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Article

A Generalization of the Fractional Stockwell Transform

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Abstract: This paper presents a generalized fractional Stockwell transform (GFST), extending the classical Stockwell transform and fractional Stockwell transform, which are widely used tools in time–frequency analysis. The GFST on $L^2(\mathbb{R}, \mathbb{C})$ is defined as a convolution consistent with the classical Stockwell transform, and the fundamental properties of GFST such as linearity, translation, scaling, etc., are discussed. We focus on establishing an orthogonality relation and derive an inversion formula as a direct application of this relation. Additionally, we characterize the range of the GFST on $L^2(\mathbb{R}, \mathbb{C})$. Finally, we prove an uncertainty principle of the Heisenberg type for the proposed GFST.

Keywords: fractional stockwell transform; generalized fractional Stockwell tranform; special affine Fourier transform; Stockwell transform; convolution; uncertainty principle

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1. Introduction

It is well known that the Fourier transform is an efficient tool in signal processing. To address the limitations of the Fourier transform, researchers have explored various integral transforms that extend the classical Fourier transform. In [1], Namias introduced the fractional Fourier transform, which depends on a real parameter α. When $\alpha = \frac{\pi}{2}$, this fractional Fourier transform reduces to the Fourier transform. The fractional Fourier transform has applications in optics [2], quantum mechanics [1,3], and signal processing [4]. The linear canonical transform [5], a generalization of both the Fourier and fractional Fourier transforms, is used in quantum mechanics [6] and optical systems [7].

The special affine Fourier transform (SAFT), introduced by Abe and Sheridan [8–10], stands out as a more general and versatile approach among the generalizations of the Fourier transform. The SAFT presents a time–frequency representation of a signal in quantum mechanics and optics, characterized by six real parameters a, b, c, d, p, and q, with $ad - bc = 1$ and $b \neq 0$. Notably, the SAFT includes the generalizations of the Fourier transform, such as fractional Fourier transform [1,2] and linear canonical transform [5,11], as special cases determined by specific parameter choices. It also includes pivotal transforms such as Laplace, Fresnel [12], Bargmann, Gauss–Weierstrass, offset Fourier, and offset fractional Fourier [13] within its framework. For more details, one can refer to [14]. Also, the special affine Fourier transform has been discussed on quaternion-valued functions [15] and on octonian-valued functions [16]. Using the SAFT, there are many hybrid transforms in the literature—for instance, see windowed SAFT [17], short-time SAFT [18], special affine wavelet transform [19], and special affine Stockwell transform [20].

In this paper, we introduce a hybrid integral transform involving SAFT and Stockwell transform. We now briefly recall the theory of Stockwell transform from the literature. Stockwell et al. [21] introduced the Stockwell transform, commonly referred as the S transform, which employs a Gaussian window function that dilates and translates to achieve localization of the signal $f \in L^2(\mathbb{R})$. For a one-dimensional signal f, it is defined as follows.

$$S(\tau,t) = \int_{\mathbb{R}} f(x)\frac{|t|}{\sqrt{2\pi}} e^{-\frac{(\tau-x)^2 t^2}{2}} e^{-i2\pi tx}\, dx,\ \forall(\tau,t) \in \mathbb{R}\times\mathbb{R}^*.$$

In 2008, Du et al. [22] introduced a Stockwell transform using a general window function $\phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ instead of a Gaussian function and obtained an inversion formula. The Stockwell transform with a general window function is given as followers:

$$S_\phi(b,\xi) = \frac{|\xi|}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x)\overline{\phi(\xi(x-b))}\, dx,\ \forall(b,\xi) \in \mathbb{R}\times\mathbb{R}^*. \quad (1)$$

The Stockwell transform is widely applied in geophysics [23] to provide more detailed information about spectral (frequency) components.

Hybrid versions of the S transform with the kernels of the generalized Fourier transforms have been extensively studied. Xu and Guo [24] extended the classical Stockwell transform by introducing the fractional S transform, which incorporates a single real parameter. Du et al. [25] also explored the applications of fractional S transform in seismic data analysis. The paper in [26] introduced a fractional Stockwell transform with a general window function instead of a particular window function as in [24,25]. More recently, the fractional S transform has been investigated in higher-dimensional Euclidean spaces and quaternion function spaces [27,28]. In this direction, the linear canonical Stockwell transform (LCST) [29–32] and offset linear canonical Stockwell transform [33] have also been recently investigated.

This paper aims to develop a novel integral transform called generalized fractional Stockwell transform (GFST), by utilizing the kernel of the special affine Fourier transform (SAFT). Hence, the transform may also be referred as the special affine Stockwell transform (SAST). To achieve this, we apply the special affine convolution introduced in [18]. Using the SAFT kernel in the GFST definition, the transform gains greater flexibility and broadens the applicability of the Stockwell transform across various fields. It is worth noting that [20] addresses a form of the special affine Stockwell transform. As one of the SAFT parameters, q, is not included and the parameter p is involved only in the constant factor $e^{\frac{i}{2B}Dp^2}$, the transform introduced in [20] is essentially a linear canonical S transform with an additional constant factor involving p.

The structure of this paper is organized as follows. Section 2 introduces the fundamental notations, definitions, and results. In Section 3, the definition of the generalized fractional Stockwell transform (GFST) and its key theorems are presented. This section emphasizes proving Parseval's identity and, based on that, deriving the inversion formula for the GFST. Additionally, a theorem that characterizes the range of the GFST is included. Section 4 then addresses another key topic in the paper: the uncertainty principle. In this section, we demonstrate the uncertainty principle related to the GFST, using the uncertainty principle of the special affine Fourier transform and several supporting lemmas.

2. Preliminaries

Throughout this article, the symbol $\mathbf{A}$ denotes the six real parameters a, b, c, d, p, and q such that $ad - bc = 1$ with $b \neq 0$ and the symbol $\boldsymbol{\lambda}$ denotes the number $bq - dp$. We denote any $\mathbf{A}$ that satisfies the above-stated condition as the parameter matrix with a representation

$\mathbf{A} = \left(\begin{array}{cc|c} a & b & p \\ c & d & q \end{array}\right)$. Furthermore, we make use of the following notations from the paper [18] for easier computation.

$$\mu_{p,b}(x) = \exp\left(i\frac{p}{b}x\right) \quad ; \quad \Omega_{\mathbf{A}}(x) = \exp\left(\frac{i}{2b}[dx^2 + 2\lambda x]\right), \tag{2}$$

$$\Phi_{a,b}(x) = \exp\left(i\frac{a}{2b}x^2\right) \quad ; \quad \Psi_{\mathbf{A}}(x) = \exp\left(\frac{i}{2b}[dx^2 - 2px + 2\lambda x]\right), \tag{3}$$

As usual, the set of reals, non-zero reals, and complex numbers are denoted by $\mathbb{R}, \mathbb{R}^*$, and $\mathbb{C}$, respectively. The space of integrable and square-integrable complex valued functions on $\mathbb{R}$ are given by $L^1(\mathbb{R}, \mathbb{C})$ and $L^2(\mathbb{R}, \mathbb{C})$, respectively, with the norm and inner product

$$\|f\|_1 = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |f(x)|\, dx \,; \; \langle f, h\rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\overline{h(x)}\, dx.$$

We denote the Hilbert space of square-integrable complex valued functions on $\mathbb{R} \times \mathbb{R}^*$ by $L^2(\mathbb{R} \times \mathbb{R}^*, \mathbb{C})$ with the inner product given by

$$\langle F, H\rangle = \frac{1}{2\pi} \int_{\mathbb{R}^*} \int_{\mathbb{R}} F(y,\xi)\overline{H(y,\xi)}\, dy \frac{d\xi}{|\xi|}.$$

For $\kappa > 0$ and $t \in \mathbb{R}$, the dilation and translation of a function f are given by

$$(D_\kappa f)(x) = \kappa f(\kappa x) \,; \; (T_t f)(x) = f(x - t),$$

respectively.

Next, we recall necessary definitions and required results. If the Fourier transform of f is given by

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-ix\xi}\, dx, \; \forall \xi \in \mathbb{R},$$

then we have the following well-known results [34]:

1. If $g(x) = \check{\bar{f}}(x) = \overline{f(-x)}$, then $\hat{g}(\xi) = \overline{\hat{f}(\xi)}$.
2. If $g(x) = f(\kappa x)$ and $\kappa > 0$, then $\hat{g}(\xi) = (D_{1/\kappa}\hat{f})(\xi)$.

According to Ref. [14], for $f \in L^1(\mathbb{R}, \mathbb{C})$, the special affine Fourier transform (SAFT) of f, denoted by $\mathcal{S}_{\mathbf{A}} f$, is given by

$$(\mathcal{S}_{\mathbf{A}} f)(y) = \int_{\mathbb{R}} f(x) K_{\mathbf{A}}(x,y)\, dx, \; y \in \mathbb{R},$$

where $K_{\mathbf{A}}(x,y) = \frac{1}{\sqrt{2\pi|b|}}\, \Phi_{a,b}(x)\, \Omega_{\mathbf{A}}(y)\, \mu_{(p-y),b}(x)$.

