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Fixed Points of Exponential-Type Contractions in Fuzzy Metric Spaces with Applications to Nonlinear Fractional Boundary Value Problems

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Abstract

In this paper, we introduce the notion of fuzzy exponential contractions within the framework of fuzzy metric spaces. These mappings, which involve point-dependent exponential terms, are studied under the assumptions of either fuzzy continuity or the weaker condition of fuzzy Picard continuity. We establish corresponding existence and uniqueness theorems, and we further demonstrate the scope of the theory through illustrative examples and by applying it to prove an existence and uniqueness result for a class of nonlinear fractional differential equations.

Keywords: fuzzy metric spaces; fuzzy logic; fuzzy exponential contractions; fuzzy continuous and Picard continuous mappings; fixed point

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1. Introduction

The Banach Contraction Principle (BCP) [1] stands as a cornerstone of nonlinear functional analysis, providing a simple yet powerful method for establishing the existence and uniqueness of fixed points for self-mappings on complete metric spaces. Its profound utility in the theory of differential and integral equations has motivated a vast and enduring research program aimed at its generalization and extension. A significant direction of this research involves relaxing the stringent uniformity of the contraction condition, leading to concepts such as Kannan contractions [2], Chatterjea contractions [3], and Ćirić-type quasi-contractions [4].

Parallel to these developments, the introduction of fuzzy set theory by Zadeh [5] offered a new paradigm for handling uncertainty and imprecision. This progress led to the emergence of the concept of fuzzy metric spaces (see, e.g., [6–8]). The most widely accepted formulation of a fuzzy metric space was proposed by Kramosil and Michalek [9], and was later refined into a more topologically consistent framework by George and

Veeramani [10]. In these spaces, the classical notion of distance is replaced by a fuzzy set $M(x, y, t)$, interpreted as the degree of nearness between x and y with respect to a parameter $t > 0$. Grabiec [11] successfully extended the BCP to the fuzzy setting in the sense of Kramosil and Michalek by introducing the notions of G-Cauchy sequences and G-completeness, and by establishing a fixed point theorem for fuzzy contractions; that is, for mappings T, satisfying

$$M(Tx, Ty, \lambda t) \geq M(x, y, t),$$

for some $\lambda \in (0, 1)$. This seminal work laid the foundation for a rich fixed point theory in fuzzy metric spaces. Building on Grabiec's foundational results, Fang [12] developed new fixed point theorems for contractive-type mappings in G-complete fuzzy metric spaces. Subsequently, Mishra et al. [13] derived several common fixed point results for asymptotically commuting mappings within the same framework. For more recent fixed point results in fuzzy metric spaces, we refer the reader to [14–19].

Despite the abundance of generalizations, a common limitation of many existing results is the reliance on a uniform contraction constant or a uniformly bounded scaling factor. This can be overly restrictive when analyzing operators arising from differential or integral equations with non-uniform nonlinearities. To address this, recent studies have explored contractions involving functions of the points themselves, such as those of polynomial type [20]. However, a comprehensive framework that incorporates variable exponents within a multiplicative (t-norm-based) structure in fuzzy metric spaces remains largely unexplored.

In this paper, we introduce and systematically analyze a new and broader class of mappings termed fuzzy exponential contractions. The central idea is to generalize the standard fuzzy contraction inequality by considering a finite product, under a continuous t-norm $*$, of terms of the form $[M(Tx, Ty, t)]^{a_i(Tx,Ty)}$. Here, the exponents $a_i : X \times X \to [1, \infty)$ are functions that allow the contraction condition to adapt to the local geometry of the space. Formally, a mapping T is a fuzzy exponential contraction if there exist $\lambda \in (0, 1)$, $k \in \mathbb{N}$, and functions a_i, such that for all $x, y \in X$:

$$*\prod_{i=1}^{k}[M(Tx, Ty, t)]^{a_i(Tx,Ty)} \geq *\prod_{i=1}^{k}[M(x, y, t/\lambda)]^{a_i(x,y)}.$$

This definition encapsulates the standard fuzzy Banach contraction when $k = 1$ and $a_1 \equiv 1$, but it significantly expands the class of admissible mappings.

The main contributions of this work include the formal definition and framework for fuzzy exponential contractions, situating them within the existing hierarchy of fuzzy contractive mappings and demonstrating that they constitute a genuine extension. We establish fixed point theorems for these contractions in complete fuzzy metric spaces under both fuzzy continuity and the weaker assumption of fuzzy Picard continuity, which we introduce herein. To validate our theoretical developments, we present concrete examples of mappings that are fuzzy exponential contractions but not standard fuzzy contractions, illustrating the necessity and robustness of our approach. Finally, we demonstrate the practical utility of our framework by applying it to a nonlinear fractional boundary value problem with Caputo derivative, showing how the flexibility of fuzzy exponential contractions allows us to handle generalized Lipschitz conditions involving T-invariant functions that fall outside the scope of classical contraction principles.

The rest of this paper is structured as follows. Section 2 revisits essential preliminaries on fuzzy metric spaces. Sections 3 are dedicated to presenting the main fixed point results and illustrative examples. In Section 4, we apply our findings to prove the existence and

uniqueness of solutions for a fractional boundary value problem. Finally, Section 5 contains concluding remarks.

2. Mathematical Preliminaries

This section is devoted to recalling some fundamental definitions and results from the theory of fuzzy metric spaces, which will serve as essential tools in the sequel. For a more comprehensive discussion, we refer the reader to [9–11].

Definition 1 ([21]). *A triangular norm (or the t-norm) is a binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *that satisfies the following conditions for all* $a, b, c \in [0,1]$*:*

1. *Commutativity:* $a * b = b * a$;
2. *Associativity:* $(a * b) * c = a * (b * c)$;
3. *Monotonicity: if* $a \leq c$ *and* $b \leq d$*, then* $a * b \leq c * d$;
4. *Identity:* $a * 1 = a$.

A t-norm is called continuous if it is continuous as a function on $[0,1]^2$.

Common examples of continuous t-norms include the product t-norm ($a * b = a \cdot b$), the minimum t-norm ($a * b = \min\{a, b\}$), and the Łukasiewicz t-norm ($a * b = \max\{a + b - 1, 0\}$).

