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Successive Approximation and Stability Analysis of Fractional Stochastic Differential Systems with Non-Gaussian Process and Poisson Jumps

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Abstract: This paper investigates a new class of fractional stochastic differential systems with non-Gaussian processes and Poisson jumps. Firstly, we examine the solvability results for the considered system. Furthermore, new stability results for the proposed system are derived. The findings are established through the application of Grönwall's inequality, the successive approximation method, and the corollary of the Bihari inequality. Finally, the validity of the results is proved through an example.

Keywords: fractional stochastic system; stability; successive approximation method; Bihari inequality

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1. Introduction

Fractional differential equations (FDEs for short) extend classical differential equations by incorporating derivatives of non-integer order. These equations can describe more complex phenomena such as memory, hereditary processes, and anomalous diffusion, which cannot be effectively captured by integer-order derivatives. FDEs have a wide range of applications across various fields, such as economics, finance, material science, environmental science, signal processing, engineering, and control theory. For further information on FDEs, see [1,2] and the references cited within those works. The Caputo derivative is one of the most commonly used definition [3] of the fractional derivative. It is particularly useful in modeling physical processes where the initial conditions are specified in terms of integer-order derivatives, such as in the case of initial-value problems. Shu and Wang [4] investigated the existence and uniqueness results for FDEs with nonlocal initial conditions. Fractional stochastic differential equations (FSDEs for short) are a class of differential equations that combine the concepts of fractional calculus and stochastic processes. They are used to model systems that exhibit both random fluctuations (stochasticity) and memory effects (fractionality). FSDEs are often employed in fields such as physics, finance, biology, and engineering, where systems experience both uncertainty and long-range dependencies, such as anomalous diffusion or hereditary behaviors. For more details on FSDEs, see [5–8] and references therein. FSDEs with Rosenblatt processes and Poisson jumps extend traditional FSDEs by incorporating more complex forms of randomness, i.e., long-range dependence (through the Rosenblatt process) and discontinuous jumps (via the Poisson process), which are often seen in financial models, biological systems, and

physical processes. For more details on such systems, see [9–13] and references therein. Ren and Sakthivel [14] discussed the stability outcomes for stochastic systems of the second order with Poisson jumps. The stability concept of fractional stochastic systems presents a challenging but crucial area of research for modeling real-world phenomena. These systems capture the complex interplay of long-range dependencies, non-Gaussian randomness, and discrete jumps, all of which are essential to understanding the behavior of systems subject to both continuous fluctuations and rare, disruptive events. Investigating their stability requires advanced mathematical techniques, but the insights gained have broad applications in fields like finance, network theory, and engineering. For more basic facts on stability and its applications, see [15–18] and the references therein. Sathiyaraj et al. [19] investigated the stability of Hilfer FSDs. Agarwal et al. [20] discussed the stability results for impulsive fractional systems with delay. Despite the growing interest in fractional stochastic systems, the analysis of their solutions, especially under non-Lipschitz conditions, remains a challenging task. The existence and uniqueness of mild solutions are fundamental to the study of such systems, but standard methods like fixed-point approaches may not always apply in this context. Therefore, motivated by the above facts, we used new techniques, such as the successive approximation and Grönwall's inequality, to establish these existence and stability results.

Based on the above discussions, we consider the fractional stochastic system

$$\begin{cases} {}^c D_{\kappa}^{\theta} y(\kappa) = \mathcal{A}y(\kappa) + \Delta_1(\kappa, y(\kappa)) + \Delta_2(\kappa, y(\kappa)) \frac{d\hat{\mathcal{R}}^{\mathcal{H}}(\kappa)}{d\kappa} + \int_{\mathcal{Y}} \Delta_3(\kappa, y(\kappa), \xi) \hat{\mathcal{N}}(d\kappa, d\xi), \kappa \in (0, b], \\ y(0) = y_0, \quad y'(0) = y_1, \end{cases} \quad (1)$$

where ${}^c D_{\kappa}^{\theta}$ is the Caputo derivative of order $1 < \theta < 2$ and the state $\kappa(\cdot) \in \mathcal{K}$. $\mathcal{A} : \mathfrak{D}(\mathcal{A}) \subseteq \mathcal{K} \rightarrow \mathcal{K}$ is a sectorial operator of type $(\mathcal{M}, \chi, \theta, \rho)$ defined on $\mathcal{K}$. $\mathcal{J}_1 = [0, b]$. $\hat{\mathcal{R}}^{\mathcal{H}} = \{\hat{\mathcal{R}}^{\mathcal{H}}(\kappa) : \kappa \geq 0\}$ is a Rosenblatt process that takes values on $\mathcal{Y}$ with $\mathcal{H} \in (0.5, 1)$. The functions Δ_1 , Δ_2 , and Δ_3 satisfy certain conditions, which will be detailed later.

The paper is organized as follows: In Section 2, we define various preliminaries that will be used to prove the results. In Section 3, we investigate the existence and stability results for the considered systems by using successive approximation techniques and Bihari's inequality. Finally, in Section 4, an example is provided to demonstrate the validity of the results.

Remark 1. *The key contribution of this work is the exploration of existence and stability results for a novel class of fractional stochastic differential systems driven by non-Gaussian processes and Poisson jumps. This is achieved through the application of successive approximation methods and the Bihari inequality, offering a distinct approach compared with the fixed-point techniques and stability results available in the literature, as shown in [18–21].*

2. Preliminaries

Let $\mathcal{K}$ and $\mathcal{Y}$ be real separable Hilbert spaces and $L(\mathcal{K}, \mathcal{Y})$ be the space of all operators from $\mathcal{K}$ to $\mathcal{Y}$, which are bounded and linear. For simplicity, for $\mathcal{K}$, $\mathcal{Y}$, and $L(\mathcal{K}, \mathcal{Y})$, we used the same norm $\|\cdot\|$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and for $\kappa \geq 0$, $\mathcal{F}_{\kappa}$, denote the σ -field generated by P -null sets and $\{\hat{\mathcal{R}}_{\mathcal{H}}(\kappa) : \beta \in [0, \kappa]\}$.

Let $\bar{q} = \bar{q}(\kappa), \kappa \in D_{\bar{q}}$ be a stationary $\mathcal{F}_{\kappa}$ -Poisson point process with a characteristic measure η . Let $\hat{\mathcal{N}}_1(d\kappa, d\xi)$ be the Poisson counting measure associated with $\bar{q}$, i.e., $\hat{\mathcal{N}}_1(\kappa, \mathcal{V}) = \sum_{\beta \in D_{\bar{q}}, \beta \leq \kappa} I_{\mathcal{V}}(\bar{q}(\beta))$, with measurable set $\mathcal{V} \in B(\mathcal{K} \setminus \{0\})$, which represents the Borel σ -field of $\mathcal{K} \setminus \{0\}$. Let $\hat{\mathcal{N}}(d\kappa, d\xi) = \hat{\mathcal{N}}_1(d\kappa, d\xi) - d\kappa\eta(d\xi)$ be the compensated

Poisson measure that is independent of the Rosenblatt process. Let $p_2([0, b] \times \mathcal{V}; \mathcal{K})$ signify the space of all predictable mappings $\Delta : [0, b] \times \mathcal{V} \rightarrow \mathcal{K}$ for which

$$\int_0^b \int_{\mathcal{V}} \mathbb{E} \|\Delta(\kappa, \xi)\|^2 \eta(d\xi) d\kappa < \infty.$$

