *and the condition (22) hold. Then, the system (1) and (2) is Ulam–Hyers-stable and hence generalized Ulam–Hyers-stable in* $\mathbb{X} \times \mathbb{X}$.

Proof. According to Lemma 2, we can write the solution to (34) as

$$
\begin{aligned}
\hat{x}(t) \;=\; & \int_1^t \left[\frac{\left(\log \frac{t}{s}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{(f_1(s,\hat{x}(s),\hat{y}(s))+\lambda_1(s))}{s} - \chi_1 \frac{\left(\log \frac{t}{s}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{\hat{x}(s)}{s} \right] ds \\
& + \mathbb{M}_1(t) \left\{ \sum_{i=1}^{m_1} \lambda_i \int_1^{\eta_i} \left[\frac{\left(\log \frac{\eta_i}{s}\right)^{\alpha_i+\kappa_3+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_3+\kappa_4)} \frac{(f_2(s,\hat{x}(s),\hat{y}(s))+\lambda_2(s))}{s} - \chi_2 \frac{\left(\log \frac{\eta_i}{s}\right)^{\alpha_i+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_4)} \frac{\hat{y}(s)}{s} \right] ds \right. \\
& \left. - \int_1^{\mu_1} \left[\frac{\left(\log \frac{\mu_1}{s}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{(f_1(s,\hat{x}(s),\hat{y}(s))+\lambda_1(s))}{s} - \chi_1 \frac{\left(\log \frac{\mu_1}{s}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{\hat{x}(s)}{s} \right] ds \right\} \\
& + \mathbb{M}_2(t) \left\{ \sum_{j=1}^{n} \rho_j \int_{\zeta_j}^{\omega_j} \frac{1}{s} \int_1^s \left[\frac{\left(\log \frac{s}{u}\right)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{(f_2(u,\hat{x}(u),\hat{y}(u))+\lambda_2(u))}{u} - \chi_2 \frac{\left(\log \frac{s}{u}\right)^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{\hat{y}(u)}{u} \right] du ds \right. \\
& \left. - \int_1^{T} \left[\frac{\left(\log \frac{T}{s}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{(f_1(s,\hat{x}(s),\hat{y}(s))+\lambda_1(s))}{s} - \chi_1 \frac{\left(\log \frac{T}{s}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{\hat{x}(s)}{s} \right] ds \right\} \\
& + \mathbb{M}_3(t) \left\{ \sum_{l=1}^{m_2} \sigma_l \int_1^{\nu_l} \left[\frac{\left(\log \frac{\nu_l}{s}\right)^{\beta_l+\kappa_1+\kappa_2-1}}{\Gamma(\beta_l+\kappa_1+\kappa_2)} \frac{(f_1(s,\hat{x}(s),\hat{y}(s))+\lambda_1(s))}{s} - \chi_1 \frac{\left(\log \frac{\nu_l}{s}\right)^{\beta_l+\kappa_2-1}}{\Gamma(\beta_l+\kappa_2)} \frac{\hat{x}(s)}{s} \right] ds \right. \\
& \left. - \int_1^{\mu_2} \left[\frac{\left(\log \frac{\mu_2}{s}\right)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{(f_2(s,\hat{x}(s),\hat{y}(s))+\lambda_2(s))}{s} - \chi_2 \frac{\left(\log \frac{\mu_2}{s}\right)^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{\hat{y}(s)}{s} \right] ds \right\} \\
& + \mathbb{M}_4(t) \left\{ \sum_{j=1}^{n} \tau_j \int_{\zeta_j}^{\omega_j} \frac{1}{s} \int_1^s \left[\frac{\left(\log \frac{s}{u}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{(f_1(u,\hat{x}(u),\hat{y}(u))+\lambda_1(u))}{u} - \chi_1 \frac{\left(\log \frac{s}{u}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{\hat{x}(u)}{u} \right] du ds \right. \\
& \left. - \int_1^{T} \left[\frac{\left(\log \frac{T}{s}\right)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{(f_2(s,\hat{x}(s),\hat{y}(s))+\lambda_2(s))}{s} - \chi_2 \frac{\left(\log \frac{T}{s}\right)^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{\hat{y}(s)}{s} \right] ds \right\},
\end{aligned}
$$