The concept of a fuzzy metric space provides a generalization of a classical metric space by introducing a notion of distance that is gradual rather than absolute. The following definition of fuzzy metric spaces is presented by Kramosil and Michalek [9]:

Definition 2 ([9]). *A fuzzy metric space is an ordered triple* $(X, M, *)$*, where* X *is a nonempty set,* $*$ *is a continuous t-norm, and* $M : X \times X \times (0, \infty) \to (0,1]$ *is a function (called a fuzzy metric) satisfying the following conditions:*

1. $M(x, y, 0) = 0, \forall\ x, y \in X$;
2. $M(x, y, t) = 1$ *for all* $t > 0$ *if and only if* $x = y$;
3. $M(x, y, t) = M(y, x, t)\ \forall\ x, y \in X,\ t > 0$;
4. $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)\ \forall\ x, y \in X,\ t, s > 0$;
5. *The function* $t \mapsto M(x, y, t)$ *is left continuous on* $[0, \infty)$.

The development of fixed point theory in fuzzy metric spaces relies on appropriate notions of convergence and completeness. We recall the concepts of G-Cauchy sequences and G-completeness introduced by Grabiec [11], which are essential for the following fundamental result: the fuzzy analogue of the Banach Contraction Principle.

Definition 3 ([11]). *Let* $(X, M, *)$ *be a fuzzy metric space.*

(i) A sequence $\{x_n\}$ *in* X *is said to be G-Cauchy if*

$$\forall_{t>0} \forall_{p \in \mathbb{N}} \{ \lim_{n \to \infty} M(x_n, x_{n+p}, t) = 1 \}.$$

(ii) A sequence $\{x_n\}$ *in* X *is said to converge to* $x \in X$ *if*

$$\forall_{t>0} \{ \lim_{n \to \infty} M(x_n, x, t) = 1 \}.$$

(iii) A fuzzy metric space $(X, M, *)$ *is called* G-complete *if every G-Cauchy sequence in* X *is convergent.*

Theorem 1 ([11]). *Let $(X, M, *)$ be a G-complete fuzzy metric space such that*

$$\lim_{t\to\infty} M(x, y, t) = 1 \quad \text{for all } x, y \in X.$$

If $T : X \to X$ is a mapping satisfying

$$M(Tx, Ty, \lambda t) \geqslant M(x, y, t) \quad \text{for all } x, y \in X,\ t > 0,$$

where $\lambda \in (0, 1)$; then, T has a unique fixed point.

The original notion of fuzzy metric spaces due to Kramosil and Michálek, was later refined by George and Veeramani [10] to achieve better topological properties. In their formulation, the fuzzy metric satisfies $M(x, y, t) > 0$ for all t, and the function $t \mapsto M(x, y, t)$ is continuous.

Definition 4 ([10]). *A fuzzy metric space is an ordered triple $(X, M, *)$, where X is a nonempty set, $*$ is a continuous t-norm, and $M : X \times X \times (0, \infty) \to (0, 1]$ is a function (called a fuzzy metric), satisfying the following conditions for all $x, y, z \in X$ and $s, t > 0$:*

1. $M(x, y, t) > 0$;
2. $M(x, y, t) = 1 \iff x = y$;
3. $M(x, y, t) = M(y, x, t)$;
4. $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;
5. $M(x, y, \cdot) : (0, \infty) \to (0, 1]$ *is continuous.*

Definition 5 ([10]). *In a fuzzy metric space $(X, M, *)$:*

1. *A sequence $\{x_n\}$ is called M-Cauchy if*

$$\forall_{\epsilon>0}\forall_{t>0}\exists_{n_0\in\mathbb{N}}\forall_{n>m\geq n_0}\{M(x_m, x_n, t) > 1 - \epsilon\}.$$

2. *A sequence $\{x_n\}$ converges to a point $x \in X$ (denoted $x_n \to x$) if*

$$\forall_{\epsilon>0}\forall_{t>0}\exists_{n_0\in\mathbb{N}}\forall_{n\geq n_0}\{M(x_n, x, t) > 1 - \epsilon\}.$$

3. *The space is M-complete if every M-Cauchy sequence in X is convergent.*

Definition 6 ([22]). *A sequence (t_n) of positive real numbers is called s-increasing if there exists $n_0 \in \mathbb{N}$, such that*

$$t_{n+1} \geq t_n + 1 \quad \text{for all } n \geq n_0.$$

Theorem 2 ([22]). *Let $(X, M, *)$ be an M-complete fuzzy metric space. Suppose that for every $\varepsilon > 0$ and every s-increasing sequence (t_n) there exists $n_0 \in \mathbb{N}$, such that*

$$* \prod_{n\geq n_0} M(x, y, t_n) > 1 - \varepsilon,$$

or, equivalently, the infinite product $* \prod_{n=1}^{\infty} M(x, y, t_n)$ converges whenever $*$ is the usual product on $(0, 1]$. If $\lambda \in (0, 1)$ and $T : X \to X$ is a self-mapping satisfying

$$M(Tx, Ty, \lambda t) \geq M(x, y, t), \quad \forall\, x, y \in X,\ t > 0,$$

then T has a unique fixed point in X.

Definition 7. *A mapping $T : X \to X$ is called fuzzy continuous if for every sequence $\{x_n\}$ in X with $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$, we have $\lim_{n\to\infty} M(Tx_n, Tx, t) = 1$ for all $t > 0$.*

Lemma 1. *Let $(X, M, *)$ be a fuzzy metric space and let $T : X \to X$ be a mapping. For a constant $\lambda \in (0, 1)$, the following two contraction conditions are equivalent:*

(1) $M(Tw, Tz, t) \geq M(w, z, t/\lambda)$ *for all* $w, z \in X$, $t > 0$,
(2) $M(Tw, Tz, t\lambda) \geq M(w, z, t)$ *for all* $w, z \in X$, $t > 0$.

Proof. Assume (1) holds. For any $t > 0$, set $s = t/\lambda$. Then, $t = s\lambda$, and applying (1) gives

$$M(Tw, Tz, s\lambda) \geq M(w, z, s\lambda/\lambda) = M(w, z, s).$$

Since $s > 0$ is arbitrary, this is exactly condition (2). Conversely, assume (2) holds. For any $t > 0$, set $s = t\lambda$. Then, $t = s/\lambda$, and applying (2) yields

$$M(Tw, Tz, s) \geq M(w, z, s/\lambda),$$

which is condition (1). Thus, the two forms are equivalent. □

Throughout this work, the notation $*\prod_{i=1}^{k}$ denotes the iterated application of the t-norm $*$. Formally, for a finite sequence of values $u_1, u_2, \ldots, u_k \in [0, 1]$, it is defined as

$$*\prod_{i=1}^{k} u_i = u_1 * u_2 * \cdots * u_k.$$

For example, if the t-norm $*$ is the product ($a * b = a \cdot b$), then $*\prod_{i=1}^{k} u_i = u_1 \cdot u_2, \ldots, \cdot u_k$. If $*$ is the minimum t-norm ($a * b = \min\{a, b\}$), then $*\prod_{i=1}^{k} u_i = \min\{u_1, u_2, \ldots, u_k\}$. Similarly, for an infinite sequence of values $u_1, u_2, \ldots, u_k, \cdots \in [0, 1]$, the iterated application of the t-norm $*$ is defined as

$$*\prod_{i=1}^{\infty} u_i = u_1 * u_2 * \cdots * u_k * \cdots.$$

3. Main Results

3.1. Class of Fuzzy Exponential Contractions

We first introduce the class of fuzzy exponential contractions that will be used to establish our main results.