Next, we define the $\mathcal{K}$ -valued stochastic integral $\int_0^b \int_{\mathcal{V}} \Delta(\kappa, \xi) \hat{\mathcal{N}}(d\kappa, d\xi)$, which is a centered square-integrable martingale. Let $\mathcal{Q}_1 \in L(\mathcal{Y}, \mathcal{Y})$ represents a nonnegative self-adjoint operator. Let $\mathcal{L}_2^0 = \mathcal{L}_2^0(\mathcal{Q}_1^{1/2} \mathcal{Y}, \mathcal{K})$ signify the separable Hilbert space of all Hilbert–Schmidt operators from $\mathcal{Q}_1^{1/2} \mathcal{Y}$ to $\mathcal{K}$, equipped with the norm

$$\|\varphi\|_{\mathcal{L}_2^0}^2 = \|\varphi \mathcal{Q}_1^{1/2}\|^2 = \text{Tr}(\varphi \mathcal{Q}_1 \varphi^*).$$

Definition 1 ([9]). Let $\varphi : [0, b] \rightarrow \mathcal{L}_2^0(\mathcal{Q}_1^{1/2} \mathcal{Y}, \mathcal{K})$ such that $\sum_{j=1}^{\infty} \|K_{\mathcal{H}}^*(\varphi \mathcal{Q}_1^{1/2} e_j)\|_{L^2([0, b], \mathcal{K})} < \infty$; then, the stochastic integral corresponding to the Rosenblatt process is defined as follows:

$$\int_0^{\kappa} \varphi(\beta) d\hat{\mathcal{R}}_{\mathcal{Q}_1}(\beta) = \sum_{j=1}^{\infty} \int_0^{\kappa} \varphi(\beta) \mathcal{Q}_1^{1/2} e_j dr_j(\beta) = \sum_{j=1}^{\infty} \int_0^{\kappa} \int_0^{\kappa} (K_{\mathcal{H}}^*(\varphi \mathcal{Q}_1^{1/2} e_j))(x_1, x_2) d\mathcal{W}(x_1) d\mathcal{W}(x_2).$$

Lemma 1 ([11]). For $\varphi : [0, b] \rightarrow \mathcal{L}_2^0(\mathcal{Q}_1^{1/2} \mathcal{Y}, \mathcal{K})$ such that $\sum_{j=1}^{\infty} \|\varphi \mathcal{Q}_1^{1/2} e_j\|_{L^{1/\mathcal{H}}([0, b], \mathcal{K})} < \infty$ holds and for any $x_1, x_2 (x_2 > x_1) \in [0, b]$, we obtain

$$\mathbb{E} \left\| \int_{x_1}^{x_2} \varphi(\beta) d\hat{\mathcal{R}}_{\mathcal{Q}_1}(\beta) \right\|^2 \leq c_{\mathcal{H}} (x_2 - x_1)^{2\mathcal{H}-1} \sum_{j=1}^{\infty} \int_{x_1}^{x_2} \|\varphi(\beta) \mathcal{Q}_1^{1/2} e_j\|^2 d\beta.$$

If $\sum_{j=1}^{\infty} \|\varphi \mathcal{Q}_1^{1/2} e_j\|$ is uniformly convergent for $\kappa \in [0, b]$, then

$$\mathbb{E} \left\| \int_{x_1}^{x_2} \varphi(\beta) d\hat{\mathcal{R}}_{\mathcal{Q}_1}(\beta) \right\|^2 \leq c_{\mathcal{H}} (x_2 - x_1)^{2\mathcal{H}-1} \int_{x_1}^{x_2} \|\varphi(\beta)\|_{\mathcal{L}_2^0}^2 d\beta.$$

The Banach space $\mathcal{C}(\mathcal{K})$ formed by all $\mathcal{F}_{\kappa}$ -adapted measurable, $\mathcal{K}$ -valued stochastic processes $\{y(\kappa) : \kappa \in [0, b]\}$ such that $y(\kappa)$ is càdlàg on $\kappa \in [0, b]$, and the norm defined by

$$\|y\|_{\mathcal{C}} = \left(\sup_{0 \leq \kappa \leq b} \mathbb{E} \|y(\kappa)\|^2 \right)^{1/2}.$$

For $\delta_2 > 0$, we take $\mathcal{A}_{\mathcal{C}} = \{y \in \mathcal{C}(\mathcal{K}) : \|y\|_{\mathcal{C}}^2 \leq \delta_2\}$.

Definition 2 ([4]). The Caputo derivative of order θ for a function $\mathcal{F} : (0, \infty) \rightarrow \mathbb{R}$ can be written as

$${}^c D_{\kappa}^{\theta} \mathcal{F}(\kappa) = \frac{1}{\Gamma(m - \theta)} \int_0^{\kappa} (\kappa - \beta)^{m-\theta-1} \mathcal{F}^m(\beta) d\beta, \quad \kappa > 0, \quad (2)$$

where $m - 1 < \theta \leq m$, $m \in \mathbb{N}$, and $\mathcal{F}(\kappa) \in C^m(0, \infty)$.

The Laplace transform of the Caputo derivative of order $\theta > 0$ is given as

$$\mathcal{L}({}^c D_{\kappa}^{\theta} \mathcal{F}(\kappa))(\lambda) = \lambda^{\theta} (\mathcal{L}\mathcal{F})(\lambda) - \sum_{j=0}^{m-1} \lambda^{\theta-j-1} (D^j \mathcal{F})(0), \quad m - 1 < \theta \leq m.$$

Definition 3 ([4]). A linear and closed operator $\mathcal{A} : \mathfrak{D}(\mathcal{A}) \subseteq \mathcal{K} \rightarrow \mathcal{K}$ is called the sectorial operator of type $(\mathcal{M}, \chi, \theta, \rho)$ if there exists $\mathcal{M} > 0, \chi \in (0, \pi/2), \rho \in \mathbb{R}$, such that the θ -resolvent of $\mathcal{A}$ exists outside the sector

$$\rho + \mathcal{V}_\chi = \{ \rho + \lambda^\theta : \lambda \in \mathbb{C}, |\operatorname{Arg}(-\lambda^\theta)| < \chi \},$$

and

$$\|(\lambda^\theta I - \mathcal{A})^{-1}\| \leq \mathcal{M}/|\lambda^\theta - \rho|, \lambda^\theta \notin \rho + \mathcal{V}_\chi.$$

Moreover, $\mathcal{A}$ is the generator of $\{\mathcal{T}_\theta(\kappa)\}_{\kappa \geq 0}$ on $\mathcal{K}$, where

$$\mathcal{T}_\theta(\kappa) = \frac{1}{2\pi i} \int_c e^{\lambda \kappa} \mathcal{R}(\lambda^\theta, \mathcal{A}) d\lambda.$$

Next, the estimates for $\|\mathcal{S}_\theta(\kappa)\|$, $\|\mathcal{Q}_\theta(\kappa)\|$, and $\|\mathcal{T}_\theta(\kappa)\|$ are given in the following lemmas.

Lemma 2 ([4]). Let $\mathcal{A}$ be an operator of type $(\mathcal{M}, \chi, \theta, \rho)$. The estimates for $\|\mathcal{S}_\theta(\kappa)\|$ are defined as follows:

1. Let $\rho \geq 0$. Given $\omega \in (0, \pi)$, we obtain

$$\begin{aligned} \|\mathcal{S}_\theta(\kappa)\| \leq & \frac{\mathcal{E}_1(\chi, \omega) \mathcal{M} e^{[\mathcal{E}_1(\chi, \omega)(1+\rho\kappa^\theta)]^{1/\theta}} \left[\left(1 + \frac{\sin \omega}{\sin(\omega - \chi)} \right)^{1/\theta} - 1 \right]}{\pi \sin^{(1+1/\theta)\theta}} (1 + \rho\kappa^\theta) \\ & + \frac{\Gamma(\theta) \mathcal{M}}{\pi(1 + \rho\kappa^\theta) \left| \cos \frac{\pi - \omega}{\theta} \right|^\theta \sin \chi \sin \omega}, \end{aligned}$$

for $\kappa > 0$, where $\mathcal{E}_1(\chi, \omega) = \max \left\{ 1, \frac{\sin \chi}{\sin(\omega - \chi)} \right\}$.