$$
\begin{aligned}
\hat{y}(t) \;=\; & \int_1^t \left[\frac{\left(\log \frac{t}{s}\right)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{(f_2(s,\hat{x}(s),\hat{y}(s))+\lambda_2(s))}{s} - \chi_2 \frac{\left(\log \frac{t}{s}\right)^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{\hat{y}(s)}{s} \right] ds \\
& + \mathbb{N}_1(t) \left\{ \sum_{i=1}^{m_1} \lambda_i \int_1^{\eta_i} \left[\frac{\left(\log \frac{\eta_i}{s}\right)^{\alpha_i+\kappa_3+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_3+\kappa_4)} \frac{(f_2(s,\hat{x}(s),\hat{y}(s))+\lambda_2(s))}{s} - \chi_2 \frac{\left(\log \frac{\eta_i}{s}\right)^{\alpha_i+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_4)} \frac{\hat{y}(s)}{s} \right] ds \right. \\
& \left. - \int_1^{\mu_1} \left[\frac{\left(\log \frac{\mu_1}{s}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{(f_1(s,\hat{x}(s),\hat{y}(s))+\lambda_1(s))}{s} - \chi_1 \frac{\left(\log \frac{\mu_1}{s}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{\hat{x}(s)}{s} \right] ds \right\} \\
& + \mathbb{N}_2(t) \left\{ \sum_{j=1}^{n} \rho_j \int_{\zeta_j}^{\omega_j} \frac{1}{s} \int_1^s \left[\frac{\left(\log \frac{s}{u}\right)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{(f_2(u,\hat{x}(u),\hat{y}(u))+\lambda_2(u))}{u} - \chi_2 \frac{\left(\log \frac{s}{u}\right)^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{\hat{y}(u)}{u} \right] du ds \right. \\
& \left. - \int_1^{T} \left[\frac{\left(\log \frac{T}{s}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{(f_1(s,\hat{x}(s),\hat{y}(s))+\lambda_1(s))}{s} - \chi_1 \frac{\left(\log \frac{T}{s}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{\hat{x}(s)}{s} \right] ds \right\} \\
& + \mathbb{N}_3(t) \left\{ \sum_{l=1}^{m_2} \sigma_l \int_1^{\nu_l} \left[\frac{\left(\log \frac{\nu_l}{s}\right)^{\beta_l+\kappa_1+\kappa_2-1}}{\Gamma(\beta_l+\kappa_1+\kappa_2)} \frac{(f_1(s,\hat{x}(s),\hat{y}(s))+\lambda_1(s))}{s} - \chi_1 \frac{\left(\log \frac{\nu_l}{s}\right)^{\beta_l+\kappa_2-1}}{\Gamma(\beta_l+\kappa_2)} \frac{\hat{x}(s)}{s} \right] ds \right. \\
& \left. - \int_1^{\mu_2} \left[\frac{\left(\log \frac{\mu_2}{s}\right)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{(f_2(s,\hat{x}(s),\hat{y}(s))+\lambda_2(s))}{s} - \chi_2 \frac{\left(\log \frac{\mu_2}{s}\right)^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{\hat{y}(s)}{s} \right] ds \right\} \\
& + \mathbb{N}_4(t) \left\{ \sum_{j=1}^{n} \tau_j \int_{\zeta_j}^{\omega_j} \frac{1}{s} \int_1^s \left[\frac{\left(\log \frac{s}{u}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)} \frac{(f_1(u,\hat{x}(u),\hat{y}(u))+\lambda_1(u))}{u} - \chi_1 \frac{\left(\log \frac{s}{u}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)} \frac{\hat{x}(u)}{u} \right] du ds \right. \\
& \left. - \int_1^{T} \left[\frac{\left(\log \frac{T}{s}\right)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)} \frac{(f_2(s,\hat{x}(s),\hat{y}(s))+\lambda_2(s))}{s} - \chi_2 \frac{\left(\log \frac{T}{s}\right)^{\kappa_4-1}}{\Gamma(\kappa_4)} \frac{\hat{y}(s)}{s} \right] ds \right\}.
\end{aligned}
$$

