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Coefficient Bounds for Alpha-Convex Functions Involving the Linear q-Derivative Operator Connected with the Cardioid Domain

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Abstract: Scholars from several disciplines have recently expressed interest in the field of fractional q-calculus based on fractional integrals and derivative operators. This article mathematically applies the fractional q-differential and q-integral operators in geometric function theory. The linear q-derivative operator $\mathbf{S}_{\mu,\delta,q}^{n,m}$ and subordination are used in this study to define and construct new classes of α-convex functions associated with the cardioid domain. Additionally, this paper explores acute inequality problems for newly defined classes $\mathcal{R}_q^{\alpha}(a,c,m,L,P)$, of α-convex functions in the open unit disc $\mathcal{U}_s$, such as initial coefficient bounds, coefficient inequalities, Fekete–Szegö problems, the second Hankel determinants, and logarithmic coefficients. The results presented in this paper are simple to comprehend and demonstrate how current research relates to earlier research. We found all of the estimates, and they are sharp.

Keywords: analytic functions; alpha-convex functions; fractional q-calculus; cardioid domain; coefficients bounds; Fekete–Szegö inequality; Hankel determinants; inverse coefficients; logarithmic coefficients

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1. Introduction and Preliminaries

The mathematical field explored in this paper is called quantum calculus, or q-calculus, which integrates quantum mechanics concepts into classical calculus by introducing a new parameter, q. Euler introduced the concept, and later, Jackson [1] generalized it into q-integrals and q-derivatives. Moreover, numerous researchers have studied geometric function theory by exploring new subclasses based on the q approach due to its applications in many quantitative fields, which piqued their interest and motivation.

An extension of conventional fractional calculus, fractional q-calculus finds use in a variety of fields, including q-difference, q-integral equations, ordinary fractional calculus, and optimal control problems. Refer to published work [2] and recent literature, which may contain references such as [3], to learn more. Fractional calculus operators are widely employed to address problems in the applied sciences and geometric functions [4,5]. Srivastava [6] presented the concept of q-calculus in a review article regarding the applications of a q-calculus operator. For more details on q-calculus and its application, see [7–9].

In geometric function theory, estimating the coefficient bounds for functions belonging to different classes is an essential problem, since it helps us to find the geometric properties

of functions associated with the class. Recently, Srivastava et al. [10] considered the class of multivalent functions associated with the cardioid domain and determined the coefficient bounds for the first two coefficients. Khan and AbaOud [11] studied the fractional q-calculus operator for subclasses of q-starlike functions related to the cardioid domain. Also, Al-Shaikh [12] introduced certain classes of q-starlike and q-convex functions and estimated the bounds for the first and second coefficients.

The class of the holomorphic functions $f(z)$ normalized in the open unit disk $\mathcal{U}_s = \{z \in \mathbb{C} : |z| < 1\}$ is represented by $\mathcal{A}$. Its form is

$$f(z) := z + \sum_{j=2}^{\infty} a_j z^j. \tag{1}$$

The family of univalent functions in $\mathcal{A}$ are represented by $\mathcal{S}$. We also consider $\mathcal{S}^*$ and $\mathcal{K}$, the classes of starlike and convex functions (see [13]) by

$$\mathcal{S}^* = \left\{ f : f \in \mathcal{A} \text{ and } Re\left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \right\},$$

and

$$\mathcal{K} = \left\{ f : f \in \mathcal{A} \text{ and } Re\left\{ 1 + \frac{zf'(z)}{f(z)} \right\} > 0 \right\}.$$

Mocanu [14] introduced the classes of α-convex functions $\mathcal{N}_\alpha$, which are defined on $\mathcal{A}$ with $f(0) = 0$ and $f'(0) \neq 0$, if it satisfies

$$Re\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf''(z)}{f'(z)} + 1 \right) \right] > 0, \quad (\alpha \in \mathbb{R}; z \in \mathcal{U}_s).$$

Note that $\mathcal{N}_0 = \mathcal{S}^*$ and $\mathcal{N}_1 = \mathcal{K}$. We say that f is subordinate to g if, given two functions f and $g \in \mathcal{U}_s$, there exists a Schwarz function κ such that $f(z) = g(\kappa(z))$ for $z \in \mathcal{U}_s$. It can be written as $f \prec g$. Whenever g is univalent in $\mathcal{U}_s$, a necessary and sufficient condition for $f(z) \prec g(z)$ is that $f(0) = g(0)$ and $f(\mathcal{U}_s) \subset g(\mathcal{U}_s)$ (see [15]).

Let $S^*(\phi)$ be the class of starlike functions defined as

$$S^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \right\},$$

where $\phi(z)$ is analytic in $\mathcal{U}_s$. Assume that $\mathcal{P}$ is the Carathéodory class of all analytic functions p in $\mathcal{U}_s$ with $Re(p(z)) > 0$ and that

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \qquad (z \in \mathbb{U}_s).$$

Paprocki and Sokól [16] considered a subclass of $S^*(\phi)$, that is the class $\mathcal{SL}$ of shell-like functions:

$$\mathcal{SL} = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec J(z) \right\},$$

where

$$J(z) = \frac{1+\nu^2 z^2}{1-\nu z-\nu^2 z^2}, \quad \nu = \frac{1-\sqrt{5}}{2} \approx -0.618.$$

Remark 1. *$J(z)$ is univalent in $|z| < \nu^2 (\approx 0.38)$ and $Re(J'(z)) < 0, z \in \mathcal{U}_s$. The functions $J(e^{it})$ in the class $\mathcal{SL}$ form a shell-like curve with $\{t \in [o, 2\pi) \setminus \{\pi\}\}$.*

Further, the coefficients of shell-like function $J(z)$ are expressed in terms of Fibonacci numbers

$$J(z) = 1 + \sum_{n=1}^{\infty} (u_{n-1} + u_{n+1}) \nu^n z^n,$$

where

$$u_n = \frac{(1-\nu)^n-(\nu)^n}{\sqrt{5}}.$$

We notice that the function $J(z)$ maps the unit circle to a curve, denoted by the conchoid of Maclaurin, which is

$$J(e^{\iota\phi}) = \frac{\sqrt{5}}{2(3-2\cos\phi)} + \frac{\iota\sin\phi(4\cos\phi-1)}{2(3-2\cos\phi)(1+\cos\phi)} \quad (0 \leq \phi < 2\pi).$$

Various notable subclasses on the subordination of analytic function have been extensively studied by many authors based upon their image domain (see [17–20]).

Goodman [21] shows that $J(\mathcal{U}_s)$ consisting of functions $J(z) = \frac{1+z}{1-z}$ lies in the right half plane and for $J(\mathcal{U}_s)$, which consists of functions $J(z) = \frac{1+(1-2\alpha)z}{1-z}$ lying to the right of line $Re(w) = \alpha$ $(0 \leq \alpha < 1)$. Janowski [22] introduced the class of functions $J(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq A < B \leq 1)$, such that their image domain lies in the circular disk. Noor and Malik [23] considered the analytic functions $J(z) = \frac{A+1J_j(z)-(A-1)}{B+1J_j(z)-(B-1)}$ $(-1 \leq A < B \leq 1)$ in the unit disk $\mathcal{U}_s$ that maps to oval and petal-type regions. Paprocki and Sokól [16] showed that the functions $J(z) = \left(\frac{1+z}{1+\frac{1-\beta}{\beta}z}\right)^{\frac{1}{\alpha}}$ in the disk $\mathcal{U}_s$ map onto a leaf-like domain. For more details, (see [24–26]).

Definition 1. *The function $J(z)$ belongs to the class $\mathcal{T}_L^P$, if it satisfies*

$$J(z) \prec \widetilde{J}(L,P,z) \quad (-1 < P < L \leq 1),$$

where

$$\widetilde{J}(L,P,z) = \frac{2L^2\nu^2z^2+(L-1)\nu z+2}{2P\nu^2z^2+(P-1)\nu z+2}, \quad \left(\nu = \frac{1-\sqrt{5}}{2},\ z \in \mathcal{U}_s\right). \tag{2}$$

We observed that the function $\widetilde{J}(z)$ maps the unit circle to the cardioid-like curve $C_{L,P}$ with the parametric equation

$$u = \frac{4+(L-1)(P-1)\nu^2+4LP\nu^4+2\eta cos\theta+4(L+p)\nu^2cos2\theta}{4(P-1)(\nu+P\nu^3)cos\theta+8P\nu^2cos2\theta+4+(P-1)^2\nu^2+4P^2\nu^4},$$

$$v = \frac{(L-P)(\nu-\nu^3)sin\theta+2\nu^2sin2\theta}{4(P-1)(\nu+P\nu^3)cos\theta+8P\nu^2cos2\theta+4+(P-1)^2\nu^2+4P^2\nu^4},$$

where

$$\eta = (L+P-2)\tau+(2LP-L-P)\nu^3.$$

Recently, Malik et al. [27] considered the subclass of an analytic function $J(z)$ with a positive real part in $\mathcal{U}_s$, which maps onto the cardioid domain. For $-1 < P < L \leq 1$, they defined a class $\mathcal{CP}[\mathcal{A},\mathcal{B}]$ concerning the geometrical interpretation of the function as cardioid, like the curve on their image domain. In addition, they also obtain Fekete–Szegö inequalities for functions belonging to this class (see [28–30]).

The class $\mathcal{S}$ has long been a subject of interest for mathematicians, particularly with regard to coefficient estimates. A seminal moment in this area came in 1916, when Bieberbach [31] proposed the famous coefficient conjecture. Nearly seven decades later, De Branges [32] finally settled the conjecture, proving that $a_n \leq n$ for all $n = 2,3,...$, with equality holding only for rotations of the Koebe function. This conjecture and related coefficient problems have been the focus of numerous research papers.

The kth Hankel determinant $\mathcal{Z}_{k,d}(f)$ with $d \geqq 1$ and $k \geqq 1$ is composed of the coefficients of the Maclaurin expansion of $f(z)$, a concept presented by Pommerenke [33,34] which can be defined as follows:

$$\mathcal{Z}_{k,d}(f) = \begin{vmatrix} l_d & l_{d+1} & \cdots & l_{d+k-1} \\ l_{d+1} & l_{d+2} & \cdots & l_{d+1} \\ l_{d+2} & l_{d+3} & \cdots & l_{d+k+1} \\ \vdots & \vdots & \ddots & \vdots \\ l_{d+k-1} & l_{d+k} & \cdots & l_{d+2(k-1)} \end{vmatrix}.$$

When $k = 2$ and $d = 1$, we can see that $\mathcal{Z}_{2,1}(f)$ equals $l_3 - l_2^2$. The second Hankel determinant $\mathcal{Z}_{2,2}(f)$, which is equal to $l_2 l_4 - l_3^2$, is commonly used to present a certain class in this area of research [35–37]. Fekete–Szegö [38] analyzed the Hankel determinant of the function $f(z)$ to obtain

$$|\mathcal{Z}_{2,1}(f)| = \begin{vmatrix} l_1 & l_2 \\ l_2 & l_3 \end{vmatrix} = l_1 l_3 - l_2^2.$$

They built upon previous research on estimates of $|l_3 - I_2^2|$, with the condition $l_1 = 1$ and $I \in \mathbb{R}$. They also introduced the concept of maximizing the non-linear functional $|a_3 - \mathcal{T} a_2^2|$ for the class of univalent functions, popularly known as the Fekete–Szegö problem:

$$\left|l_3 - \mathcal{T} l_2^2\right| \leq \begin{cases} 3 - 4\mathcal{T} & \text{if } \mathcal{T} \leq 0, \\ 1 + 2e^{\frac{-2\mathcal{T}}{1-\mathcal{T}}} & \text{if } 0 \leq \mathcal{T} \leq 1, \\ 4\mathcal{T} - 3 & \text{if } \mathcal{T} \geq 1. \end{cases}$$

The above problem is sharp for the function in the class with each "$\mathcal{T}$".