2. Let $\rho < 0$. Given $\omega \in (0, \pi)$, we obtain

$$\|\mathcal{S}_\theta(\kappa)\| \leq \left(\frac{e \mathcal{M} \left[(1 + \sin \omega)^{1/\theta} - 1 \right]}{\pi |\cos \omega|^{(1+1/\theta)}} + \frac{\Gamma(\theta) \mathcal{M}}{\pi |\cos \omega| \left| \cos \frac{\pi - \omega}{\theta} \right|^\theta} \right) \frac{1}{1 + |\rho| \kappa^\theta},$$

for $\kappa > 0$.

Lemma 3 ([4]). Let $\mathcal{A}$ be an operator of type $(\mathcal{M}, \chi, \theta, \rho)$. The estimates for $\|\mathcal{T}_\theta(\kappa)\|$ and $\|\mathcal{Q}_\theta(\kappa)\|$ are defined as follows:

1. Let $\rho \geq 0$. Given $\omega \in (0, \pi)$, we obtain

$$\begin{aligned} \|\mathcal{T}_\theta(\kappa)\| \leq & \frac{\mathcal{M} \left[\left(1 + \frac{\sin \omega}{\sin(\omega - \chi)} \right)^{1/\theta} - 1 \right]}{\pi \sin \chi} (1 + \rho\kappa^\theta)^{1/\theta} \kappa^{\theta-1} e^{[\mathcal{E}_1(\chi, \omega)(1+\rho\kappa^\theta)]^{1/\theta}} \\ & + \frac{\mathcal{M} \kappa^{\theta-1}}{\pi(1 + \rho\kappa^\theta) \left| \cos \frac{\pi - \omega}{\theta} \right|^\theta \sin \chi \sin \omega}, \end{aligned}$$

$$\|Q_\theta(\kappa)\| \leq \frac{\mathcal{M} \left[\left(1 + \frac{\sin \varpi}{\sin(\varpi - \chi)} \right)^{1/\theta} - 1 \right] \mathcal{E}_1(\chi, \varpi)}{\frac{(\theta + 2)}{\pi \sin \chi} \frac{\theta}{\theta}} (1 + \rho \kappa^\theta)^{\frac{\theta - 1}{\theta}} \kappa^{\theta - 1} e^{[\mathcal{E}_1(\chi, \varpi)(1 + \rho \kappa^\theta)]^{1/\theta}} \\ + \frac{\mathcal{M} \theta \Gamma(\theta)}{\pi (1 + \rho \kappa^\theta) \left| \cos \frac{\pi - \varpi}{\theta} \right|^\theta \sin \chi \sin \varpi},$$

for $\kappa > 0$, where $\mathcal{E}_1(\chi, \varpi) = \max \left\{ 1, \frac{\sin \chi}{\sin(\varpi - \chi)} \right\}$.

2. Let $\rho < 0$. Given $\varpi \in (0, \pi)$, we obtain

$$\|T_\theta(\kappa)\| \leq \left(\frac{e \mathcal{M} \left[(1 + \sin \varpi)^{1/\theta} - 1 \right]}{\pi |\cos \varpi|} + \frac{\mathcal{M}}{\pi |\cos \varpi| \left| \cos \frac{\pi - \varpi}{\theta} \right|} \right) \frac{\kappa^{\theta - 1}}{1 + |\rho| \kappa^\theta},$$

$$\|Q_\theta(\kappa)\| \leq \left(\frac{e \mathcal{M} \left[(1 + \sin \varpi)^{1/\theta} - 1 \right] \kappa}{\pi |\cos \varpi|^{1 + \frac{2}{\theta}}} + \frac{\theta \Gamma(\theta) \mathcal{M}}{\pi |\cos \varpi| \left| \cos \frac{\pi - \varpi}{\theta} \right|} \right) \frac{1}{1 + |\rho| \kappa^\theta},$$

for $\kappa > 0$.

Lemma 4 ([4]). Let $\mathcal{A}$ be an operator of type $(\mathcal{M}, \chi, \theta, \rho)$. The unique solution of the Cauchy problem

$$\begin{cases} {}^c D_\kappa^\theta y(\kappa) = \mathcal{A}y(\kappa) + \mathcal{G}(\kappa), \quad \kappa \in [0, b], \quad 1 < \theta < 2, \\ y(0) = y_0, \quad y'(0) = y_1, \end{cases} \quad (3)$$

is given by

$$y(\kappa) = \mathcal{S}_\theta(\kappa)y_0 + \mathcal{Q}_\theta(\kappa)y_1 + \int_0^\kappa \mathcal{T}_\theta(\kappa - \beta)\mathcal{G}(\beta)d\beta,$$

where

$$\mathcal{S}_\theta(\kappa) = \frac{1}{2\pi i} \int_c e^{\lambda \kappa} \lambda^{\theta - 1} \mathcal{R}(\lambda^\theta, \mathcal{A}) d\lambda, \quad \mathcal{Q}_\theta(\kappa) = \frac{1}{2\pi i} \int_c e^{\lambda \kappa} \lambda^{\theta - 2} \mathcal{R}(\lambda^\theta, \mathcal{A}) d\lambda, \\ \mathcal{T}_\theta(\kappa) = \frac{1}{2\pi i} \int_c e^{\lambda \kappa} \mathcal{R}(\lambda^\theta, \mathcal{A}) d\lambda,$$

where c is a suitable path, with $\lambda^\theta \notin \rho + \mathcal{V}_\chi$ for $\lambda \in \mathbb{C}$.

Proof. By utilizing the Laplace transform, we obtain

$$\mathcal{L}({}^c D_\kappa^\theta y(\kappa))(\lambda) = \lambda^\theta (\mathcal{L}y)(\lambda) - \lambda^{\theta - 1} y(0) - \lambda^{\theta - 2} y'(0).$$

It follows that

$$\lambda^\theta (\mathcal{L}y)(\lambda) - \lambda^{\theta - 1} y(0) - \lambda^{\theta - 2} y'(0) = \mathcal{A}(\mathcal{L}y)(\lambda) + (\mathcal{L}\mathcal{G})(\lambda). \quad (4)$$

From Equation (4), we obtain

$$(\mathcal{L}y)(\lambda) = (\lambda^\theta I - \mathcal{A})^{-1} [\lambda^{\theta - 1} y(0) + \lambda^{\theta - 2} y'(0) + (\mathcal{L}\mathcal{G})(\lambda)]. \quad (5)$$

By utilizing the inverse Laplace transform of Equation (5), we obtain

$$y(\kappa) = \mathcal{S}_\theta(\kappa)y_0 + \mathcal{Q}_\theta(\kappa)y_1 + \int_0^\kappa \mathcal{T}_\theta(\kappa - \beta)\mathcal{G}(\beta)d\beta.$$

□

Definition 4 ([22]). An $\mathcal{F}_\kappa$ -adapted stochastic process $y \in \mathcal{C}(\mathcal{K})$ is said to be a solution of (1), if $y(\kappa)$ is càdlàg on $\kappa \in [0, b]$ and, for every $\kappa \in [0, b]$, y satisfies $y(0) = y_0$, $y'(0) = y_1$, and

$$\begin{aligned} y(\kappa) &= \mathcal{S}_\theta(\kappa)y_0 + \mathcal{Q}_\theta(\kappa)y_1 + \int_0^\kappa \mathcal{T}_\theta(\kappa - \beta)\Delta_1(\beta, y(\beta))d\beta + \int_0^\kappa \mathcal{T}_\theta(\kappa - \beta)\Delta_2(\beta, y(\beta))d\mathcal{R}^H(\beta) \\ &+ \int_0^\kappa \int_{\mathcal{V}} \mathcal{T}_\theta(\kappa - \beta)\Delta_3(\beta, y(\beta), \xi)\hat{\mathcal{N}}(d\beta, d\xi). \end{aligned}$$

Definition 5 ([14]). A solution y of (1) is called stable in mean square if, for any $\epsilon > 0$, there exists $\Theta > 0$ such that

$$\mathbb{E}\|y(\kappa) - \hat{y}(\kappa)\|^2 < \epsilon, \quad \text{whenever} \quad \mathbb{E}\|y_0 - \hat{y}_0\|^2 + \mathbb{E}\|y_1 - \hat{y}_1\|^2 < \Theta,$$

where $\hat{y}$ is another solution of (1), with $\hat{y}(0) = \hat{y}_0$ and $\hat{y}'(0) = \hat{y}_1$.