Notably, if $\mathbf{A}^\star = \left(\begin{array}{cc|c} d & -b & bq - dp \\ -c & a & cp - aq \end{array}\right)$, then $\mathbf{A}^\star$ is also a parameter matrix whenever $\mathbf{A}$ is a parameter matrix and the inverse of $\mathcal{S}_{\mathbf{A}}$ is given by $\mathcal{S}_{\mathbf{A}^\star}$. Precisely, the inversion formula for SAFT is given as follows.

Theorem 1 (Inversion Formula for SAFT [10])**.** *If $f, \mathcal{S}_{\mathbf{A}} f \in L^1(\mathbb{R}, \mathbb{C})$, then*

$$f(x) = \int_{\mathbb{R}} (\mathcal{S}_{\mathbf{A}} f)(y) K_{\mathbf{A}^\star}(y,x)\, dy = \int_{\mathbb{R}} (\mathcal{S}_{\mathbf{A}} f)(y)\overline{K_{\mathbf{A}}(x,y)}\, dy \; a.e \text{ on } \mathbb{R}.$$

Theorem 2 (Parseval's identity for SAFT [14]). *The special affine Fourier transform $\mathcal{S}_{\mathbf{A}}$ on $L^2(\mathbb{R},\mathbb{C})$ satisfies the following:*

$$\langle f,g\rangle_{L^2(\mathbb{R},\mathbb{C})} = \langle \mathcal{S}_{\mathbf{A}}f,\mathcal{S}_{\mathbf{A}}g\rangle_{L^2(\mathbb{R}^2,\mathbb{C})},$$

$\forall f,g \in L^2(\mathbb{R}^2,\mathbb{C})$.

Definition 1 ([18]). *Suppose that $f \in L^p(\mathbb{R},\mathbb{C})$, $p = 1$ or 2, $g \in L^1(\mathbb{R},\mathbb{C})$, and $\mathbf{A}$ is a parameter matrix. We define a novel special affine convolution or a generalized fractional convolution as follows.*

$$(f \circledast_{\mathbf{A}} g)(x) = \frac{\overline{\Phi_{a,b}(x)}}{\sqrt{|b|}}\big(\Phi_{a,b}f * \Psi_{\mathbf{A}}g\big)(x),\ \forall x \in \mathbb{R},$$

where $(h_1 * h_2)(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} h_1(y)h_2(x-y)\,dy$.

Theorem 3 (Convolution Theorem for SAFT [18]). *Let $f \in L^p(\mathbb{R},\mathbb{C})$, $p = 1,2$ and $g \in L^1(\mathbb{R},\mathbb{C})$. Then, $\mathcal{S}_{\mathbf{A}}(f \circledast_{\mathbf{A}} g) = \overline{\Omega_{\mathbf{A}}}\ (\mathcal{S}_{\mathbf{A}}f)\ (\mathcal{S}_{\mathbf{A}}g_{\mathbf{A}})$, where $g_{\mathbf{A}} = g\ \overline{\Phi_{a,b}}\ \Psi_{\mathbf{A}}$.*

3. Generalized Fractional Stockwell Transform

For a given parameter matrix $\mathbf{A}$, $(y,\xi) \in \mathbb{R}\times\mathbb{R}^*$, and $0 \neq g \in L^1(\mathbb{R},\mathbb{C}) \cap L^2(\mathbb{R},\mathbb{C})$, we let $g^{\mathbf{A}}_{y,\xi}(x) = |\xi|\ g(\xi(x-y))\ \Phi_{a,b}(y)\ \overline{\Phi_{a,b}(x)}\ \mu_{p-\lambda,b}(y-x)\ \mu_{\xi,b}(x)$, $\forall x \in \mathbb{R}$, and let $\boldsymbol{\lambda} = bq - dp$.

Definition 2. *Let $0 \neq g \in L^1(\mathbb{R},\mathbb{C}) \cap L^2(\mathbb{R},\mathbb{C})$ and $f \in L^2(\mathbb{R},\mathbb{C})$. For a parameter matrix $\mathbf{A}$, we define the generalized fractional Stockwell transform (GFST) of f as follows.*

$$\begin{aligned}(\mathcal{S}^s_{\mathbf{A},g}f)(y,\xi) &\\ &= \frac{1}{\sqrt{2\pi|b|}}\int_{\mathbb{R}} f(x)\overline{g^{\mathbf{A}}_{y,\xi}(x)}\,dx,\ \forall (y,\xi) \in \mathbb{R}\times\mathbb{R}^*,\\ &= \frac{|\xi|}{\sqrt{2\pi|b|}}\int_{\mathbb{R}} f(x)\overline{g(\xi(x-y))}\exp\Big(\tfrac{-i}{2b}\big[a(y^2-x^2)+2((p-[bq-dp])(y-x)+\xi x)\big]\Big)\,dx\\ &\quad \forall (y,\xi) \in \mathbb{R}\times\mathbb{R}^*.\end{aligned}$$

Remark 1. *Applying the generalized fractional convolution introduced in Definition 1, the generalized fractional Stockwell transform can also be rewritten as follows.*

1. $(\mathcal{S}^s_{\mathbf{A},g}f)(y,\xi) = [\mu_{-\xi,b}f \circledast_{\mathbf{A}} \overline{\Phi_{d,b}}D_{\xi}\check{\bar{g}}](y),\ \forall (y,\xi) \in \mathbb{R}\times\mathbb{R}^*.$
2. $(\mathcal{S}^s_{\mathbf{A},g}f)(y,\xi) = [\mathcal{S}_A(F)](y)\overline{\Phi_{a+d}(y)\mu_{p,b}(y)},\ \forall (y,\xi) \in \mathbb{R}\times\mathbb{R}^*,$
   where $F(t) = \mu_{y-\xi-\lambda,b}(t)(T_y(D_{\xi}\check{\bar{g}}))(t)$.

Remark 2. *For particular choices of the matrix $\mathbf{A}$, the generalized fractional Stockwell transform reduces to the existing transforms in the literature.*

1. *If* $\mathbf{A} = \left(\begin{array}{cc|c} 0 & 1 & 0\\ 1 & 0 & 0\end{array}\right)$, *then $\mathcal{S}^s_{\mathbf{A},g}$ is the classical Stockwell transform given in (1).*
2. *If* $\mathbf{A} = \left(\begin{array}{cc|c} \cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\end{array}\right)$, *where $\theta \in \mathbb{R}$, then $\mathcal{S}^s_{\mathbf{A},g}$ is the fractional Stockwell transform [26], with a constant factor given by*

$$(\mathcal{F}^s_{\theta,g}f)(y,z) = \frac{|z|}{\sqrt{2\pi|\sin\theta|}}e^{-i\frac{\cot\theta}{2}y^2}\int_{\mathbb{R}} f(x)\overline{g(z(x-y))}e^{i\frac{\cot\theta}{2}x^2}e^{-i\csc\theta zx}\,dx,$$

   where $(y,z) \in \mathbb{R}\times\mathbb{R}^*$.

3. *If* $\mathbf{A} = \left(\begin{array}{cc|c} a & b & 0 \\ c & d & 0 \end{array}\right)$*, where* $a, b, c, d \in \mathbb{R}$ *with* $b \neq 0$*, then we obtain the linear canonical Stockwell transform [30], which is given below. For* $(y, z) \in \mathbb{R} \times \mathbb{R}^*$*,*

$$(\mathcal{L}^s_{\mathbf{A},g} f)(y,z) = \frac{1}{\sqrt{2\pi|b|}} \overline{\Phi_{a,b}(y)} \int_{\mathbb{R}} f(x)|z|\overline{g(z(x-y))}\, \Phi_{a,b}(x)\, \mu_{-z,b}(x)dx,$$

where g is a Gaussian window function.

Lemma 1 (Properties of $\mathcal{S}^s_{\mathbf{A},g}$)**.** *Let* $0 \neq g \in L^1(\mathbb{R}, \mathbb{C}) \cap L^2(\mathbb{R}, \mathbb{C})$*. If* $f_1, f_2 \in L^2(\mathbb{R}, \mathbb{C})$*,* $z \in \mathbb{C}$*, and* $0 \neq \kappa, t \in \mathbb{R}$*, then*

1. *Linearity: The generalized fractional Stockwell transform is a linear map on* $L^2(\mathbb{R}, \mathbb{C})$*.*
2. *Translation:*
$[\mathcal{S}^s_{\mathbf{A},g}(T_t f)](y,\xi) = \overline{\mu_{y+\xi,b}(t)}\, \Phi_{a,b}(\sqrt{2}t)[\mathcal{S}^s_{\mathbf{A},g}(\mu_{t,b} f)](y-t,\xi)$.
3. *Parity: If* $\check{f}(x) = f(-x)$*, then* $[\mathcal{S}^s_{\check{\mathbf{A}},g}\check{f}](y,\xi) = [\mathcal{S}^s_{\check{\mathbf{A}},g} f](-y,-\xi)$*, where* $\check{\mathbf{A}} = \left[\begin{array}{cc|c} a & b & -p \\ c & d & -q \end{array}\right]$*,*
4. *Scaling:*
If $\Theta_{\mathbf{A},\kappa}(x) = \exp\left(\frac{i}{2b}\left[2\lambda\left(\frac{1}{\kappa} - \kappa\right)x\right]\right)$ *and* $\mathbf{A}_\kappa = \left[\begin{array}{cc|c} \frac{a}{\kappa^2} & b & \frac{p}{\kappa} \\ c & d\kappa^2 & q\kappa \end{array}\right]$*, then*
$$[\mathcal{S}^s_{\mathbf{A},g}(D_\kappa f)](y,\xi) = k\, \Theta_{\mathbf{A},\kappa}(\kappa y)\, [\mathcal{S}^s_{\mathbf{A}_\kappa,g}(\Theta_{\mathbf{A},\kappa} f)](\kappa y, \xi/\kappa),\ \forall (y,\xi) \in \mathbb{R} \times \mathbb{R}^*.$$

Proof. Let $z \in \mathbb{C}$ and $0 \neq \kappa, t \in \mathbb{R}$.