In view of $|\lambda_1|<\epsilon_1$, $|\lambda_2|<\epsilon_2$ and (16), we have

$$\sup_{t\in\mathcal{J}}\Bigg|\hat{x}(t)-\int_1^t\left[\frac{\left(\log\frac{t}{s}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)}\frac{f_1(s,\hat{x}(s),\hat{y}(s))}{s}-\chi_1\frac{\left(\log\frac{t}{s}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)}\frac{\hat{x}(s)}{s}\right]ds$$
$$-\mathbb{M}_1(t)\Bigg\{\sum_{i=1}^{m_1}\lambda_i\int_1^{\eta_i}\left[\frac{\left(\log\frac{\eta_i}{s}\right)^{\alpha_i+\kappa_3+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_3+\kappa_4)}\frac{f_2(s,\hat{x}(s),\hat{y}(s))}{s}-\chi_2\frac{\left(\log\frac{\eta_i}{s}\right)^{\alpha_i+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_4)}\frac{\hat{y}(s)}{s}\right]ds$$
$$-\int_1^{\mu_1}\left[\frac{\left(\log\frac{\mu_1}{s}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)}\frac{f_1(s,\hat{x}(s),\hat{y}(s))}{s}-\chi_1\frac{\left(\log\frac{\mu_1}{s}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)}\frac{\hat{x}(s)}{s}\right]ds\Bigg\}$$
$$-\mathbb{M}_2(t)\Bigg\{\sum_{j=1}^{n}\rho_j\int_{\zeta_j}^{\omega_j}\frac{1}{s}\int_1^{s}\left[\frac{\left(\log\frac{s}{u}\right)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)}\frac{f_2(u,\hat{x}(u),\hat{y}(u))}{u}-\chi_2\frac{\left(\log\frac{s}{u}\right)^{\kappa_4-1}}{\Gamma(\kappa_4)}\frac{\hat{y}(u)}{u}\right]duds$$
$$-\int_1^{T}\left[\frac{\left(\log\frac{T}{s}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)}\frac{f_1(s,\hat{x}(s),\hat{y}(s))}{s}-\chi_1\frac{\left(\log\frac{T}{s}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)}\frac{\hat{x}(s)}{s}\right]ds\Bigg\}$$
$$-\mathbb{M}_3(t)\Bigg\{\sum_{l=1}^{m_2}\sigma_l\int_1^{\nu_l}\left[\frac{\left(\log\frac{\nu_l}{s}\right)^{\beta_l+\kappa_1+\kappa_2-1}}{\Gamma(\beta_l+\kappa_1+\kappa_2)}\frac{f_1(s,\hat{x}(s),\hat{y}(s))}{s}-\chi_1\frac{\left(\log\frac{\nu_l}{s}\right)^{\beta_l+\kappa_2-1}}{\Gamma(\beta_l+\kappa_2)}\frac{\hat{x}(s)}{s}\right]ds$$
$$-\int_1^{\mu_2}\left[\frac{\left(\log\frac{\mu_2}{s}\right)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)}\frac{f_2(s,\hat{x}(s),\hat{y}(s))}{s}-\chi_2\frac{\left(\log\frac{\mu_2}{s}\right)^{\kappa_4-1}}{\Gamma(\kappa_4)}\frac{\hat{y}(s)}{s}\right]ds\Bigg\}$$
$$-\mathbb{M}_4(t)\Bigg\{\sum_{j=1}^{n}\tau_j\int_{\zeta_j}^{\omega_j}\frac{1}{s}\int_1^{s}\left[\frac{\left(\log\frac{s}{u}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)}\frac{f_1(u,\hat{x}(u),\hat{y}(u))}{u}-\chi_1\frac{\left(\log\frac{s}{u}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)}\frac{\hat{x}(u)}{u}\right]duds$$
$$-\int_1^{T}\left[\frac{\left(\log\frac{T}{s}\right)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)}\frac{f_2(s,\hat{x}(s),\hat{y}(s))}{s}-\chi_2\frac{\left(\log\frac{T}{s}\right)^{\kappa_4-1}}{\Gamma(\kappa_4)}\frac{\hat{y}(s)}{s}\right]ds\Bigg\}\Bigg|\le\Psi_1\epsilon_1+\Psi_2\epsilon_2,$$