3. Stability Results

We make the following assumptions:

[A1]: The operators $\mathcal{S}_\theta(\kappa)$, $\mathcal{Q}_\theta(\kappa)$, and $\mathcal{T}_\theta(\kappa)$ are compact in $\overline{\mathfrak{D}(\mathcal{A})}$ such that

$$\sup_{\kappa \in \mathcal{J}_1} \|\mathcal{S}_\theta(\kappa)\| \leq \mathcal{M}, \quad \sup_{\kappa \in \mathcal{J}_1} \|\mathcal{Q}_\theta(\kappa)\| \leq \mathcal{M}, \quad \sup_{\kappa \in \mathcal{J}_1} \|\mathcal{T}_\theta(\kappa)\| \leq \mathcal{M}.$$

[A2]: The functions $\Delta_1 : [0, b] \times \mathcal{K} \rightarrow \mathcal{K}$ and $\Delta_2 : [0, b] \times \mathcal{K} \rightarrow \mathcal{L}_2^0(\mathcal{Q}_1^{1/2}\mathcal{Y}, \mathcal{K})$ are measurable and satisfy, for all $\kappa \in [0, b]$, $y_1, y_2 \in \mathcal{K}$,

$$\mathbb{E}\|\Delta_1(\kappa, y_1) - \Delta_1(\kappa, y_2)\|^2 \vee \mathbb{E}\|\Delta_2(\kappa, y_1) - \Delta_2(\kappa, y_2)\|_{\mathcal{L}_2^0}^2 \leq \Psi(\mathbb{E}\|y_1 - y_2\|^2),$$

where $\Psi(\cdot)$ is a concave, non-decreasing, and continuous function from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that $\Psi(0) = 0$, $\Psi(\beta) > 0$ for $\beta > 0$, and $\int_{0+} \frac{d\beta}{\Psi(\beta)} = \infty$.

[A3]: The function $\Delta_3 : [0, b] \times \mathcal{K} \times \mathcal{V} \rightarrow \mathcal{K}$ is measurable and satisfies, for $\kappa \in [0, b]$, $y_1, y_2 \in \mathcal{K}$,

$$\begin{aligned} &\int_{\mathcal{V}} \mathbb{E}\|\Delta_3(\kappa, y_1, \xi) - \Delta_3(\kappa, y_2, \xi)\|^2 \eta(d\xi) \\ &\vee \left(\int_{\mathcal{V}} \mathbb{E}\|\Delta_3(\kappa, y_1, \xi) - \Delta_3(\kappa, y_2, \xi)\|^4 \eta(d\xi) \right)^{1/2} \leq \Psi(\mathbb{E}\|y_1 - y_2\|^2), \end{aligned}$$

and

$$\left(\int_{\mathcal{V}} \mathbb{E}\|\Delta_3(\kappa, y, \xi)\|^4 \eta(d\xi) \right)^{1/2} \leq \Psi(\mathbb{E}\|y\|^2).$$

[A4]: For all $\kappa \in [0, b]$, there exists a constant $\Lambda_\Delta > 0$ such that

$$\mathbb{E}\|\Delta_1(\kappa, 0)\|^2 \vee \mathbb{E}\|\Delta_2(\kappa, 0)\|_{\mathcal{L}_2^0}^2 \vee \left(\int_{\mathcal{V}} \mathbb{E}\|\Delta_3(\kappa, 0, \xi)\|^2 \eta(d\xi) \right) \leq \Lambda_\Delta.$$

Let us define the successive approximation sequence as follows:

$$\begin{aligned} y^0(\kappa) &= \mathcal{S}_\theta(\kappa)y_0 + \mathcal{Q}_\theta(\kappa)y_1 \\ y^m(\kappa) &= \mathcal{S}_\theta(\kappa)y_0 + \mathcal{Q}_\theta(\kappa)y_1 \\ &\quad + \int_0^\kappa \mathcal{T}_\theta(\kappa - \beta)\Delta_1(\beta, y^{m-1}(\beta))d\beta + \int_0^\kappa \mathcal{T}_\theta(\kappa - \beta)\Delta_2(\beta, y^{m-1}(\beta))d\hat{\mathcal{R}}^\mathcal{H}(\beta) \\ &\quad + \int_0^\kappa \int_{\mathcal{V}} \mathcal{T}_\theta(\kappa - \beta)\Delta_3(\beta, y^{m-1}(\beta), \xi)\hat{\mathcal{N}}(d\beta, d\xi), \kappa \in [0, b], \quad m = 1, 2, \dots \end{aligned} \quad (6)$$

Lemma 5. Suppose that assumptions [A1]–[A4] hold. Then, there exists a constant $\delta_2 > 0$ such that $\|y^m\|_{\mathcal{C}}^2 \leq \delta_2$, for all $m \geq 0$.

Proof. For $\kappa \in [0, b]$, we obtain

$$\begin{aligned} \mathbb{E}\|y^0(\kappa)\|^2 &\leq \mathbb{E}\|\mathcal{S}_\theta(\kappa)y_0 + \mathcal{Q}_\theta(\kappa)y_1\|^2 \\ &\leq 2\mathcal{M}^2\mathbb{E}\|y_0\|^2 + 2\mathcal{M}^2\mathbb{E}\|y_1\|^2. \end{aligned}$$

Hence, we obtain

$$\|y^0\|_{\mathcal{C}}^2 \leq 2\mathcal{M}^2\mathbb{E}\|y_0\|^2 + 2\mathcal{M}^2\mathbb{E}\|y_1\|^2 = \delta_1.$$

This shows that $y^0 \in \mathcal{C}(\mathcal{K})$. Now, for $m \geq 1$, we obtain, by utilizing Hölder's inequality,