We now recall some essential definitions and concepts of the basic or q-calculus. Throughout this article, we suppose that $0 < q < 1$.

Definition 2 ([1,7]). *The q-derivative (or q-difference) operator $\mathcal{D}_q$ of a function $f(z)$ is defined as follows:*

$$\mathcal{D}_q f(z) := \begin{cases} \frac{f(z) - f(qz)}{(1-q)z} & \text{if } q \in (0,1);\ z \neq 0, \\ f'(0) & \text{if } z = 0;\ q \to 1^-, \\ f'(z) & \text{if } z \neq 0;\ q \to 1^-. \end{cases}$$

It readily follows from the hypothesis of the above definition that

$$\mathcal{D}_q z^n = [n]_q z^{n-1}, \mathcal{D}_q \left(\sum_{j=2}^{\infty} a_j z^j \right) = \sum_{j=2}^{\infty} [j]_q a_j z^{j-1} \quad (z \in \mathcal{U}_s),$$

where $[\lambda]_q$ is the so-called q-number, which is defined in [39] by

$$[\lambda]_q := \begin{cases} \frac{1-q^\lambda}{1-q} & (\lambda \in \mathbb{C}), \\ \sum_{j=0}^{n-1} q^j = 1 + q + q^2 + \cdots + q^{n-1} & (\lambda = n \in \mathbb{N} := 1, 2, \cdots). \end{cases}$$

For two functions f and $g \in \mathcal{A}$ is defined by the Hadamard product (or convolution)

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z).$$

Recently, for a function belonging to the class $\mathcal{A}$ with $\mu > -1, n > 0$, Raina and Sharma [40] considered the following integral operator:

$$\mathcal{I}_{n,\mu}f(z) = \frac{\mu+1}{n} z^{1-\frac{\mu+1}{n}} \int_0^z t^{\frac{\mu+1}{n}-2} f(t)dt.$$

which is the generalization of a linear integral operator studied in [41].

By using simple calculations on the above integration, we get

$$\mathcal{I}_{n,\mu}f(z) = z + \sum_{j=2}^{\infty} \left(\frac{\mu+1}{\mu+1+n(j-1)} \right) a_j z^j.$$

We generalized this operator as follows:

$$\mathbf{H}^m_{\mu,\delta}f(z) = z + \sum_{j=2}^{\infty} \left(\frac{\mu+\delta}{\mu+\delta+n(j-1)} \right)^m a_j z^j \quad (n \geq 0,\ \mu+\delta > 0).$$

Now, we define the operator $\mathbf{S}^{n,m}_{\mu,\delta}f(z) : \mathcal{A} \to \mathcal{A}$ as

$$\mathbf{S}^{n,m}_{\mu,\delta}f(z) = \mathbf{H}^m_{\mu,\delta}f(z) * \Phi(b,\tau,z) = z + \sum_{j=2}^{\infty} \left(\tfrac{\mu+\delta}{\mu+\delta+n(j-1)} \right)^m \tfrac{(b)_{j-1}}{(\tau)_{j-1}} a_j z^j,$$

where

$$\Phi(b,\tau,z) = \sum_{j=1}^{\infty} \frac{(b)_{j-1}}{(\tau)_{j-1}} z^j \quad (\tau \neq 0, -1, -2, \cdots; z \in \mathcal{U}_s)$$

and $(x)_k$ is the Pochhammer symbol defined by

$$(x)_j := \begin{cases} 1 & j = 0, \\ x(x+1)(x+2)\cdots(x+j-1) & j \in \mathbb{N} := 1, 2, \cdots. \end{cases}$$

Definition 3. *For $f(z) \in \mathcal{A}$, the extended q-derivative operator $\mathbf{S}^{n,m}_{\mu,\delta,q}f(z) : \mathcal{A} \to \mathcal{A}$ is defined as follows*

$$\mathbf{S}^{n,m}_{\mu,\delta,q}f(z) = \mathbf{H}^m_{\mu,\delta,q}f(z) * \Phi_q(b,\tau,z) = z + \sum_{j=2}^{\infty} \gamma_q(j) a_j z^j,$$

where

$$\gamma_q(j) := \left(\frac{[\mu+\delta]_q}{[\mu+\delta+n(j-1)]_q} \right)^m \frac{[b]_{j-1,q}}{[\tau]_{j-1,q}} \tag{3}$$

and

$$[x]_{j,q} := \begin{cases} 1 & j = 0, \\ [x]_q[x+1]_q[x+2]_q \cdots [x+j-1]_q & j \in \mathbb{N} := 1, 2, \cdots. \end{cases}$$

Numerous researchers have investigated various coefficient problems in the literature, including estimating coefficients of functions, inverse functions, Fekete–Szegö problems, Hankel determinants, and logarithmic coefficients. As mentioned earlier, the research on the cardioid domain in the image plane motivated us to define a new subclass of analytic functions involving the linear q-derivative operator $\mathbf{S}^{n,m}_{\mu,\delta,q}$ and find a bound for the coefficient of this class.

Now, we define a new subclass of analytic functions associated with the region bounded by the cardioid domain as follows:

Definition 4. *The function $f(z)$ of the form (1) is said to be in the class $\mathcal{R}_q^{\alpha}(b,\tau,m,L,P)$, if it satisfies*

$$\alpha\left(1+\frac{z\left(D_qD_q\mathbf{S}_{\mu,\delta,q}^{n,m}f(z)\right)}{D_q\mathbf{S}_{\mu,\delta,q}^{n,m}f(z)}\right)+(1-\alpha)\left(\frac{z\left(D_q\mathbf{S}_{\mu,\delta,q}^{n,m}f(z)\right)}{\mathbf{S}_{\mu,\delta,q}^{n,m}f(z)}\right)\prec\widetilde{J}(L,P,z), \tag{4}$$

where $\widetilde{J}(L,P,z)$ is given by (2) and $-1<P<L\leq 1,\ 0\leq\alpha\leq 1$ and $m\in\mathbb{N}_0$.

Remark 2. *If we set the parameters $\alpha,\ m,\ b,\ \tau$, and $q\to 1$ in (4), we obtain several novel subclasses of the class $\mathcal{R}_q^{\alpha}(b,\tau,m,L,P)$:*

(i) *For $\alpha=0,m=0,b=\tau$, we have $\mathcal{R}_q^{\alpha}(b,\tau,0,L,P)=\mathcal{CS}^*[L,P]$ ([42]).*

(ii) *For $\alpha=0,m=0,b=\tau,L=1,P=-1$, we have $\mathcal{R}_q^{\alpha}(b,\tau,0,1,-1)=\mathcal{SL}$ ([16]).*

(iii) *For $\alpha=1,m=0,b=\tau$, we obtain $\mathcal{R}_q^{\alpha}(b,\tau,0,L,P)$, the class of convex functions connected with the cardioid domain.*

(iv) *For $\alpha=1,m=0,b=\tau,L=1,P=-1$, we have $\mathcal{R}_q^{\alpha}(b,\tau,0,1,-1)=\mathcal{KSL}$, the class of convex shell-like functions connected with the Fibonacci numbers ([18]).*

(v) *For $m=0,b=\tau,L=1,P=-1$, we have $\mathcal{R}_q^{\alpha}(b,\tau,0,1,-1)=\mathcal{SLM}_{\alpha}$ ([17]).*

The lemmas provided below effectively demonstrate the theorems in primary results.

Lemma 1 ([27]). *Consider the function $\widetilde{J}(L,P;z)$ is defined by (2). Then*

(i) *The function $\widetilde{J}(L,P;z)$ is univalent in the disk $|z|<\nu^2$.*

(ii) *If $J(z)\prec\widetilde{J}(L,P,z)$, then $Re(J(z))>\xi$, where*

$$\xi=\frac{2(L+P-2)\nu+2(2LP-L-P)\nu^3+16(L+P)\nu^2\varsigma}{4(P-1)(\nu+P\nu^3)+32P\nu^2\varsigma}$$

and

$$\varsigma=\frac{4+\nu^2-P^2\nu^2-4P^2\nu^4-\left(1-P\nu^2\right)\vartheta(P)}{4\nu(1+P^2\nu^2)}$$

with

$$\vartheta(P)=\sqrt{5(2P\nu^2+2-(P-1)\nu)(2+2P\nu^2+(P-1)\nu)},$$

$$-1<P\leq 1\ \ \text{and}\ \ \nu=\tfrac{1-\sqrt{5}}{2}.$$

(iii) *If $\widetilde{J}(L,P;z)=z+\sum_{j=1}^{\infty}\widetilde{J}_jz^j$, then*

$$\widetilde{J}_j:=\begin{cases}(L-P)\frac{\nu}{2} & \text{for } j=1,\\ (L-P)(5-P)\frac{\nu^2}{2^2} & \text{for } j=2,\\ \left(\frac{1-P}{2}\right)\nu\widetilde{J}_{k-1}-P\nu^2\widetilde{J}_{k-2} & \text{for } k=3,4\cdots,\end{cases}$$

where $-1<P<L\leq 1$.

(iv) *Let $J(z)\prec\widetilde{J}(L,P;z)$ and $J(z)=1+\sum_{j=1}^{\infty}J_jz^j$. Then,*

$$\left|J_2-vJ_1^2\right|\leq\frac{\nu(L-P)}{2}\max\left\{1,\left|\frac{\nu}{2}(P-5+v(L-P))\right|\right\}\quad(v\in\mathbb{C}).$$

Lemma 2 ([43]). *Let $p \in \mathcal{P}$ and*

$$p(z) = 1 + \sum_{j=1}^{\infty} c_j z^j. \tag{5}$$

Then,

$$\left|c_2 - \tfrac{v}{2}c_1^2\right| \leq \max\{2, 2|v-1|\} = \begin{cases} 2 & if\ 0 \leq v \leq 2, \\ 2|v-1| & elsewhere \end{cases}$$

and

$$|c_j| \leq 2 \quad (j \geq 1).$$

Lemma 3 ([44]). *Let $p \in \mathcal{P}$ be represented as in (5) and if $\mathcal{K} \in [0,1]$ with $\mathcal{K}(2\mathcal{K}-1) \leq \mathcal{Q} \leq \mathcal{K}$, then*

$$|c_3 - 2\mathcal{K}c_1c_2 + \mathcal{Q}c_1^3| \leq 2.$$

Lemma 4 ([45]). *Let $p \in \mathcal{P}$ be represented as in (5) with $c_1 \geq 0$, then,*

$$\begin{aligned} 2c_2 &= c_1^2 + \left(4 - c_1^2\right) r, \\ 4c_3 &= c_1^3 + 2\left(4 - c_1^2\right) rc_1 - \left(4 - c_1^2\right) r^2 c_1 + 2\left(4 - c_1^2\right)(1-|r|^2)\delta, \end{aligned}$$

for some $r,\ \delta \in \overline{\mathcal{U}_s} = \mathcal{U}_s \cup \{1\}$.

Lemma 5 ([46]). *For any real number $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$, let*

$$\Psi(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = \max\{\mathcal{G}_1 + \mathcal{G}_2\, r + \mathcal{G}_3\, r^2 + 1 - |r|^2\}.$$

If $\mathcal{G}_1\mathcal{G}_3 \geq 0$, then

$$\Psi(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = \begin{cases} |\mathcal{G}_1| + |\mathcal{G}_2| + |\mathcal{G}_3|, & |\mathcal{G}_2| \geq 2(1 - |\mathcal{G}_3|), \\ 1 + |\mathcal{G}_1| + \dfrac{\mathcal{G}_2^2}{4(1-|\mathcal{G}_3|)}, & |\mathcal{G}_2| < 2(1 - |\mathcal{G}_3|). \end{cases}$$

In this paper, we investigate some geometrical aspects of analytic functions in the class $\mathcal{R}_q^{\alpha}(b, \tau, m, L, P)$ connected with the cardioid domain defined in (4). Moreover, we estimate that the coefficients of functions, the inverse function, Fekete–Szegö problems, Hankel determinants, and logarithmic coefficients belong to the class $\mathcal{R}_q^{\alpha}(b, \tau, m, L, P)$.

