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Well-Posedness of the Mild Solutions for Incommensurate Systems of Delay Fractional Differential Equations

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Abstract: Systems of incommensurate delay fractional differential equations (DFDEs) with non-vanishing constant delay of retarded type are investigated. It is shown that the mild solutions are well-posed in Hadamard sense on the space of continuous functions. The analysis is local and carried out for finite intervals. The strong results are obtained with weak conditions by using state-of-the-art new methods. No condition on the Lipschitz parameter is added for well-posedness results. Application of this theorem for the Hopfield neural network is carried out.

Keywords: fractional differential equation; well-posed in Hadamard sense; non-vanishing constant delay of retarded type; incommensurate systems; Hopfield neural networks

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1. Introduction

Fractional differential equations (FDEs) have found extensive use in the modeling of complex systems, primarily due to their ability to represent memory in systems dynamics [1,2]. Fortunately, there are various fractional derivatives, each having different memory kernels. This diversity, provide us with a rich set of operators capable of capturing the intricate dynamics of complex natural systems [3].

Similar to FDEs, delay differentiate equations (DDEs) have been widely applied in mathematical models, especially in biological phenomena [4]. However, DDEs are rarely studied and well-analyzed in connection to FDEs. The analysis of the linear delay fractional differential equation (DFDE),

$$\mathfrak{D}^{\psi}y(t) = ay(\theta(t)) + by(t) + f(t), \quad t > 0, \quad \psi \in (0,1), \quad a,b \in \mathbb{R}, \quad f \in \mathcal{C}[0,\mathfrak{T}], \quad (1)$$

has been explored in [5]. In their paper, the lag term is defined by $\theta(t) := t - \tau$ with a constant time delay $\tau > 0$. This study covers the uniqueness of the mild solution and the dependency of the mild solution on its parameters. The stability of the problem (1) has also been studied [6].

We note that such DFDEs are known as retarded ones since the lag term does not involve fractional derivatives. If the equation involves terms like $\mathfrak{D}^{\psi}y(\theta(t))$, the equation becomes a neutral type, which we do not study here. The existence result of fractional-order neutral time-delay systems can be found in [7].

Most applied problems can not be modeled by a simple one-D equations like (1). A state of natural systems involves more than one component that are connected to each other to achieve an aim. In mathematics, a system is described by more than one equation. If the dynamics of each state depend on other states, it is called a coupled system. FDEs and DFDEs are utilized in diverse modeling. Currently, there is abundant work in biology, mechanics, electronics, and other branch of science that use systems of FDEs in their modeling.

However, they usually use a common order for all equations. But there is no need for all states to have the same memory and thus the same order of derivatives [8,9]. Therefore, the systems that may have different orders for each equation must be important for modeling. Such systems are known as incommensurate systems [10,11]. Incommensurate FDEs have been used in the structure of fractional Hopfield neural networks (FHNNs) [11–13].

A linear incommensurate system of DFDEs with multiple delays (multi-delay) can be described by

$$\begin{cases} {}^{C}\mathfrak{D}^{\psi_1} y_1(t) & = a_{11}y_1(t) + \ldots + a_{1\nu}y_\nu(t) + b_{11}y_1(\theta_1(t)) + \ldots + b_{1\nu}y_\nu(\theta_\nu(t)) + f_1(t), \\ & \vdots \\ {}^{C}\mathfrak{D}^{\psi_\nu} y_\nu(t) & = a_{\nu 1}y_1(t) + \ldots + a_{\nu\nu}y_\nu(t) + b_{\nu 1}y_1(\theta_1(t)) + \ldots + b_{\nu\nu}y_\nu(\theta_\nu(t)) + f_\nu(t), \end{cases} \tag{2}$$

where $\psi_i \in (0,1)$, $a_{ij}, b_{i,j} \in \mathbb{R}$ and $f_i : [0,\mathfrak{T}] \to \mathbb{R} \in \mathcal{C}[0,\mathfrak{T}]$, $y_i : [0,\mathfrak{T}] \to \mathbb{R} \in \mathcal{C}[0,\mathfrak{T}]$, for $i,j = 1,\ldots,\nu$. This system can be expressed in shorthand as

$$\mathfrak{D}^{\Psi} Y(t) = AY(t) + BY(\Theta(t)) + F(t) \tag{3}$$

where $\Psi = [\psi_1,\ldots,\psi_\nu]^T \in (0,1)^\nu$, $Y = [y_1,\ldots,y_\nu]^T \in (\mathcal{C}[0,\mathfrak{T}])^\nu$, $F = [f_1,\ldots,f_\nu]^T \in (\mathcal{C}[0,\mathfrak{T}])^\nu$, $\Theta = [\theta_1,\ldots,\theta_\nu]^T$, $\theta_i(t) = t - \tau_i$, $\tau_i \in \mathbb{R}$, $A = (a_{ij})$ and $B = (b_{ij})$. The meaning of such vector operators later will be recalled. Problem (2) has been studied subject to delay condition

$$y_i(t) = g_i(t), \quad t \in (-\tau_i, 0]. \tag{4}$$

If we assume that all delays are equal (i.e., $\tau_i = \tau$ where $\tau > 0$ is a positive constant), then the initial condition and prehistoric conditions (4) can be expressed as

$$Y(t) = G(t), \quad t \in (-\tau, 0]. \tag{5}$$

Such delays are also referred to as a single/constant delay. In this case, the lag term is a single function, and we simply use θ to represent it, where $\theta(t) := t - \tau$. The nonlinear DFDE with a constant delay can be described by

$$\begin{cases} {}^{C}\mathfrak{D}^{\psi_1} y_1(t) & = u_1(t, y_1(t),\ldots,y_\nu(t), y_1(\theta(t)),\ldots,y_\nu(\theta(t))), \\ & \vdots \\ {}^{C}\mathfrak{D}^{\psi_\nu} y_\nu(t) & = u_\nu(t, y_1(t),\ldots,y_\nu(t), y_1(\theta(t)),\ldots,y_\nu(\theta(t))), \end{cases} \tag{6}$$

subject to condition (5). It can be written in a compact form as

$$\mathfrak{D}^{\Psi} Y(t) = U(t, Y(t), Y(\Theta(t))). \tag{7}$$

where $\Theta(t) = [\theta,\ldots,\theta]^T$. We note that the related operations of $Y(\Theta(t))$ are explained for the algebra of the vector-valued function in the next sections, and they are not composite functions.

There are studies on systems with various delays for each state [14]. Specifically, ref. [14] addresses the stability of a class of incommensurate DFDEs with multiple delays. A numerical method for nonlinear systems of DFDE with a single/constant delay $\tau_i = \tau$ and commensurate order $\psi_i = \psi$ has been studied in [15]. The stability of a class of commensurate systems (3) with a constant delay, where A is a zero matrix, is studied in [16].