$$\begin{aligned} \mathbb{E}\|y^m(\kappa)\|^2 &\leq 4\mathbb{E}\|\mathcal{S}_\theta(\kappa)y_0 + \mathcal{Q}_\theta(\kappa)y_1\|^2 + 4\mathbb{E}\left\|\int_0^\kappa \mathcal{T}_\theta(\kappa - \beta)\Delta_1(\beta, y^{m-1}(\beta))d\beta\right\|^2 \\ &\quad + 4\mathbb{E}\left\|\int_0^\kappa \mathcal{T}_\theta(\kappa - \beta)\Delta_2(\beta, y^{m-1}(\beta))d\hat{\mathcal{R}}^\mathcal{H}(\beta)\right\|^2 \\ &\quad + 4\mathbb{E}\left\|\int_0^\kappa \int_{\mathcal{V}} \mathcal{T}_\theta(\kappa - \beta)\Delta_3(\beta, y^{m-1}(\beta), \xi)\hat{\mathcal{N}}(d\beta, d\xi)\right\|^2 \\ &\leq 4\delta_1 + 4\mathcal{M}^2b \int_0^\kappa \mathbb{E}\|\Delta_1(\beta, y^{m-1}(\beta))\|^2 d\beta \\ &\quad + 4\mathcal{M}^2c_{\mathcal{H}}b^{2\mathcal{H}-1} \int_0^\kappa \mathbb{E}\|\Delta_2(\beta, y^{m-1}(\beta))\|_{\mathcal{L}_2^0}^2 d\beta \\ &\quad + 4C\mathcal{M}^2 \left(\int_0^\kappa \int_{\mathcal{V}} \mathbb{E}\|\Delta_3(\beta, y^{m-1}(\beta), \xi)\|^2 \eta(d\xi) d\beta \right. \\ &\quad \left. + \int_0^\kappa \Psi(\mathbb{E}\|y^{m-1}(\beta)\|^2) d\beta \right) \\ &\leq 4\delta_1 + 8\mathcal{M}^2b \int_0^\kappa \mathbb{E}\|\Delta_1(\beta, y^{m-1}(\beta)) - \Delta_1(\beta, 0)\|^2 d\beta + 8\mathcal{M}^2b \int_0^\kappa \mathbb{E}\|\Delta_1(\beta, 0)\|^2 d\beta \\ &\quad + 8\mathcal{M}^2c_{\mathcal{H}}b^{2\mathcal{H}-1} \int_0^\kappa \mathbb{E}\|\Delta_2(\beta, y^{m-1}(\beta)) - \Delta_2(\beta, 0)\|_{\mathcal{L}_2^0}^2 d\beta \\ &\quad + 8\mathcal{M}^2c_{\mathcal{H}}b^{2\mathcal{H}-1} \int_0^\kappa \mathbb{E}\|\Delta_2(\beta, 0)\|_{\mathcal{L}_2^0}^2 d\beta \\ &\quad + 4C\mathcal{M}^2 \left(2 \int_0^\kappa \int_{\mathcal{V}} \mathbb{E}\|\Delta_3(\beta, y^{m-1}(\beta), \xi) - \Delta_3(\beta, 0, \xi)\|^2 \eta(d\xi) d\beta \right. \\ &\quad \left. + 2 \int_0^\kappa \int_{\mathcal{V}} \mathbb{E}\|\Delta_3(\beta, 0, \xi)\|^2 \eta(d\xi) d\beta + \int_0^\kappa \Psi(\mathbb{E}\|y^{m-1}(\beta)\|^2) d\beta \right) \\ &\leq 4\delta_1 + 8\mathcal{M}^2b \int_0^\kappa \Psi(\mathbb{E}\|y^{m-1}(\beta)\|^2) d\beta + 8\mathcal{M}^2b^2\Lambda_\Delta \\ &\quad + 8\mathcal{M}^2c_{\mathcal{H}}b^{2\mathcal{H}-1} \int_0^\kappa \Psi(\mathbb{E}\|y^{m-1}(\beta)\|^2) d\beta + 8\mathcal{M}^2c_{\mathcal{H}}b^{2\mathcal{H}}\Lambda_\Delta \end{aligned}$$

$$+ 4C\mathcal{M}^2 \left(3 \int_0^\kappa \Psi \left(\mathbb{E} \|y^{m-1}(\beta)\|^2 \right) d\beta + 2b\Lambda_\Delta \right).$$

Hence, we obtain

$$\mathbb{E} \|y^m(\kappa)\|^2 \leq h_0 + h_1 \int_0^\kappa \Psi \left(\mathbb{E} \|y^{m-1}(\beta)\|^2 \right) d\beta,$$

where $h_0 = 4\delta_1 + 8\mathcal{M}^2(b^2 + c_{\mathcal{H}}b^{2\mathcal{H}} + Cb)\Lambda_\Delta$, $h_1 = \mathcal{M}^2(8b + 8c_{\mathcal{H}}b^{2\mathcal{H}-1} + 12C)$. Because $\Psi(\cdot)$ is concave and $\Psi(0) = 0$, there are constants μ_1 and μ_2 such that

$$\Psi(\beta) = \mu_1 + \mu_2\beta, \text{ for all } \beta \geq 0.$$

Hence, we obtain

$$\begin{aligned} \mathbb{E} \|y^m(\kappa)\|^2 &\leq h_0 + h_1 \int_0^\kappa (\mu_1 + \mu_2 \mathbb{E} \|y^{m-1}(\beta)\|^2) d\beta \\ &\leq h_2 + h_1\mu_2 \int_0^\kappa \mathbb{E} \|y^{m-1}(\beta)\|^2 d\beta, \end{aligned}$$

and

$$\sup_{0 \leq s \leq \kappa} \mathbb{E} \|y^m(s)\|^2 \leq h_2 + h_1\mu_2 \int_0^\kappa \sup_{0 \leq r \leq \beta} \mathbb{E} \|y^{m-1}(r)\|^2 d\beta,$$

where $h_2 = h_0 + h_1\mu_1 b$. On the other hand, for any $n \geq 1$,

$$\max_{1 \leq m \leq n} \sup_{0 \leq s \leq \kappa} \mathbb{E} \|y^{m-1}(s)\|^2 \leq \mathbb{E} \|y^0(s)\|^2 + \max_{1 \leq m \leq n} \sup_{0 \leq s \leq \kappa} \mathbb{E} \|y^m(s)\|^2,$$

and we obtain

$$\begin{aligned} \max_{1 \leq m \leq n} \sup_{0 \leq x \leq \kappa} \mathbb{E} \|y^m(x)\|^2 &\leq h_2 + h_1\mu_2 b\delta_1 + h_1\mu_2 \int_0^\kappa \max_{1 \leq m \leq n} \sup_{0 \leq r \leq x} \mathbb{E} \|y^m(r)\|^2 dx \\ &\leq h_3 + h_4 \int_0^\kappa \max_{1 \leq m \leq n} \sup_{0 \leq r \leq x} \mathbb{E} \|y^m(r)\|^2 dx, \end{aligned}$$

where $h_3 = h_2 + h_1\mu_2 b\delta_1$, $h_4 = h_1\mu_2$. Since n is arbitrary, by Grönwall's inequality, we obtain

$$\sup_{0 \leq x \leq \kappa} \mathbb{E} \|y^m(x)\|^2 \leq h_3 e^{h_4 b} = \delta_2 < \infty.$$

Hence, we obtain $\|y^m\|_{\mathcal{C}}^2 \leq \delta_2$, for all $m \geq 0$. Thus, $y^m \in \mathcal{A}_{\mathcal{C}}$. $\square$

Lemma 6. Suppose that the assumptions of Lemma 5 hold; then, there exist constants $\delta_3 > 0$ and $\delta_4 > 0$ such that for all $\kappa \in [0, b]$ and $m, h \geq 1$,

$$\sup_{0 \leq x \leq \kappa} \mathbb{E} \|y^{m+h}(x) - y^m(x)\|^2 \leq \delta_3 \int_0^\kappa \Psi \left(\sup_{0 \leq r \leq x} \mathbb{E} \|y^{m+h-1}(r) - y^{m-1}(r)\|^2 \right) dx.$$

and

$$\sup_{0 \leq x \leq \kappa} \mathbb{E} \|y^{m+h}(x) - y^m(x)\|^2 \leq \delta_4 \kappa.$$