Definition 8. *Let $(X, M, *)$ be a fuzzy metric space and $T : X \to X$ be a self-mapping. We say that T is a fuzzy exponential contraction if there exist a constant $\lambda \in (0, 1)$, a positive integer k, and a set of functions $a_i : X \times X \to [1, \infty)$ for $i = 1, \ldots, k$, such that for every $x, y \in X$, the following inequality holds:*

$$*\prod_{i=1}^{k} [M(Tx, Ty, t)]^{a_i(Tx,Ty)} \geq *\prod_{i=1}^{k} [M(x, y, t/\lambda)]^{a_i(x,y)}. \tag{1}$$

This definition generalizes the standard fuzzy contraction ($M(Tx, Ty, t) \geq M(x, y, t/\lambda)$) by incorporating an exponential of fuzzy distance terms, each weighted by functions a_i that may depend on the points themselves.

Remark 1. *The class of fuzzy Banach contractions is a specific instance of the more general fuzzy exponential contractions. This can be seen by setting the parameters in the definition of a fuzzy*

exponential contraction as follows: let the integer k be 1, and let the function a_1 be the constant function $a_1(x, y) = 1$ for all x, y in the space X.

We now turn to the first main fixed point result for fuzzy exponential contractions. This theorem generalizes the fuzzy Banach contraction principle in the framework of G-complete fuzzy metric spaces due to Grabiec [11].

Theorem 3. *Let $(X, M, *)$ be a G-complete fuzzy metric space satisfying*

$$\lim_{t\to\infty} M(x, y, t) = 1, \ \forall\, x, y \in X. \tag{2}$$

Suppose that $T : X \to X$ is a fuzzy exponential contraction and is fuzzy continuous. Then, T possesses a unique fixed point.

Proof. Let $z_0 \in X$ be arbitrary. Define the Picard sequence $\{z_n\}$ by $z_{n+1} = Tz_n$ for all $n \geq 0$. Applying the contraction condition (1) with $x = z_0$, $y = z_1$ yields

$$*\prod_{i=1}^{k}[M(Tz_0, Tz_1, t)]^{a_i(Tz_0,Tz_1)} \geq *\prod_{i=1}^{k}[M(z_0, z_1, t/\lambda)]^{a_i(z_0,z_1)},$$

which is equivalent to

$$*\prod_{i=1}^{k}[M(z_1, z_2, t)]^{a_i(z_1,z_2)} \geq *\prod_{i=1}^{k}[M(z_0, z_1, t/\lambda)]^{a_i(z_0,z_1)}.$$

Proceeding inductively, we obtain for all $n \geq 0$

$$*\prod_{i=1}^{k}[M(z_n, z_{n+1,t})]^{a_i(z_n,z_{n+1})} \geq *\prod_{i=1}^{k}[M(z_0, z_1, t/\lambda^n)]^{a_i(z_0,z_1)}. \tag{3}$$

Noting that the t-norm is non-decreasing and that each term $[M(\cdot,\cdot,\cdot)]^{a_i(\cdot,\cdot)}$ lies in the interval $[0, 1]$, we can deduce that any single term on the left-hand side is greater than or equal to the entire product on the left-hand side. Therefore, for some $j \in \{1, \ldots, k\}$, we have

$$[M(z_n, z_{n+1}, t)]^{a_j(z_n,z_{n+1})} \geq *\prod_{i=1}^{k}[M(z_0, z_1, t/\lambda^n)]^{a_i(z_0,z_1)}. \tag{4}$$

Since $a_j(z_n, z_{n+1}) \geq 1$ and $M(z_n, z_{n+1}, t) \in [0, 1]$, it follows that

$$[M(z_n, z_{n+1}, t)] \geq [M(z_n, z_{n+1}, t)]^{a_j(z_n,z_{n+1})}. \tag{5}$$

Combining (4) and (5), we get

$$[M(z_n, z_{n+1}, t)] \geq *\prod_{i=1}^{k}[M(z_0, z_1, t/\lambda^n)]^{a_i(z_0,z_1)}.$$

Since $t/\lambda^n \to \infty$ as $n \to \infty$, the condition (2) implies that

$$\lim_{n\to\infty} M(z_0, z_1, t/\lambda^n) = 1. \tag{6}$$

Therefore, due to the continuity of t-norm, the right-hand side of the last inequality converges to 1 (the identity for the t-norm). This implies

$$\lim_{n\to\infty} M(z_n, z_{n+1}, t) = 1 \quad \text{for all } t > 0.$$

To prove $\{z_n\}$ is G-Cauchy, we use the fuzzy triangle inequality repeatedly. For any $n \in \mathbb{N}$ and fixed $p \in \mathbb{N}$, we have

$$\begin{aligned} M(z_n, z_{n+p}, t) &\geq M\left(z_n, z_{n+1}, \frac{t}{p}\right) * M\left(z_{n+1}, z_{n+2}, \frac{t}{p}\right) * \cdots \\ &\quad * M\left(z_{n+p-1}, z_{n+p}, \frac{t}{p}\right) \\ &\to \underbrace{1 * 1 * \cdots * 1}_{p \text{ times}} = 1 \quad \text{as } n \to \infty \quad \text{(due to the continuity of t-norm).} \end{aligned}$$

Hence, $\{z_n\}$ is a G-Cauchy sequence.

Since $(X, M, *)$ is G-complete, the Cauchy sequence $\{z_n\}$ converges. Thus, there exists $z^* \in X$, such that

$$\lim_{n\to\infty} M(z_n, z^*, t) = 1 \quad \text{for all } t > 0.$$

Since T is continuous and $z_n \to z^*$, it follows that $Tz_n \to Tz^*$ and $Tz_n \to z^*$,i.e.,

$$\lim_{n\to\infty} M(Tz^*, Tz_n, t/2) = 1 \text{ and } \lim_{n\to\infty} M(z_{n+1}, z^*, t/2) = 1.$$

Using the triangle inequality of the G-metric M and the continuity of the t-norm $*$, we have for any $n \in \mathbb{N}$

$$\begin{aligned} M(Tz^*, z^*, t) &\geq \lim_{n\to\infty} [M(Tz^*, Tz_n, t/2) * M(Tz_n, z^*, t/2)] \\ &= \left(\lim_{n\to\infty} M(Tz^*, Tz_n, t/2)\right) * \left(\lim_{n\to\infty} M(z_{n+1}, z^*, t/2)\right) \\ &= 1 * 1 \\ &= 1. \end{aligned}$$

Since $M(Tz^*, z^*, t) \leq 1$ by definition, it follows that $M(Tz^*, z^*, t) = 1$ for all $t > 0$. This implies $Tz^* = z^*$.