The objective of this paper is to study System (6) with constant delay. Such systems may appear in research papers, but fundamental questions regarding these systems, such

as the existence of a unique solution and well-posedness are still unstudied. Existence results for systems of fractional equations typically utilize fixed-point theorems like the Banach fixed-point or Schauder fixed-point theorems. However, as discussed in [17] and other relative works, these types of theorems impose additional conditions on U or even $\mathfrak{T}$. Therefore, we apply the state-of-the-art method used in [17] to obtain strong results under weaker conditions. In the main theorems, we will find that continuity of the functions U and G and Lipschitz continuity of the function U with respect to Y is sufficient to guaranty the existence of the unique mild solution. Moreover, we show it is sufficient to ensure the continuous dependency of the mild solution to G, if we assume U is also Lipschitz continuous with respect to W.

Remark 1. *The fractional derivative and integral operators have two parameters. For example, in RL integral*

$$\mathfrak{I}^{\psi} y(t)(s) = \frac{1}{\Gamma(\psi)} \int_0^s (s-t)^{1-\psi} y(t) dt.$$

the variable t is the dummy variable, and s is the active variable. Usually, in the literature, the active variable is deleted. However, in this paper, for clarity, we may add the second variable to separate the dummy variable and active variable.

2. Vector-Valued Operational Algebra

Here, we clarify the algebra of vector-valued functions and their related operations through exact definitions. Suppose $\mathcal{Y}$ is a space of functions, and

$$F = [f_1, \dots, f_\nu]^T : [0, \mathfrak{T}] \to \mathbb{R}^\nu \in \mathcal{Y}^\nu$$

is a vector-valued function. For operators $o_i : \mathcal{Y} \to \mathcal{Y}$, we define the vector-operator

$$\mathfrak{O} = [o_1, \dots, o_\nu]^T : \mathcal{Y}^\nu \to \mathcal{Y}^\nu$$

by

$$\mathfrak{O}(F) = [o_1(f_1), \dots, o_\nu(f_\nu)]^T = \mathrm{Diag}([o_1, \dots, o_\nu])F.$$

Such an $\mathfrak{O}$ can be ${}^C D^{\Psi} = [{}^C D^{\psi_1}, \dots, {}^C D^{\psi_\nu}]^T$ or $\mathcal{I}^{\Psi} = [\mathcal{I}^{\psi_1}, \dots, \mathcal{I}^{\psi_\nu}]^T$.

The elementary operations such as addition, difference, product, division, and power operations of vector-valued functions are performed element-wise. For example, if $G = [g_1, \dots, g_\nu] : [0, \mathfrak{T}] \to \mathbb{R}^\nu$ is another vector-valued function

$$F \pm G = [f_1 \pm g_1, \dots, f_\nu \pm g_\nu]^T.$$

Let $\Theta = [\theta_1, \dots, \theta_\nu]^T : [0, \mathfrak{T}] \to \mathbb{R}^\nu$ be a vector-lag term. Clearly, $F(\Theta)$ is not well-defined as a composite function. However, we define it as vector-wise composite functional,

$$F(\Theta) = [f_1(\theta_1), \dots, f_\nu(\theta_\nu)]^T. \tag{8}$$

We note that, in our study $\Theta = [\theta, \dots, \theta]^T$ can be substituted by θ and serve as a composite function! Consequently, in our study, $F(\Theta) = F(\theta)$. However, in multi-delay cases, the use of element-wise composition is indispensable.

The operation of one-dimensional functions and operators with a scalar is inherited by all elements of the vector. For example, if $f : [0, \mathfrak{T}] \to \mathbb{R}$ is a scalar function, as well as $\Psi = [\psi_1, \dots, \psi_\nu]$, then $G = f^{\Psi}$ is defined by

$$G = [f^{\psi_1}, \dots, f^{\psi_\nu}]^T.$$

Example 1. *For* $\Psi = [\psi_1, \dots, \psi_\nu]^T$ *and* $P = [p_1, \dots, p_\nu]^T$, *we have*

$$\mathfrak{I}^{\Psi} t^{P} = \frac{\Gamma(P+1)}{\Gamma(P+\Psi+1)} t^{\Psi+P}. \tag{9}$$

We note that this equation is exactly similar to the computation of the non-vector case of $I^{\alpha} t^{\beta}$. *However, we should note that the notation here has a different meaning. To be more clear, the right-hand side of Equation* (9) *is*

$$[\frac{\Gamma(p_1+1)}{\Gamma(p_1+\psi_1+1)} t^{p_1+\psi_1}, \dots, \frac{\Gamma(p_\nu+1)}{\Gamma(p_\nu+\psi_\nu+1)} t^{p_\nu+\psi_\nu}]^T!$$

Theorem 1. *Let* $0 < \psi_i \leq 1$, $i = 1, \dots, \nu$ *and* $Y \in (\mathcal{AC}[0, \mathfrak{T}])^{\nu}$. *Then,*

$$\mathfrak{I}^{\Psi}\, {}^{C}D^{\Psi} Y = Y(t) - Y(0) \tag{10}$$

and

$${}^{C}D^{\Psi} \mathfrak{I}^{\Psi} Y(t) = Y(t). \tag{11}$$

Theorem 2 (Generalized Gronwall inequality [18,19]). *Suppose* $q, \mathfrak{T} > 0$, $c \geq 0$ *and* $y : [0, \mathfrak{T}] \to \mathbb{R}$ *is a locally integrable and no-negative function satisfying*

$$y(t) \leq c + M \int_0^t (t-a)^{\psi-1} y(s) ds \tag{12}$$

Then, $y(t) \leq c E_{\psi}(M\Gamma(\psi) t^{\psi})$.

3. Existence of a Unique Continuous Mild Solution with Single Delay

Applying the RL integral $\mathfrak{I}^{\Psi}$ to both sides of (3), we obtain

$$Y(t) = Y(0) + \mathfrak{I}^{\Psi} AY(t) + \mathfrak{I}^{\Psi} BY(\Theta(t)). \tag{13}$$

It is well-known that the solution to (13) may not be differentiated, or may not be in $\mathcal{AC}$ (see, for example, [20]). So, the solution of (13) may not satisfies the original Equation (3). But, in most applied mathematics, we need a solution of the model based on (3). Therefore, we use the adjective "mild" before the word solution to emphasize that the solution of associate integral may not be the solution of the original Equation (3).

We should note that interchanging the place of the fractional integral and constant matrix in System (13) can be problematic. While this interchange is valid for commensurate systems, it is incorrect for incommensurate systems. In particular, we have

$$\mathfrak{I}^{\Psi} AY \neq A\mathfrak{I}^{\Psi} Y. \tag{14}$$

This leads to additional complexities for incommensurate FDEs compared to commensurate FDEs.

Similarly, and in general, from (7), we can infer that the mild solution satisfies

$$Y(t) = Y(0) + \mathfrak{I}^{\Psi} U(t, Y(t), Y(\Theta(t))). \tag{15}$$

Definition 1. *We define* Y *as a mild solution of* (7) *under Condition* (5) *if it fulfills System* (15).

Remark 2. *A solution to* (15) *might not be differentiable and, therefore, might not satisfy the original Problem* (7) *under Condition* (5). *However, the solution to the original problem satisfies* (15).