1. One can observe that the special affine convolution $\circledast_{\mathbf{A}}$ satisfies
$(f_1 + f_2) \circledast_{\mathbf{A}} f_3 = (f_1 \circledast_{\mathbf{A}} f_3) + (f_2 \circledast_{\mathbf{A}} f_3)$ and $(zf_1) \circledast_{\mathbf{A}} f_2 = z(f_1 \circledast_{\mathbf{A}} f_2)$,
for all $f_1, f_2 \in L^2(\mathbb{R}, \mathbb{C})$, $f_3 \in L^1(\mathbb{R}, \mathbb{C}) \cap L^2(\mathbb{R}, \mathbb{C})$, and $z \in \mathbb{C}$. Using these facts, we obtain that the GFST is a linear map on $L^2(\mathbb{R}, \mathbb{C})$.
Assume that $f \in L^1(\mathbb{R}, \mathbb{C}) \cap L^2(\mathbb{R}, \mathbb{C})$. Since $L^1(\mathbb{R}, \mathbb{C})) \cap L^2(\mathbb{R}, \mathbb{C})$ is dense in $L^2(\mathbb{R}, \mathbb{C})$ and the SAFT is continuous on $L^2(\mathbb{R}, \mathbb{C})$, the results remain true for $f \in L^2(\mathbb{R}, \mathbb{C})$.

2. $[\mathcal{S}^s_{\mathbf{A},g}(T_t f)](y,\xi)$

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} (T_t f)(x)\overline{g^{\mathbf{A}}_{y,\xi}(x)}\, dx \\
&= \frac{1}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} f(x-t)\, |\xi|\, \overline{g(\xi(x-y))}\, \overline{\Phi_{a,b}(y)}\, \overline{\mu_{p-\lambda,b}(y-x)}\, \Phi_{a,b}(x)\, \overline{\mu_{\xi,b}(x)}\, dx \\
&= \frac{1}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} f(s)\, |\xi|\, \overline{g(\xi((s+t)-y))}\, \overline{\Phi_{a,b}(y)}\, \overline{\mu_{p-\lambda,b}(y-(s+t))} \\
&\qquad \Phi_{a,b}(s+t)\overline{\mu_{\xi,b}(s+t)}\, ds \quad (\text{putting } x = s+t) \\
&= \frac{1}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} f(s)\, |\xi|\, \overline{g(\xi(s-(y-t))}\, \overline{\Phi_{a,b}(y-t)}\, \overline{\mu_{y,b}(t)}\, \overline{\mu_{p-\lambda,b}((y-t)-s)} \\
&\qquad \Phi_{a,b}(s)\, \Phi_{a,b}(\sqrt{2}t)\, \mu_{s,b}(t)\, \overline{\mu_{\xi,b}(s)}\, \overline{\mu_{\xi,b}(t)}\, ds \\
&= \Phi_{a,b}(\sqrt{2}t)\, \overline{\mu_{y,b}(t)}\, \overline{\mu_{\xi,b}(t)} \frac{1}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} f(s)\, \mu_{t,b}(s)\, |\xi|\, \overline{g(\xi(s-(y-t))}\, \overline{\Phi_{a,b}(y-t)} \\
&\qquad \overline{\mu_{p-\lambda,b}((y-t)-s)}\, \Phi_{a,b}(s)\, \overline{\mu_{\xi,b}(s)}\, ds \\
&= \Phi_{a,b}(\sqrt{2}t)\, \overline{\mu_{y+\xi,b}(t)}\, [\mathcal{S}^s_{\mathbf{A},g}(\mu_{t,b} f)](y-t,\xi).
\end{aligned}$$

3. If $\check{\mathbf{A}} = \left[\begin{array}{cc|c} a & b & -p \\ c & d & -q \end{array}\right]$, then $\check{\mathbf{A}}$ is a parameter matrix. Using the easy identity $\overline{\mu_{p-\lambda,b}(s)} = \overline{\Phi_{d,b}(s)}\,\Psi_{\mathbf{A}}(s)$, $\forall s \in \mathbb{R}$, we have that

$$\begin{aligned}
&[\mathcal{S}^s_{\mathbf{A},g}(\check{f})](y,\xi) \\
&= \frac{1}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} \check{f}(x)\overline{g^{\mathbf{A}}_{y,\xi}(x)}\,dx \\
&= \frac{1}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} f(-x)\,|\xi|\,\overline{g(\xi(x-y))}\;\overline{\Phi_{a,b}(y)}\;\overline{\mu_{p-\lambda,b}(y-x)}\Phi_{a,b}(x)\overline{\mu_{\xi,b}(x)}\,dx \\
&= \frac{1}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} f(-x)\,|\xi|\,\overline{g(\xi(x-y))}\;\overline{\Phi_{a,b}(y)}\;\overline{\Phi_{d,b}(y-x)}\,\Psi_{\mathbf{A}}(y-x)\Phi_{a,b}(x)\overline{\mu_{\xi,b}(x)}\,dx \\
&= \frac{1}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} f(s)\,|\xi|\,\overline{g(\xi(-s-y))}\;\overline{\Phi_{a,b}(y)}\;\overline{\Phi_{d,b}(y+s)}\,\Psi_{\mathbf{A}}(y+s)\Phi_{a,b}(-s)\overline{\mu_{-\xi,b}(s)}\,ds \\
&\quad \text{(by the change of variable } x=-s) \\
&= \frac{1}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} f(s)\,|\xi|\,\overline{g(-\xi(s-(-y)))}\;\overline{\Phi_{a,b}(-y)}\;\overline{\Phi_{d,b}(-y-s)}\,\Psi_{\check{\mathbf{A}}}(-y-s)\Phi_{a,b}(s)\overline{\mu_{-\xi,b}(s)}\,ds \\
&= [\mathcal{S}^s_{\check{\mathbf{A}},g}(\check{f})](-y,-\xi).
\end{aligned}$$

4. If $\mathbf{A}_\kappa = \left[\begin{array}{cc|c} \frac{a}{\kappa^2} & b & \frac{p}{\kappa} \\ c & d\kappa^2 & q\kappa \end{array}\right]$, then $\mathbf{A}_\kappa$ is a parameter matrix. Hence,

$$\begin{aligned}
&[\mathcal{S}^s_{\mathbf{A},g}(D_\kappa f)](y,\xi) \\
&= \frac{1}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} (D_\kappa f)(x)\overline{g^{\mathbf{A}}_{y,\xi}(x)}\,dx \\
&= \frac{1}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} \kappa f(\kappa x)\,|\xi|\,\overline{g(\xi(x-y))}\;\overline{\Phi_{a,b}(y)}\;\overline{\mu_{p-\lambda,b}(y-x)} \\
&\qquad \Phi_{a,b}(x)\overline{\mu_{\xi,b}(x)}\,dx \\
&= \frac{1}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} \kappa f(\kappa x)\,|\xi|\,\overline{g(\xi(x-y))}\;\overline{\Phi_{a,b}(y)}\;\overline{\Phi_{d,b}(y-x)}\,\Psi_{\mathbf{A}}(y-x) \\
&\qquad \Phi_{a,b}(x)\overline{\mu_{\xi,b}(x)}\,dx \\
&= \frac{1}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} \kappa f(s)\,|\xi|\,\overline{g\left(\xi\left(\frac{s}{\kappa}-y\right)\right)}\;\overline{\Phi_{a,b}(y)}\;\overline{\Phi_{d,b}\left(y-\frac{s}{\kappa}\right)}\,\Psi_{\mathbf{A}}\left(y-\frac{s}{\kappa}\right) \\
&\qquad \Phi_{a,b}\left(\frac{s}{\kappa}\right)\overline{\mu_{\xi,b}\left(\frac{s}{\kappa}\right)}\,\frac{ds}{\kappa} \\
&\quad \left(\text{by the change of variable } x=\frac{s}{\kappa}\right) \\
&= \frac{1}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} \kappa f(s)\,\frac{|\xi|}{\kappa}\,\overline{g\left(\frac{\xi}{\kappa}(s-\kappa y)\right)}\;\overline{\Phi_{a/\kappa^2,b}(\kappa y)}\;\overline{\Phi_{d\kappa^2,b}(\kappa y-s)}\,\Psi_{\mathbf{A}_\kappa}(\kappa y-s) \\
&\qquad \Phi_{a/\kappa^2,b}(s)\overline{\mu_{\xi/\kappa,b}(s)}\,\exp\left(\frac{i}{2b}\left[2\lambda(\kappa y-s)\left(\kappa-\frac{1}{\kappa}\right)\right]\right)ds \\
&= \frac{\kappa}{\sqrt{2\pi|b|}}\,\exp\left(\frac{i}{2b}\left[2\lambda\left(\kappa-\frac{1}{\kappa}\right)\kappa y\right]\right) \int_{\mathbb{R}} f(s)\exp\left(\frac{i}{2b}\left[2\lambda\left(\kappa-\frac{1}{\kappa}\right)s\right]\right) \\
&\qquad \frac{|\xi|}{\kappa}\,\overline{g\left(\frac{\xi}{\kappa}(s-\kappa y)\right)}\;\overline{\Phi_{a/\kappa^2,b}(\kappa y)}\;\overline{\Phi_{d\kappa^2,b}(\kappa y-s)}\,\Psi_{\mathbf{A}_\kappa}(\kappa y-s)\,\Phi_{a/\kappa^2,b}(s)\overline{\mu_{\xi/\kappa,b}(s)}\,ds \\
&= k\,\Theta_{\mathbf{A},\kappa}(\kappa y)\,[\mathcal{S}^s_{\mathbf{A}_\kappa,g}(\Theta_{\mathbf{A},\kappa}f)](\kappa y,\xi/\kappa),\ \forall (y,\xi)\in\mathbb{R}\times\mathbb{R}^*.
\end{aligned}$$