The authors organize this paper in the following way. In Section 1, we extend the linear operator into the q-derivative operator, define the class, and introduce the paper with some basic definitions and lemmas. The coefficients of functions to the class $\mathcal{R}_q^{\alpha}(b, \tau, m, L, P)$, Fekete–Szegö problems, and Hankel determinants are estimated in Section 2. In addition to determining the Fekete–Szegö problems, the coefficients of the inverse function that belongs to the class $\mathcal{R}_q^{\alpha}(b, \tau,\ m, L, P)$ are also determined in Section 3. Section 4 calculates the logarithmic coefficients of f for the class, and the paper presents the concluding sections in Section 5.

2. Coefficient Estimates for the Class $\mathcal{R}_q^{\alpha}(b, \tau, m, L, P)$

In this section, we determine some coefficient bounds, Fekete–Szegö problems, and Hankel determinants for functions belonging to the class $\mathcal{R}_q^{\alpha}(b, \tau, m, L, P)$. Throughout the paper, unless otherwise mentioned, we suppose that for $\beta(n) = [n]_q + \alpha - 1,\ n \in \mathbb{N}$.

Theorem 1. *If* $f(z) \in \mathcal{R}_q^{\alpha}(b,\tau,m,L,P)$ *and* $-1 \le P < L \le 1$, *then*

$$|a_2| \le \frac{(L-P)\,|\nu|}{2\beta(2)\gamma_q(2)}, \tag{6}$$

$$|a_3| \le \frac{(L-P)|\nu|^2}{4\gamma_q(3)\mathbb{A}_1}\left|\frac{(L-P)\sigma}{\beta(2)^2}+5-P\right| \tag{7}$$

and

$$|a_4| \le \frac{(L-P)\,|\nu|}{2\gamma_q(4)\mathbb{A}_2}, \tag{8}$$

where $\mathbb{A}_1 = \beta(3)+\alpha\left([3]_q[2]_q-[3]_q\right)$, $\mathbb{A}_2 = \beta(4)+\alpha\left([4]_q[3]_q-[4]_q\right)$, $\sigma = \beta(2)+\alpha\left([2]_q^2-[2]_q\right)$ *and* $\gamma_q(j)$ *is given in (3). The result is sharp for the function given in (20).*

Proof. Let $f \in \mathcal{R}_q^{\alpha}(b,\tau,m,L,P)$. Then, by the subordination principle, there exists an analytic function $\kappa(z)$ with $\kappa(0)=0$ and $|\kappa(z)|<1$ such that

$$\alpha\left(1+\frac{z\left(D_qD_q\mathbf{S}^{n,m}_{\mu,\delta,q}f(z)\right)}{D_q\mathbf{S}^{n,m}_{\mu,\delta,q}f(z)}\right)+(1-\alpha)\left(\frac{z\left(D_q\mathbf{S}^{n,m}_{\mu,\delta,q}f(z)\right)}{\mathbf{S}^{n,m}_{\mu,\delta,q}f(z)}\right)=\widetilde{J}(L,P,\kappa(z)). \tag{9}$$

We define the function $\kappa(z)$ that

$$\begin{aligned}\kappa(z)&=(J(z)-1)(J(z)+1)^{-1}\\&=\frac{1}{2}c_1z+\frac{1}{2}\left(c_2-\frac{1}{2}c_1^2\right)z^2+\frac{1}{2}\left(c_3-c_1c_2+\frac{1}{4}c_1^3\right)z^3+\cdots,\end{aligned} \tag{10}$$

where $J(z)$ is given in (5).

Equation (2) can be written as

$$\begin{aligned}\widetilde{J}\quad&(L,P,z)=\tfrac{2L\nu^2z^2+(L-1)\nu z+2}{2P\nu^2z^2+(P-1)\nu z+2}\\&=1+\tfrac{1}{2}(L-P)\nu z+\tfrac{1}{4}(L-P)(5-P)\nu^2z^2+\tfrac{1}{8}(L-P)\left(P^2-10P+5\right)\nu^3z^3+\cdots.\end{aligned} \tag{11}$$

Then, by simple calculations, we obtain

$$\begin{aligned}\widetilde{J}(L,P,\kappa(z))\quad&=1+\tfrac{1}{4}(L-P)\nu c_1z+\left(\tfrac{1}{4}(L-P)\nu c_2-\tfrac{(L-P)(2-(5-P))\nu^2}{16}c_1^2\right)z^2\\&+\left(\begin{array}{l}\tfrac{\nu(L-P)}{4}c_3+\tfrac{\nu(L-P)}{8}(-2+(5-P)\nu)c_1c_2\\+\tfrac{(L-P)}{64}\left(4\nu-4(5-P)\nu^2+\left(P^2-10P+5\right)\nu^3\right)c_1^3\end{array}\right)z^3+\cdots.\end{aligned} \tag{12}$$

Now

$$\begin{aligned}&\alpha\left(1+\tfrac{z\left(D_qD_q\mathbf{S}^{n,m}_{\mu,\delta,q}f(z)\right)}{D_q\mathbf{S}^{n,m}_{\mu,\delta,q}f(z)}\right)+(1-\alpha)\left(\tfrac{z\left(D_q\mathbf{S}^{n,m}_{\mu,\delta,q}f(z)\right)}{\mathbf{S}^{n,m}_{\mu,\delta,q}f(z)}\right)=1+\gamma_q(2)\beta(2)a_2z+\\&\left(\gamma_q(3)\left(\beta(3)+\alpha\left([3]_q[2]_q-[3]_q\right)\right)a_3-\gamma_q(2)^2\left(\beta(2)+\alpha\left([2]_q^2-[2]_q\right)\right)a_2^2\right)z^2+\\&\left(\begin{array}{l}\gamma_q(4)\left(\beta(4)+\alpha\left([3]_q[4]_q-[4]_q\right)\right)a_4\\+\gamma_q(2)\gamma_q(3)\left(\alpha\left([2]_q+[3]_q-[2]_q[3]_q-[2]_q^2[3]_q\right)-\beta(3)-\beta(2)\right)a_2a_3\\+\gamma_q(2)^3\left(\beta(2)+\alpha\left([2]_q^3-[2]_q\right)\right)a_2^3\end{array}\right)z^3+\cdots.\end{aligned} \tag{13}$$

Comparing the coefficients from (12) and (13) with (9), we get

$$a_2=\frac{(L-P)\nu c_1}{4\beta(2)\gamma_q(2)} \tag{14}$$

and

$$a_3 = \frac{(L-P)\nu}{4\gamma_q(3)\mathbb{A}_1}\left(c_2 - \frac{1}{2}\rho c_1^2\right), \tag{15}$$

where

$$\rho = 1 - \nu\left(\frac{(L-P)\left(\beta(2) + \alpha\left([2]_q^2 - [2]_q\right)\right)}{2\beta(2)^2} + \frac{5-P}{2}\right) \tag{16}$$

and

$$a_4 = \frac{(L-P)\nu}{4\gamma_q(4)\mathbb{A}_2}\left(c_3 - c_1c_2B(\alpha,q) - c_1^3D(\alpha,q)\right) \tag{17}$$

where,

$$\begin{aligned} B(\alpha,q) = \; & 1 - \tfrac{(5-P)\nu}{2} \\ & - \left(\tfrac{(L-P)\nu}{4\beta(2)\mathbb{A}_1}\right)\left(\beta(3) + \beta(2) + \alpha\left([2]_q[3]_q\left(1 + [2]_q\right) - [3]_q - [2]_q\right)\right), \end{aligned} \tag{18}$$

$$\begin{aligned} D(\alpha,q) = \; & \left(\tfrac{(L-P)\nu\rho}{8\beta(2)\mathbb{A}_1}\right)\left(\beta(3) + \beta(2) + \alpha\left([2]_q[3]_q\left(1 + [2]_q\right) - [3]_q - [2]_q\right)\right) \\ & + \tfrac{((L-P)\nu)^2}{16\beta(2)^3}\left(\beta(2) + \alpha\left([2]_q^3 - [2]_q\right)\right) + \left(\tfrac{(L-P)\nu-1}{4}\right) \\ & - \left(\tfrac{\nu}{16(L-P)}\right)\left(P^2 - 10P + 5\right). \end{aligned} \tag{19}$$

If $P < L$, then $\rho > 2$. Therefore, taking Lemma 2 into account, we reach the required conclusion of (6) and (7).

Let

$$\mathcal{K} = \frac{B(\alpha,q)}{2} \quad \text{and} \quad \mathcal{Q} = -D(\alpha,q).$$

It is clear that

$$\begin{aligned} \mathcal{K} - \mathcal{Q} = \; & \tfrac{1}{2} - \tfrac{(5-P)\nu}{4} - \tfrac{(L-P)\nu}{8\beta(2)\mathbb{A}_1}\left(\beta(3) + \beta(2) + \alpha\left([2]_q[3]_q\left(1 + [2]_q\right) - [3]_q - [2]_q\right)\right) \\ & (1-\rho) + \tfrac{((L-P)\nu)^2}{16(\beta(2))^3\left(\beta(2) + \alpha\left([2]_q^3 - [2]_q\right)\right)} + \tfrac{(L-P)\nu-1}{4}\tfrac{\nu\left(P^2-10P+5\right)}{16(L-P)} \geq 0, \end{aligned}$$

and

$$\mathcal{Q} - \mathcal{K}(2\mathcal{K} - 1) \geq 0.$$

Thus, by using the Lemma 3, we have

$$a_4 \leq \tfrac{(L-P)\,|\nu|}{2\gamma_q(4)\mathbb{A}_2}.$$

If we consider the function $\widetilde{f}_1 : \mathcal{U} \to \mathbb{C}$ as

$$\widetilde{f}_1(z) = z + \frac{(L-P)\nu}{2\beta(2)\gamma_q(2)}z^2 + \frac{(L-P)\nu}{2\gamma_q(3)\mathbb{A}_1}(1-\rho)z^3 + \cdots. \tag{20}$$

Then, we obtain the following result with normalized conditions on $\widetilde{f}_1 \in \mathcal{R}_q^{\alpha}(b,\tau,m,L,P)$

$$\alpha\left(1 + \frac{z\left(D_qD_q\mathbf{S}_{\mu,\delta,q}^{n,m}f(z)\right)}{D_q\mathbf{S}_{\mu,\delta,q}^{n,m}f(z)}\right) + (1-\alpha)\left(\frac{z\left(D_q\mathbf{S}_{\mu,\delta,q}^{n,m}f(z)\right)}{\mathbf{S}_{\mu,\delta,q}^{n,m}f(z)}\right) = \widetilde{J}(L,P,z).$$

This shows that the result is sharp for the function $\widetilde{f}_1$ given in (20). □

Remark 3. *The normalized extremal function f_t yields the equalities, which are defined by*

$$\alpha\left(1+\frac{z\left(D_qD_q\mathbf{S}^{n,m}_{\mu,\delta,q}f_t(z)\right)}{D_q\mathbf{S}^{n,m}_{\mu,\delta,q}f_t(z)}\right)+(1-\alpha)\left(\frac{z\left(D_q\mathbf{S}^{n,m}_{\mu,\delta,q}f_t(z)\right)}{\mathbf{S}^{n,m}_{\mu,\delta,q}f_t(z)}\right)=\widetilde{J}\left(L,P,z^t\right)\quad \text{for } t=1,2,3.$$