Proof. By the definition of y^m , for any $m, h \geq 1$ and $\kappa \in [0, b]$, we obtain

$$\begin{aligned}
& \sup_{0 \leq x \leq \kappa} \mathbb{E} \|y^{m+h}(x) - y^m(x)\|^2 \\
& \leq 3\mathcal{M}^2 b \int_0^\kappa \sup_{0 \leq r \leq x} \mathbb{E} \|\Delta_1(r, y^{m+h-1}(r)) - \Delta_1(r, y^{m-1}(r))\|^2 dx \\
& + 3\mathcal{M}^2 c_{\mathcal{H}} b^{2\mathcal{H}-1} \int_0^\kappa \sup_{0 \leq r \leq x} \mathbb{E} \|\Delta_2(r, y^{m+h-1}(r)) - \Delta_2(r, y^{m-1}(r))\|_{\mathcal{L}_2^0}^2 dx \\
& + 3C\mathcal{M}^2 \left(\int_0^\kappa \int_{\mathcal{V}} \sup_{0 \leq r \leq x} \mathbb{E} \|\Delta_3(r, y^{m+h-1}(r), \xi) - \Delta_3(r, y^{m-1}(r), \xi)\|^2 \eta(d\xi) dx \right. \\
& \left. + \int_0^\kappa \left(\int_{\mathcal{V}} \sup_{0 \leq r \leq x} \mathbb{E} \|\Delta_3(r, y^{m+h-1}(r), \xi) - \Delta_3(r, y^{m-1}(r), \xi)\|^4 \eta(d\xi) \right)^{1/2} dx \right) \\
& \leq 3\mathcal{M}^2 b \int_0^\kappa \Psi \left(\sup_{0 \leq r \leq x} \mathbb{E} \|y^{m+h-1}(r) - y^{m-1}(r)\|^2 \right) dx \\
& + 3\mathcal{M}^2 c_{\mathcal{H}} b^{2\mathcal{H}-1} \int_0^\kappa \Psi \left(\sup_{0 \leq r \leq x} \mathbb{E} \|y^{m+h-1}(r) - y^{m-1}(r)\|^2 \right) dx \\
& + 6C\mathcal{M}^2 \int_0^\kappa \Psi \left(\sup_{0 \leq r \leq x} \mathbb{E} \|y^{m+h-1}(r) - y^{m-1}(r)\|^2 \right) dx \\
& \leq \delta_3 \int_0^\kappa \Psi \left(\sup_{0 \leq r \leq x} \mathbb{E} \|y^{m+h-1}(r) - y^{m-1}(r)\|^2 \right) dx,
\end{aligned}$$

where $\delta_3 = 3\mathcal{M}^2 b + 3\mathcal{M}^2 c_{\mathcal{H}} b^{2\mathcal{H}-1} + 6C\mathcal{M}^2$. Next, we obtain

$$\begin{aligned}
\sup_{0 \leq x \leq \kappa} \mathbb{E} \|y^{m+h}(x) - y^m(x)\|^2 & \leq \delta_3 \int_0^\kappa \Psi \left(\sup_{0 \leq r \leq x} \mathbb{E} \|y^{m+h-1}(r) - y^{m-1}(r)\|^2 \right) dx \\
& \leq \delta_3 \int_0^\kappa \Psi(2\delta_2) dx \\
& = \delta_4 \kappa,
\end{aligned}$$

where $\delta_4 = \delta_3 \Psi(2\delta_2)$. $\square$

Theorem 1. Suppose that the assumptions of Lemmas 5 and 6 are satisfied; then, the successive approximation sequence y^m ($m \in \mathbb{N}$) converges to a unique mild solution $y(\kappa) \in \mathcal{A}_C$.

Proof. Step 1. Prove that $\{y^m(\kappa) : \kappa \in [0, b]\}$ is a Cauchy sequence. Let $u_1(v) = \delta_3 \Psi(v)$. Choose $c_1 \in [0, b]$ such that $u_1(\delta_4 v) \leq \delta_4$ for $v \in [0, c_1]$. We first introduce two sequences of functions $\{\omega_m(\kappa)\}_{m \in \mathbb{N}}$ and $\{\omega_{m,h}(\kappa)\}_{m,h \in \mathbb{N}}$ as

$$\begin{aligned}
\omega_1(\kappa) &= \delta_4 \kappa, \quad \omega_{m+1}(\kappa) = \int_0^\kappa u_1(\omega_m(v)) dv, \\
\omega_{m,h}(\kappa) &= \sup_{0 \leq v \leq \kappa} \mathbb{E} \|y^{m+h}(v) - y^m(v)\|^2.
\end{aligned}$$

Next, we obtain

$$\omega_{1,h}(\kappa) = \sup_{0 \leq v \leq \kappa} \mathbb{E} \|y^{1+h}(v) - y^1(v)\|^2 \leq \delta_4 \kappa = \omega_1(\kappa),$$

and

$$\begin{aligned}\omega_{2,h}(\kappa) &= \sup_{0 \leq \nu \leq \kappa} \mathbb{E} \|y^{2+h}(\nu) - y^2(\nu)\|^2 \\ &\leq \int_0^\kappa u_1 \left(\sup_{0 \leq \nu \leq x} \mathbb{E} \|y^{1+h}(\nu) - y^1(\nu)\|^2 \right) dx \\ &\leq \int_0^\kappa u_1(\omega_1(x)) dx = \omega_2(\kappa) \leq \delta_4 \kappa = \omega_1(\kappa).\end{aligned}$$

Hence, we obtain

$$\omega_{2,h}(\kappa) \leq \omega_2(\kappa) \leq \omega_1(\kappa).$$

Let the results be true for m ; then,

$$\begin{aligned}\omega_{m+1,h}(\kappa) &= \sup_{0 \leq \nu \leq \kappa} \mathbb{E} \|y^{m+h+1}(\nu) - y^{m+1}(\nu)\|^2 \\ &\leq \int_0^\kappa u_1 \left(\sup_{0 \leq \nu \leq x} \mathbb{E} \|y^{m+h}(\nu) - y^m(\nu)\|^2 \right) dx \\ &\leq \int_0^\kappa u_1(\omega_m(x)) dx = \omega_{m+1}(\kappa) \leq \int_0^\kappa u_1(\omega_{m-1}(x)) dx = \omega_m(\kappa).\end{aligned}$$

This implies that $\omega_m(\gamma)$ is a decreasing, continuous and nonnegative function on $[0, c_1]$ by induction on m . Hence, the function $\omega(\kappa)$ is defined as

$$\omega(\kappa) = \lim_{m \rightarrow \infty} \omega_m(\kappa) = \lim_{m \rightarrow \infty} \delta_3 \int_0^\kappa \Psi(\psi_{m-1}(x)) dx = \delta_3 \int_0^\kappa \Psi(\omega(x)) dx, \quad 0 \leq \kappa \leq c_1.$$

By using the Bihari inequality [5], we obtain that $\omega(\kappa) = 0$ for all $0 \leq \kappa \leq b$. Thus,

$$0 \leq \omega_{m,m}(\kappa) \leq \omega_m(c_1) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This shows that $\{y^m(\kappa) : \kappa \in [0, b]\}$ is a Cauchy sequence on $\mathcal{A}_C$. Now, by using the Borel–Cantelli Lemma, we obtain that as $m \rightarrow \infty$ $y^m(\kappa) \rightarrow y(\kappa)$ holds uniformly for $\kappa \in [0, b]$. By taking limits on both sides of Equation (6), we obtain

$$\begin{aligned}y(\kappa) &= \mathcal{S}_\theta(\kappa)y_0 + \mathcal{Q}_\theta(\kappa)y_1 + \int_0^\kappa \mathcal{T}_\theta(\kappa - \beta)\Delta_1(\beta, y(\beta))d\beta + \int_0^\kappa \mathcal{T}_\theta(\kappa - \beta)\Delta_2(\beta, y(\beta))d\hat{\mathcal{R}}^H(\beta) \\ &+ \int_0^\kappa \int_{\mathcal{V}} \mathcal{T}_\theta(\kappa - \beta)\Delta_3(\beta, y(\beta), \xi)\hat{\mathcal{N}}(d\beta, d\xi).\end{aligned}$$

Step 2. Let y_1 and y_2 be two solutions of system (1). For $\kappa \in [0, b]$, we obtain, similar to Lemma 6,

$$\sup_{0 \leq x \leq \kappa} \mathbb{E} \|y_1(x) - y_2(x)\|^2 \leq \delta_3 \int_0^\kappa \Psi \left(\sup_{0 \leq r \leq x} \mathbb{E} \|y_1(r) - y_2(r)\|^2 \right) dx.$$

By using the Bihari inequality [5], we obtain $\sup_{0 \leq x \leq \kappa} \mathbb{E} \|y_1(x) - y_2(x)\|^2 = 0$. Thus, $y_1(\kappa) = y_2(\kappa)$ for $\kappa \in [0, b]$. $\square$

Theorem 2. Suppose that the assumptions of Theorem 1 are satisfied; then, the mild solution of system (1) is stable in mean square.