□

Lemma 2. *For $f \in L^2(\mathbb{R},\mathbb{C})$, $t \in \mathbb{R}$, and $\boldsymbol{\lambda} = bq - dp$, we have*

1. $[\mathcal{S}_{\mathbf{A}}f](y) = \frac{1}{\sqrt{|b|}}\Omega_{\mathbf{A}}(y)(f_{a,b,p})^{\wedge}\left(\frac{y}{b}\right)$, *where* $f_{a,b,p}(x) = f(x)\Phi_{a,b}(x)\mu_{p,b}(x)$,

2. $[\mathcal{S}_{\mathbf{A}} f](y-t) = \exp\left(\frac{i}{2b}[d(t^2-2yt)-2\lambda t]\right)[\mathcal{S}_{\mathbf{A}}(\mu_{t,b} f)](y)$,
3. $[\mathcal{S}_{\mathbf{A}}(\mu_{\xi,b} f)](y) = \exp\left(\frac{-i}{2b}[d(\xi^2-2y\xi)-2\lambda\xi]\right)[\mathcal{S}_{\mathbf{A}} f](y-\xi)$.

Proof. Suppose $f \in L^1(\mathbb{R},\mathbb{C}) \cap L^2(\mathbb{R},\mathbb{C})$.

1. The proof of the first claim can be found in [18].
2. For $t \in \mathbb{R}$,

$$
\begin{aligned}
&[\mathcal{S}_{\mathbf{A}} f](y-t)\\
&= \int_{\mathbb{R}} f(x) K_{\mathbf{A}}(x, y-t)\, dx\\
&= \int_{\mathbb{R}} f(x) \frac{1}{\sqrt{2\pi|b|}} \Phi_{a,b}(x) \Omega_{\mathbf{A}}(y-t) \mu_{(p-y+t),b}(x)\, dx\\
&= \int_{\mathbb{R}} f(x) \frac{1}{\sqrt{2\pi|b|}} \Phi_{a,b}(x) \Omega_{\mathbf{A}}(y) \mu_{(p-y),b}(x)\\
&\qquad \mu_{t,b}(x) \exp\left(\frac{i}{2b}[d(-2yt+t^2)-2\lambda t]\right) dx\\
&= \exp\left(\frac{i}{2b}[d(-2yt+t^2)-2\lambda t]\right) \int_{\mathbb{R}} \mu_{t,b}(x) f(x) K_{\mathbf{A}}(x, y-t)\, dx\\
&= \exp\left(\frac{i}{2b}[d(t^2-2yt)-2\lambda t]\right)[\mathcal{S}_{\mathbf{A}}(\mu_{t,b} f)](y).
\end{aligned}
$$

3. From [35,36], we have that

$$[\mathcal{S}_{\mathbf{A}}(f(x)e^{i\xi x})](y) = [\mathcal{S}_{\mathbf{A}} f](y-b\xi) \exp\left(-i\frac{bd\xi^2}{2} + id\xi(y-p) + ib\xi q\right).$$

Using this result, we can obtain the following. For $0 \neq b, \xi \in \mathbb{R}$,

$$
\begin{aligned}
&[\mathcal{S}_{\mathbf{A}}(\mu_{\xi,b} f)](y)\\
&= [\mathcal{S}_{\mathbf{A}}(f(x)e^{i\xi x/b})](y)\\
&= [\mathcal{S}_{\mathbf{A}} f](y-\xi) \exp\left(-i\frac{d\xi^2}{2b} + id\frac{\xi}{b}(y-p) + i\xi q\right)\\
&= [\mathcal{S}_{\mathbf{A}} f](y-\xi) \exp\left(\frac{-i}{2b}[d\xi^2 - 2d\xi y + 2dp\xi - 2bq\xi]\right)\\
&= \exp\left(\frac{-i}{2b}[d(\xi^2-2y\xi)-2\lambda\xi]\right)[\mathcal{S}_{\mathbf{A}} f](y-\xi).
\end{aligned}
$$

□

Lemma 3. *Let $0 \neq g \in L^1(\mathbb{R},\mathbb{C}) \cap L^2(\mathbb{R},\mathbb{C})$, $f \in L^2(\mathbb{R},\mathbb{C})$ and $\boldsymbol{\lambda} = bq - dp$. Then, we have*

1. $[\mathcal{S}_{\mathbf{A}}(\overline{\Phi_{d,b}} D_\xi \check{g})_{\mathbf{A}}](y) = \frac{1}{\sqrt{|b|}} \Omega_{\mathbf{A}}(y) \overline{\hat{g}\left(\frac{y-\lambda}{b\xi}\right)}$,
2. $[\mathcal{S}_{\mathbf{A}}(\overline{\mu_{p,b}} \Phi_{a,b} D_\xi g)](y) = \frac{1}{\sqrt{|b|}} \Omega_{\mathbf{A}}(y) (D_\xi g)\hat{}\left(\frac{y}{b}\right)$,
3. $\Omega_{\mathbf{A}}(y) \overline{[\mathcal{S}_{\mathbf{A}}(\overline{\Phi_{d,b}} D_\xi \check{g})_{\mathbf{A}}](y)} = \overline{\Omega_{\mathbf{A}}(y)} [\mathcal{S}_{\mathbf{A}}(\overline{\Phi_{d,b}} D_\xi g)_{\mathbf{A}}](y)$.

Proof. We know that

$$\Psi_{\mathbf{A}}(x) = \Phi_{d,b}(x) \overline{\mu_{p,b}(x)} \mu_{\lambda,b}(x). \tag{4}$$

For an arbitrary $(y,\xi) \in \mathbb{R} \times \mathbb{R}^*$, we have

1. $[\mathcal{S}_{\mathbf{A}}(\overline{\Phi_{d,b}}D_{\xi}\check{\bar{g}})_{\mathbf{A}}](y)$

$$
\begin{aligned}
&= \int_{\mathbb{R}} (\overline{\Phi_{d,b}}D_{\xi}\check{\bar{g}})_{\mathbf{A}}(x)K_{\mathbf{A}}(x,y)\,dx \\
&= \int_{\mathbb{R}} \overline{\Phi_{d,b}(x)}(D_{\xi}\check{\bar{g}})(x)\overline{\Phi_{a,b}(x)}\Psi_{\mathbf{A}}(x)\,\frac{1}{\sqrt{2\pi|b|}}\Phi_{a,b}(x)\Omega_{\mathbf{A}}(y)\mu_{(p-y),b}(x)\,dx \\
&= \int_{\mathbb{R}} (D_{\xi}\check{\bar{g}})(x)\frac{1}{\sqrt{2\pi|b|}}\overline{\Phi_{d,b}(x)}\Psi_{\mathbf{A}}(x)\Omega_{\mathbf{A}}(y)\mu_{p,b}(x)\mu_{-y,b}(x)\,dx \quad \text{(by (4))} \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi|b|}}\Omega_{\mathbf{A}}(y)|\xi|\check{\bar{g}}(\xi x)\mu_{\lambda,b}(x)\mu_{-y,b}(x)\,dx \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi|b|}}\Omega_{\mathbf{A}}(y)\check{\bar{g}}(t)\mu_{\lambda,b}(t/\xi)\mu_{-y,b}(t/\xi)\,dt \\
&= \frac{1}{\sqrt{|b|}}\Omega_{\mathbf{A}}(y)\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}}\check{\bar{g}}(t)\exp -i\left(\frac{y-\lambda}{b\xi}\right)t\,dt \\
&= \frac{1}{\sqrt{|b|}}\Omega_{\mathbf{A}}(y)\hat{\check{\bar{g}}}\left(\frac{y-\lambda}{b\xi}\right) \\
&= \frac{1}{\sqrt{|b|}}\Omega_{\mathbf{A}}(y)\overline{\hat{g}\left(\frac{y-\lambda}{b\xi}\right)}.
\end{aligned}
$$

2. Applying Lemma 2 with $f = \overline{\mu_{p,b}\Phi_{a,b}}D_{\xi}g$, we obtain

$$[\mathcal{S}_{\mathbf{A}}(\overline{\mu_{p,b}\Phi_{a,b}}D_{\xi}g)](y) = \frac{1}{\sqrt{|b|}}\Omega_{\mathbf{A}}(y)(D_{\xi}g)\hat{}\left(\frac{y}{b}\right).$$