In other words, normalized extremal functions are supplied by

$$\widetilde{J}(L,P,z)=1+\frac{(L-P)\nu}{2\beta(2)\gamma_q(2)}z+\frac{(L-P)\nu}{2\gamma_q(3)\mathbb{A}_1}(1-\rho)z^2+\cdots, \tag{21}$$

$$\widetilde{J}\left(L,P,z^2\right)=1+\frac{(L-P)\nu}{2\beta(2)\gamma_q(2)}z^2+\frac{(L-P)\nu}{2\gamma_q(3)\mathbb{A}_1}(1-\rho)z^4+\cdots, \tag{22}$$

$$\widetilde{J}\left(L,P,z^3\right)=1+\frac{(L-P)\nu}{2\beta(2)\gamma_q(2)}z^3+\frac{(L-P)\nu}{2\gamma_q(3)\mathbb{A}_1}(1-\rho)z^6+\cdots. \tag{23}$$

By comparing the coefficients of the series in (13) with the coefficients of the series in (21) and using Definition 4, we can demonstrate that the function described by (21) is the extremal function of the sharpness of (6), as follows:

$$a_2=\frac{(L-P)\nu c_1}{4\beta(2)\gamma_q(2)}.$$

Therefore, below, we present the normalized extremal function defined by (21) as follows:

$$\widetilde{f}_1(z)=z+\frac{(L-P)\nu}{2\beta(2)\gamma_q(2)}z^2+\frac{(L-P)\nu}{2\gamma_q(3)\mathbb{A}_1}(1-\rho)z^3+\cdots.$$

The extremal functions denoted by (22) and (23) can also be obtained in this way as follows:

$$\widetilde{f}_2(z)=z+\frac{(L-P)\nu}{2\beta(2)\gamma_q(2)}z^3+\cdots. \tag{24}$$

$$\widetilde{f}_3(z)=z+\frac{(L-P)\nu}{2\beta(2)\gamma_q(2)}z^4+\cdots. \tag{25}$$

Example 1. *Let $L=0.5, P=0.25, m=1, \alpha=1, k=1, \mu+\delta=1, b=\tau$ and $q=\frac{1}{2}$ and using the definition of q-number, we get*

$$[2]_{0.5}=1.5, [3]_{0.5}=1.75 \quad \text{and} \quad [4]_{0.5}=1.875.$$

Hence, Theorem 1 yields

$$|a_2|\leq 0.07803, \quad |a_3|\leq 0.0795, \quad \text{and} \quad |a_4|\leq 0.04443,$$

where

$$\gamma_q(2)=0.66, \quad \gamma_q(3)=0.57, \quad \text{and} \quad \gamma_q(4)=0.53.$$

If we set $\alpha=0$ in Theorem 1, then we deduce the following result.

Corollary 1. *If $f(z)\in\mathcal{R}^0_q(b,\tau,m,L,P)$ and $-1\leq P<L\leq 1$, then*

$$|a_2|\leq\frac{(L-P)\,|\nu|}{2\left([2]_q-1\right)\gamma_q(2)},$$

$$|a_3| \leq \frac{(L-P)|\nu|^2}{4\gamma_q(3)\left([3]_q-1\right)}\left(\frac{(L-P)}{[2]_q-1}+5-P\right)$$

and

$$|a_4| \leq \frac{(L-P)\,|\nu|}{2\gamma_q(4)\left([4]_q-1\right)}.$$

By taking $q \to 1,\ m=0,\ b=\tau$ in Corollary 1, we deduce the result of [28] for a_2, a_3 and the coefficient a_4 is fresh.

Corollary 2. *If $f(z) \in \mathcal{R}_q^{\alpha}(\tau,\tau,0,L,P)$ and $-1 \leq P < L \leq 1$, then*

$$|a_2| \leq \frac{(L-P)\,|\nu|}{2},\quad |a_3| \leq \frac{(L-P)|\nu|^2}{8}(L-2P+5)$$

and

$$|a_4| \leq \frac{(L-P)\,|\nu|}{6}.$$

Theorem 2. *If $f(z) \in \mathcal{R}_q^{\alpha}(b,\tau,m,L,P)$ and $-1 \leq P < L \leq 1$, then*

$$|a_3-\mathcal{M}a_2^2| \leq \frac{(L-P)\,|\nu|}{4\gamma_q(3)\mathbb{A}_1}\max\left\{2,\left|\nu\left(\frac{\beta(2)^2(P-5)-(L-P)\sigma}{\beta(2)^2}+\frac{\mathcal{M}(L-P)\mathbb{A}_1\gamma_q(3)}{\beta(2)^2\gamma_q(2)^2}\right)\right|\right\}.$$

This result is sharp for the function in (24).

Proof. Let $f \in \mathcal{R}_q^{\alpha}(b,\tau,m,L,P)$. We note from (9) that

$$\alpha\left(1+\frac{z\left(D_qD_q\mathbf{S}_{\mu,\delta,q}^{n,m}f(z)\right)}{D_q\mathbf{S}_{\mu,\delta,q}^{n,m}f(z)}\right)+(1-\alpha)\left(\frac{z\left(D_q\mathbf{S}_{\mu,\delta,q}^{n,m}f(z)\right)}{\mathbf{S}_{\mu,\delta,q}^{n,m}f(z)}\right)=\widetilde{J}(L,P,\kappa(z)).$$

Therefore,

$$\begin{aligned}1+\gamma_q(2)\beta(2)a_2z+\quad &\left(\gamma_q(3)\mathbb{A}_1a_3-\gamma_q(2)^2\sigma a_2^2\right)z^2+\cdots\\ &=1+h_1z+h_2z^2+h_3z^3+\cdots.\end{aligned} \tag{26}$$

By making use of a comparison on coefficient in (26), we deduce that

$$\begin{aligned}a_2&=\tfrac{h_1}{\beta(2)\gamma_q(2)}\\ a_3&=\tfrac{1}{\gamma_q(3)\mathbb{A}_1}\left(h_2+\tfrac{\sigma h_1^2}{\beta(2)^2}\right).\end{aligned}$$

Consequently, using basic computations, we get

$$|a_3-\mathcal{M}a_2^2|=\left(\tfrac{1}{\gamma_q(3)\mathbb{A}_1}\right)|h_2-\rho_1h_1^2|,$$

where

$$\rho_1=\frac{1}{\beta(2)^2}\left[-\sigma+\mathcal{M}\frac{\gamma_q(3)\mathbb{A}_1}{\gamma_q(2)^2}\right].$$

The result follows according to Lemma 1. The equality

$$|a_3-\mathcal{M}a_2^2|=\frac{(L-P)|\nu|^2}{4\gamma_q(3)\mathbb{A}_1}\left(\frac{(L-P)\sigma}{\beta(2)^2}+(5-P)-\mathcal{M}\frac{(L-P)\gamma_q(3)\mathbb{A}_1}{\beta_2^2\gamma_q(2)^2}\right).$$

holds for the function $\widetilde{f}_2$ which is defined by (24). We have $\widetilde{f}_2(0) = 0,\ \widetilde{f}_2'(0) = 1$ and

$$\alpha\left(1+\frac{z\left(D_qD_q\mathbf{S}^{n,m}_{\mu,\delta,q}\widetilde{f}(z)\right)}{D_q\mathbf{S}^{n,m}_{\mu,\delta,q}\widetilde{f}(z)}\right)+(1-\alpha)\left(\frac{z\left(D_q\mathbf{S}^{n,m}_{\mu,\delta,q}\widetilde{f}(z)\right)}{\mathbf{S}^{n,m}_{\mu,\delta,q}\widetilde{f}(z)}\right)=\widetilde{J}\left(L,P,z^2\right).$$

Therefore, $\widetilde{f}_2 \in \mathcal{R}^{\alpha}_q(b,\tau,m,L,P)$. This shows that the result is sharp for the function $\widetilde{f}_2$ given in (24). □

If we set $\alpha = 0$ in Theorem 2, then we deduce the following result.

Corollary 3. *If $f(z) \in \mathcal{R}^0_q(b,\tau,m,L,P)$ and $-1 \le P < L \le 1$, then*

$$\begin{aligned}|a_3-\mathcal{M}a_2^2| \quad &\le \frac{(L-P)\,|\nu|}{4\gamma_q(3)\left([3]_q-1\right)}\times\\ &\max\left\{2,\left|\nu\left(\frac{\left([2]_q-1\right)(P-5)-(L-P)}{[2]_q-1}+\mathcal{M}\frac{(L-P)\left([3]_q-1\right)\gamma_q(3)}{\left([2]_q-1\right)^2\gamma_q(2)^2}\right)\right|\right\}.\end{aligned}$$

By taking $q \to 1,\ m = 0,\ b = \tau,$ in Corollary 3, we deduce the following corollary.

Corollary 4 ([28]). *If $f(z) \in \mathcal{R}^{\alpha}_q(\tau,\tau,0,L,P)$ and $-1 \le P < L \le 1$, then*

$$|a_3-\mathcal{M}a_2^2| \le \frac{(L-P)\,|\nu|}{8}\max\{2,|\nu(-(L-2P+5)+2(L-P)\mathcal{M})|\}.$$

Theorem 3. *If $f(z) \in \mathcal{R}^{\alpha}_q(b,\tau,m,L,P)$ and $-1 \le P < L \le 1$, then*

$$|a_3-a_2^2| \le \frac{(L-P)\,|\nu|}{4\mathbb{A}_1\gamma_q(3)}\max\left\{2,2\left|\rho-\frac{(L-P)\nu\mathbb{A}_1\gamma_q(3)}{2\beta(2)^2\gamma_q(2)^2}-1\right|\right\}.$$

This result is the best result possible. The function given in (24) holds equality.

Proof. Applying (14) and (15), we arrived at

$$a_3-a_2^2=\left(\frac{(L-P)\nu}{4\mathbb{A}_1\gamma_q(3)}\right)\left(c_2-\frac{c_1^2}{2}\left(\rho-\frac{(L-P)\nu\mathbb{A}_1\gamma_q(3)}{2\beta(2)^2\gamma_q(2)^2}\right)\right).$$

Then, by utilizing the modulus in the resulting equation, we obtain

$$|a_3-a_2^2|=\left|\left(\frac{(L-P)\nu}{4\mathbb{A}_1\gamma_q(3)}\right)\left(c_2-\frac{c_1^2}{2}\rho_2\right)\right|,$$

where

$$\rho_2=\rho-\frac{(L-P)\nu\mathbb{A}_1\gamma_q(3)}{2\beta(2)^2\gamma_q(2)^2}.$$

Hence, by Lemma 1, we obtain

$$|a_3-a_2^2| \le \frac{(L-P)\,|\nu|}{4\mathbb{A}_1\gamma_q(3)}max\left\{2,2\left|\rho-\frac{(L-P)\nu\mathbb{A}_1\gamma_q(3)}{2\beta(2)^2\gamma_q(2)^2}-1\right|\right\}.$$

□

Theorem 4. *If* $f(z) \in \mathcal{R}_q^{\alpha}(b,\tau,m,L,P)$ *and* $-1 \leq P < L \leq 1$, *then*

$$|a_2a_3 - a_4| \leq \frac{(L-P)\,|\nu|}{2\mathbb{A}_2\gamma_q(4)}.$$

The above outcome is the best possible. Equality is attained for the function provided in (25).