and

$$\sup_{t\in\mathcal{J}}\Bigg|\hat{y}(t)-\int_1^t\left[\frac{\left(\log\frac{t}{s}\right)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)}\frac{f_2(s,\hat{x}(s),\hat{y}(s))}{s}-\chi_2\frac{\left(\log\frac{t}{s}\right)^{\kappa_4-1}}{\Gamma(\kappa_4)}\frac{\hat{y}(s)}{s}\right]ds$$
$$-\mathbb{N}_1(t)\Bigg\{\sum_{i=1}^{m_1}\lambda_i\int_1^{\eta_i}\left[\frac{\left(\log\frac{\eta_i}{s}\right)^{\alpha_i+\kappa_3+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_3+\kappa_4)}\frac{f_2(s,\hat{x}(s),\hat{y}(s))}{s}-\chi_2\frac{\left(\log\frac{\eta_i}{s}\right)^{\alpha_i+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_4)}\frac{\hat{y}(s)}{s}\right]ds$$
$$-\int_1^{\mu_1}\left[\frac{\left(\log\frac{\mu_1}{s}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)}\frac{f_1(s,\hat{x}(s),\hat{y}(s))}{s}-\chi_1\frac{\left(\log\frac{\mu_1}{s}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)}\frac{\hat{x}(s)}{s}\right]ds\Bigg\}$$
$$-\mathbb{N}_2(t)\Bigg\{\sum_{j=1}^{n}\rho_j\int_{\zeta_j}^{\omega_j}\frac{1}{s}\int_1^{s}\left[\frac{\left(\log\frac{s}{u}\right)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)}\frac{f_2(u,\hat{x}(u),\hat{y}(u))}{u}-\chi_2\frac{\left(\log\frac{s}{u}\right)^{\kappa_4-1}}{\Gamma(\kappa_4)}\frac{\hat{y}(u)}{u}\right]duds$$
$$-\int_1^{T}\left[\frac{\left(\log\frac{T}{s}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)}\frac{f_1(s,\hat{x}(s),\hat{y}(s))}{s}-\chi_1\frac{\left(\log\frac{T}{s}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)}\frac{\hat{x}(s)}{s}\right]ds\Bigg\}$$
$$-\mathbb{N}_3(t)\Bigg\{\sum_{l=1}^{m_2}\sigma_l\int_1^{\nu_l}\left[\frac{\left(\log\frac{\nu_l}{s}\right)^{\beta_l+\kappa_1+\kappa_2-1}}{\Gamma(\beta_l+\kappa_1+\kappa_2)}\frac{f_1(s,\hat{x}(s),\hat{y}(s))}{s}-\chi_1\frac{\left(\log\frac{\nu_l}{s}\right)^{\beta_l+\kappa_2-1}}{\Gamma(\beta_l+\kappa_2)}\frac{\hat{x}(s)}{s}\right]ds$$
$$-\int_1^{\mu_2}\left[\frac{\left(\log\frac{\mu_2}{s}\right)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)}\frac{f_2(s,\hat{x}(s),\hat{y}(s))}{s}-\chi_2\frac{\left(\log\frac{\mu_2}{s}\right)^{\kappa_4-1}}{\Gamma(\kappa_4)}\frac{\hat{y}(s)}{s}\right]ds\Bigg\}$$
$$-\mathbb{N}_4(t)\Bigg\{\sum_{j=1}^{n}\tau_j\int_{\zeta_j}^{\omega_j}\frac{1}{s}\int_1^{s}\left[\frac{\left(\log\frac{s}{u}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)}\frac{f_1(u,\hat{x}(u),\hat{y}(u))}{u}-\chi_1\frac{\left(\log\frac{s}{u}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)}\frac{\hat{x}(u)}{u}\right]duds$$
$$-\int_1^{T}\left[\frac{\left(\log\frac{T}{s}\right)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)}\frac{f_2(s,\hat{x}(s),\hat{y}(s))}{s}-\chi_2\frac{\left(\log\frac{T}{s}\right)^{\kappa_4-1}}{\Gamma(\kappa_4)}\frac{\hat{y}(s)}{s}\right]ds\Bigg\}\Bigg|\le\Phi_1\epsilon_1+\Phi_2\epsilon_2.$$