The existence of a mild solution of incommensurate systems of FDEs has been established in [17]. Based on the discussion in that paper, the existence of a mild solution based on the Banach fixed-point theorem and the Schauder fixed-point theorem require stronger conditions. To obtain the weaker condition, the authors of [17] proposed a direct method using Cauchy sequences. They first established the result on the interval $[0, 1]$ and then extended the global existence using the tail part of the RL integral.

Now, consider a single delay, τ. Let $t \in (0, \tau]$, which implies that $t - \tau \in (-\tau, 0]$. Then, $Y(\Theta(t)) = G(\Theta(t))$, and Equation (15) becomes equivalent to

$$Y(t) = G(0) + \mathfrak{I}^{\Psi} U(t, Y(t), G(\Theta(t))), \quad t \in [0, \tau]. \tag{16}$$

We note that $G(\Theta(0)) = G(-\tau)$ is not defined. We assume $G(\Theta(0^+)) = G(-\tau^+) < \infty$, and extend G on $[-\tau, 0]$ by $G(-\tau) = G(-\tau^+)$ to use compactness of interval. It follows immediately that $G \in \mathcal{C}[-\tau, 0]$. Also, it is clear the value of $\mathfrak{I}^{\Psi}$ at $t = 0$ is zero for any continuous function.

We use the absolute value notation as a norm in $\mathbb{R}^{\nu}$, defined by

$$|Y| = \max_{i=1,\dots,\nu} |y_i|.$$

However, we use the norm notation for spaces of functions, especially for $F \in (C[0, \mathfrak{T}])^{\nu}$

$$\|F\| = \sup_{t \in [0,\mathfrak{T}]} |F(t)| = \sup_{t \in [0,\mathfrak{T}]} \max_{i=1,\dots,\nu} |f_i(t)|.$$

Now, consider the following hypotheses:

(H1) U is globally Lipschitz continuous with respect to Y, i.e., $\exists\, L_i \in \mathbb{R}$, such that

$$|u_i(t, Y, W) - u_i(t, Z, W)| \leq L_i |Y - Z|$$

for all $t \in [0, \mathfrak{T}]$, and for all $Z, Y, W \in \mathbb{R}^{\nu}$.

(H2) $U : [0, \mathfrak{T}] \times \mathbb{R}^{\nu} \times \mathbb{R}^{\nu} \to \mathbb{R}^{\nu}$ is continuous with respect to its domain.

(H3) G is a continuous function on $(-\tau, 0]$ and $\lim_{t \to \tau} g_i(t) < \infty$.

Remark 3. *Letting $L = \max_{i=1,\dots,\nu} |L_i|$, it follows from (H1) that*

$$|U(t, Y, W) - U(t, Z, W)| \leq L|Y - Z|, \quad \forall t \in [0, \mathfrak{T}], \forall Z, Y, W \in \mathbb{R}^{\nu}.$$

Remark 4. *More precisely, (H3) states that if we extend G to $[-\tau, 0]$ by $G(-\tau) = \lim_{t \to \tau} G(t)$, then it is continuous on $[-\tau, 0]$. Thus, the condition (H3) can be expressed as "G is continuous on $[-\tau, 0]$".*

Theorem 3. *Let Hypotheses (H1)–(H3) hold. Then, System (7) subject to Condition (5) processes a mild continuous solution on $[0, \tau]$.*

Proof. The proof is similar to Theorem 7 of [17], so we provide a proof sketch here. First, we assume $\tau \leq 1$, introduce a Picard operator $\mathfrak{P}$ by

$$\mathfrak{P}Y = G(0) + \mathfrak{I}^{\Psi} U(t, Y(t), G(\Theta(t))) \tag{17}$$

and we show that the functions

$$Y_{n+1} = \mathfrak{P}Y_n, \quad Y_0 = G_0, \quad n = 0, 1, \dots$$

are Cauchy sequences in $(\mathcal{C}[0,\tau])^{\nu}$. Consequently, they have a uniform limit, say Y. By the uniform convergence theorem for fractional integrals, we have

$$\begin{aligned}\lim_{N\to\infty} Y_{n+1} &= \lim_{N\to\infty} \mathfrak{P}Y_n = G(0) + \mathfrak{I}^{\Psi} U(t, Y_N(t), G(\Theta(t))) \\ &= G(0) + \mathfrak{I}^{\Psi} \lim_{N\to\infty} U(t, Y_N(t), G(\Theta(t))).\end{aligned} \tag{18}$$

Since U is continuous, we can rewrite the above as

$$\lim_{N\to\infty} Y_{n+1} = G(0) + \mathfrak{I}^{\Psi} U(t, \lim_{N\to\infty} Y_N(t), G(\Theta(t))). \tag{19}$$

Thus, Y satisfies Equation (16). Now, assume that $1 < \tau \leq 2$. Given that a solution exists on $[0,1]$, w can decompose the $\mathfrak{I}^{\Psi}$ into two operators: the Fredholm operator $\mathfrak{F}^{\Psi}$ and the Volterra operator $\mathfrak{V}^{\Psi}$. The Fredholm operator is defined as

$$\mathfrak{F}^{\Psi} Y(t) = \frac{1}{\Gamma(\Psi)} \int_0^1 (t-z)^{\Psi-1} Y(z) dz \tag{20}$$

and the Volterra operator is defined as

$$\mathfrak{V}^{\Psi} Y(t) = \frac{1}{\Gamma(\Psi)} \int_1^t (t-z)^{\Psi-1} Y(z) dz. \tag{21}$$

We know that $\mathfrak{I}^{\Psi} = \mathfrak{F}^{\Psi} + \mathfrak{V}^{\Psi}$. Therefore, Equation (16) can be written as

$$Y(t) = G(0) + \mathfrak{F}^{\Psi} U(., Y(.), G(\Theta(.)))(t) + \mathfrak{V}^{\Psi} U(., Y(.), G(\Theta(.)))(t), \quad t \in [0,\tau]. \tag{22}$$

We have already established the existence of a solution of (16) on $[0,1]$. For convenience, we rename Y on $[0,1]$ by Y_1. Then, the tail of Equation (22) is known function say Z, i.e.,

$$Z(t) := G(0) + \mathfrak{F}^{\Psi} U(., Y_1(.), G(\Theta(.)))(t).$$

Thus,

$$Y(t) = Z(t) + \mathfrak{V}^{\Psi} U(., Y(.), G(\Theta(.)))(t), \quad t \in [1,\tau]. \tag{23}$$

Substituting $t = s+1$ gives

$$Y(s+1) = Z(s+1) + \frac{1}{\Gamma(\Psi)} \int_1^{s+1} (s+1-z)^{\Psi-1} U(z, Y(z), G(\Theta(z))) dz, \tag{24}$$

where $s \in [0, \tau-1]$. If we substitute $z = r+1$, we obtain

$$Y(s+1) = Z(s+1) + \frac{1}{\Gamma(\Psi)} \int_0^s (s-r)^{\Psi-1} U(r+1, Y(r+1), G(\Theta(r+1))) dr. \tag{25}$$