Proof. After simplifying with (14), (15), and (17), we conclude that

$$\begin{aligned} a_2a_3 - a_4 &= \left(\frac{(L-P)\nu}{4\mathbb{A}_2\gamma_q(4)}\right)\left(-c_3 + \left(\frac{(L-P)\nu\mathbb{A}_2\gamma_q(4)}{4\mathbb{A}_1\beta\left([2]_q\right)\left(\gamma_q(2)\right)\left(\gamma_q(3)\right)} + B(\alpha,q)\right)c_1c_2 \right. \\ &\quad \left. -\left(\frac{\rho(L-P)\nu\mathbb{A}_2\gamma_q(4)}{8\mathbb{A}_1\beta\left([2]_q\right)\left(\gamma_q(2)\right)\left(\gamma_q(3)\right)} - D(\alpha,q)\right)c_1^3\right). \end{aligned}$$

By making use of the modulus in the resulting equation, we have

$$\begin{aligned} |a_2a_3 - a_4| &= \left|\frac{(L-P)\nu}{4\mathbb{A}_2\gamma_q(4)}\right|\left|c_3 - \left(\frac{(L-P)\nu\mathbb{A}_2\gamma_q(4)}{4\mathbb{A}_1\beta(2)\left(\gamma_q(2)\right)\left(\gamma_q(3)\right)} + B(\alpha,q)\right)c_1c_2 \right. \\ &\quad \left. +\left(\frac{\rho(L-P)\nu\mathbb{A}_2\gamma_q(4)}{8\mathbb{A}_1\beta(2)\left(\gamma_q(2)\right)\left(\gamma_q(3)\right)} - D(\alpha,q)\right)c_1^3\right|. \end{aligned}$$

Let

$$\mathcal{K} = \left(\frac{(L-P)\nu\mathbb{A}_2\gamma_q(4) + 4B(\alpha,q)\mathbb{A}_1\beta(2)\left(\gamma_q(2)\right)\left(\gamma_q(3)\right)}{8\mathbb{A}_1\beta(2)\left(\gamma_q(2)\right)\left(\gamma_q(3)\right)}\right)$$

and

$$\mathcal{Q} = \left(\frac{\rho(L-P)\nu\mathbb{A}_2\gamma_q(4) - 8D(\alpha,q)\mathbb{A}_1\beta(2)\left(\gamma_q(2)\right)\left(\gamma_q(3)\right)}{8\mathbb{A}_1\beta(2)\left(\gamma_q(2)\right)\left(\gamma_q(3)\right)}\right).$$

It is clear that

$$\mathcal{K} - \mathcal{Q} = \frac{(L-P)\nu\mathbb{A}_2\gamma_q(4)(1-\rho)+4\mathbb{A}_1\beta(2)\left(\gamma_q(2)\right)\left(\gamma_q(3)\right)(B(\alpha,q)-2D(\alpha,q)}{8\mathbb{A}_1\beta(2)\left(\gamma_q(2)\right)\left(\gamma_q(3)\right)} \geq 0,$$

and

$$\begin{aligned} \mathcal{Q} - \mathcal{K}(2\mathcal{K}-1) &= \frac{1}{32\left(\mathbb{A}_1\beta\left([2]_q\right)\left(\gamma_q(2)\right)\left(\gamma_q(3)\right)^2\right.} \\ &\quad \left\{\begin{array}{l} 4\mathbb{A}_1\beta(2)\left(\gamma_q(2)\right)\left(\gamma_q(3)\right)[\rho(L-P)\nu\,\mathbb{A}_2\gamma_q(4) \\ -8D(\alpha,q)\mathbb{A}_1\beta(2)\left(\gamma_q(2)\right)\left(\gamma_q(3)\right) \\ +(L-P)\nu\mathbb{A}_2\gamma_q(4) + 4B(\alpha,q)\mathbb{A}_1\beta(2)\left(\gamma_q(2)\right)\left(\gamma_q(3)\right)] \\ -\left((L-P)\nu\mathbb{A}_2\gamma_q(4) + 4B(\alpha,q)\mathbb{A}_1\beta(2)\left(\gamma_q(2)\right)\left(\gamma_q(3)\right)\right)^2. \end{array}\right. \end{aligned}$$

Since $\mathcal{Q} - \mathcal{K}(2\mathcal{K}-1) \geq 0$. Hence, using the Lemma 3, we have

$$|a_2a_3 - a_4| \leq \frac{(L-P)\,|\nu|}{2\mathbb{A}_2\gamma_q(4)}.$$

□

Theorem 5. *If* $f(z) \in \mathcal{R}_q^{\alpha}(b,\tau,m,L,P)$ *and* $-1 \leq P < L \leq 1$, *then*

$$|\mathcal{H}_{2,2}(f)| \leq \left(\frac{(L-P)\nu}{2\mathbb{A}_1\gamma_q(3)}\right)^2.$$

Proof. For the functions $f \in \mathcal{R}_q^{\alpha}(b,\tau,m,L,P)$, we obtain the second Hankel determinant

$$\mathcal{H}_{2,2}(f) = a_2a_4 - a_3^2.$$

As a result, by applying (14), (15), and (17), we get

$$\begin{aligned}\mathcal{H}_{2,2}(f) = \; & \left[\tfrac{(L-P)\nu}{4}\right]^2\Bigg(-\tfrac{4D(\alpha,q)\mathbb{A}_1^2\left(\gamma_q(3)\right)^2+\rho^2\mathbb{A}_2\beta(2)\gamma_q(2)\gamma_q(4)}{4\mathbb{A}_2\beta(2)\gamma_q(2)\gamma_q(4)\mathbb{A}_1^2\left(\gamma_q(3)\right)^2}c_1^4 \\ & -\tfrac{B(\alpha,q)\mathbb{A}_1^2\left(\gamma_q(3)\right)^2-\rho\mathbb{A}_1\beta(2)\gamma_q(2)\gamma_q(4)}{\mathbb{A}_2\beta(2)\gamma_q(2)\gamma_q(4)\mathbb{A}_1^2\left(\gamma_q(3)\right)^2}c_1^2c_2 - \tfrac{1}{\mathbb{A}_1^2\left(\gamma_q(3)\right)^2}c_2^2 \\ & +\tfrac{1}{\mathbb{A}_2\beta(2)\gamma_q(2)\gamma_q(4)}c_1c_3\Bigg).\end{aligned} \tag{27}$$

According to the rotation invariant characteristic for the family $\mathcal{R}_q^{\alpha}(b,\tau,m,L,P)$ and determinant $\mathcal{H}_{2,2}(f)$, $c_1 = c \in [0,2]$ is assumed to be feasible. Using Lemma 4, the coefficients c_2 and c_3 can be represented in terms of c_1. After that, the modulus is used in the resultant Equation (27) to give

$$\begin{aligned}|\mathcal{H}_{2,2}(f)| \; = & \left|\tfrac{(L-P)\nu}{4}\right|^2 \\ & \Bigg|-\tfrac{\mathbb{A}_1^2\left(\gamma_q(3)\right)^2(4D(\alpha,q)+2B(\alpha,q)-1)-\left(\rho^2\mathbb{A}_2+2\rho\mathbb{A}_1-\mathbb{A}_2\right)\beta(2)\gamma_q(2)\gamma_q(4)}{4\mathbb{A}_2\beta(2)\gamma_q(2)\gamma_q(4)\mathbb{A}_1^2\left(\gamma_q(3)\right)^2}c^4 \\ & -\tfrac{\left(4-c^2\right)}{4\mathbb{A}_2\gamma_q(2)\mathbb{A}_2\gamma_q(4)\beta(2)}r^2c^2 - \tfrac{\left(4-c^2\right)^2}{4\mathbb{A}_1^2\left(\gamma_q(3)\right)^2}r^2 + \tfrac{\left(4-c^2\right)(1-|r|^2)}{2\mathbb{A}_2\beta(2)\gamma_q(2)\gamma_q(4)}c\delta \\ & +\tfrac{\left(4-c^2\right)}{2}\left(\tfrac{\mathbb{A}_1^2\left(\gamma_q(3)\right)^2(1-B(\alpha,q))+\beta(2)\gamma_q(2)\gamma_q(4)(\rho\mathbb{A}_1-\mathbb{A}_2)}{\mathbb{A}_2\beta(2)\gamma_q(2)\gamma_q(4)\mathbb{A}_1^2\left(\gamma_q(3)\right)^2}\right)rc^2\Bigg|.\end{aligned}$$

For $c = 0$, it is clear that $|\mathcal{H}_{2,2}(f)| \leq \left|\tfrac{(L-P)\nu}{2\mathbb{A}_1\left(\gamma_q(3)\right)}\right|^2$ and for $c = 2$,

$$\begin{aligned}|\mathcal{H}_{2,2}(f)| \; = & \left|\tfrac{(L-P)\nu}{2}\right|^2 \\ & \left|\tfrac{\mathbb{A}_1^2\left(\gamma_q(3)\right)^2(4D(\alpha,q)+2B(\alpha,q)-1)-\left(\rho^2\mathbb{A}_2+2\rho\mathbb{A}_1-\mathbb{A}_2\right)\beta(2)\gamma_q(2)\gamma_q(4)}{\mathbb{A}_2\beta(2)\gamma_q(2)\gamma_q(4)\mathbb{A}_1^2\left(\gamma_q(3)\right)^2}\right|.\end{aligned}$$

With the case $c \in (0,2)$,$|\delta| \leq 1$, then

$$\begin{aligned}|\mathcal{H}_{2,2}(f)| \; \leq & \tfrac{((L-P)\nu)^2\left(4-c^2\right)c}{32\mathbb{A}_2\beta(2)\gamma_q(2)\gamma_q(4)}\Bigg(\\ & \Bigg|-\tfrac{\mathbb{A}_1^2\left(\gamma_q(3)\right)^2(4D(\alpha,q)+2B(\alpha,q)-1)-\left(\rho^2\mathbb{A}_2+2\rho\mathbb{A}_1-\mathbb{A}_2\right)\beta(2)\gamma_q(2)\gamma_q(4)}{2(4-c^2)\mathbb{A}_1^2\left(\gamma_q(3)\right)^2}c^3 \\ & +\tfrac{\mathbb{A}_1^2\left(\gamma_q(3)\right)^2(1-B(\alpha,q))+\beta(2)\gamma_q(2)\gamma_q(4)(\rho\mathbb{A}_1-\mathbb{A}_2)}{\mathbb{A}_1^2\left(\gamma_q(3)\right)^2}rc \\ & -\tfrac{c^2\left(\mathbb{A}_1^2\left(\gamma_q(3)\right)^2-\mathbb{A}_2\beta(2)\gamma_q(2)\gamma_q(4)\right)+4\mathbb{A}_2\beta(2)\gamma_q(2)\gamma_q(4)}{2c\mathbb{A}_1^2\left(\gamma_q(3)\right)^2}r^2\Bigg|+1-|r|^2\Bigg).\end{aligned}$$

We can express the aforementioned inequality in terms of $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$, in light of Lemma 5 as

$$|\mathcal{H}_{2,2}(f)| \leq \tfrac{((L-P)\nu)^2\left(4-c^2\right)c}{32\mathbb{A}_2\beta(2)\gamma_q(2)\gamma_q(4)}\Psi(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3),$$

where

$$\Psi(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3) = |\mathcal{G}_1 + \mathcal{G}_2r + \mathcal{G}_3r^2| + 1 - |r|^2,$$

with

$$\mathcal{G}_1 = -\frac{\mathbb{A}_1^2\left(\gamma_q(3)\right)^2(4D(\alpha,q)+2B(\alpha,q)-1)-\left(\rho^2\mathbb{A}_2+2\rho\mathbb{A}_1-\mathbb{A}_2\right)\beta(2)\gamma_q(2)\gamma_q(4)}{2(4-c^2)\mathbb{A}_1^2\left(\gamma_q(3)\right)^2}c^3,$$

$$\mathcal{G}_2 = \frac{\mathbb{A}_1^2\left(\gamma_q(3)\right)^2(1-B(\alpha,q))+\beta(2)\gamma_q(2)\gamma_q(4)(\rho\mathbb{A}_1-\mathbb{A}_2)}{\mathbb{A}_1^2\left(\gamma_q(3)\right)^2}c,$$

$$\mathcal{G}_3 = -\frac{c^2\left(\mathbb{A}_1^2\left(\gamma_q(3)\right)^2-\mathbb{A}_2\beta(2)\gamma_q(2)\gamma_q(4)\right)+4\mathbb{A}_2\beta(2)\gamma_q(2)\gamma_q(4)}{2c\mathbb{A}_1^2\left(\gamma_q(3)\right)^2}.$$