Proof. For initial data y_0 and $\hat{y}_0$, let $y(\kappa)$ and $\hat{y}(\kappa)$ be the solution of (1), respectively. For $\kappa \in [0, b]$, we obtain

$$\begin{aligned}
\sup_{0 \leq x \leq \kappa} \mathbb{E} \|y(x) - \hat{y}(x)\|^2 &\leq 8\mathcal{M}^2 \mathbb{E} \|y_0 - \hat{y}_0\|^2 + 8\mathcal{M}^2 \mathbb{E} \|y_1 - \hat{y}_1\|^2 \\
&+ 4\mathcal{M}^2 b \int_0^\kappa \Psi \left(\sup_{0 \leq r \leq x} \mathbb{E} \|y(r) - \hat{y}(r)\|^2 \right) dx \\
&+ 4\mathcal{M}^2 c_{\mathcal{H}} b^{2\mathcal{H}-1} \int_0^\kappa \Psi \left(\sup_{0 \leq r \leq x} \mathbb{E} \|y(r) - \hat{y}(r)\|^2 \right) dx \\
&+ 8C\mathcal{M}^2 \int_0^\kappa \Psi \left(\sup_{0 \leq r \leq x} \mathbb{E} \|y(r) - \hat{y}(r)\|^2 \right) dx \\
&\leq 8\mathcal{M}^2 (\mathbb{E} \|y_0 - \hat{y}_0\|^2 + \mathbb{E} \|y_1 - \hat{y}_1\|^2) + e_0 \int_0^\kappa \Psi \left(\sup_{0 \leq r \leq x} \mathbb{E} \|y(r) - \hat{y}(r)\|^2 \right) dx,
\end{aligned}$$

where $e_0 = 4\mathcal{M}^2 b + 4\mathcal{M}^2 c_{\mathcal{H}} b^{2\mathcal{H}-1} + 8C\mathcal{M}^2$. Let $\Psi_1(w) = e_0 \Psi(w)$; here, Ψ is a increasing and concave function from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that $\Psi(0) = 0$, $\Psi(w) > 0$ for $w > 0$ and $\int_0^+ \frac{dw}{\Psi(w)} = +\infty$. Hence, $\Psi_1(w)$ is also a concave function from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that $\Psi_1(0) = 0$, $\Psi(w) \geq \Psi_1(w)$, for any $0 \leq w \leq 1$ and $\int_0^+ \frac{dw}{\Psi_1(w)} = +\infty$. Therefore, for any $\epsilon > 0$, $\epsilon_1 = \epsilon/2$, we have $\lim_{x \rightarrow 0} \int_x^{\epsilon_1} \frac{dw}{\Psi_1(w)} = +\infty$. Thus, there is a positive constant $\Theta < \epsilon_1$ such that

$$\int_{\Theta}^{\epsilon_1} \frac{dw}{\Psi_1(w)} \geq b.$$

From the Corollary of the Bihari inequality [5], let

$$w_0 = 8\mathcal{M}^2 (\mathbb{E} \|y_0 - \hat{y}_0\|^2 + \mathbb{E} \|y_1 - \hat{y}_1\|^2), \text{ and } w(\kappa) = \sup_{0 \leq x \leq \kappa} \mathbb{E} \|y(x) - \hat{y}(x)\|^2,$$

and $v(\kappa) = 1$. Therefore, when $w_0 \leq \Theta \leq \epsilon_1$, the Corollary of the Bihari inequality [5] shows that

$$\int_{w_0}^{\epsilon_1} \frac{dw}{\Psi_1(w)} \geq \int_{\delta}^{\epsilon_1} \frac{dw}{\Psi_1(w)} \geq b = \int_0^b v(x) dx.$$

Thus, for any $\kappa \in [0, b]$, the estimate $w(\kappa) \leq \epsilon$, holds. Hence, system (1) is stable in mean square. $\square$

4. Applications

4.1. Numerical Example

Consider the following system:

$$\begin{cases}
{}^c D_{\kappa}^{\theta} y(\kappa, \varrho) = \frac{\partial^2 y(\kappa, \varrho)}{\partial \varrho^2} + \frac{e^{-\kappa} (1 + y(\kappa, \varrho))}{2 + |y(\kappa, \varrho)|} + \frac{2(e^{-8\kappa} + y(\kappa, \varrho))}{1 + 2e^{\kappa}} \frac{d\hat{\mathcal{R}}^{\mathcal{H}}(\kappa)}{d\kappa} \\
\quad + \int_{\mathcal{V}} e^{-2\kappa} \cos(\kappa) (1 + y(\kappa, \varrho)) \xi \hat{\mathcal{N}}(d\kappa, d\xi), \quad \kappa \in [0, 1] \quad \varrho \in [0, \pi], \\
y(\kappa, 0) = 0 = y(\kappa, \pi), \\
y(0, \varrho) = y_0(\varrho), \quad y'(0, \varrho) = y_1(\varrho), \quad \varrho \in [0, \pi],
\end{cases} \quad (7)$$

where ${}^c D_{\kappa}^{\theta}$ is the Caputo derivative of order $1 < \theta < 2$ and $\hat{\mathcal{R}}^{\mathcal{H}}(\kappa)$ is the Rosenblatt process with Hurst index $1/2 < \mathcal{H} < 1$. Let $Q_1 = 1$, $\mathcal{V} = \mathbb{R}$, $\lambda_1 = 1$, $\lambda_j = 0$, and $j > 1$. Let $\mathcal{K} = L^2[0, \pi]$, and define the operator $\mathcal{A} : \mathfrak{D}(\mathcal{A}) \subseteq \mathcal{K} \rightarrow \mathcal{K}$ by

$$\mathcal{A}p = p'' \text{ with } \mathfrak{D}(\mathcal{A}) = \{p \in \mathcal{K} : p, p' \text{ are absolutely continuous, } p'' \in \mathcal{K} \text{ and } p(0) = p(\pi) = 0\}.$$

$\mathcal{A}$ is generator of a resolvent family $\{\mathcal{T}_\theta(\kappa) : \kappa \geq 0\}$ and dense in the space $\mathcal{K}$. By Theorems 3.3 and 3.4 (see [4]), we obtain $\mathcal{M} = 3$. Define the functions as follows:

$$\Delta_1(\kappa, y) = \frac{e^{-\kappa}(2 + y(\kappa, \varrho))}{2 + |y(\kappa, \varrho)|}, \quad \Delta_2(\kappa, y) = \frac{2(e^{-8\kappa} + y(\kappa, \varrho))}{1 + 2e^\kappa}, \quad \Delta_3(\kappa, y, \xi) = e^{-2\kappa} \cos(\kappa)(1 + y(\kappa, \varrho))\xi.$$