According to (H_2), we have

$$
\begin{aligned}
\parallel \hat{x}-x \parallel = & \sup_{t\in\mathcal{J}}|\hat{x}(t)-x(t)| \leq \Psi_1\epsilon_1+\Psi_2\epsilon_2 \\
& +\sup_{t\in\mathcal{J}}\Bigg\{\int_1^t\Bigg[\frac{\left(\log\frac{t}{s}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)}\frac{|f_1(s,\hat{x}(s),\hat{y}(s))-f_1(s,x(s),y(s))|}{s} \\
& +\chi_1\frac{\left(\log\frac{t}{s}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)}\frac{|\hat{x}(s)-x(s)|}{s}\Bigg]ds \\
& +|\mathbb{M}_1(t)|\Bigg\{\sum_{i=1}^{m_1}|\lambda_i|\int_1^{\eta_i}\Bigg[\frac{\left(\log\frac{\eta_i}{s}\right)^{\alpha_i+\kappa_3+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_3+\kappa_4)}\frac{|f_2(s,\hat{x}(s),\hat{y}(s))-f_2(s,x(s),y(s))|}{s} \\
& +\chi_2\frac{\left(\log\frac{\eta_i}{s}\right)^{\alpha_i+\kappa_4-1}}{\Gamma(\alpha_i+\kappa_4)}\frac{|\hat{y}(s)-y(s)|}{s}\Bigg]ds \\
& +\int_1^{\mu_1}\Bigg[\frac{\left(\log\frac{\mu_1}{s}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)}\frac{|f_1(s,\hat{x}(s),\hat{y}(s))-f_1(s,x(s),y(s))|}{s} \\
& +\chi_1\frac{\left(\log\frac{\mu_1}{s}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)}\frac{|\hat{x}(s)-x(s)|}{s}\Bigg]ds\Bigg\} \\
& +|\mathbb{M}_2(t)|\Bigg\{\sum_{j=1}^{n}|\rho_j|\int_{\zeta_j}^{\omega_j}\frac{1}{s}\int_1^{s}\Bigg[\frac{\left(\log\frac{s}{u}\right)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)}\frac{|f_2(u,\hat{x}(u),\hat{y}(u))-f_2(u,x(u),y(u))|}{u} \\
& +\chi_2\frac{\left(\log\frac{s}{u}\right)^{\kappa_4-1}}{\Gamma(\kappa_4)}\frac{|\hat{y}(u)-y(u)|}{u}\Bigg]duds \\
& +\int_1^{T}\Bigg[\frac{\left(\log\frac{T}{s}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)}\frac{|f_1(s,\hat{x}(s),\hat{y}(s))-f_1(s,x(s),y(s))|}{s} \\
& +\chi_1\frac{\left(\log\frac{T}{s}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)}\frac{|\hat{x}(s)-x(s)|}{s}\Bigg]ds\Bigg\} \\
& +|\mathbb{M}_3(t)|\Bigg\{\sum_{l=1}^{m_2}|\sigma_l|\int_1^{\nu_l}\Bigg[\frac{\left(\log\frac{\nu_l}{s}\right)^{\beta_l+\kappa_1+\kappa_2-1}}{\Gamma(\beta_l+\kappa_1+\kappa_2)}\frac{|f_1(s,\hat{x}(s),\hat{y}(s))-f_1(s,x(s),y(s))|}{s} \\
& +\chi_1\frac{\left(\log\frac{\nu_l}{s}\right)^{\beta_l+\kappa_2-1}}{\Gamma(\beta_l+\kappa_2)}\frac{|\hat{x}(s)-x(s)|}{s}\Bigg]ds \\
& +\int_1^{\mu_2}\Bigg[\frac{\left(\log\frac{\mu_2}{s}\right)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)}\frac{|f_2(s,\hat{x}(s),\hat{y}(s))-f_2(s,x(s),y(s))|}{s} \\
& +\chi_2\frac{\left(\log\frac{\mu_2}{s}\right)^{\kappa_4-1}}{\Gamma(\kappa_4)}\frac{|\hat{y}(s)-y(s)|}{s}\Bigg]ds\Bigg\} \\
& +|\mathbb{M}_4(t)|\Bigg\{\sum_{j=1}^{n}|\tau_j|\int_{\zeta_j}^{\omega_j}\frac{1}{s}\int_1^{s}\Bigg[\frac{\left(\log\frac{s}{u}\right)^{\kappa_1+\kappa_2-1}}{\Gamma(\kappa_1+\kappa_2)}\frac{|f_1(u,\hat{x}(u),\hat{y}(u))-f_1(u,x(u),y(u))|}{u} \\
& +\chi_1\frac{\left(\log\frac{s}{u}\right)^{\kappa_2-1}}{\Gamma(\kappa_2)}\frac{|\hat{x}(u)-x(u)|}{u}\Bigg]duds \\
& +\int_1^{T}\Bigg[\frac{\left(\log\frac{T}{s}\right)^{\kappa_3+\kappa_4-1}}{\Gamma(\kappa_3+\kappa_4)}\frac{|f_2(s,\hat{x}(s),\hat{y}(s))-f_2(s,x(s),y(s))|}{s} \\
& +\chi_2\frac{\left(\log\frac{T}{s}\right)^{\kappa_4-1}}{\Gamma(\kappa_4)}\frac{|\hat{y}(s)-y(s)|}{s}\Bigg]ds\Bigg\}\Bigg\} \\
\leq & \Psi_1\epsilon_1+\Psi_2\epsilon_2+(\mathbb{L}_1\Psi_1+\mathbb{L}_2\Psi_2)(\parallel\hat{x}-x\parallel+\|\hat{y}-y\|)+\Psi_3\parallel\hat{x}-x\parallel+\Psi_4\|\hat{y}-y\|,
\end{aligned}
$$