3. Note that

$$\overline{\Omega_{\mathbf{A}}(y-\lambda)} = \exp\left(\frac{-i}{2b}\left[dy^2 + d(-2y\lambda+\lambda^2) + 2\lambda y - 2\lambda^2\right]\right). \tag{5}$$

Therefore,

$\Omega_{\mathbf{A}}(y)\overline{[\mathcal{S}_{\mathbf{A}}(\overline{\Phi_{d,b}}D_{\xi}\check{\bar{g}})_{\mathbf{A}}](y)}$

$$
\begin{aligned}
&= \frac{1}{\sqrt{|b|}}\hat{g}\left[\frac{1}{\xi}\left(\frac{y-\lambda}{b}\right)\right] \text{(by Lemma 3 (1))} \\
&= \frac{1}{\sqrt{|b|}}(D_{\xi}g)\hat{}\left(\frac{y-\lambda}{b}\right) \\
&= \overline{\Omega_{\mathbf{A}}(y-\lambda)}[\mathcal{S}_{\mathbf{A}}(\overline{\mu_{p,b}\Phi_{a,b}}D_{\xi}g)](y-\lambda) \quad \text{(by Lemma 3 (2))} \\
&= \overline{\Omega_{\mathbf{A}}(y-\lambda)}\exp\left(\frac{i}{2b}[d(\lambda^2-2y\lambda)-2\lambda^2]\right)[\mathcal{S}_{\mathbf{A}}(\mu_{\lambda,b}\overline{\mu_{p,b}\Phi_{a,b}}D_{\xi}g)](y) \\
&\qquad \text{(by Lemma 2 (2))} \\
&= \overline{\Omega_{\mathbf{A}}(y)}[\mathcal{S}_{\mathbf{A}}(\mu_{\lambda,b}\overline{\mu_{p,b}\Phi_{a,b}}D_{\xi}g)](y) \quad \text{(by (5))} \\
&= \overline{\Omega_{\mathbf{A}}(y)}[\mathcal{S}_{\mathbf{A}}(\overline{\Phi_{d,b}}\Psi_{\mathbf{A}}\overline{\Phi_{a,b}}D_{\xi}g)](y) \quad \text{(by (4))} \\
&= \overline{\Omega_{\mathbf{A}}(y)}[\mathcal{S}_{\mathbf{A}}(\overline{\Phi_{d,b}}D_{\xi}g)_{\mathbf{A}}](y).
\end{aligned}
$$

□

Lemma 4. *If* $0 \neq g \in L^1(\mathbb{R},\mathbb{C}) \cap L^2(\mathbb{R},\mathbb{C})$ *and* $0 \neq b, t, \lambda \in \mathbb{R}$, *then*

$$\int_{\mathbb{R}^*}\left|\hat{g}\left(\frac{t-\lambda-\xi}{b\xi}\right)\right|^2\frac{d\xi}{|\xi|} = \int_{\mathbb{R}^*}\left|\hat{g}\left(x-\frac{1}{b}\right)\right|^2\frac{dx}{|x|}.$$

Proof. By using the change of variables $\xi = \frac{t-\lambda}{bx}$, we can obtain the desired result. □

Theorem 4. *(Parseval's identity) Let $0 \neq g \in L^1(\mathbb{R},\mathbb{C}) \cap L^2(\mathbb{R},\mathbb{C})$ such that*

$$0 \neq C_{b,g} = \frac{1}{|b|\sqrt{2\pi}} \int_{\mathbb{R}} \left| \hat{g}\left(x - \frac{1}{b} \right) \right|^2 \frac{dx}{|x|} < \infty. \tag{6}$$

Then, for $f,h \in L^2(\mathbb{R},\mathbb{C})$, we have $\left\langle \mathcal{S}^s_{\mathbf{A},g} f, \mathcal{S}^s_{\mathbf{A},g} h \right\rangle = C_{b,g}\langle f,h\rangle$.

Proof. Let $f,h \in L^2(\mathbb{R},\mathbb{C})$.

$$\begin{aligned}
&\left\langle \mathcal{S}^s_{\mathbf{A},g} f, \mathcal{S}^s_{\mathbf{A},g} h \right\rangle \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^*} \int_{\mathbb{R}} (\mathcal{S}^s_{\mathbf{A},g} f)(y,\xi) \overline{(\mathcal{S}^s_{\mathbf{A},g} f)(y,\xi)}\, dy\, \frac{d\xi}{|\xi|} \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^*} \int_{\mathbb{R}} [\mu_{-\xi,b} f \circledast_{\mathbf{A}} \overline{\Phi_{d,b}} D_\xi \check{g}](y) \overline{[\mu_{-\xi,b} h \circledast_{\mathbf{A}} \overline{\Phi_{d,b}} D_\xi \check{g}](y)}\, dy\, \frac{d\xi}{|\xi|} \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^*} \int_{\mathbb{R}} [\mathcal{S}_{\mathbf{A}}(\mu_{-\xi,b} f \circledast_{\mathbf{A}} \overline{\Phi_{d,b}} D_\xi \check{g})](t) \overline{[\mathcal{S}_{\mathbf{A}}(\mu_{-\xi,b} h \circledast_{\mathbf{A}} \overline{\Phi_{d,b}} D_\xi \check{g})](t)}\, dt\, \frac{d\xi}{|\xi|} \\
&\quad \text{(by Parseval's identity for SAFT)} \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^*} \int_{\mathbb{R}} \overline{\Omega_{\mathbf{A}}(t)} [\mathcal{S}_{\mathbf{A}}(\mu_{\xi,b} f)](t) [\mathcal{S}_{\mathbf{A}}(\overline{\Phi_{d,b}} D_\xi \check{g})_{\mathbf{A}}](t) \\
&\quad \overline{\overline{\Omega_{\mathbf{A}}(t)} [\mathcal{S}_{\mathbf{A}}(\mu_{\xi,b} h)](t) [\mathcal{S}_{\mathbf{A}}(\overline{\Phi_{d,b}} D_\xi \check{g})_{\mathbf{A}}](t)}\, dt\, \frac{d\xi}{|\xi|} \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^*} \int_{\mathbb{R}} [\mathcal{S}_{\mathbf{A}}(\mu_{-\xi,b} f)](t) \overline{[\mathcal{S}_{\mathbf{A}}(\mu_{-\xi,b} h)](t)} |[\mathcal{S}_{\mathbf{A}}(\overline{\Phi_{d,b}} D_\xi \check{g})_{\mathbf{A}}](t)|^2\, dt\, \frac{d\xi}{|\xi|} \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^*} \int_{\mathbb{R}} [\mathcal{S}_{\mathbf{A}} f](t+\xi) \overline{[\mathcal{S}_{\mathbf{A}} h](t+\xi)} |[\mathcal{S}_{\mathbf{A}}(\overline{\Phi_{d,b}} D_\xi \check{g})_{\mathbf{A}}](t)|^2\, dt\, \frac{d\xi}{|\xi|} \\
&\quad \text{(by Lemma 2 (2))} \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^*} \int_{\mathbb{R}} [\mathcal{S}_{\mathbf{A}} f](t) \overline{[\mathcal{S}_{\mathbf{A}} h](t)} |[\mathcal{S}_{\mathbf{A}}(\overline{\Phi_{d,b}} D_\xi \check{g})_{\mathbf{A}}](t-\xi)|^2\, dt\, \frac{d\xi}{|\xi|} \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}^*} [\mathcal{S}_{\mathbf{A}} f](t) \overline{[\mathcal{S}_{\mathbf{A}} h](t)} \frac{1}{|b|} \left| \hat{g}\left(\frac{t-\xi-\lambda}{b\xi} \right) \right|^2 \frac{d\xi}{|\xi|}\, dt \\
&\quad \text{(by Lemma 3 (1) and Fubini's theorem)} \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}^*} [\mathcal{S}_{\mathbf{A}} f](t) \overline{[\mathcal{S}_{\mathbf{A}} h](t)} \frac{1}{|b|} \left| \hat{g}\left(x - \frac{1}{b} \right) \right|^2 \frac{dx}{|x|}\, dt \\
&\quad \text{(by Lemma 4)} \\
&= \frac{C_{b,g}}{\sqrt{2\pi}} \int_{\mathbb{R}} [\mathcal{S}_{\mathbf{A}} f](t) \overline{[\mathcal{S}_{\mathbf{A}} h](t)}\, dt \\
&= \frac{C_{b,g}}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y) \overline{h(y)}\, dy \quad \text{(by Parseval's identity for SAFT)} \\
&= C_{b,g}\langle f,h\rangle.
\end{aligned}$$

□

Theorem 5 (The Inversion Formula)**.** *If $0 \neq g \in L^1(\mathbb{R},\mathbb{C}) \cap L^2(\mathbb{R},\mathbb{C})$ satisfies the admissibility condition* (6)*, then for each $f \in L^2(\mathbb{R},\mathbb{C})$, we have*

$$f(x) = \frac{1}{C_{b,g}\sqrt{2\pi}} \int_{\mathbb{R}^*} \mu_{\xi,b}(x)[\mathcal{S}^s_{\mathbf{A},g} f \circledast_{\mathbf{A}} \overline{\Phi_{d,b}} D_\xi g](x) \frac{d\xi}{|\xi|},$$

weakly in $L^2(\mathbb{R},\mathbb{C})$.