To prove the uniqueness, suppose z^{**} is another fixed point of T. Applying the contraction condition (1) with $x = z^*$, $y = z^{**}$

$$\begin{aligned} *\prod_{i=1}^{k} [M(z^*, z^{**}, t)]^{a_i(z^*, z^{**})} &= *\prod_{i=1}^{k} [M(Tz^*, Tz^{**}, t)]^{a_i(Tz^*, Tz^{**})} \\ &\geq *\prod_{i=1}^{k} [M(z^*, z^{**}, t/\lambda)]^{a_i(z^*, z^{**})} \\ &= *\prod_{i=1}^{k} [M(Tz^*, Tz^{**}, t/\lambda)]^{a_i(Tz^*, Tz^{**})} \\ &\geq *\prod_{i=1}^{k} \left[M(z^*, z^{**}, t/\lambda^2)\right]^{a_i(z^*, z^{**})} \\ &\cdots \\ &\geq *\prod_{i=1}^{k} [M(z^*, z^{**}, t/\lambda^n)]^{a_i(z^*, z^{**})} \to 1 * \cdots * 1 \\ &\quad \text{as } n \to \infty \text{ (by (2) and t-norm continuity).} \end{aligned}$$

Since each term $[M(\cdot,\cdot,\cdot)]^{a_i(\cdot,\cdot)}$ is in $[0,1]$, we must have

$$[M(z^*,z^{**},t)]^{a_i(z^*,z^{**})} = 1 \text{ for each } i \in \{1,\cdots,k\}.$$

Therefore, $z^* = z^{**}$, and the fixed point is unique. □

Remark 2. *It is well-known that fuzzy contractions are necessarily fuzzy continuous. In contrast, fuzzy exponential contractions are generally not fuzzy continuous. However, the continuity of fuzzy exponential contractions can be established under additional conditions on the coefficient functions a_i (see Proposition 1).*

Proposition 1. *Let $(X, M, *)$ be a fuzzy metric space and $T : X \to X$ be a fuzzy exponential contraction. Assume that the following condition holds:*

(i) For all $i \in \{1,\ldots,k\}$, there exists $B_i > 0$, such that $a_i(x,y) \leq B_i$ for all $x, y \in X$.

Then, T is fuzzy continuous.

Proof. Let $\{u_n\}$ be a sequence in X, such that $\lim_{n\to\infty} u_n = u$ for some $u \in X$, i.e., $\lim_{n\to\infty} M(u_n,u,t) = 1$ for all $t > 0$. We need to show that $\lim_{n\to\infty} M(Tu_n,Tu,t) = 1$ for all $t > 0$. Using the contraction condition (1) with $x = u_n$, $y = u$, we get

$$*\prod_{i=1}^{k}[M(Tu_n,Tu,t)]^{a_i(Tu_n,Tu)} \geq *\prod_{i=1}^{k}[M(u_n,u,t/\lambda)]^{a_i(u_n,u)}. \tag{7}$$

Given that the t-norm is non-decreasing and all terms $[M(\cdot,\cdot,\cdot)]^{a_i(\cdot,\cdot)}$ are bounded by $[0,1]$, we can infer that the value of any individual term on the left-hand side is an upper bound for the complete product. Thus, for each $j \in \{1,\ldots,k\}$, it follows that

$$[M(Tu_n,Tu,t)]^{a_j(Tu_n,Tu)} \geq *\prod_{i=0}^{k}[M(u_n,u,t/\lambda)]^{a_i(u_n,u)}. \tag{8}$$

Since $a_j(Tu_n,Tu) \geq 1$ and $M(Tu_n,Tu,t) \in [0,1]$, we have

$$[M(Tu_n,Tu,t)] \geq [M(Tu_n,Tu,t)]^{a_j(Tu_n,Tu)} \tag{9}$$

By assumption (i), $a_i(u_n,u) \leq B_i$ for all $i \in \{1,\cdots k\}$. Since $0 \leq M(u_n,u,t/\lambda) \leq 1$, we have

$$*\prod_{i=1}^{k}[M(u_n,u,t/\lambda)]^{a_i(u_n,u)} \geq *\prod_{i=1}^{k}[M(u_n,u,t/\lambda)]^{B_i} \tag{10}$$

Combining (8)–(10), we obtain

$$[M(Tu_n,Tu,t)] \geq *\prod_{i=1}^{k}[M(u_n,u,t/\lambda)]^{B_i}. \tag{11}$$

As $n \to \infty$, $M(u_n,u,t/\lambda) \to 1$ for any fixed $t > 0$. Since the t-norm is continuous and 1 is its identity, the right-hand side of (11) converges to 1. Hence

$$\lim_{n\to\infty} M(Tu_n,Tu,t) = 1 \quad \text{for all } t > 0.$$

This proves that T is fuzzy continuous. □

Corollary 1. *Let $(X, M, *)$ be an G-complete fuzzy metric space such that for all $x, y \in X$, $\lim_{t\to\infty} M(x, y, t) = 1$ and $T : X \to X$ is a fuzzy exponential contraction. Assume that condition (i) of Proposition 1 holds. Then, T admits a unique fixed point.*

Proof. By Proposition 1, T is fuzzy continuous. The result then follows directly from Theorem 3. □

Corollary 2. *Let $(X, M, *)$ be an G-complete fuzzy metric space such that for all $x, y \in X$, $\lim_{t\to\infty} M(x, y, t) = 1$ and $T : X \to X$ is a mapping. Assume there exist $\lambda \in (0, 1)$, a positive integer k, and $\{a_i\}_{i=1}^{k} \subset [1, \infty)$, such that for all $x, y \in X$ and any t:*

$$*\prod_{i=1}^{k}[M(Tx, Ty, t)]^{a_i} \geq *\prod_{i=1}^{k}[M(x, y, t/\lambda)]^{a_i}.$$

Then, T admits a unique fixed point.