which implies that

$$
\parallel \hat{x}-x \parallel\leq\Psi_1\epsilon_1+\Psi_2\epsilon_2+(\mathbb{L}_1\Psi_1+\mathbb{L}_2\Psi_2)(\parallel\hat{x}-x\parallel+\|\hat{y}-y\|)+\Psi_3\parallel\hat{x}-x\parallel+\Psi_4\|\hat{y}-y\|.
$$

In a similar fashion, one can find that

$$
\parallel \hat{y}-y \parallel\leq\Phi_1\epsilon_1+\Phi_2\epsilon_2+(\mathbb{L}_1\Phi_1+\mathbb{L}_2\Phi_2)(\parallel\hat{x}-x\parallel+\|\hat{y}-y\|)+\Phi_3\parallel\hat{x}-x\parallel+\Phi_4\|\hat{y}-y\|.
$$

Thus, according to condition (22), we obtain

$$
\begin{aligned}
& \parallel(\hat{x},\hat{y})-(x,y)\|\leq\|\hat{x}-x\parallel+\|\hat{y}-y\parallel \\
& \leq(\Psi_1+\Phi_1)\epsilon_1+(\Psi_2+\Phi_2)\epsilon_2+(\mathbb{L}_1(\Psi_1+\Phi_1)+\mathbb{L}_2(\Psi_2+\Phi_2))(\parallel\hat{x}-x\parallel+\|\hat{y}-y\|) \\
& +\max(\Psi_3+\Phi_3,\Psi_4+\Phi_4)(\parallel\hat{x}-x\parallel+\|\hat{y}-y\|).
\end{aligned}
$$