It is interesting to note that when considering the Volterra operator at the peak of dynamical System (25), it becomes an RL integral operate, and by renaming $Y_2(s) = Y(s+1)$, we obtain

$$Y_2(s) = Z(s+1) + \mathfrak{J}^{\Psi} U(r+1, Y_2(r), G(\Theta(r+1)))(s), \quad s \in [0, \tau-1]. \tag{26}$$

Equation (26) is of the same form as (16). Consequently, the existence of a unique $Y_2 \in (\mathcal{C}[0,\tau-1])^{\nu}$ is guaranteed in the same way. Conclusively,

$$Y(t) = \begin{cases} Y_1(t), & t \in [0,1], \\ Y_2(t-1), & t \in [1,\tau]. \end{cases} \tag{27}$$

Clearly, $Y_2(1-1) = Y_2(0) = Z(1) = Y_1(1)$ and Y is a continuous solution to (16) on $[0, \tau]$. By induction, (16) has a unique continuous solution for any $\tau > 0$. □

The uniqueness of the solution follows from the generalized Gronwall inequality.

Theorem 4. *Let Hypotheses (H1)–(H3) hold. Then, System* (7) *subject to condition* (5) *has a unique mild continuous solution on* $[0, \tau]$.

Proof. The proof is similar to Theorem 10 of [17]. First, we show the uniqueness for $\tau \leq 1$. Let X and Y be two solutions. Then,

$$|x_i(t) - y_i(t)| \leq |\mathfrak{I}^{\psi_i} u_i(t, x_i(t), g_i(\theta_i(t))) - \mathfrak{I}^{\psi_i} u_i(t, y_i(t), g_i(\theta_i(t)))|, \quad t \in [0, \tau] \subseteq [0, 1].$$

Since $t \leq 1$, it follows from Hypotheses (H1) that

$$\begin{aligned} |x_i(t) - y_i(t)| &\leq \frac{L_i}{\Gamma(\alpha_i)} \int_0^t (t-z)^{\psi_i - 1} |X(z) - Y(z)| dz \\ &\leq M^* \int_0^t (t-z)^{\psi^* - 1} |X(z) - Y(z)| dz \end{aligned} \tag{28}$$

where $M* = \max_{i=1,\dots,\nu} \frac{L_i}{\Gamma(\alpha_i)}$ and $\psi^* = \min_{i=1,\dots,\nu} \psi_i$. We note that Inequality (28) is independent of i. Thus,

$$|X(t) - Y(t)| \leq M^* \int_0^t (t-z)^{\psi^* - 1} |X(z) - Y(z)| dz. \tag{29}$$

Immediately, it follows from the generalized Gronwall inequity that $|X(t) - Y(t)| = 0$, and $X(t) = Y(t)$ for all $t \in [0, 1]$. Now, let $\tau \in [0, 2]$. From a previous argument, we already know that $X(t) = Y(t)$ for $t \in [0, 1]$. From an argument similar to the proof of Theorem (3), Y and X satisfies Equation (25) while the definition of Z only uses the information of $Y(t)$ on $[0, 1]$. Since Y is unique on $[0, 1]$, the tail function Z is unique for all $t \in [0, \tau]$. It follows from (25) that

$$|X(s+1) - Y(s+1)| \leq M^* \int_0^s (s-z)^{\psi^* - 1} |X(r+1) - Y(r+1)| dr, \tag{30}$$

for $s \in [0, \tau - 1] \subseteq [0, 1]$. Therefore, from the generalized Gronwall inequality,

$$X(s+1) = Y(s+1), \quad s \in [0, \tau - 1]$$

or, equivalently, $X(t) = Y(t)$ on $[1, \tau]$ and, thus, on $[0, \tau]$. Similar induction can be used to prove that $X(t) = Y(t)$ for any delay $\tau > 0$. □

Up to this point, we have determined that System (1) has a unique mild solution within the interval $[0, \tau]$, and this solution is continuous. Now, we establish the existence of a unique continuous solution for any arbitrary interval $[0, \mathfrak{T}]$.

Theorem 5. *Let Hypotheses (H1)–(H3) hold. Then, System* (7) *subject to Condition* (5) *processes a unique mild continuous solution on* $[0, \mathfrak{T}]$.

Proof. If $\mathfrak{T} < \tau$, the claim follows from Theorem 4. Suppose $\tau < \mathfrak{T} \leq 2\tau$. We already know that there exists a solution on $[0, \tau]$. Denote this solution as Y_1. Then, System (15) is equivalent to

$$Y(t) = G(0) + \mathfrak{F}_\tau^\Psi U(z, Y_1(z), G(z-\tau))(t) + \mathfrak{V}_\tau^\Psi U(z, Y(z), Y_1(z-\tau))(t) \tag{31}$$

for $t \in [\tau, \mathfrak{T}]$, where

$$\mathfrak{F}_{\tau}^{\Psi} Y(t) = \frac{1}{\Gamma(\Psi)} \int_0^{\tau} (t-z)^{\Psi-1} Y(z) dz \tag{32}$$

is a Fredholm operator, and

$$\mathfrak{V}_{\tau}^{\Psi} Y(t) = \frac{1}{\Gamma(\Psi)} \int_{\tau}^{t} (t-z)^{\Psi-1} Y(z) dz \tag{33}$$

is a Volterra operator. For the tail of System (31) $Z(t) = G(0) + \mathfrak{F}_{\tau}^{\Psi} U(s, Y_1(s), G(s-\tau))(t)$ the Fredholm operator only depends on the values of $Y_1(s)$ and $G_1(s-\tau)$ on $s \in [0, \tau]$. Since Y_1 is known to be a unique, Z is well-defined unique function. Conclusively, by substituting $t = r + \tau$, we have (31),

$$Y(r+\tau) = Z(r+\tau) + \mathfrak{V}_{\tau}^{\Psi} U(s, Y(s), Y_1(s-\tau))(r+\tau). \tag{34}$$

Regarding the peak operator, by substituting $z = s + \tau$, we obtain

$$\begin{aligned} \mathfrak{V}_{\tau}^{\Psi} U(s, Y(s), Y_1(s-\tau))(r+\tau) =& \frac{1}{\Gamma(\Psi)} \int_{\tau}^{r+\tau} (r+\tau-z)^{\Psi-1} U(z, Y(z), Y_1(z-\tau)) dz \\ =& \frac{1}{\Gamma(\Psi)} \int_0^{r} (t-z)^{\Psi-1} U(s+\tau, Y(s+\tau), Y_1(s)) ds \\ =& \mathfrak{J}^{\Psi} U(s+\tau, Y(s+\tau), Y_1(s))(r). \end{aligned} \tag{35}$$

If we rename $Y_2(t) := Y(t+\tau)$ and consider Equation (35), then Equation (34) can be written as

$$Y_2(r) = Z(r+\tau) + \mathfrak{J}^{\Psi} U(s+\tau, Y_2(s), Y_1(s))(r), \quad r \in [0, \mathfrak{T}-\tau]. \tag{36}$$

Noticeably, we convert an equation on $[\tau, \mathfrak{T}]$ into a similar type of equation on the interval $[0, \mathfrak{T}-\tau] \subseteq [0, \tau]$. Therefore, we can use the same reasoning as in Theorems 3 and 4 to show that System (36) has a unique continuous solution on $[0, \mathfrak{T}-\tau]$.