Proof. This is a special case of Corollary 1 where $a_i(x, y) = a_i$ for all $x, y \in X$ (constant functions) and $B_i = a_i$. □

Remark 3. *Corollary 2 recovers the standard fuzzy Banach contraction principle (Theorem 1) when $k = 1$ and $a_1 = 1$.*

The following example demonstrates that fuzzy exponential contractions form a genuine extension of standard fuzzy contractions:

Example 1. *Let $X = \{x_1, x_2, x_3, x_4\}$ be equipped with the fuzzy metric*

$$M(x, y, t) = \begin{cases} e^{-d(x,y)/t} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases},$$

and the product t-norm $a * b = ab$, where d is the discrete metric. Therefore, the fuzzy metric simplifies to

$$M(x_i, x_j, t) = \begin{cases} 1 & \text{if } i = j \\ e^{-1/t} & \text{if } i \neq j \\ 0 & \text{if } t = 0 \end{cases}.$$

Since X is a finite set, it is easy to verify that every G-Cauchy sequence in this space is eventually constant, and every eventually constant sequence converges to a point in X, we conclude that the fuzzy metric space is G-complete.

Define $T : X \to X$ by

$$Tx_1 = x_1, \quad Tx_2 = x_3, \quad Tx_3 = x_4, \quad Tx_4 = x_1.$$

T is not standard fuzzy contraction. In fact , the standard fuzzy contraction requires a $\lambda \in (0, 1)$, such that for all $x, y \in X$,

$$M(Tx, Ty, t) \geq M(x, y, t/\lambda)$$

Take $(x, y) = (x_1, x_2)$, the above inequality becomes

$$e^{-1/t} \geq e^{-\lambda/t}$$

For this to hold for all $t > 0$, we need $\lambda \geq 1$, which contradicts $\lambda \in (0, 1)$. Therefore, T is not a standard fuzzy contraction.

To show that T is a fuzzy exponential contraction. Let $k = 2$, $\lambda = \frac{3}{4}$, $a_2(x, y) \equiv 1$, and define $a_1(x, y)$ symmetrically as

$$\begin{aligned} \text{for all } i, \quad a_1(x_i, x_i) &= 1, \\ a_1(x_1, x_2) = a_1(x_2, x_1) = a_1(x_2, x_3) &= a_0(x_3, x_2) = 3, \\ a_1(x_1, x_3) = a_1(x_3, x_1) = a_1(x_3, x_4) &= a_1(x_4, x_3) = 2, \\ a_1(x_1, x_4) = a_1(x_4, x_1) &= 1, \\ a_1(x_2, x_4) = a_1(x_4, x_2) &= 6. \end{aligned}$$

The contraction condition to verify is

$$[M(Tx, Ty, t)]^{a_1(Tx,Ty)+1} \geq [M(x, y, 4t/3)]^{a_1(x,y)+1}.$$

We distinguish the following two cases:
Case I ($\boldsymbol{t = 0}$): For all $x, y \in X$, both sides are 0. The inequality $0 \geq 0$ holds.
Case II ($\boldsymbol{t \neq 0}$): For $x \neq y$, $M(\cdot, \cdot, s) = e^{-1/s}$, so the inequality simplifies to

$$e^{-(a_1(Tx,Ty)+1)/t} \geq e^{-3(a_1(x,y)+1)/(4t)} \iff 4(a_1(Tx, Ty) + 1) \leq 3(a_1(x, y) + 1). \tag{12}$$

The following table verifies condition (12) for all distinct unordered pairs (x, y):

(x, y)	$a_1(x, y)$	(Tx, Ty)	$a_1(Tx, Ty)$	$4(a_1(Tx, Ty) + 1)$	$3(a_1(x, y) + 1)$
(x_1, x_2)	3	(x_1, x_3)	2	$4(2 + 1) = 12$	$3(3 + 1) = 12$
(x_1, x_3)	2	(x_1, x_4)	1	$4(1 + 1) = 8$	$3(2 + 1) = 9$
(x_1, x_4)	1	(x_1, x_1)	0	$4(0 + 1) = 4$	$3(1 + 1) = 6$
(x_2, x_3)	3	(x_3, x_4)	2	$4(2 + 1) = 12$	$3(3 + 1) = 12$
(x_2, x_4)	6	(x_3, x_1)	2	$4(2 + 1) = 12$	$3(6 + 1) = 21$
(x_3, x_4)	2	(x_4, x_1)	1	$4(1 + 1) = 8$	$3(2 + 1) = 9$

For all cases, the inequality $4(a_1(Tx, Ty) + 1) \leq 3(a_1(x, y) + 1)$ holds. In the case where $x = y$, both sides of the original inequality equal 1. Therefore, T is a fuzzy polynomial contraction. The mapping T is continuous (as any mapping on a discrete space is continuous), and x_1 is its unique fixed point by Theorem 3.

Remark 4. *The illustrative Example 1 provided herein utilizes a finite set to demonstrate the mechanics of the G-Cauchy definition, while the G-Cauchy definition is general, the interaction between it and the proposed fuzzy exponential contraction condition has only been explicitly justified in this finite setting.*

Theorem 4. *Let $(X, M, *)$ be an M-complete fuzzy metric space such that for each $\varepsilon > 0$ and each s-increasing sequence $\{t_n\}$ of positive real numbers, there exists n_0, $p \in \mathbb{N}$, such that*

$$* \prod_{n \geq n_0} [M(x, y, t_n)]^p > 1 - \varepsilon.$$

Suppose that $T : X \to X$ is fuzzy continuous and there exist a constant $\lambda \in (0, 1)$, and a function $a : X \times X \to [1, \infty)$, such that for every $x, y \in X$, the following inequality holds:

$$[M(Tx, Ty, t)]^{a(Tx,Ty)} \geq [M(x, y, t/\lambda)]^{a(x,y)}. \tag{13}$$

Then, T has a unique fixed point.