It is evident that the maximum of Ψ can be found and $\mathcal{G}_1\mathcal{G}_3 \geq 0$ by using Lemma 5. Note that $|\mathcal{G}_2| \geq 2(1-|\mathcal{G}_3|)$ is equal to

$$\wp(c,\alpha) = \frac{\left(\mathbb{A}_1^2\left(\gamma_q(3)\right)^2(2-B(\alpha,q))+\beta(2)\gamma_q(2)\gamma_q(4)(\rho\mathbb{A}_1-2\mathbb{A}_2)\right)c^2-}{\mathbb{A}_1^2\left(\gamma_q(3)\right)^2c}$$
$$\frac{2\mathbb{A}_1^2\left(\gamma_q(3)\right)^2c-4\beta(2)\gamma_q(2)\gamma_q(4)\mathbb{A}_2}{\mathbb{A}_1^2\left(\gamma_q(3)\right)^2c} \geq 0.$$

To prove that $\wp(c,\alpha) \geq 2$, we must show that the minima of $\wp\ (c,\ \alpha)$ is positive for every $c \in [0,2]$ and $0 \leq \alpha \leq 1$. We can quickly ascertain using simple math

$$\min \wp(c,\alpha) = \wp(2,\alpha) = \frac{2\left[\mathbb{A}_1^2\left(\gamma_q(3)\right)^2(1-B(\alpha,q))+\beta(2)\gamma_q(2)\gamma_q(4)(\rho\mathbb{A}_1-\mathbb{A}_2)\right]}{\mathbb{A}_1^2\left(\gamma_q(3)\right)^2} > 0.$$

Thus, using Lemma 5, we have

$$\Psi(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3) = (|\mathcal{G}_1|+|\mathcal{G}_2|+|\mathcal{G}_3|),$$

and

$$|\mathcal{H}_{2,2}(f)| \leq \frac{((L-P)\nu)^2\left(4-c^2\right)c}{32\mathbb{A}_2\beta(2)\gamma_q(2)\gamma_q(4)}(|\mathcal{G}_1|+|\mathcal{G}_2|+|\mathcal{G}_3|) = \left[\frac{(L-P)\nu}{4}\right]^2$$
$$\left[\left(\frac{\mathbb{A}_1^2\left(\gamma_q(3)\right)^2(4D(\alpha,q)+2B(\alpha,q)-1)-\beta(2)\gamma_q(2)\gamma_q(4)\left(\left(\rho^2-1\right)\mathbb{A}_2+2\rho\mathbb{A}_1\right)-\mathbb{A}_1^2\left(\gamma_q(3)\right)^2}{4\mathbb{A}_1^2\left(\gamma_q(3)\right)^2\beta(2)\gamma_q(2)\gamma_q(4)\mathbb{A}_2}\right.\right.$$
$$\left.-\frac{2\left[\mathbb{A}_1^2\left(\gamma_q(3)\right)^2(1-B(\alpha,q))+\beta(2)\gamma_q(2)\gamma_q(4)(\rho\mathbb{A}_1-\mathbb{A}_2)\right]+\mathbb{A}_2\beta(2)\gamma_q(2)\gamma_q(4)}{4\mathbb{A}_1^2\left(\gamma_q(3)\right)^2\beta(2)\gamma_q(2)\gamma_q(4)\mathbb{A}_2}\right)c^4+$$
$$\left(\frac{2\left[-\mathbb{A}_1^2\left(\gamma_q(3)\right)^2B(\alpha,q)+\beta(2)\gamma_q(2)\gamma_q(4)(\rho\mathbb{A}_1-\mathbb{A}_2)-\mathbb{A}_2\beta(2)\gamma_q(2)\gamma_q(4)\right]+\mathbb{A}_1^2\left(\gamma_q(3)\right)^2}{\mathbb{A}_1^2\left(\gamma_q(3)\right)^2\beta(2)\gamma_q(2)\gamma_q(4)\mathbb{A}_2}\right)c^2$$
$$\left.+\frac{16\mathbb{A}_2\beta(2)\gamma_q(2)\gamma_q(4)}{4\mathbb{A}_1^2\left(\gamma_q(3)\right)^2\beta(2)\gamma_q(2)\gamma_q(4)\mathbb{A}_2}\right] = \varkappa(c).$$

The maxima of $\varkappa$ can be easily found from $\left(\frac{(L-P)\nu}{2\mathbb{A}_1\gamma_q(3)}\right)^2$ at $c=0$, then

$$|\mathcal{H}_{2,2}(f)| \leq \left(\frac{(L-P)\nu}{2\mathbb{A}_1\gamma_q(3)}\right)^2.$$

Thus, the desired inequality is established. □

Example 2. *Let* $L = 0.5, P = 0.25, m = 1, \alpha = 1, k = 1, \mu+\delta = 1, b = \tau$ *and* $q = 0.5$ *and using the definition of q-number, we have*

$$[2]_{0.5} = 1.5, [3]_{0.5} = 1.75 \ \ and \ \ [4]_{0.5} = 1.875.$$

Therefore, using the above Theorem 5, we obtain

$$|\mathcal{H}_{2,2}(f)| \leq 0.00267.$$

3. Inverse Coefficients for the Class $\mathcal{R}_q^{\alpha}(b, \tau, m, L, P)$

A univalent function $f(z)$ given in (1) defined based on $\mathcal{U}$ has an inverse at least on a disc of with a radius of $\frac{1}{4}$ if it satisfies

$$f\left(f^{-1}(w)\right) = w$$

and of the form

$$f^{-1}(w) = w + \mathcal{D}_2 w^2 + \mathcal{D}_3 w^3 + \mathcal{D}_4 w^4 + \cdots \quad \left(|w| < \tfrac{1}{4}\right). \tag{28}$$

According to a simple calculation, we have

$$\mathcal{D}_2 = -a_2, \tag{29}$$

$$\mathcal{D}_3 = 2a_2^2 - a_3, \tag{30}$$

$$\mathcal{D}_4 = -5a_2^3 + 5a_2 a_3 - a_4, \tag{31}$$

and so on.

Theorem 6. *If $f(z) \in \mathcal{R}_{q,\alpha}(b, \tau, m, L, P)$ and $-1 \leq P < L \leq 1$, then*

$$|\mathcal{D}_2| \leq \frac{(L-P)|\nu|}{2\beta(2)\gamma_q(2)} \tag{32}$$

and

$$|\mathcal{D}_3| \leq \frac{(L-P)|\nu|}{4\mathbb{A}_1\gamma_q(3)} \max\left\{2, \left|\nu\left(\frac{(L-P)\sigma}{\beta(2)^2} + (5-P) - \frac{2(L-P)\mathbb{A}_1\gamma_q(3)}{\beta(2)^2\gamma_q(2)^2}\right)\right|\right\}. \tag{33}$$

$$|\mathcal{D}_4| \leq \tfrac{(L-P)|\nu|}{2\mathbb{A}_2\gamma_q(4)} \tag{34}$$

Proof. Let $f \in \mathcal{R}_q^{\alpha}(b, \tau, m, L, P)$. In light of (29), we can deduce from (14) that

$$\mathcal{D}_2 = -\frac{(L-P)\nu c_1}{4\beta(2)\gamma_q(2)},$$

and by using the modulus to get

$$|\mathcal{D}_2| \leq \frac{(L-P)|\nu|}{2\beta(2)\gamma_q(2)}.$$

Similarly, in light of (30), we can deduce from (14) and (15) that

$$\mathcal{D}_3 = -\tfrac{(L-P)\nu}{4\mathbb{A}_1\gamma_q(3)}\left(c_2 - \tfrac{c_1^2}{2}\right) - \tfrac{(L-P)\nu^2 c_1^2}{16\gamma_q(3)\mathbb{A}_1}\left[(5-P) + \tfrac{(L-P)\sigma}{(\beta(2))^2} - \tfrac{2(L-P)\gamma_q(3)\mathbb{A}_1}{\beta(2)^2(\gamma_q(2))^2}\right].$$

Thus, by using the modulus, we get

$$|\mathcal{D}_3| = \tfrac{(L-P)\nu}{4\mathbb{A}_1\gamma_q(3)}\left|c_2 - \tfrac{c_1^2}{2}\rho_3\right|,$$

where

$$\rho_3 = 1 - \tfrac{\nu}{2}\left((5-P) + \tfrac{(L-P)\sigma}{\beta(2)^2} - \tfrac{2(L-P)\mathbb{A}_1\gamma_q(3)}{\beta(2)^2(\gamma_q(2))^2}\right).$$

The following results are due to Lemma 2; that is,

$$|\mathcal{D}_3| \leq \frac{(L-P)\nu}{4\mathbb{A}_1\gamma_q(3)} \max\left\{2, \left|\nu\left[(5-P) + \frac{(L-P)\sigma}{\beta(2)^2} - \frac{2(L-P)\gamma_q(3)\mathbb{A}_1}{\beta(2)^2(\gamma_q(2))^2}\right]\right|\right\}.$$

From (34), after simplification, we have

$$\mathcal{D}_4 = \left(\tfrac{(L-P)\nu}{4\mathbb{A}_2\gamma_q(4)}\right)(\mathcal{O}_1 c_1^3 + \mathcal{O}_2 c_1 c_2 - c_3),$$

where

$$\mathcal{O}_1 = \frac{-5((L-P)\nu)^2\mathbb{A}_1\mathbb{A}_2\gamma_q(3)\gamma_q(4) - 10(L-P)\nu\rho\gamma_q(4)(\gamma_q(2)\beta(2)^2\mathbb{A}_2}{16\mathbb{A}_1(\gamma_q(2)\beta(2)^3\gamma_q(3)} + \frac{D(\alpha,q)}{\gamma_q(3)},$$

$$\mathcal{O}_2 = \frac{5(L-P)\nu\gamma_q(4)\mathbb{A}_2 + 4B(\alpha,q)\mathbb{A}_1\gamma_q(2)\beta(2)\gamma_q(3)}{\gamma_q(3)\gamma_q(2)\beta(2)\mathbb{A}_1}.$$

Taking $\mathcal{K} = \mathcal{O}_2/2$ and $\mathcal{Q} = -\mathcal{O}_1$, it is clear that $\mathcal{K} - \mathcal{Q} \geq 0$ and $\mathcal{Q} - \mathcal{K}(2\mathcal{K}-1) \geq 0$. Hence, using Lemma 3, we get

$$|\mathcal{D}_4| \leq \tfrac{(L-P)\,|\nu|}{2\mathbb{A}_3\gamma_q(4)}.$$

□

By taking $q \to 1,\ m = 0,\ b = \tau,$ in Theorem 6, we deduce the following corollary.