We note that the only difference between Equations (16) and (36) is that $G(0)$ is replaced by $Z(r+\tau)$, which does not effect the proof. Thus, System (15) has a unique solution, given by

$$y(t) = \begin{cases} G(t), & t \in (-\tau, 0], \\ Y_1(t), & t \in (0, \tau], \\ Y_2(t-\tau), & t \in (\tau, \mathfrak{T}]. \end{cases} \tag{37}$$

It is clear that $Y(\tau^+) = Y_2(0^+) = Z(\tau^+) = Z(\tau) = Y_1(\tau)$, indicating that Y is continuous at τ. Moreover, $Y(0^+) = Y_1(0^+) = Y_1(0) = G(0) = Y(0)$, showing that Y is also continuous at 0. Conclusively, $Y \in (\mathcal{C}[0, \mathfrak{T}])^{\nu}$. This completes the proof for this case. The same recursive argument can be applied to prove the cases when $\mathfrak{T}$ belongs to $[2\tau, 3\tau]$, $[3\tau, 4\tau]$ and so on, thereby completing the proof. □

4. Transforming DFDEs into Fractional Integral Equations Without Delays

The existence analysis above presents a constructive approach to study DFDEs. Let $n = \text{Ciel}\left(\frac{T}{\tau}\right)$, meaning that $n-1 < \frac{T}{\tau} \leq n$. Let Y_j represent the solution on the interval $[(j-1)\tau, j\tau]$ for $j = 1, \ldots, n$. As stated previously, for $j < n$, we can define the Fredholem operators related to the system's memory on $[(j-1)\tau, j\tau]$ as follows:

$$\mathfrak{F}_{j\tau}^{\Psi} Y(t) = \frac{1}{\Gamma(\Psi)} \int_{(j-1)\tau}^{j\tau} (t-z)^{\Psi-1} Y_j(z) dz. \tag{38}$$

It should be noted that while this operator utilizes the information of Y at $[(j-1)\tau, j\tau]$, the variable t can take on arbitrary values. In particular, this operator is well-defined for $[0, \infty)$. Consequently, the contribution of Y within the jth interval regarded to the memory of the Fredholm operator (38) is distributed over future time.

Theorem 6. *The memory associated with the fixed interval j is fading as t increases.*

Proof. From Theorem (5), Y_j is continuous. Assume $t_2 > t_1 \geq 1 + j\tau$. Then, $t_2 - z > t_1 - z > 1$ and $(t_2 - z)^{1-\psi_i} > (t_1 - z)^{1-\psi_i} \geq 1$ for $z \in [(j-1)\tau, j\tau]$. Therefore,

$$(t_2 - z)^{\Psi-1}|Y(z)| < (t_1 - z)^{\Psi-1}|Y(z)| \leq 1$$

where the inequities are component-wise. Thus,

$$\mathfrak{F}_{j\tau}^{\Psi}|Y|(t_2) \leq \mathfrak{F}_{j\tau}^{\Psi}|Y|(t_1).$$

Therefore, $\mathfrak{F}_{j\tau}^{\Psi}|Y|$ is decreasing component-wise for $t > 1 + j\tau$. Since Y is continuous, and $(t - z)^{\Psi-1}Y(z) \to [0, \ldots, 0]^T$ uniformly as $t \to \infty$, we conclude that

$$\mathfrak{F}_{j\tau}^{\Psi}Y(t) \to [0, \ldots, 0]^T$$

as $t \to \infty$. This indicates that the memory of jth interval fades to zero. □

Finally, we can define the peak information of $\mathfrak{J}^{\Psi}$ by Volterra operator,

$$\mathfrak{V}_{j\tau}^{\Psi}Y(t) = \frac{1}{\Gamma(\Psi)}\int_{j\tau}^{t}(t - z)^{\Psi-1}Y(z)dz \tag{39}$$

where $t \in [j\tau, (j+1)\tau]$, for $j = 0, \ldots, n-2$, and $t \in [(n-1)\tau, \mathfrak{T}]$ when $j = n-1$. We show that the translation of the peak operator is precisely the RL operator of a translation of Y.

Theorem 7. *For $j = 0, \ldots, n-1$, we have*

$$\mathfrak{V}_{j\tau}^{\Psi}Y(t + j\tau) = \mathfrak{J}^{\Psi}Y(s + j\tau)(t). \tag{40}$$

Proof. Substitute $s = z - j\tau$ into (39). This gives

$$\mathfrak{V}_{j\tau}^{\Psi}Y(t) = \frac{1}{\Gamma(\Psi)}\int_{0}^{t-j\tau}(t - s - j\tau)^{\Psi-1}Y(s + j\tau)ds. \tag{41}$$

Now, consider the value of this operator when t is replaced by $t + j\tau$,

$$\mathfrak{V}_{j\tau}^{\Psi}Y(t + j\tau) = \frac{1}{\Gamma(\Psi)}\int_{0}^{t}(t - s)^{\Psi-1}Y(s + j\tau)ds = \mathfrak{J}^{\Psi}Y(s + j\tau)(t) \tag{42}$$

where $t \in [0, \tau]$ for $j < n-1$ and $t \in [0, \mathfrak{T} - (n-1)\tau]$ if $j = n-1$. □

For convenience, we define $Y_0(t) = G(t)$ on the interval $(-\tau, 0]$. Then, we have the following series of equations: For $t \in (0, \tau]$, Y_1 is given by

$$Y_1(t) = G(0) + \mathfrak{J}^{\Psi}U(s, Y_1(s), Y_0(\Theta(s)))(t), \tag{43}$$

for $t \in (\tau, 2\tau]$, Y_2 is expressed as

$$Y_2(t) = G(0) + \mathfrak{F}_{\tau}^{\Psi}U(s, Y_1(s), Y_0(\Theta(s)))(t) + \mathfrak{V}_{\tau}^{\Psi}U(s, Y_2(s), Y_1(\Theta(s)))(t),$$

and for $j < n$, and $t \in ((j-1)\tau, j\tau]$, Y_j is obtained by

$$\begin{aligned} Y_j(t) =& G(0) + \sum_{i=1}^{j-1} \mathfrak{F}_{i\tau}^{\Psi} U(s, Y_i(s), Y_{i-1}(\Theta(s)))(t) \\ &+ \mathfrak{V}_{(j-1)\tau}^{\Psi} U(s, Y_j(s), Y_{j-1}(\Theta(s)))(t), \end{aligned} \tag{44}$$