Therefore,

$$\parallel (\hat{x},\hat{y})-(x,y)\parallel \leq \frac{(\Psi_1+\Phi_1)\epsilon_1+(\Psi_2+\Phi_2)\epsilon_2}{1-[\mathbb{L}_1(\Psi_1+\Phi_1)+\mathbb{L}_2(\Psi_2+\Phi_2)]-\max(\Psi_3+\Phi_3,\Psi_4+\Phi_4)}. \quad (35)$$

Letting $c=\left(c_{f_1},c_{f_2}\right)=\left(\frac{\Psi_1+\Phi_1}{1-\mathbb{T}},\frac{\Psi_2+\Phi_2}{1-\mathbb{T}}\right)$, we obtain

$$\parallel (\hat{x},\hat{y})-(x,y)\parallel \leq c\epsilon^T,$$

where $\mathbb{T}=[\mathbb{L}_1(\Psi_1+\Phi_1)+\mathbb{L}_2(\Psi_2+\Phi_2)]+\max(\Psi_3+\Phi_3,\Psi_4+\Phi_4)<1$, according to condition (22). Hence, the system (1) and (2) is Ulam–Hyers-stable. Furthermore, it is generalized Ulam–Hyers-stable since $\parallel (\hat{x},\hat{y})-(x,y)\parallel \leq \Omega(\epsilon)$ with $\Omega(\epsilon)=c\epsilon^T$, $\Omega(0)=0$. □

Example 2. *Problem (29) and (30) with f_1 and f_2 given in (32) is Ulam–Hyers-stable, as well as generalized Ulam–Hyers-stable, since* $\mathbb{T}\approx 0.52186379<1$.

6. Conclusions

We have presented the criteria ensuring the existence and Ulam–Hyers stability of solutions to a fully coupled nonlocal integral boundary value problem of nonlinear fractional Hilfer–Hadamard Langevin equations. The results derived in this paper are not only novel in the given configuration but also correspond to some new configurations as special cases. For example, letting $\rho_j=\tau_j=0$ for $j=1,\ldots,n$, the present results become those for boundary conditions

$$\begin{cases} x(1)=0, \quad x(\mu_1)=\sum\limits_{i=1}^{m_1}\lambda_i {}^H I_{1^+}^{\alpha_i} y(\eta_i), \quad x(T)=0, \\ y(1)=0, \quad y(\mu_2)=\sum\limits_{l=1}^{m_2}\sigma_l {}^H I_{1^+}^{\beta_l} x(\nu_l), \quad y(T)=0. \end{cases}$$

If we take $\lambda_i=0$ for $i=1,\ldots,m_1$ and $\sigma_l=0$ for $l=1,\ldots,m_2$ in the present results, we obtain results for the given system with the boundary conditions

$$\begin{cases} x(1)=0, \quad x(\mu_1)=0, \quad x(T)=\sum\limits_{j=1}^{n}\rho_j\int_{\zeta_j}^{\omega_j}\frac{1}{s}y(s)ds, \\ y(1)=0, \quad y(\mu_2)=0, \quad y(T)=\sum\limits_{j=1}^{n}\tau_j\int_{\zeta_j}^{\omega_j}\frac{1}{s}x(s)ds. \end{cases}$$

Our results correspond to the four-point nonlocal boundary conditions

$$x(1)=0,\ x(\mu_1)=0,\ x(T)=0,\ y(1)=0,\ y(\mu_2)=0,\ y(T)=0,$$

by taking all $\lambda_i=0\ (i=1,\ldots,m_1), \sigma_l=0\ (l=1,\ldots,m_2), \rho_j=\tau_j=0\ (j=1,\ldots,n)$.

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