Corollary 5 ([28]). *If $f(z) \in \mathcal{R}_q^{\alpha}(b,\tau,0,L,P)$ and $-1 \leq P < L \leq 1$, then*

$$|\mathcal{D}_2| \leq \frac{(L-P)\,|\nu|}{2}$$

and

$$|\mathcal{D}_3| \leq \frac{(L-P)\,|\nu|}{8} \max\{2, \nu|3L-2P-5|\}.$$

Theorem 7. *If $f(z) \in \mathcal{R}_q^{\alpha}(b,\tau,m,L,P)$ and $-1 \leq P < L \leq 1$ with f^{-1} given in (28), then*

$$|\mathcal{D}_3 - \mathcal{M}\mathcal{D}_2^2| \leq \frac{(L-P)\,|\nu|}{4\gamma_q(3)\mathbb{A}_1} \max\left\{2, \left|\nu\left(P - 5 + \frac{(2-\mathcal{M})\gamma_q(3)\mathbb{A}_1(L-P)}{\beta(2)^2\gamma_q(2)^2} - \frac{(L-P)\sigma}{\beta(2)^2}\right)\right|\right\}.$$

Proof. Let $f \in \mathcal{R}_q^{\alpha}(b,\tau,m,L,P)$. Thus, in view of (26), we have

$$a_2 = \frac{h_1}{\beta(2)\gamma_q(2)} \quad \text{and} \quad a_3 = \frac{1}{\gamma_q(3)\mathbb{A}_1}\left(h_2 + \frac{\sigma h_1^2}{\beta(2)^2}\right).$$

From (29) and (30), we find the Fekete–Szegö inequality that

$$|\mathcal{D}_3 - \mathcal{M}\mathcal{D}_2^2| = \tfrac{1}{\gamma_q(3)\mathbb{A}_1}|h_2 - \rho_4 h_1^2|,$$

where

$$\rho_4 = \tfrac{1}{\beta(2)^2}\left[\tfrac{(2-\mathcal{M})\gamma_q(3)\mathbb{A}_1}{\gamma_q(2)^2} - \sigma\right].$$

Thus, the result follows according to Lemma 1

$$|\mathcal{D}_3 - \mathcal{M}\mathcal{D}_2^2| \leq \frac{(L-P)\,|\nu|}{4\gamma_q(3)\mathbb{A}_1} \max\left\{2, \left|\nu\left(P - 5 + \frac{(2-\mathcal{M})(L-P)\gamma_q(3)\mathbb{A}_1}{\beta(2)^2\gamma_q(2)^2} - \frac{(L-P)\sigma}{\beta(2)^2}\right)\right|\right\}.$$

This shows the result is sharp. □

By taking $q \to 1,\ m = 0,\ b = \tau,$ in Theorem 7, we deduce the following corollary.

Corollary 6 ([28]). *If $f(z) \in \mathcal{R}_q^{\alpha}(b, \tau, 0, L, P)$ and $-1 \leq P < L \leq 1$ with f^{-1} given in (28), then*

$$|\mathcal{D}_3 - \mathcal{M}\mathcal{D}_2^2| \leq \frac{(L-P)\,|\nu|}{8} \max\{2, |\nu(3L - 2P - 5 - 2\mathcal{M}(L-P))|\}.$$

Theorem 8. *If $f(z) \in \mathcal{R}_q^{\alpha}(b, \tau, m, L, P)$, then*

$$|\mathcal{D}_1\mathcal{D}_3 - \mathcal{D}_2^2| \leq \max\left\{2, 2\left|\frac{(L-P)\nu\mathbb{A}_1\gamma_q(3) + 2\beta(2)^2\gamma_q(2)^2(\rho-1)}{2\beta(2)^2\gamma_q(2)^2}\right|\right\}.$$

Proof. By using (29) and (30), we obtain

$$\mathcal{D}_1\mathcal{D}_3 - \mathcal{D}_2^2 = \frac{-(L-P)\nu}{4\mathbb{A}_1\gamma_q(3)}\left[c_2 - \frac{c_1^2}{2}\rho_5\right], \tag{35}$$

where

$$\rho_5 = \frac{(L-P)\nu\mathbb{A}_1\gamma_q(3) + 2\rho\beta(2)^2\gamma_q(2)^2}{2\beta(2)^2\gamma_q(2)^2}.$$

By applying Lemma 2 and the modulus to the above Equation (35), we get

$$|\mathcal{D}_1\mathcal{D}_3 - \mathcal{D}_2^2| \leq \max\left\{2, 2\left|\frac{(L-P)\nu\mathbb{A}_1\gamma_q(3) + 2\beta(2)^2\gamma_q(2)^2(\rho-1)}{2\beta(2)^2\gamma_q(2)^2}\right|\right\}.$$

□

Theorem 9. *If $f(z) \in \mathcal{R}_q^{\alpha}(b, \tau, m, L, P)$, then*

$$|\mathcal{D}_2\mathcal{D}_3 - \mathcal{D}_4| \leq \frac{(L-P)\,|\nu|}{2\mathbb{A}_2\gamma_q(4)}.$$

Proof. Using (29), (30) and (31), we have

$$\mathcal{D}_2\mathcal{D}_3 - \mathcal{D}_4 = \left(\frac{(L-P)\nu}{4\mathbb{A}_2\gamma_q(4)}\right)\left(-c_3 + G_1c_1c_2 + G_2c_1^3\right), \tag{36}$$

where

$$\begin{aligned} G_1 &= -\frac{5(L-P)\nu\gamma_q(4)\mathbb{A}_2 + 16B(\alpha,q)\mathbb{A}_1\gamma_q(2)\gamma_q(3)\beta(2) - (L-P)\nu\gamma_q(4)\mathbb{A}_1\mathbb{A}_2\beta(2)}{16\gamma_q(2)\gamma_q(3)\mathbb{A}_1}, \\ G_2 &= \frac{-5((L-P)\nu)^2\mathbb{A}_1\mathbb{A}_2\gamma_q(3)\gamma_q(4) - 10(L-P)\nu\mathbb{A}_2\gamma_q(4)\left(\gamma_q(2)\beta(2)\right)^2}{16\left(\gamma_q(2)\beta(2)^3\gamma_q(3)\mathbb{A}_1\right.} \\ &\quad + \frac{16\mathbb{A}_1\left(\gamma_q(2)\beta\left([2]_q\right)\right)^3\gamma_q(3)D(\alpha,q) + 2(L-P)\nu\rho\mathbb{A}_2\left(\gamma_q(2)\beta(2)\right)^2\gamma_q(4)}{16\left(\gamma_q(2)\beta(2)^3\gamma_q(3)\mathbb{A}_1\right.}. \end{aligned}$$

By making use of modulus in the resulting equation (36), we obtain

$$|\mathcal{D}_2\mathcal{D}_3 - \mathcal{D}_4| \leq \left(\frac{(L-P)\,|\nu|}{2\mathbb{A}_2\gamma_q(4)}\right)|c_3 - G_1c_1c_2 - G_2c_1^3|.$$

Taking $\mathcal{K} = G_1/2$ and $\mathcal{Q} = -G_2$. It is clear that $\mathcal{K} - \mathcal{Q} \geq 0$ and $\mathcal{Q} - \mathcal{K}(2\mathcal{K} - 1) \geq 0$. Hence, using Lemma 3, we get

$$|\mathcal{D}_2\mathcal{D}_3 - \mathcal{D}_4| \leq \frac{(L-P)\,|\nu|}{2\mathbb{A}_2\gamma_q(4)}.$$

□

4. Logarithmic Coefficients for the Class $\mathcal{R}_q^{\alpha}(b, \tau, m, L, P)$

For $f \in \mathcal{S}$, the logarithmic coefficient t_r is defined as

$$\frac{1}{2}\log\left(\frac{f(z)}{z}\right) = \sum_{r=1}^{\infty} t_r z^r, \tag{37}$$

t_r represents the logarithmic coefficients of f, which are crucial for researching univalent functions. Logarithmic coefficient inputs for the Hankel determinant are a logical consideration. The Hankel determinant was first introduced by Kowalczyk et al. in [47,48] using logarithmic coefficients and deriving

$$\mathcal{Y}_{k,r}\left(F_f/2\right) := \begin{vmatrix} t_r & t_{r+1} & \cdots & t_{r+k-1} \\ t_{r+1} & t_{r+2} & \cdots & t_{r+k} \\ t_{r+2} & t_{r+3} & \cdots & t_{r+k+1} \\ \vdots & \vdots & \ddots & \vdots \\ t_{r+k-1} & t_{r+k} & \cdots & t_{r+2(k-1)} \end{vmatrix}.$$

In particular, it is mentioned that

$$\mathcal{Y}_{2,1}\left(F_f/2\right) = \begin{vmatrix} t_1 & t_2 \\ t_2 & t_3 \end{vmatrix} = \left|t_1 t_3 - t_2^2\right|.$$

For other findings about the logarithmic coefficients, see [49–51].

Note that (1) provides the logarithmic coefficients of f, which are given in the following form

$$t_1 = \frac{1}{2}a_2, \tag{38}$$

$$t_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right), \tag{39}$$

$$t_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right). \tag{40}$$

We state and prove the following theorems.

Theorem 10. *If $f(z) \in \mathcal{R}_q^{\alpha}(b, \tau, m, L, P)$, then*

$$|t_1| \leq \frac{(L-P)\,|\nu|}{4\beta(2)\gamma_q(2)},$$

$$|t_2| \leq \frac{(L-P)\,|\nu|}{64\gamma_q(3)\mathbb{A}_1}\max\left\{2, 2\left|\frac{\left(4\left(\beta(2)\gamma_q(2)\right)^2\rho + (L-P)\gamma_q(3)\mathbb{A}_1\nu\right)}{4\left(\beta(2)\gamma_q(2)\right)^2} - 1\right|\right\},$$

$$|t_3| \leq \frac{(L-P)\nu\beta(2)\gamma_q(2)}{4\mathbb{A}_2}.$$

Proof. By using (14), (15) and (16) in (38), (39) and (40), we have

$$t_1 = \frac{(L-P)\nu c_1}{8\beta(2)\gamma_q(2)}, \tag{41}$$

$$t_2 = \frac{(L-P)\nu}{64\gamma_q(3)\mathbb{A}_1}\left(c_2 - \frac{\left(4\left(\beta(2)\gamma_q(2)\right)^2\rho+(L-P)\gamma_q(3)\mathbb{A}_1\nu\right)}{8\left(\beta(2)\gamma_q(2)\right)^2}c_1^2\right), \tag{42}$$

$$t_3 = \frac{(L-P)\nu\beta(2)\gamma_q(2)}{8\mathbb{A}_2}\left(c_3 - \frac{M_2c_1c_2}{4\gamma_q(3)\gamma_q(4)\gamma_q(2)\beta(2)\mathbb{A}_1} + \frac{M_1c_1^3}{48\left(\beta(2)\gamma_q(2)\right)^3\gamma_q(3)\gamma_q(4)\mathbb{A}_1}\right), \tag{43}$$

where M_1 and M_2 are given by

$$\begin{aligned} M_1 = {} & ((L-P)\nu)^3\mathbb{A}_1\mathbb{A}_2\gamma_q(3)\gamma_q(4) + 6((L-P)\nu)^2\rho\mathbb{A}_2\gamma_q(4)\left(\beta(2)\gamma_q(2)\right)^2 \\ & -48(L-P)\nu D(\alpha,q)\mathbb{A}_1\mathbb{A}_2\gamma_q(2)\gamma_q(3)\gamma_q(4)\beta(2)^3, \\ M_2 = {} & 4(L-P)\nu B(\alpha,q)\mathbb{A}_1\gamma_q(2)\gamma_q(3)\beta(2) + ((L-P)\nu)^2\mathbb{A}_2\gamma_q(4). \end{aligned}$$

By applying Lemma 2 and modulus in (41), we get

$$|t_1| \leq \frac{(L-P)\,|\nu|}{4\beta(2)\gamma_q(2)}.$$

From (42), this can be written as

$$t_2 = \frac{(L-P)\nu}{64\gamma_q(3)\mathbb{A}_1}\left(c_2 - \frac{c_1^2}{2}\rho_6\right),$$

where

$$\rho_6 = \frac{\left(4\left(\beta(2)\gamma_q(2)\right)^2\rho+(L-P)\gamma_q(3)\mathbb{A}_1\nu\right)}{4\left(\beta(2)\gamma_q(2)\right)^2}$$

and, taking the modulus, we have

$$|t_2| \leq \frac{(L-P)\,|\nu|}{64\gamma_q(3)\mathbb{A}_1}\max\left\{2, 2\left|\frac{\left(4\left(\beta(2)\gamma_q(2)\right)^2\rho+(L-P)\gamma_q(3)\mathbb{A}_1\nu\right)}{4\left(\beta(2)\gamma_q(2)\right)^2} - 1\right|\right\}.$$