For final interval $t \in ((n-1)\tau, \mathfrak{T}]$, where $j = n$, we have

$$\begin{aligned} Y_n(t) =& G(0) + \sum_{i=1}^{n-1} \mathfrak{F}_{i\tau}^{\Psi} U(s, Y_i(s), Y_{i-1}(\Theta(s)))(t) \\ &+ \mathfrak{V}_{(n-1)\tau}^{\Psi} U(s, Y_n(s), Y_{n-1}(\Theta(s)))(t). \end{aligned} \tag{45}$$

We define the tail function as

$$Z_j(t) = G(0) + \sum_{i=1}^{j-1} \mathfrak{F}_{i\tau}^{\Psi} U(s, Y_i(s), Y_{i-1}(\Theta(s)))(t).$$

By using Theorem 7, we obtain integral equations

$$Y_j(t + j\tau) = Z_j(t + j\tau) + \mathfrak{J}^{\Psi} U(s + j\tau, Y_j(s + j\tau), Y_{j-1}(\Theta(s + j\tau)))(t) \tag{46}$$

for $j = 1, \ldots, n-1$, $t \in (0, \tau]$, and $j = n$, $t \in (0, \mathfrak{T} - (n-1)\tau] \subseteq (0, \tau]$. It is clear that Equation (46) is an integral equation with respect to $Y_j(t + j\tau)$.

5. Well-Posedness

The well-posedness of a mathematical problem in the Hadamard sense requires three conditions: existence, uniqueness, and stability. Until now, we have provided the first two conditions. In this section, we investigated the third condition. The stability for the well-posedness of a problem implies that the solution depends continuously to the data. In particular, for initial value problems, the data consists of initial conditions. Here, the given data are a function G which is well-defined at $(-\tau, 0]$. To ensure the stability, let emphasize the dependence of Y to G through $Y(t, G)$. Let $\tilde{Y}(t, \tilde{G})$ be a solution to (46) corresponding with $\tilde{G}$. This implies $\tilde{Y}_0(t) = \tilde{G}(t)$ for $t \in (0, \tau]$ and $\tilde{Y}_j$ satisfies the systems described by (46). To establish well-posedness we need a slightly stronger assumption than (H1). This assumption involves adding Lipschitz continuity for the third parameter of U.

(H4) U is globally Lipschitz continuous with respect to Y and W, i.e., $\exists\, L_i \in \mathbb{R}$, such that

$$|u_i(t, Y, W) - u_i(t, \hat{Y}, \hat{W})| \leq L_i \max\{|Y - \hat{Y}|, |W - \hat{W}|\}$$

for all $t \in [0, \mathfrak{T}]$ and for all $Y, W, \hat{Y}, \hat{W} \in \mathbb{R}^{\nu}$.

Theorem 8. *Let Hypotheses (H2)–(H4) hold. Then, the mild solution of System (7) subject to Condition (5) is Lipschitz continuous with respect to G on the finite interval* $[0, \mathfrak{T}]$.

Proof. Let $M = \max \frac{L_i}{\Gamma(\alpha_i)}$. Then,

$$\begin{aligned} |Y_1(t) - \tilde{Y}_1(t)| \leq& |G(0) - \tilde{G}(0)| \\ &+ M \int_0^t (t-z)^{\psi^*-1} \max\{|Y_1(z) - \hat{Y}_1(z)|, |G(z) - \hat{G}(z)|\} \end{aligned} \tag{47}$$

for $t \leq 1$, where $\psi^* = \min \psi_i$. Therefore,

$$\begin{aligned}|Y_1(t) - \tilde{Y}_1(t)| \leq & |G(0) - \tilde{G}(0)| \\ & + M \int_0^t (t-z)^{\psi^*-1} |Y_1(z) - \hat{Y}_1(z)| dz \\ & + M \int_0^t (t-\tau)^{\psi^*-1} |G(z) - \hat{G}(z)| dz\end{aligned}$$

for $t \leq 1$. Thus,

$$\begin{aligned}|Y_1(t) - \tilde{Y}_1(t)| \leq & |G(0) - \tilde{G}(0)| + M \int_0^t (t-z)^{\psi^*-1} |Y_1(z) - \hat{Y}_1(z)| dz \\ & + M \int_0^t (t-\tau)^{\psi^*-1} |G(z) - \hat{G}(z)| dz.\end{aligned}$$

It follows that

$$|Y_1(t) - \tilde{Y}_1(t)| \leq \left(1 + \frac{Mt^{\psi^*}}{\psi^*}\right) \|G - \tilde{G}\| + M \int_0^t (t-z)^{\psi^*-1} |Y_1(z) - \hat{Y}_1(z)| dz \tag{48}$$

and, thus,

$$|Y_1(t) - \tilde{Y}_1(t)| \leq \left(1 + \frac{Mt^{\psi^*}}{\psi^*}\right) E_{\psi^*}(M\Gamma(\psi^*) t^{\psi^*}) \|G - \tilde{G}\|. \tag{49}$$

Consequently,

$$|Y_1(t) - \tilde{Y}_1(t)| = C_1 \|G - \tilde{G}\|, \quad t \in [0,1], \tag{50}$$

where $C_1 := \max_{t \in [0,1]} \left(1 + \frac{Mt^{\psi^*}}{\psi^*}\right) E_{\psi^*}(M\Gamma(\psi^*) t^{\psi^*})$. This completes the proof on $[0, \tau]$ when $\tau \leq 1$. Now, consider $\tau \in [0,2]$. For $t > 1$, we have

$$\begin{aligned}|Y_1(t) - \tilde{Y}_1(t)| \leq & |G(0) - \tilde{G}(0)| + M \int_0^t |(t-z)^{\Psi-1}| |Y_1(z) - \hat{Y}_1(z)| dz \\ & + M \int_0^t |(t-z)^{\Psi-1}| |G(z) - \hat{G}(z)| dz \\ \leq & |G(0) - \tilde{G}(0)| + M \int_0^1 |(t-z)^{\Psi-1}| |Y_1(z) - \hat{Y}_1(z)| dz \\ & + M \int_1^t |(t-z)^{\Psi-1}| |Y_1(z) - \hat{Y}_1(z)| dz + M \int_0^t |(t-z)^{\Psi-1}| |G(z) - \hat{G}(z)| dz \\ \leq & \|G - \tilde{G}\| + M \max_{z \in [0,1]} |Y_1(z) - \hat{Y}_1(z)| \Omega + M \int_1^t |(t-z)^{\Psi-1}| |Y_1(z) - \hat{Y}_1(z)| dz \\ & + M \|G - \hat{G}\| \Omega \\ \leq & (1 + MC_1\Omega + M\Omega) \|G - \hat{G}\| + M \int_1^t |(t-z)^{\Psi-1}| |Y_1(z) - \hat{Y}_1(z)| dz\end{aligned} \tag{51}$$

where

$$\Omega := \max_{t \in [0,2]} \int_0^t |(t-z)^{\Psi-1}| dz$$

and, clearly, $\Omega < \infty$. It immediately follows that

$$|Y_1(t+1) - \tilde{Y}_1(t+1)| \leq C_2 \|G - \hat{G}\| + M \int_0^t (t-s)^{\psi^*-1} |Y_1(s+1) - \hat{Y}_1(s+1)| ds \tag{52}$$

for $t \in [0,1]$ where $C_2 = 1 + MC_1\Omega + M\Omega$. By applying the generalized Gronwall inequality, we obtain

$$|Y_1(t+1) - \tilde{Y}_1(t+1)| \leq C_2 E_{\psi^*}(M\Gamma(\psi^*)t^{\psi^*})\|G - \hat{G}\|, \quad t \in [0,1] \tag{53}$$

and, thus,

$$|Y_1(t) - \tilde{Y}_1(t)| \leq C_2 E_{\psi^*}(M\Gamma(\psi^*)(t-1)^{\psi^*})\|G - \hat{G}\|, \quad t \in [1,2]. \tag{54}$$