Proof. Let $h \in L^2(\mathbb{R},\mathbb{C})$. For a fixed $f \in L^2(\mathbb{R},\mathbb{C})$, by Parseval's identity, we have that

$$C_{b,g}\langle f, h\rangle = \langle \mathcal{S}^s_{\mathbf{A},g} f, \mathcal{S}^s_{\mathbf{A},g} h\rangle.$$

$C_{b,g}\langle f, h\rangle$

$$= \frac{1}{2\pi}\int_{\mathbb{R}^*}\int_{\mathbb{R}} [\mathcal{S}^s_{\mathbf{A},g} f](y,\xi)\overline{[\mathcal{S}^s_{\mathbf{A},g} h](y,\xi)}\, dy \frac{d\xi}{|\xi|}$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}^*}\int_{\mathbb{R}} [\mathcal{S}^s_{\mathbf{A},g} f](y,\xi)\overline{[\mu_{-\xi,b} h \circledast_{\mathbf{A}} \overline{\Phi_{d,b}} D_\xi \check{g}](y)}\, dy \frac{d\xi}{|\xi|}$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}^*}\int_{\mathbb{R}} [\mathcal{S}_{\mathbf{A}}([\mathcal{S}^s_{\mathbf{A},g} f](\cdot,\xi))](t)\overline{[\mathcal{S}_{\mathbf{A}}(\mu_{-\xi,b} h \circledast_{\mathbf{A}} \overline{\Phi_{d,b}} D_\xi \check{g})](t)}\, dt \frac{d\xi}{|\xi|}$$

(by Parseval's identity for SAFT)

$$= \frac{1}{2\pi}\int_{\mathbb{R}^*}\int_{\mathbb{R}} [\mathcal{S}_{\mathbf{A}}([\mathcal{S}^s_{\mathbf{A},g} f](\cdot,\xi))](t)\Omega_{\mathbf{A}}(t)\overline{[\mathcal{S}_{\mathbf{A}}(\mu_{-\xi,b} h)](t)[\mathcal{S}_{\mathbf{A}}(\overline{\Phi_{d,b}} D_\xi \check{g})_{\mathbf{A}}](t)}\, dt \frac{d\xi}{|\xi|}$$

(by Theorem 3)

$$= \frac{1}{2\pi}\int_{\mathbb{R}^*}\int_{\mathbb{R}} [\mathcal{S}_{\mathbf{A}}([\mathcal{S}^s_{\mathbf{A},g} f](\cdot,\xi))](t)\overline{[\mathcal{S}_{\mathbf{A}}(\mu_{-\xi,b} h)](t)\Omega_{\mathbf{A}}(y)}[\mathcal{S}_{\mathbf{A}}(\overline{\Phi_{d,b}} D_\xi g)_{\mathbf{A}}](y)\, dt \frac{d\xi}{|\xi|}$$

(by Lemma 3 (3))

$$= \frac{1}{2\pi}\int_{\mathbb{R}^*}\int_{\mathbb{R}} [\mathcal{S}_{\mathbf{A}}(\mathcal{S}^s_{\mathbf{A},g} f \circledast_{\mathbf{A}} \overline{\Phi_{d,b}} D_\xi g)](t)\overline{[\mathcal{S}_{\mathbf{A}}(\mu_{-\xi,b} h)](t)}\, dt \frac{d\xi}{|\xi|}$$

(by Theorem 3)

$$= \frac{1}{2\pi}\int_{\mathbb{R}^*}\int_{\mathbb{R}} [\mathcal{S}^s_{\mathbf{A},g} f \circledast_{\mathbf{A}} \overline{\Phi_{d,b}} D_\xi g](y)\overline{\mu_{-\xi,b}(y) h(y)}\, dy \frac{d\xi}{|\xi|}$$

(by Parseval's identity for SAFT)

$$= \frac{1}{2\pi}\int_{\mathbb{R}}\left(\int_{\mathbb{R}^*} \mu_{\xi,b}(y)[\mathcal{S}^s_{\mathbf{A},g} f \circledast_{\mathbf{A}} \overline{\Phi_{d,b}} D_\xi g](y) \frac{d\xi}{|\xi|}\right)\overline{h(y)}\, dy$$

(by Fubini's theorem)

$$= \left\langle \frac{1}{\sqrt{2\pi}}\mu_{\xi,b}[\mathcal{S}^s_{\mathbf{A},g} f \circledast_{\mathbf{A}} \overline{\Phi_{d,b}} D_\xi g], h\right\rangle, \forall h \in L^2(\mathbb{R},\mathbb{C}).$$

Hence ,

$$f(x) = \frac{1}{C_{b,g}\sqrt{2\pi}} \int_{\mathbb{R}^*} \mu_{\xi,b}(x)[\mathcal{S}^s_{\mathbf{A},g} f \circledast_{\mathbf{A}} \overline{\Phi_{d,b}} D_\xi g](x) \frac{d\xi}{|\xi|},$$

weakly in $L^2(\mathbb{R},\mathbb{C})$. □

Let us recall a definition and a result from [37] which will be useful in discussing the range characterization of $\mathcal{S}^s_{\mathbf{A},g}$.

Definition 3 ([37], p. 43)**.** *If Λ is a function on $\mathbb{R} \times \mathbb{R}^*$ that takes values in $L^2(\mathbb{R},\mathbb{C})$, then $\Theta = \int_{\mathbb{R}^*}\int_{\mathbb{R}} \Lambda(y,\xi)\, dy \frac{d\xi}{|\xi|}$ means*

$$\langle \Theta, \phi \rangle = \int_{\mathbb{R}^*} \int_{\mathbb{R}} \langle \Lambda(y,\xi), \phi \rangle dy \frac{d\xi}{|\xi|}, \ \forall \phi \in L^2(\mathbb{R}, \mathbb{C}).$$

Lemma 5 ([37], p. 43)**.** *If the mapping* $T(g) = \int_{\mathbb{R}^*} \int_{\mathbb{R}} \langle \Lambda(y,\xi), g \rangle \, dy \, \frac{d\xi}{|\xi|}$ *is a bounded conjugate-linear functional on* $L^2(\mathbb{R}, \mathbb{C})$*, then* T *defines a unique element* $\Theta \in L^2(\mathbb{R}, \mathbb{C})$ *such that*

$$T(g) = \int_{\mathbb{R}^*} \int_{\mathbb{R}} \langle \Lambda(y,\xi), g \rangle \, dy \, \frac{d\xi}{|\xi|} = \langle \Theta, g \rangle, \forall g \in L^2(\mathbb{R}, \mathbb{C}).$$

Theorem 6 (Characterization of range of $\mathcal{S}^s_{\mathbf{A},g}$)**.** *Let* $h \in L^2(\mathbb{R} \times \mathbb{R}^*)$*. Then,* $h \in \mathcal{S}_{\mathbf{A},g}(L^2(\mathbb{R}, \mathbb{C}))$ *if and only if*

$$h(y', \xi') = \frac{1}{C_{b,g}} \int_{\mathbb{R}^*} \int_{\mathbb{R}} h(y,\xi) J^{\mathbf{A}}_g(y,\xi; y', \xi') \, dy \, \frac{d\xi}{|\xi|}, \tag{7}$$

where $J^{\mathbf{A}}_g(y,\xi; y', \xi') = \langle g^{\mathbf{A}}_{y,\xi}, g^{\mathbf{A}}_{y',\xi'} \rangle = \int_{\mathbb{R}} g^{\mathbf{A}}_{y,\xi}(x) \overline{g^{\mathbf{A}}_{y',\xi'}(x)} \, dx.$

Proof. Suppose that $h \in \mathcal{S}^s_{\mathbf{A},g}(L^2(\mathbb{R}^N, \mathbb{C}))$. Then, there exists $f \in L^2(\mathbb{R}^N)$ such that $h = \mathcal{S}^s_{\mathbf{A},g} f$. Therefore, applying the inversion formula of the generalized fractional Stockwell transform (Theorem 5), we obtain

$$h(y', \xi') = \langle f, g^{\mathbf{A}}_{y',\xi'} \rangle = \left\langle \left(\frac{1}{C_{b,g}} \int_{\mathbb{R}^*} \int_{\mathbb{R}} (\mathcal{S}^s_{\mathbf{A},g} f)(y,\xi) g^{\mathbf{A}}_{y,\xi} \, dy \, \frac{d\xi}{|\xi|} \right), g^{\mathbf{A}}_{y',\xi'} \right\rangle.$$

Now, consider the linear functional $T(\phi) = \int_{\mathbb{R}^*} \int_{\mathbb{R}} \langle (\mathcal{S}^s_{\mathbf{A},g} f)(y,\xi) g^{\mathbf{A}}_{y,\xi}, \phi \rangle \, dy \, \frac{d\xi}{|\xi|}, \ \forall \phi \in L^2(\mathbb{R}^N, \mathbb{C})$. Clearly, T is a conjugate linear functional. We claim that T is bounded. For,

$$\begin{aligned}
|T(\phi)| &= \left| \int_{\mathbb{R}^*} \int_{\mathbb{R}} \langle (\mathcal{S}^s_{\mathbf{A},g} f)(y,\xi) g^{\mathbf{A}}_{y,\xi}, \phi \rangle \, dy \, \frac{d\xi}{|\xi|} \right| \\
&= \left| \int_{\mathbb{R}^*} \int_{\mathbb{R}} (\mathcal{S}^s_{\mathbf{A},g} f)(y,\xi) \langle g^{\mathbf{A}}_{y,\xi}, \phi \rangle \, dy \, \frac{d\xi}{|\xi|} \right| \\
&= \left| \int_{\mathbb{R}^*} \int_{\mathbb{R}} (\mathcal{S}^s_{\mathbf{A},g} f)(y,\xi) \overline{\langle \phi, g^{\mathbf{A}}_{y,\xi} \rangle} \, dy \, \frac{d\xi}{|\xi|} \right| \\
&= \left| \int_{\mathbb{R}^*} \int_{\mathbb{R}} (\mathcal{S}^s_{\mathbf{A},g} f)(y,\xi) \overline{(\mathcal{S}^s_{\mathbf{A},g} \phi)(y,\xi)} \, dy \, \frac{d\xi}{|\xi|} \right| \\
&= \left| C_{b,g} \langle f, \phi \rangle \right| \text{ (by Theorem 4)} \\
&\leq |C_{b,g}| \, \|f\|_2 \|\phi\|_2.
\end{aligned}$$

Hence, T is bounded.