We have a concave, continuous, and non-decreasing function $\Psi = \mathcal{I} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\mathcal{I}(\varpi) = \varpi > 0$, for $\varpi > 0$, $\mathcal{I}(0) = 0$, and $\int_0^+ \frac{d\varpi}{\varpi} = \infty$. Next, we obtain

$$\begin{aligned} \mathbb{E}\|\Delta_1(\kappa, y_1(\kappa, \varrho)) - \Delta_1(\kappa, y_2(\kappa, \varrho))\|^2 &\leq \frac{e^{-2\kappa}}{4} \mathbb{E}\|y_1(\kappa, \varrho) - y_2(\kappa, \varrho)\|^2 \\ &\leq \frac{1}{4} \mathbb{E}\|y_1(\kappa, \varrho) - y_2(\kappa, \varrho)\|^2, \end{aligned}$$

$$\begin{aligned} \mathbb{E}\|\Delta_2(\kappa, y_1(\kappa, \varrho)) - \Delta_2(\kappa, y_2(\kappa, \varrho))\|_{\mathcal{L}_2^0}^2 &\leq \frac{4}{(1 + 2e^\kappa)^2} \mathbb{E}\|y_1(\kappa, \varrho) - y_2(\kappa, \varrho)\|^2 \\ &\leq 4\mathbb{E}\|y_1(\kappa, \varrho) - y_2(\kappa, \varrho)\|^2, \end{aligned}$$

and

$$\mathbb{E}\|\Delta_1(\kappa, 0)\|^2 \leq e^{-2\kappa} \leq 1, \quad \text{and} \quad \mathbb{E}\|\Delta_2(\kappa, 0)\|_{\mathcal{L}_2^0}^2 \leq \frac{4e^{-16\kappa}}{(1 + 2e^\kappa)^2} \leq 4.$$

We assume that $\int_{\mathcal{V}} \xi^2 \eta(d\xi) < 1$ and $\int_{\mathcal{V}} \xi^4 \eta(d\xi) < 1$. Next, we obtain

$$\begin{aligned} &\int_{\mathcal{V}} \mathbb{E}\|\Delta_3(\kappa, y_1(\kappa, \varrho), \xi) - \Delta_3(\kappa, y_2(\kappa, \varrho), \xi)\|^2 \eta(d\xi) \vee \\ &\left(\int_{\mathcal{V}} \mathbb{E}\|\Delta_3(\kappa, y_1(\kappa, \varrho), \xi) - \Delta_3(\kappa, y_2(\kappa, \varrho), \xi)\|^4 \eta(d\xi) \right)^{1/2} \leq \mathbb{E}\|y_1(\kappa, \varrho) - y_2(\kappa, \varrho)\|^2, \end{aligned}$$

and $\int_{\mathcal{V}} \mathbb{E}\|\Delta_3(\kappa, 0, \xi)\|^2 \eta(d\xi) \leq 1$. We obtain $\Delta_\Delta = 1$. All the assumptions of Theorem (2) are satisfied. Hence, the successive approximations sequence y^m , $m = 0, 1, 2, \dots$ converges to the unique stable mild solution of system (7) on $[0, 1]$.

4.2. Application

Motivated by the designs presented in references [23,24], we introduced a new filter pattern for our system (1), which is displayed as follows.

Figure 1 illustrates the basic structure of the block diagram, which helps improve the solution's efficiency while minimizing input requirements and is detailed as follows:

1. Product Modulator (PM for short)-1 receives the inputs $y(\beta)$ and Δ_1 and produces the output as $\Delta_1(\beta, y(\beta))$.
2. PM-2 receives the inputs $y(\beta)$ and Δ_2 and produces the output as $\Delta_2(\beta, y(\beta))$.
3. PM-3 receives the inputs $\int_{\mathcal{V}} \hat{\mathcal{N}}(d\beta, d\xi)$ and Δ_3 and produces the output as $\int_{\mathcal{V}} \Delta_3(\beta, y(\beta), \xi) \hat{\mathcal{N}}(d\beta, d\xi)$.
4. PM-4 receives the inputs y_1 and $\mathcal{Q}_\theta(\kappa)$ at $\kappa = 0$ and produces the output as $\mathcal{Q}_\theta(\kappa)y_1$.
5. PM-5 receives the inputs y_0 and $\mathcal{S}_\theta(\kappa)$ at $\kappa = 0$ and produces the output as $\mathcal{S}_\theta(\kappa)y_0$.
6. Here, the integrators compute the integral of

$$\mathcal{T}_\theta(\kappa - \beta) \left(\Delta_1(\beta, y(\beta)) + \Delta_2(\beta, y(\beta)) + \int_{\mathcal{V}} \Delta_3(\beta, y(\beta), \xi) \hat{\mathcal{N}}(d\beta, d\xi) \right),$$

over the time period κ .

Furthermore,

1. The inputs $\mathcal{T}_\theta(\kappa - \beta)$ and $\Delta_1(\beta, y(\beta))$ are combined and multiplied with an output of integrator over $(0, \kappa)$.
2. The inputs $\mathcal{T}_\theta(\kappa - \beta)$ and $\Delta_2(\beta, y(\beta))$ are combined and multiplied with an output of integrator over $(0, \kappa)$ with respect to $\hat{\mathcal{R}}^{\mathcal{H}}(\beta)$.
3. The inputs $\mathcal{T}_\theta(\kappa - \beta)$ and $\int_{\mathcal{V}} \Delta_3(\beta, y(\beta), \xi) \hat{\mathcal{N}}(d\beta, d\xi)$ are combined and multiplied with an output of integrator over $(0, \kappa)$.

Finally, all the outputs from the integrators are sent to the summing network. As a result, the output $y(\kappa)$ is obtained, which is bounded and stable in mean square.

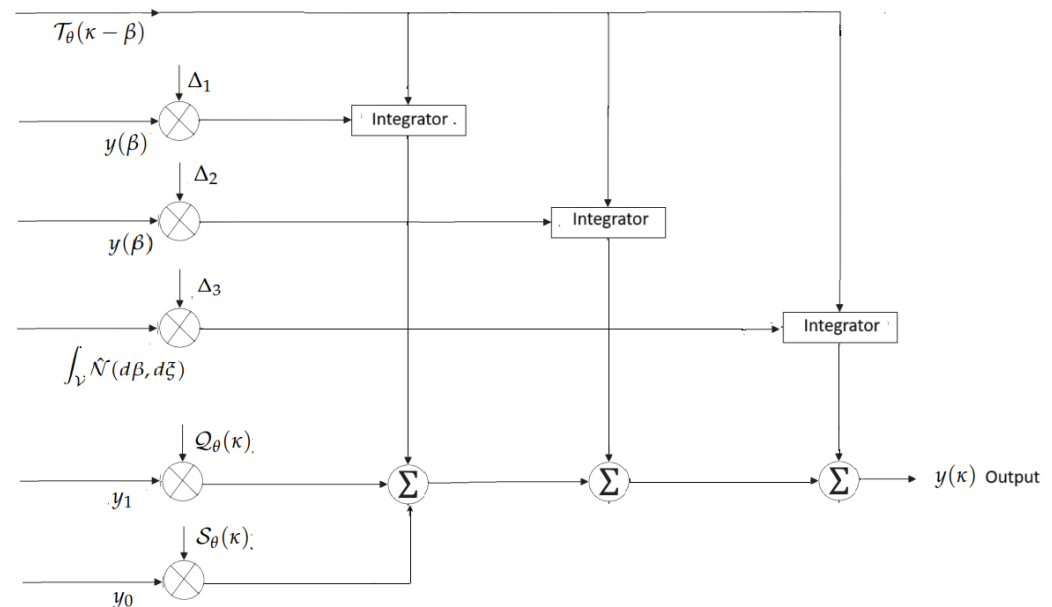


Figure 1. Filter system.

5. Conclusions

This paper presents a novel class of fractional stochastic differential systems driven by non-Gaussian processes and Poisson jumps. The successive approximation technique is employed to derive new constructive results regarding the existence and stability of the proposed system. To demonstrate the effectiveness of these results, an illustrative example is provided. As a potential avenue for future research, we encourage further exploration of fractional stochastic differential systems incorporating non-Gaussian processes, Poisson jumps, impulsive effects, and delays.

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