Particularly, by defining $C_3 = \max\{C_1, \max_{t\in[0,1]} C_2 E_{\psi^*}(M\Gamma(\psi^*)(t-1)^{\psi^*})\}$, we have

$$|Y_1(t) - \tilde{Y}_1(t)| \leq C_3\|G - \hat{G}\|, \quad t \in [0,\tau]. \tag{55}$$

Using the same induction, Equation (55) holds for any $\tau < \infty$. Now we consider the dynamics of the system for $t > \tau$. Suppose that

$$|Y(t) - \tilde{Y}(t)| \leq C\|G - \hat{G}\|, \quad t \in [0,(j-1)\tau], j \geq 1 \tag{56}$$

for some constant C. We provide the same result on $[(j-1)\tau, j\tau]$. We note that, by definition of Z_j,

$$Z_j(t) = G(0) + \frac{1}{\Gamma(\Psi)}\sum_{i=1}^{j-1}\int_{(i-1)\tau}^{i\tau}(t-s)^{\Psi-1}U(s, Y_i(s), Y_{i-1}(\Theta(s)))ds. \tag{57}$$

Thus,

$$\begin{aligned}|Z_j(t) - \tilde{Z}_j(t)| \leq & |G(0) - \tilde{G}(0)| \\ & + M\sum_{i=1}^{j-1}\int_{(i-1)\tau}^{i\tau}|(t-s)^{\Psi-1}|\max\{|Y_i(s) - \tilde{Y}_i|, |Y_{i-1}(\Theta(s)) - \tilde{Y}_{i-1}(\Theta(s))|\}ds\end{aligned} \tag{58}$$

For $i \leq j-1$, based on the assumption, $|Y_i(s) - \tilde{Y}_i| \leq C\|G - \hat{G}\|$ and $|Y_{i-1}(\Theta(s)) - Y_{i-1}(\Theta(s))| \leq C\|G - \hat{G}\|$. Therefore, for $t \in [(j-1)\tau, j\tau]$,

$$\begin{aligned}|Z_j(t) - \tilde{Z}_j(t)| \leq & |G(0) - \tilde{G}(0)| + MC\|G - \hat{G}\|\sum_{i=1}^{j-1}\int_{(i-1)\tau}^{i\tau}|(t-s)^{\Psi-1}|ds \\ \leq & \|G - \tilde{G}\| + MC\|G - \hat{G}\|\int_0^{(j-1)\tau}|(t-s)^{\Psi-1}|ds, \\ \leq & (1 + MC\Omega_j)\|G - \tilde{G}\|\end{aligned} \tag{59}$$

where

$$\Omega_j := \max_{t\in[(j-1)\tau, j\tau]}\int_0^t |(t-z)^{\Psi-1}|dz \leq \infty.$$

Using (46), we obtain

$$\begin{aligned}&|Y_j(j\tau + t) - \tilde{Y}_j(j\tau + t)| \leq |Z_j(j\tau + t) - \tilde{Z}_j(j\tau + t)| \\ &\quad + M\int_0^t |(t-s)^{\Psi-1}|\max\{|Y_i(j\tau + s) - \tilde{Y}_i(j\tau + s)|, |Y_{i-1}(\Theta(s)) - \tilde{Y}_{i-1}(\Theta(j\tau + s))|\}ds \\ &\leq (1 + MC\Omega_j)\|G - \tilde{G}\| \\ &\quad + M\int_0^t |(t-s)^{\Psi-1}|\max\{|Y_i(j\tau + s) - \tilde{Y}_i(j\tau + s)|, |Y_{i-1}(\Theta(j\tau + s)) - \tilde{Y}_{i-1}(\Theta(j\tau + s))|\}ds\end{aligned} \tag{60}$$

This equation is similar to (47). A similar analysis shows that $|Y_j(j\tau + t) - \tilde{Y}_j(j\tau + t)|$ is proportional to $\|G - \tilde{G}\|$. We need prove this separately for τ in $[0,1]$, $[1,2]$, and so

on. We briefly carry the proof for the case $\tau \in [0,1]$, while it is also similar to previous discussions. From (60), we have

$$\begin{aligned}|Y_j(j\tau+t)-\tilde{Y}_j(j\tau+t)| \leq & (1+MC\Omega_j)\|G-\tilde{G}\| \\ & + M\int_0^t |(t-s)^{\psi^*-1}||Y_i(j\tau+s)-\tilde{Y}_i(j\tau+s)|ds \\ & + M\int_0^t |(t-s)^{\psi^*-1}||Y_{i-1}(\Theta(j\tau+s))-\tilde{Y}_{i-1}(\Theta(j\tau+s))|\}ds \\ \leq & (1+MC\Omega_j+\frac{MCt^{\psi^*}}{\psi^*})\|G-\tilde{G}\| \\ & + M\int_0^t |(t-s)^{\psi^*-1}||Y_i(j\tau+s)-\tilde{Y}_i(j\tau+s)|ds\end{aligned} \tag{61}$$

By applying the generalized Gronwall inequality, we obtain

$$|Y_j(j\tau+t)-\tilde{Y}_j(j\tau+t)| \leq (1+MC\Omega_j+\frac{MCt^{\psi^*}}{\psi^*})E_{\psi^*}(M\Gamma(\psi^*)t^{\psi^*})\|G-\tilde{G}\|. \tag{62}$$

This shows that

$$|Y_j(t)-\tilde{Y}_j(t)| \leq C_4\|G-\tilde{G}\|, \quad t \in j\tau+[0,\tau]=[j\tau,(j+1)\tau], \quad \tau \leq 1, \tag{63}$$

where C_4 is a finite constant, defined by

$$C_4 = \max_{t\in[0,1]}(1+MC\Omega_j+\frac{MCt^{\psi^*}}{\psi^*})E_{\psi^*}(M\Gamma(\psi^*)t^{\psi^*}).$$

□

6. Examples and Applications

6.1. Applications in Hopfield Neural Network

An interesting incommensurate system of DFDEs is used in [11] to describe a Hopfield neural network (HNN). HNNs, a form of recurrent artificial neural network, are particularly known for their ability to store and retrieve memory patterns efficiently, making them suitable for numerous practical uses.