Therefore, Definition 3 and Lemma 5 can be applied with $\Lambda(y,\xi) = (\mathcal{S}^s_{\mathbf{A},g} f)(y,\xi) g^{\mathbf{A}}_{y,\xi}$ and we obtain

$$\begin{aligned}
h(y', \xi') &= \frac{1}{C_{b,g}} \int_{\mathbb{R}^*} \int_{\mathbb{R}} (\mathcal{S}^s_{\mathbf{A},g} f)(y,\xi) \left\langle g^{\mathbf{A}}_{y,\xi} \overline{g^{\mathbf{A}}_{y',\xi'}} \right\rangle dx \, dy \, \frac{d\xi}{|\xi|} \\
&= \frac{1}{C_{b,g}} \int_{\mathbb{R}^*} \int_{\mathbb{R}} h(y,\xi) J^{\mathbf{A}}_g(y,\xi; y', \xi') \, dy \, \frac{d\xi}{|\xi|}.
\end{aligned}$$

For the converse part, let us assume that h satisfies Equation (7). We define

$$f(x) = \frac{1}{C_{b,g}} \int_{\mathbb{R}^*} \int_{\mathbb{R}} h(y,\xi) g^{\mathbf{A}}_{y,\xi}(x)\, dy\, \frac{d\xi}{|\xi|}.$$

Using Fubini's theorem, we obtain the following:

$$\begin{aligned}
\|f\|_2 &= \int_{\mathbb{R}} f(x)\overline{f(x)}\, dx \\
&= \frac{1}{|C_{b,g}|^2} \int_{\mathbb{R}} \left(\int_{\mathbb{R}^*} \int_{\mathbb{R}} h(y,\xi) g^{\mathbf{A}}_{y,\xi}(x)\, dy\, \frac{d\xi}{|\xi|} \right) \left(\int_{\mathbb{R}^*} \int_{\mathbb{R}} \overline{h(y',\xi') g^{\mathbf{A}}_{y',\xi'}(x)}\, dy'\, \frac{d\xi'}{|\xi'|} \right) dx \\
&= \frac{1}{|C_{b,g}|^2} \int_{\mathbb{R}^*} \int_{\mathbb{R}} \int_{\mathbb{R}^*} \int_{\mathbb{R}} \int_{\mathbb{R}} h(y,\xi)\overline{h(y',\xi')}\, g^{\mathbf{A}}_{y,\xi}(x)\overline{g^{\mathbf{A}}_{y',\xi'}(x)}\, dx\, dy\, \frac{d\xi}{|\xi|}\, dy'\, \frac{d\xi'}{|\xi'|} \\
&= \frac{1}{|C_{b,g}|^2} \int_{\mathbb{R}^*} \int_{\mathbb{R}} \int_{\mathbb{R}^*} \int_{\mathbb{R}} h(y,\xi)\overline{h(y',\xi')}\, J^{\mathbf{A}}_g(y,\xi;y',\xi')\, dy\, \frac{d\xi}{|\xi|}\, dy'\, \frac{d\xi'}{|\xi'|} \\
&= \frac{1}{|C_{b,g}|^2} \int_{\mathbb{R}^*} \int_{\mathbb{R}} h(y,\xi)\overline{h(y',\xi')}\, dy'\, \frac{d\xi'}{|\xi'|} \text{ (by Equation (7))} \\
&= \frac{1}{|C_{b,g}|^2} \|h\|_2.
\end{aligned}$$

Hence, we have $f \in L^2(\mathbb{R}^N, \mathbb{C})$. Further,

$$\begin{aligned}
(\mathcal{S}^s_{\mathbf{A},g} f)(y',\xi') &= \int_{\mathbb{R}} f(x)\overline{g^{\mathbf{A}}_{t',z'}(x)}\, dx \\
&= \int_{\mathbb{R}} \frac{1}{C_{b,g}} \int_{\mathbb{R}^*} \int_{\mathbb{R}} h(y,\xi) g^{\mathbf{A}}_{y,\xi}(x)\, dy\, \frac{d\xi}{|\xi|}\, \overline{g^{\mathbf{A}}_{y',\xi'}(x)}\, dx \\
&= \frac{1}{C_{b,g}} \int_{\mathbb{R}^*} \int_{\mathbb{R}} h(y,\xi) \int_{\mathbb{R}} g^{\mathbf{A}}_{y,\xi}(x)\overline{g^{\mathbf{A}}_{y',\xi'}(x)}\, dx\, dy\, \frac{d\xi}{|\xi|} \\
&= \frac{1}{C_{b,g}} \int_{\mathbb{R}^*} \int_{\mathbb{R}} h(y,\xi) J^{\mathbf{A}}_g(y,\xi;y',\xi')\, dy\, \frac{d\xi}{|\xi|} = h(y',\xi').
\end{aligned}$$

Hence, the theorem follows. □

4. Uncertainty Principle

In harmonic analysis, the Heisenberg uncertainty principle can be interpreted as follows: "A signal f and its Fourier transform $\hat{f}$ cannot be both time-limited and band-limited simultaneously" [38]. From a mathematical perspective, significant advancements have been made in the theory of uncertainty principles over recent decades. Various generalizations of the Fourier transform and their associated uncertainty principles have been studied extensively, with [39–41] representing some of these works.

Theorem 7 (Uncertainty Principle for SAFT [18])**.**
If $f \in L^1(\mathbb{R},\mathbb{C}) \cap L^2(\mathbb{R},\mathbb{C})$ and $xf(x), t(\mathcal{S}_{\mathbf{A}} f)(t) \in L^2(\mathbb{R},\mathbb{C})$, then

$$\left(\int_{\mathbb{R}} |xf(x)|^2\, dx \right) \left(\int_{\mathbb{R}} |t(\mathcal{S}_{\mathbf{A}} f)(t)|^2\, dy \right) \geq \frac{|b|^2}{4} \|f\|_2^4. \tag{8}$$

Remark 3. *One can observe that the generalized fractional Stockwell transform of $f \in L^2(\mathbb{R},\mathbb{C})$ can also be rewritten as follows.*

$$[\mu_{-\xi,b} f \circledast_{\mathbf{A}} \overline{\Phi_{d,b}} D_\xi \check{\bar{g}}] = \mu_{-\xi,b}[f \circledast_{\mathbf{A}} \mu_{\xi,b} \overline{\Phi_{d,b}} D_\xi \check{\bar{g}}].$$

Indeed, $[\mu_{-\xi,b} f \circledast_{\mathbf{A}} \overline{\Phi_{d,b}} D_\xi \check{\bar{g}}]$

$$\begin{aligned}
&= \frac{\overline{\Phi_{d,b}(y)}}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} \Phi_{a,b}(x) \mu_{-\xi,b}(x) f(x) \Psi_{\mathbf{A}}(y-x) \overline{\Phi_{d,b}}(y-x) D_\xi \check{\bar{g}}(y-x)\, dx \\
&= \frac{\overline{\Phi_{d,b}(y)}}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} \Phi_{a,b}(x) \mu_{-\xi,b}(x) f(x) \mu_{-\xi,b}(y-x) \mu_{\xi,b}(y-x) \\
&\qquad \Psi_{\mathbf{A}}(y-x) \overline{\Phi_{d,b}}(y-x) D_\xi \check{\bar{g}}(y-x)\, dx \\
&= \frac{\overline{\Phi_{d,b}(y)}}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} \Phi_{a,b}(x) f(x) \mu_{-\xi,b}(y) \mu_{\xi,b}(y-x) \Psi_{\mathbf{A}}(y-x) \overline{\Phi_{d,b}}(y-x) D_\xi \check{\bar{g}}(y-x)\, dx \\
&= \mu_{-\xi,b}(y) [f \circledast_{\mathbf{A}} \mu_{\xi,b} \overline{\Phi_{d,b}} D_\xi \check{\bar{g}}](y).
\end{aligned}$$