Proof. Fix an arbitrary point $x_0 \in X$ and consider the sequence defined by $x_{n+1} = Tx_n$ for $n \geq 0$. Applying the contractivity condition with $x = x_{n-1}$, $y = x_n$, we obtain

$$\big[M(x_n, x_{n+1}, t)\big]^{a(x_n, x_{n+1})} \geq \big[M(x_{n-1}, x_n, t/\lambda)\big]^{a(x_{n-1}, x_n)}.$$

By induction, this implies

$$\big[M(x_n, x_{n+1}, t)\big]^{a(x_n, x_{n+1})} \geq \big[M(x_0, x_1, t/\lambda^n)\big]^{a(x_0, x_1)}.$$

Since $a(\cdot, \cdot) \geq 1$, it follows that

$$M(x_n, x_{n+1}, t) \geq \big[M(x_0, x_1, t/\lambda^n)\big]^{a(x_0, x_1)}.$$

To show that $\{x_n\}$ is an M-Cauchy sequence, let $t > 0$ and $\varepsilon > 0$. For $m, n \in \mathbb{N}$ with $n < m$, let $\{s_j\}_{j=n}^{m-1}$ be positive reals such that

$$\sum_{j=n}^{m-1} s_j \leq 1.$$

Then

$$\begin{aligned} M(x_n, x_m, t) &\geq *\prod_{j=n}^{m-1} M(x_j, x_{j+1}, s_j t) \\ &\geq *\prod_{j=n}^{m-1} \big[M(x_0, x_1, s_j t/\lambda^j)\big]^{a(x_0, x_1)} \\ &\geq *\prod_{j=n}^{\infty} M\Big(x_0, x_1, s_j t/\lambda^j\Big)\Big]^{a(x_0, x_1)}. \end{aligned}$$

In particular, we can take $s_j = \frac{1}{j(j+1)}$, $j = n, \cdots, m-1$ (noting that $\sum_{n=1}^{\infty} s_n = 1$); then, we have

$$M(x_n, x_m, t) \geq *\prod_{j=n}^{\infty} M\Big(x_0, x_1, t/j(j+1)\lambda^j\Big)\Big]^{a(x_0, x_1)}.$$

Let $t_j = \frac{s_j t}{\lambda^j} = \frac{t}{j(j+1)\lambda^j}$. Then, $\{t_j\}$ is an s-increasing sequence since $t_{j+1} - t_j \to \infty$ as $j \to \infty$. By the hypothesis on the infinite t-norm product, for any $\varepsilon > 0$, there exists n_0 and $p \geq a(x_0, x_1)$, such that for $n \geq n_0$:

$$*\prod_{j=n}^{\infty} M\Big(x_0, x_1, t/j(j+1)\lambda^j\Big)\Big]^p > 1 - \varepsilon.$$

Hence, for $m > n \geq n_0$:

$$M(x_n, x_m, t) > 1 - \varepsilon,$$

which shows that $\{x_n\}$ is a Cauchy sequence.

By the M-completeness of $(X, M, *)$, there exists $z \in X$ with $x_n \to z$, i.e., $M(x_n, z, t) \to 1$ for all $t > 0$. By continuity of T and $*$,

$$M(Tz, z, t) \geq M(Tz, Tx_n, t/2) * M(x_{n+1}, z, t/2) \to 1,$$

hence $M(Tz, z, t) = 1$ for all $t > 0$, which yields $Tz = z$.

For the uniqueness, suppose x and z are two fixed points of T. For any $t > 0$, we obtain

$$1 > M(z, x, t) \geq [M(z, x, t)]^{a(z,x)} \geq \cdots \geq [M(z, x, t/k^n)]^{a(z,x)}, \quad \forall n \in \mathbb{N}.$$

Since (t/k^n) is an s-increasing sequence, by assumption for every $\varepsilon \in (0, 1)$ there exists $n_0 \in \mathbb{N}$, such that

$$* \prod_{n \geq n_0} [M(z, x, t/k^n)]^p \geq 1 - \varepsilon,$$

and clearly $\lim_{n\to\infty} M(z, x, t/k^n) = 1$. Hence, $M(z, x, t) = 1$ for all $t > 0$, which implies $z = x$. Thus, the fixed point of T is unique. □

Remark 5. *Note that Theorem 2 is a special case of Theorem 4 when $a \equiv 1$. This shows that our exponential contraction framework properly extends the existing theory.*

3.2. Weakening the Continuity Condition

First, we introduce the concept of fuzzy Picard continuity, which is a fuzzy analogue of Picard continuity in standard metric spaces given in [20].

Definition 9. *Let $(X, M, *)$ be a fuzzy metric space. A mapping $T : X \to X$ is called fuzzy Picard-continuous if for all $z, w \in X$, we have:*

$$\left(\lim_{n\to\infty} M(T^n z, w, t) = 1 \quad \forall t > 0\right) \Rightarrow \left(\lim_{n\to\infty} M(T(T^n z), Tw, t) = 1 \quad \forall t > 0\right),$$

where $T^0 z = z$ and $T^{n+1} z = T(T^n z)$ for all $n \geq 0$.

Remark 6. *The following relationships hold between the fuzzy and fuzzy Picard continuity concepts:*

(i) If $T : X \to X$ is fuzzy continuous, then T is fuzzy Picard-continuous.
(ii) The converse is not true: there exist fuzzy Picard-continuous mappings that are not fuzzy continuous (see Example 2).
(iii) Fuzzy Picard-continuity is a weaker condition than fuzzy continuity and is sufficient for establishing the convergence of Picard iterations to fixed points.

Example 2. *Let $X = [a, b] \subset \mathbb{R}$ with $a < b$. Consider the mapping $T : [a, b] \to [a, b]$ defined by:*

$$Tx = \begin{cases} a & \text{if } x \in [a, b), \\ \frac{a+b}{2} & \text{if } x = b. \end{cases}$$

Let $(X, M, \cdot)$ be the standard fuzzy metric space induced by the Euclidean metric d, i.e.,

$$M(x, y, t) = \begin{cases} e^{-d(x,y)/t} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases},$$

with the product t-norm $a \cdot b$.

First, we show that T is not fuzzy continuous at $x = b$. Consider the sequence $\{x_n\} = \{b - 1/n\}$, which converges to b in the fuzzy metric space since $\lim_{n\to\infty} M(x_n, b, t) = e^{-1/(nt)} = 1$. Since $Tx_n = a$ for all n and $T(b) = (a + b)/2$, the fuzzy distance between these is $M(a, (a + b)/2, t) = e^{-(b-a)/(2t)} \neq 1$. Thus, T is discontinuous at $x = b$.

Next, we prove that T is fuzzy Picard continuous. The condition for this is:
if $\lim_{n\to\infty} M(T^n z, w, t) = 1$, then $\lim_{n\to\infty} M(T^{n+1} z, Tw, t) = 1$. The Picard iterates $T^n z$, starting from any point $z \in [a, b]$, settle to the value a for all $n \geq 2$. Therefore, the hypothesis

$\lim_{n\to\infty} M(T^n z, w, t) = 1$ *implies that* $\lim_{n\to\infty} M(a, w, t) = 1$, *which forces* $w = a$. *With* $w = a$, *the conclusion of the condition becomes* $\lim_{n\to\infty} M(T^{n+1}z, T(a), t) = \lim_{n\to\infty} M(T^{n+1}z, a, t)$. *Since* $T^{n+1}z = a$ *for* $n \geq 2$, *the limit is* $\lim_{n\to\infty} M(a, a, t) = 1$. *Since the condition holds, T is fuzzy Picard continuous.*

Theorem 5. *Let* $(X, M, *)$ *be a G-complete fuzzy metric space satisfying*

$$\lim_{t\to\infty} M(x, y, t) = 1, \ \forall\, x, y \in X.$$

Suppose that $T : X \to X$ *is a fuzzy exponential contraction and is fuzzy Picard continuous. Then, T possesses a unique fixed point.*

Proof. The proof follows the same initial steps as Theorem 3 to establish that the Picard sequence $\{z_n\} = \{T^n z_0\}$ is G-Cauchy and hence converges to some $z^* \in X$, i.e., $\lim_{n\to\infty} M(T^n z_0, z^*, t) = 1$ for all $t > 0$.