Again, from (43), taking $\mathcal{K} = \frac{M_2}{8\gamma_q(3)\gamma_q(4)\gamma_q(2)\beta(2)\mathbb{A}_1}$ and $\mathcal{Q} = \frac{M_1}{48\left(\beta(2)\gamma_q(2)\right)^3\gamma_q(3)\gamma_q(4)\mathbb{A}_1}$. It is clear that $\mathcal{K} - \mathcal{Q} \geq 0$ and $\mathcal{Q} - \mathcal{K}(2\mathcal{K}-1) \geq 0$. Hence, by Lemma 3, we get

$$|t_3| \leq \frac{(L-P)\nu\beta(2)\gamma_q(2)}{4\mathbb{A}_2}.$$

□

Theorem 11. *If* $f(z) \in \mathcal{R}_q^{\alpha}(b,\tau,m,L,P)$, *then*

$$|t_2 - t_1^2| \leq \left(\frac{(L-P)\,|\nu|}{8}\right)\max\left\{2, 2\left|\left[\rho - 1 + \frac{5(L-P)\mathbb{A}_1\gamma_q(3)\nu}{4\left(\beta(2)\gamma_q(2)\right)^2}\right]\right|\right\}.$$

Proof. By using (41) and (42), we get

$$t_2 - t_1^2 = \left(\frac{(L-P)\nu}{8}\right)\left[c_2 - \frac{c_1^2}{2}\rho_7\right], \tag{44}$$

where

$$\rho_7 = \left(\rho + \frac{5(L-P)\mathbb{A}_1\gamma_q(3)\nu}{4\left(\beta(2)\gamma_q(2)\right)^2}\right).$$

By applying Lemma 2 and modulus to the above equation (44), we obtain

$$|t_2 - t_1^2| \leq \left(\frac{(L-P)\,|\nu|}{8}\right) \max\left\{2, 2\left|\left[\rho - 1 + \frac{5(L-P)\mathbb{A}_1\gamma_q(3)\nu}{4\left(\beta(2)\gamma_q(2)\right)^2}\right]\right|\right\}.$$

□

Theorem 12. *If* $f(z) \in \mathcal{R}_q^{\alpha}(b, \tau, m, L, P)$, *then*

$$|t_3 - t_1 t_2| \leq \left|\frac{(L-P)\nu\beta(2)\gamma_q(2)}{4\mathbb{A}_2}\right|.$$

Proof. By using (41), (42) and (43), we obtain

$$t_3 - t_1 t_2 = \left(\frac{(L-P)\nu\beta(2)\gamma_q(2)}{8\mathbb{A}_2}\right)\left[c_3 - F_1 c_1 c_2 + F_2 c_1^3\right], \tag{45}$$

where

$$F_1 = \frac{12M_2\beta(2)\gamma_q(2) + 6(L-P)\nu\mathbb{A}_2\gamma_q(4)}{48\mathbb{A}_2\gamma_q(3)\gamma_q(4)\left(\beta(2)\gamma_q(2)\right)^2},$$

$$F_2 = \frac{M_1\left(\beta(2)\gamma_q(2)\right) + (L-P)\nu\gamma_q(4)\mathbb{A}_2\left[3\left(\beta(2)\gamma_q(2)\right)^2\rho + (L-P)\mathbb{A}_1\gamma_q(3)\nu\right]}{48\mathbb{A}_1\gamma_q(3)\gamma_q(4)\left(\beta(2)\gamma_q(2)\right)^4}.$$

Taking the modulus in above Equation (45), we get

$$|t_3 - t_1 t_2| \leq \left|\frac{(L-P)\nu\beta(2)\gamma_q(2)}{8\mathbb{A}_2}\right| |c_3 - F_1 c_1 c_2 + F_2 c_1^3|.$$

Taking $\mathcal{K} = F_1/2$ and $\mathcal{Q} = F_2$, it is clear that $\mathcal{K} - \mathcal{Q} \geq 0$ and $\mathcal{Q} - \mathcal{K}(2\mathcal{K} - 1) \geq 0$. Hence, by Lemma 3, we get

$$|t_3 - t_1 t_2| \leq \left|\frac{(L-P)\nu\beta(2)\gamma_q(2)}{4\mathbb{A}_2}\right|.$$

□

Theorem 13. *If* $f(z) \in \mathcal{R}_q^{\alpha}(b, \tau, m, L, P)$, *then*

$$|t_1 t_3 - t_2^2| \leq \frac{(L-P)\nu F_4}{16}.$$

Proof. By using (41), (42) and (43), we obtain the second Hankel determinant

$$\mathcal{Y}_{2,1}\left(F_f/2\right) = t_1 t_3 - t_2^2.$$

That is,

$$t_1 t_3 - t_2^2 = \left(\frac{(L-P)\nu}{3072}\right)\left(F_3 c_1^4 - 48F_4 c_2^2 - 12F_5 c_1^2 c_2 + 48F_6 c_1 c_3\right), \tag{46}$$

where,

$$F_3 = \frac{M_1 - \frac{3}{4}\left((L-P)\nu\beta(2)\gamma_q(2)\gamma_q(4)\mathbb{A}_2\right)\left(4\left(\beta(2)\gamma_q(2)\right)^2\rho + (L-P)\nu\gamma_q(3)\mathbb{A}_1\right)^2}{\left(\beta(2)\gamma_q(2)\right)^3\gamma_q(3)\gamma_q(4)\mathbb{A}_1\mathbb{A}_2},$$

$$F_4 = \frac{\left((L-P)\nu\beta(2)\gamma_q(2)\right)^2}{\mathbb{A}_1\gamma_q(3)},$$

$$F_5 = (L-P)\nu\left(\frac{M_2 - (L-P)\nu\mathbb{A}_2\gamma_q(2)\gamma_q(4)(\beta(2))\left(4\beta(2)\left(\gamma_q(2)\right)^2\rho + (L-P)\nu\mathbb{A}_1\gamma_q(3)\right)}{\mathbb{A}_1\mathbb{A}_2\gamma_q(2)\gamma_q(3)\gamma_q(4)\beta(2)}\right),$$

$$F_6 = \frac{(L-P)\nu}{\mathbb{A}_2}.$$

Assuming that $c_1 = c \in [0,2]$ is feasible because of the rotation-invariant characteristic for the family $\mathcal{R}_q^{\alpha}(b,\tau,m,L,P)$ and determinant $\mathcal{Y}_{2,1}\left(F_f/2\right)$, the resulting Equation (46) is then solved by applying the modulus. Lemma 4 allows us to express the coefficients c_2 and c_3 in terms of c_1; thus, we have

$$\begin{aligned}|\mathcal{Y}_{2,1}\left(F_f/2\right)| \quad &= \tfrac{(L-P)\,|\nu|}{3072} \\ &|-(6F_5+12F_4-12F_6-F_3)c^4 - \left(12\left(4-c^2\right)F_6\right)r^2c^2 - \left(12\left(4-c^2\right)^2F_4\right)r^2 \\ &+\left(6F_5\left(4-c^2\right) - 24F_4\left(4-c^2\right) + 24F_6\left(4-c^2\right)\right)rc^2 + 24F_6\left(4-c^2\right)(1-|r|^2)c\delta|.\end{aligned}$$

For $c = 0$, it is clear that $|\mathcal{Y}_{2,1}\left(F_f/2\right)| \leq \frac{(L-P)\,|\nu|}{16}$, and for $c = 2$,

$$|\mathcal{Y}_{2,1}\left(F_f/2\right)| \leq \tfrac{(L-P)\,|\nu|}{192}|6F_5 + 12F_4 - 12F_6 - F_3|.$$

Utilizing $|\delta| \leq 1$ for the case $c \in (0,2)$, then

$$\begin{aligned}|\mathcal{Y}_{2,1}\left(F_f/2\right)| \leq \quad &\tfrac{F_6\left(4-c^2\right)(L-P)\nu c}{128}\Big(\Big|-\tfrac{6F_5+12F_4-12F_6-F_3}{24F_6\left(4-c^2\right)}c^3 + \tfrac{F_5-4F_4+4F_6}{4F_6}c\ r \\ &-\tfrac{F_6c^2+F_4\left(4-c^2\right)}{2F_6c}r^2\Big| + 1 - |r|^2\Big).\end{aligned}$$

The above inequality can now be rewritten in terms of $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$, in order to apply Lemma 5

$$|\mathcal{Y}_{2,1}\left(F_f/2\right)| \leq \tfrac{F_6\left(4-c^2\right)(L-P)\nu c}{128}\Psi(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3),$$

where

$$\Psi(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3) = |\mathcal{G}_1 + \mathcal{G}_2 r + \mathcal{G}_3 r^2| + 1 - |r|^2,$$

with

$$\mathcal{G}_1 = -\frac{6F_5+12F_4-12F_6-F_3}{24F_6(4-c^2)}c^3,$$

$$\mathcal{G}_2 = \frac{F_5-4F_4+4F_6}{4F_6}c,$$

$$\mathcal{G}_3 = -\frac{F_6c^2+F_4\left(4-c^2\right)}{2F_6c}.$$

It is evident that the maximum of Ψ and $\mathcal{G}_1\mathcal{G}_3 \geq 0$ can be determined by employing Lemma 5. Note that $|\mathcal{G}_2| \geq 2(1-|\mathcal{G}_3|)$ is equal to

$$\wp(c,\alpha) = \tfrac{(F_5-4F_4+4F_6)c^2 - 4\left(2F_6c - F_6c^2 - F_4\left(4-c^2\right)\right)}{4F_6c} \geq 0.$$

To prove that $\wp(c,\alpha) \geq 2$, we must show that the minima of $\wp(c,\alpha)$ is positive for every $c \in [0,2]$ and $0 \leq \alpha \leq 1$. We can quickly ascertain using simple math that

$$\min \wp(c,\alpha) = \wp(2,\alpha) = \tfrac{F_5 - 4F_4 + 4F_6}{2F_6} > 0.$$

Using Lemma 5, we get

$$\Psi(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = (|\mathcal{G}_1| + |\mathcal{G}_2| + |\mathcal{G}_3|),$$

and thus

$$\begin{aligned} |\mathcal{Y}_{2,1}\Big(F_f/2\Big)| &\leq \tfrac{F_6\left(4-c^2\right)(L-P)\nu c}{128}(|\mathcal{G}_1| + |\mathcal{G}_2| + |\mathcal{G}_3|) \\ &= \tfrac{(L-P)\nu}{128}\Big(\tfrac{48F_4 - 48F_6 - F_3}{3072}c^4 + \tfrac{F_5 - 8F_4 + 6F_6}{128}c^2 + \tfrac{F_4}{16}\Big) \\ &= \varkappa(c). \end{aligned}$$

The maxima value of $\varkappa$ is easily found from $\frac{(L-P)\nu F_4}{16}$ at $c = 0$, then

$$|\mathcal{Y}_{2,1}\Big(F_f/2\Big)| \leq \tfrac{(L-P)\nu F_4}{16}.$$

Thus, the desired inequality is established. □

5. Conclusions

In this investigation, using convolution, we have defined a new q-analog of the generalized integral operator. We also have defined q-analogs of the generalized integral operator of the α-convex subclass of univalent functions in the open unit disk $\mathcal{U}_s$ which are associated with the cardioid domain. Further, we have determined upper bounds for the coefficient a_2, a_3 and a_4 of functions in the class $\mathcal{R}_q^{\alpha}(b,\tau,m,L,P)$, and used them to find the maximum value of the Fekete–Szegö inequality and second Hankel determinants for this class. Additionally, our findings set the upper bounds for the inverse function f^{-1} and logarithmic functions for the newly defined subclass $\mathcal{R}_q^{\alpha}(b,\tau,m,L,P)$, including geometric aspects like the coefficient bounds, inequalities, and second Hankel determinants. Interested researchers can use (p,q) quantum calculus operators to generalize the main results of this investigation, paving the way for applications to growth and distortion results.

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