In this section, we aim to study the well-posedness of a class of such HNNs. In terms of our notation, such HNNs can be described by the incommensurate System (6), with

$$\begin{aligned}u_1(t,Y,W) &= -0.1y_1+1.5\tanh(w_3)+0.8\tanh(w_4)+1.9\tanh(w_5),\\ u_2(t,Y,W) &= -0.6y_2-0.6\tanh(w_3)-0.5\tanh(w_4)+2.2\tanh(w_5),\\ u_3(t,Y,W) &= -0.1y_3-1.5\tanh(y_1)+2.5\tanh(y_2),\\ u_4(t,Y,W) &= -0.8y_4+0.5\tanh(y_1)-0.8\tanh(y_2),\\ u_5(t,Y,W) &= -1.3y_5+\tanh(y_1)-2\tanh(y_2).\end{aligned} \tag{64}$$

The stability of such systems has been studied in [11]. The hyperbolic tangent function tanh is a Lipschitz continuous function with Lipschitz constant 1. Therefore, the functions u_i for $i=1,\ldots,5$ are also Lipschitz continuous with respect to Y and W. Thus, Hypotheses (H1) and (H4) automatically hold. Particularly, $L_1=\max\{0.1,1.5,0.8,1.9\}=1.9$, $L_2=\max\{0.6,0.5,2.2\}=2.2$, $L_3=\max\{0.1,1.5,2.5\}=2.5$, $L_4=\max\{0.8,0.5\}=0.8$, and $L_5=\max\{1.3,1,2\}=2$. Also, each u_i is continuous with respect to its variable, thus (H2) holds. Assuming that G satisfies (H3), according to Theorem 5, the Hopfield neural network (64) has a unique mild solution. According to Theorem 8, the solution is Lips-

chitz continuous with respect to G, indicating that the Hopfield neural network (64) is well-posed.

6.2. A General Linear Case

Equation (3) represents a linear incommensurate systems of DFDEs with constant coefficients. However, this equation can be generalized to the form with variable coefficients as follows:

$$\mathfrak{D}^{\Psi}Y(t) = A(t)Y(t) + B(t)Y(\Theta(t)) + F(t) \tag{65}$$

where $A \in (\mathcal{C}[0,\mathfrak{T}])^{\nu\times\nu}$ and $B \in (\mathcal{C}[0,\mathfrak{T}])^{\nu\times\nu}$.

In terms of presentations (7), we have

$$U(t,Y,W) = A(t)Y + B(t)W + F(t).$$

Since a_{ij} and b_{ij} are assumed to be continuous, they are bounded on the compact interval $[0,\mathfrak{T}]$. Therefore,

$$L_i = \max_{j=1,\dots,\nu}\{\max_{t\in[a,b]}|a_{ij}(t)|, \max_{t\in[a,b]}|b_{ij}(t)|\}$$

is a well-defined real number, and Hypothesis (H4) holds with this L_i for all $i = 1,\dots,\nu$. Consequently, according to Theorems 5 and 8, it has a unique continuous mild solution, and the solution is Lipschitz continuous with respect to any continuous G. For instance, a complex system described by

$$\begin{aligned}\mathfrak{D}^{0.5}y_1(t) &= \sin(t)y_1(t) - \exp(t)y_2(t-0.5) + \cos(t),\\ \mathfrak{D}^{0.5}y_2(t) &= \cos(t)y_1(t) - t^2(t)y_2(t-0.5) + |t|,\end{aligned} \tag{66}$$

with prehistoric conditions

$$\begin{aligned}y_1(t) &= t\sin(t), \quad t\in(-0.5,0],\\ y_2(t) &= t^2+t+1, \quad t\in(-0.5,0],\end{aligned} \tag{67}$$

has a unique solution. This example demonstrates the strength of Theorems 5–8, which imposes no further requirements for well-posedness of such systems. Due to the significance of this example, we add the following corollary for reference in future research.

Corollary 1. *Assume* $A,B,F \in (\mathcal{C}[0,\mathfrak{T}])^{\nu\times\nu}$, $G \in (\mathcal{C}[-\tau,0])^{\nu\times\nu}$, $\tau,\mathfrak{T} > 0$ *and* $\Psi \in (0,1)^{\nu}$. *Then, System* (65) *has a unique mild solution, and is Lipschitz continuous with respect to* G *on* $[0,\mathfrak{T}]$.

Proof. The proof is a straightforward conclusion of Theorems 5–8. □

7. Conclusions and Remaining Works on This Topic

We have demonstrated that the mild solutions of incommensurate systems for DFDEs satisfy delay RL integral equations. To ensure well-posedness, we divided the interval into $[0,\tau]$, $[\tau,2\tau]$, ..., $[(n-1)\tau,\mathfrak{T}]$, and transformed studied integral equations into RL integral equations without delays as described by (46). This decomposition has been utilized throughout the analysis. Then, we proceed by assuming $\tau \in [0,1]$, and extended it for $\tau < \infty$ in any interval $[n,n+1]$, through induction. To obtain the existence result, we used Picard iterations to obtain a sequence of Cauchy continuous functions. We employed the completeness of the space of continuous functions to establish uniform convergence of

such Cauchy sequences. Subsequently, we showed that the limit of the Cauchy sequence satisfies the original RL integral equation. Thus, we have established the existence of a solution. For uniqueness and stability with respect to G, we applied generalized Gronwall inequality. The results of this paper can be summarized as follows.

Theorem 9. *Assume $\psi_i \in (0,1)$, G is continuous function on $(-\tau, 0]$, $\lim_{t\to-\tau} G(t) < \infty$, and $0 \leq \tau < \infty$. Additionally, assume that U is a continuous vector-valued function and each component of U is Lipschitz continuous with respect to both the second and third variables. Then, the incommensurate system of DFDEs (6) with prehistoric condition (5) has a unique solution. Furthermore, this solution is continuously dependent on G.*

While we provided a well-posedness of the problem on the spaces of continuous functions, there are fundamental questions regarding the need to investigated in dynamic of the solution. One key question is related to the regularity of the solutions. In terms of the classical analysis, regularity speaks about the existence and behavior of derivatives of the solution. Knowledge of the differentiability of the solution is crucial for constructing efficient numerical solutions. In the context of wider analysis, regularity speaks about the existence of a solution within a specific space. Studies by Liang and Stynes [21] have investigated the regularity of a wide class of singular Volterra integral equations of the second kind in weighted space $\mathcal{C}^{m,\alpha}$. It is an ongoing area of research related to unified theories for weakly singular integrals (including logarithm singularity and singularity with power function), mainly performed in the past three decades by Vainikko and Pedas [22–24]. Recently, in a book published by Brunner [19], the regularity of the solution of a weakly singular integral equation is studied in detail. Among the related books, this book devotes a chapter to this topic. We hope to investigate the regularity of DFDEs in future studies.

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Abbreviations

The following abbreviations are used in this manuscript:

FDEs	Fractional differential equations
DFDEs	delay fractional differential equations
DDEs	delay differentiate equations
HNNs	Hopfield Neural Network

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