By the fuzzy Picard-continuity of T, this implies:

$$\lim_{n\to\infty} M(T(T^n z_0), Tz^*, t) = \lim_{n\to\infty} M(T^{n+1} z_0, Tz^*, t) = 1 \quad \text{for all } t > 0.$$

However, we also know that $\{T^{n+1}z_0\}$ is a subsequence of the convergent sequence $\{T^n z_0\}$ and must converge to the same limit z^*. Using the triangle inequality of the G-metric M and the continuity of the t-norm $*$, we have for any $n \in \mathbb{N}$

$$\begin{aligned} M(Tz^*, z^*, t) &\geq \lim_{n\to\infty} \left[M(Tz^*, T^{n+1}z_0, t/2) * M(T^{n+1}z_0, z^*, t/2) \right] \\ &= \left(\lim_{n\to\infty} M(Tz^*, T^{n+1}z_0, t/2) \right) * \left(\lim_{n\to\infty} M(z_{n+1}, z^*, t/2) \right) \\ &= 1 * 1 \\ &= 1. \end{aligned}$$

We conclude that $Tz^* = z^*$. The proof of the uniqueness remains identical to that in Theorem 3. □

We now demonstrate that Theorem 5 constitutes a proper extension of Theorems 1 and 3. This is illustrated by the following example (Example 3) and elaborated upon in the subsequent remark (Remark 7).

Example 3. *Let* $(X, M, *)$ *be the fuzzy metric space where* $X = [0, 1]$,

$$M(x, y, t) = \begin{cases} e^{-|x-y|/t} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases},$$

and the t-norm $*$ *is the product* $a \cdot b$.
Define $T : X \to X$ *by*

$$Tx = \begin{cases} 0 & \text{if } x \in [0, 1), \\ \frac{1}{2} & \text{if } x = 1. \end{cases}$$

We have already established that this mapping is fuzzy Picard continuous but not fuzzy continuous (see Example 2). Furthermore, by Remark 2, the mapping T is not fuzzy Banach contraction in the sense of Grabeic.

We now show that T is a fuzzy exponential contraction for the choice $k = 2$, $\lambda = 0.9$, and the following coefficient functions:

$$a_2(x,y) = 1 \quad \forall x,y \in X, \qquad a_1(x,y) = \begin{cases} 1, & x,y \in [0,1), x \neq y, \\ 2, & x = y, \\ \frac{2}{1-y}, & x = 1,\ y \in [0,1), \\ \frac{2}{1-x}, & y = 1,\ x \in [0,1). \end{cases}$$

The contraction condition to verify is

$$*\prod_{i=1}^{2}[M(Tx,Ty,t)]^{a_i(Tx,Ty)} \geq *\prod_{i=1}^{2}[M(x,y,t/\lambda)]^{a_i(x,y)}.$$

Since the t-norm is the product, this is equivalent to

$$[M(Tx,Ty,t)]^{a_1(Tx,Ty)+a_2(Tx,Ty)} \geq [M(x,y,t/\lambda)]^{a_1(x,y)+a_2(x,y)}.$$

Substituting $a_2(\cdot,\cdot) \equiv 1$ and $\lambda = 0.9$ gives:

$$[M(Tx,Ty,t)]^{a_1(Tx,Ty)+1} \geq [M(x,y,t/0.9)]^{a_1(x,y)+1}.$$

We distinguish the following two cases:
Case I ($\boldsymbol{t = 0}$): For all $x,y \in X$, both sides are 0. The inequality $0 \geq 0$ holds.
Case II ($\boldsymbol{t \neq 0}$): For all $x,y \in X$, $M(x,y,s) = e^{-|x-y|/s}$. Taking logarithms and simplifying yields the equivalent inequality:

$$(a_1(Tx,Ty)+1)\cdot|Tx-Ty| \leq 0.9\cdot(a_1(x,y)+1)\cdot|x-y|. \tag{14}$$

We now verify that inequality (14) holds for all $x,y \in X$. The analysis is broken into the following subcases:
Subcase 1 ($\boldsymbol{x = y}$): Both sides are 0. The inequality $0 \leq 0$ holds.
Subcase 2 ($\boldsymbol{x,y \in [0,1), x \neq y}$): LHS is 0. RHS is $1.8\cdot|x-y| > 0$. Thus, $0 \leq 1.8|x-y|$ holds.
Subcase 3 ($\boldsymbol{x \in [0,1), y = 1}$): Note that $|x-y| = |1-x| = 1-x > 0$ and $|Tx-Ty| = \frac{1}{2}$. therefore,

$$LHS = \tfrac{1}{2}(a_1(0,\tfrac{1}{2})+1) = \tfrac{1}{2}(1+1) = 1.$$
$$RHS = 0.9\cdot\left(\frac{2}{1-x}+1\right)(1-x) = 0.9(2+(1-x)) = 1.8+0.9(1-x).$$

We require $1 \leq 1.8+0.9(1-x)$. This holds for all $x \in [0,1)$.
Subcase 4 ($\boldsymbol{x = 1, y \in [0,1)}$): This case is symmetric to Case 3.

Since inequality (14) holds in all cases, T is a fuzzy exponential contraction and is fuzzy Picard-continuous. Therefore, due to Theorem 5, T has a unique fixed point, namely $z^* = 0$.

Remark 7. *In the complete fuzzy metric space $([0,1], M, *)$, we found an example where T is not fuzzy continuous at $x = 1$ but is fuzzy Picard continuous. Although T is not a fuzzy Banach contraction, it is a fuzzy exponential contraction. Thus, while Theorems 1 and 3 are not applicable, Theorem 5 successfully guarantees the unique fixed point $z^* = 0$.*

4. Application to a Nonlinear Fractional Boundary Value Problems

Fractional differential Equations (FDEs) extend the scope of classical calculus by allowing derivatives of arbitrary real or complex order, rather than limiting them to integers.