Lemma 6. *If $f, x^k(\mathcal{S}_{\mathbf{A}} f)(x) \in L^2(\mathbb{R},\mathbb{C})$, where $k = 1,2$ and $0 \neq g \in L^1(\mathbb{R},\mathbb{C}) \cap L^2(\mathbb{R},\mathbb{C})$, satisfies the admissibility condition (6), then*

$$\sqrt{2\pi} C_{b,g} \left(\int_{\mathbb{R}} |t\,(\mathcal{S}_{\mathbf{A}} f)(t)|^2\, dt \right) = \int_{\mathbb{R}^*} \int_{\mathbb{R}} \left| t\,[\mathcal{S}_{\mathbf{A}}(f \circledast_{\mathbf{A}} \mu_{\xi,b} \overline{\Phi_{d,b}} D_\xi \check{\bar{g}})](t) \right|^2 dt\, \frac{d\xi}{|\xi|}.$$

Proof. $\int_{\mathbb{R}^*} \int_{\mathbb{R}} \left| t\,[\mathcal{S}_{\mathbf{A}}(f \circledast_{\mathbf{A}} \mu_{\xi,b} \overline{\Phi_{d,b}} D_\xi \check{\bar{g}})](t) \right|^2 dt\, \frac{d\xi}{|\xi|}$

$$\begin{aligned}
&= \int_{\mathbb{R}^*} \int_{\mathbb{R}} |t|^2 \left| \overline{\Omega_{\mathbf{A}}(y)}\, [\mathcal{S}_{\mathbf{A}} f](t)\, [\mathcal{S}_{\mathbf{A}}(\mu_{\xi,b} \overline{\Phi_{d,b}} D_\xi \check{\bar{g}})_{\mathbf{A}}](t) \right|^2 dt\, \frac{d\xi}{|\xi|} \\
&\qquad \text{(by Theorem 3)} \\
&= \int_{\mathbb{R}^*} \int_{\mathbb{R}} |t|^2 \left| \overline{\Omega_{\mathbf{A}}(y)}\, [\mathcal{S}_{\mathbf{A}} f](t)\, [\mathcal{S}_{\mathbf{A}}(\mu_{\xi,b} (\overline{\Phi_{d,b}} D_\xi \check{\bar{g}})_{\mathbf{A}})](t) \right|^2 dt\, \frac{d\xi}{|\xi|} \\
&= \int_{\mathbb{R}^*} \int_{\mathbb{R}} |t|^2 \left| [\mathcal{S}_{\mathbf{A}} f](t)\, [\mathcal{S}_{\mathbf{A}}(\overline{\Phi_{d,b}} D_\xi \check{\bar{g}})_{\mathbf{A}}](t-\xi) \right|^2 dt\, \frac{d\xi}{|\xi|} \\
&\qquad \text{(by Lemma 2 (3))} \\
&= \int_{\mathbb{R}^*} \int_{\mathbb{R}} |t|^2 \left| [\mathcal{S}_{\mathbf{A}} f](t)\, \frac{1}{\sqrt{|b|}} \Omega_{\mathbf{A}}(y) \overline{\hat{g}\left(\frac{y-\lambda-\xi}{b\xi} \right)} \right|^2 dt\, \frac{d\xi}{|\xi|} \\
&\qquad \text{(by Lemma 3 (1))} \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^*} |t\,[\mathcal{S}_{\mathbf{A}} f](t)|^2 \frac{1}{|b|} \left| \hat{g}\left(\frac{y-\lambda-\xi}{b\xi} \right) \right|^2 \frac{d\xi}{|\xi|}\, dt \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^*} |t\,[\mathcal{S}_{\mathbf{A}} f](t)|^2 \frac{1}{|b|} \left| \hat{g}\left(x - \frac{1}{b} \right) \right|^2 \frac{dx}{|x|}\, dt \\
&\qquad \text{(by Lemma 4)} \\
&= \sqrt{2\pi} C_{b,g} \int_{\mathbb{R}} |t\,[\mathcal{S}_{\mathbf{A}} f](t)|^2\, dt.
\end{aligned}$$

□

Theorem 8 (Uncertainty principle for generalized fractional Stockwell transform)**.** *If $f, x^k(\mathcal{S}_{\mathbf{A}}f)(x) \in L^2(\mathbb{R},\mathbb{C})$, $(\mathcal{S}_{\mathbf{A},g}f)(\cdot,\xi) \in L^2(\mathbb{R},\mathbb{C}) \cap L^2(\mathbb{R},\mathbb{C})$ and $y^k(\mathcal{S}^s_{\mathbf{A},g}f)(y,\xi) \in L^2(\mathbb{R} \times \mathbb{R}^*,\mathbb{C})$, where $k = 1,2$, then*

$$\left(\int_{\mathbb{R}^*}\int_{\mathbb{R}}\left|y\,[\mathcal{S}^s_{\mathbf{A},g}f](y,\xi)\right|^2 dy\,\frac{d\xi}{|\xi|}\right)\left(\int_{\mathbb{R}}|t\,[\mathcal{S}_{\mathbf{A}}f](t)|^2\,dt\right) \geq \frac{C_{b,g}}{\sqrt{2\pi}}\frac{|b|^2}{4}||f||_2^4.$$

Proof. By Parseval's identity, we have that

$$C_{b,g}\frac{|b|^2}{4}||f||_2^4 = C_{b,g}\frac{|b|^2}{4}\left(\frac{1}{(C_{b,g})^2}||\mathcal{S}^s_{\mathbf{A},g}f||_2^4\right) = \frac{|b|^2}{4}\frac{1}{C_{b,g}}\left(||\mathcal{S}^s_{\mathbf{A},g}f||_2^4\right) \tag{9}$$

Replacing $f(x)$ by $\mu_{\xi,b}(x)[\mathcal{S}^s_{\mathbf{A},g}f](x,\cdot)$ in Inequality (8), we obtain the following:

$$\left(\int_{\mathbb{R}}|y\,[\mathcal{S}^s_{\mathbf{A},g}f](y,\xi)|^2\,dy\right)\left(\int_{\mathbb{R}}|t\,[\mathcal{S}_{\mathbf{A}}(f\circledast_{\mathbf{A}}\mu_{\xi,b}\overline{\Phi_{d,b}}D_{\xi}\check{g})](t)|^2\,dt\right)$$
$$\geq \frac{|b|^2}{4}\left(\int_{\mathbb{R}}\left|[\mathcal{S}^s_{\mathbf{A},g}f](y,\xi)\right|^2 dy\right)^2.$$

Using this in Equation (9),

$$\begin{aligned}
C_{b,g}\frac{|b|^2}{4}||f||_2^4 &= \frac{|b|^2}{4}\frac{1}{C_{b,g}}\left(||\mathcal{S}^s_{\mathbf{A},g}f||_2^4\right)\\
&= \frac{|b|^2}{4}\frac{1}{C_{b,g}}\left(\int_{\mathbb{R}^*}\int_{\mathbb{R}}\left|[\mathcal{S}^s_{\mathbf{A},g}f](y,\xi)\right|^2 dy\,\frac{d\xi}{|\xi|}\right)^2\\
&\leq \frac{1}{C_{b,g}}\left[\int_{\mathbb{R}^*}\left(\int_{\mathbb{R}}|y\,[\mathcal{S}^s_{\mathbf{A},g}f](y,\xi)|^2\,dy\right)^{\frac{1}{2}}\right.\\
&\qquad\left.\left(\int_{\mathbb{R}}|t\,[\mathcal{S}_{\mathbf{A}}(f\circledast_{\mathbf{A}}\mu_{\xi,b}\overline{\Phi_{d,b}}D_{\xi}\check{g})](t)|^2\,dt\right)^{\frac{1}{2}}\frac{d\xi}{|\xi|}\right]^2\\
&= \frac{1}{C_{b,g}}\left[\left(\int_{\mathbb{R}^*}\int_{\mathbb{R}}|y\,[\mathcal{S}^s_{\mathbf{A},g}f](y,\xi)|^2\,dy\,\frac{d\xi}{|\xi|}\right)^{\frac{1}{2}}\right.\\
&\qquad\left.\left(\int_{\mathbb{R}^*}\int_{\mathbb{R}}|t\,[\mathcal{S}_{\mathbf{A}}(f\circledast_{\mathbf{A}}\mu_{\xi,b}\overline{\Phi_{d,b}}D_{\xi}\check{g})](t)|^2\,dt\,\frac{d\xi}{|\xi|}\right)^{\frac{1}{2}}\right]^2\\
&\qquad\text{(by Cauchy Schwartz inequality)}\\
&= \sqrt{2\pi}\left(\int_{\mathbb{R}^*}\int_{\mathbb{R}}|y\,[\mathcal{S}^s_{\mathbf{A},g}f](y,\xi)|^2\,dy\,\frac{d\xi}{|\xi|}\right)\left(\int_{\mathbb{R}}|t\,[\mathcal{S}_{\mathbf{A}}f](t)|^2\,dt\right).\\
&\qquad\text{(by Lemma 6)}
\end{aligned}$$

Thus, the theorem follows. □

5. Conclusions

In this research article, we proposed a generalized fractional Stockwell transform as an application of a generalized fractional convolution, on the space of square-integrable complex-valued functions. This transform generalizes the classical Stockwell transform, fractional Stockwell transform, and linear canonical transform. We proved Parseval's identity for GFST and used it to derive an inversion formula for the transform. A theorem describing the range of GFST was also proved. The paper was concluded by a discussion of a Heisenberg-type uncertainty principle associated with the generalized fractional Stockwell transform. We strongly believe this transform will be an alternative tool in signal processing wherever the Stockwell transform is applied.

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