These equations naturally arise in numerous scientific and engineering contexts, including viscoelastic materials, anomalous diffusion processes, and control theory. Their intrinsic ability to represent memory and hereditary properties makes them especially effective in describing complex dynamic systems. The solutions of such equations are often obtained via fixed point techniques, where the problem is reformulated as finding a fixed point of a suitable integral operator. Recent studies by [23–26] have further advanced this approach by developing new fixed point theorems and demonstrating their effectiveness in solving various classes of fractional differential equations.

To demonstrate the utility and scope of our theoretical findings, we apply the fixed point theory for fuzzy exponential contractions to establish the existence and uniqueness of solutions for a class of nonlinear fractional differential equations. The key advantage of our approach is its ability to handle nonlinearities governed by generalized, point-dependent Lipschitz conditions, which are beyond the reach of classical fuzzy or metric contraction principles.

Consider the following nonlinear fractional boundary value problem (BVP) with integral boundary conditions:

$$\begin{cases} {}^{C}D^{\alpha}u(t) = f(t,u(t)), & t \in [0,1], \quad 1 < \alpha \leq 2, \\ u(0) = 0, \quad u(1) = \int_0^1 u(s)ds, \end{cases} \tag{15}$$

where ${}^{C}D^{\alpha}$ denotes the Caputo fractional derivative and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

It is well-known that the BVP (15) is equivalent to the nonlinear integral equation

$$u(t) = \int_0^1 G(t,s)f(s,u(s))ds, \tag{16}$$

where $G(t,s)$ is the associated Green's function. We assume the following standard property for G:

The Green's function $G(t,s)$ is continuous on $[0,1] \times [0,1]$ and satisfies

$$K = \sup_{t\in[0,1]} \int_0^1 |G(t,s)|ds < \infty. \tag{17}$$

Let $X = C([0,1],\mathbb{R})$ be the Banach space of continuous real-valued functions on $[0,1]$ equipped with the supremum norm $\|u\|_\infty = \sup_{t\in[0,1]} |u(t)|$. Define the integral operator $T : X \to X$ by

$$(Tu)(t) = \int_0^1 G(t,s)f(s,u(s))ds. \tag{18}$$

Clearly, a function $u \in X$ is a solution of the BVP (15) if and only if it is a fixed point of T.

To apply our fuzzy fixed point theory, we endow X with a fuzzy metric structure. Define the fuzzy metric M using the standard exponential form induced by the supremum norm:

$$M(u,v,t) = \exp\left(-\frac{\|u-v\|_\infty}{t}\right), \quad \text{for all } u, vs. \in X, t > 0.$$

We take the continuous t-norm $*$ to be the product t-norm, i.e., $a * b = a \cdot b$. The triple $(X, M, \cdot)$ is then a complete fuzzy metric space in the sense of George and Veeramani (M-complete).

We now present the main result of this section.

Theorem 6. *Assume the following conditions hold:*

(i) $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is continuous.*

(ii) *(Generalized Lipschitz Condition) There exists a function* $\psi : X \times X \to [1, \infty)$*, such that for all* $s \in [0,1]$ *and all* $u, vs. \in X$*,*

$$|f(s, u(s)) - f(s, v(s))| \leq \psi(u, v) \cdot |u(s) - v(s)|. \tag{19}$$

(iii) *(T-Invariance of* ψ*) The function* ψ *satisfies*

$$\psi(Tu, Tv) \leq \psi(u, v) \quad \text{for all } u, vs. \in X. \tag{20}$$

(iv) *(Parameter Constraint) The parameters satisfy*

$$K \cdot \sup_{u,v \in X} \psi(u, v) < 1, \tag{21}$$

where K *is defined in* (17)*.*

Then, the fractional boundary value problem (15) *has a unique solution in* X*.*

Proof. We will show that the operator T defined in (18) is a fuzzy exponential contraction of order $k = 1$, satisfying the conditions of Theorem 4.

For any $u, vs. \in X$ and $t > 0$, we estimate the distance between their images under T:

$$\begin{aligned} |(Tu)(t) - (Tv)(t)| &\leq \int_0^1 |G(t,s)| \cdot |f(s, u(s)) - f(s, v(s))| ds \\ &\leq \int_0^1 |G(t,s)| \cdot \psi(u, v) \cdot |u(s) - v(s)| ds \quad \text{(by (19))} \\ &\leq \psi(u, v) \cdot \|u - v\|_\infty \int_0^1 |G(t,s)| ds \\ &\leq K \cdot \psi(u, v) \cdot \|u - v\|_\infty. \end{aligned}$$

Taking the supremum over $t \in [0,1]$ yields the following inequality in the metric space $(X, \|\cdot\|_\infty)$:

$$\|Tu - Tv\|_\infty \leq K \cdot \psi(u, v) \cdot \|u - v\|_\infty. \tag{22}$$

Let us now define the contraction parameter λ and the coefficient function a for our fuzzy exponential contraction. From condition (21), we can choose a constant λ, such that

$$K \cdot \sup_{u,v \in X} \psi(u, v) < \lambda < 1.$$

Define the function $a : X \times X \to [1, \infty)$ by $a(u, v) = \psi(u, v)$. We now verify that T satisfies the fuzzy exponential contraction condition from Theorem 4

$$[M(Tu, Tv, t)]^{a(Tu,Tv)} \geq [M(u, v, t/\lambda)]^{a(u,v)}. \tag{23}$$

Recall that $M(u, v, t) = \exp(-\|u - v\|_\infty / t)$. Substituting this into (23) and taking natural logarithms, we obtain the equivalent inequality

$$-\frac{\|Tu - Tv\|_\infty}{t} \cdot a(Tu, Tv) \geq -\frac{\|u - v\|_\infty}{t/\lambda} \cdot a(u, v).$$

Multiplying both sides by $-t$ (and reversing the inequality) gives

$$\|Tu - Tv\|_\infty \cdot a(Tu, Tv) \leq \lambda \|u - v\|_\infty \cdot a(u, v). \tag{24}$$

We now prove that (24) holds. Starting from the metric inequality (22) and using the fact that $a(Tu, Tv) = \psi(Tu, Tv)$, we have

$$\begin{aligned}\|Tu - Tv\|_\infty \cdot a(Tu, Tv) &\leq (K \cdot \psi(u, v) \cdot \|u - v\|_\infty) \cdot \psi(Tu, Tv) \\ &= (K \cdot \psi(u, v))\|u - v\|_\infty \cdot \psi(Tu, Tv).\end{aligned}$$

Now, applying the T-invariance condition (20), $\psi(Tu, Tv) \leq \psi(u, v)$, we get

$$\|Tu - Tv\|_\infty \cdot a(Tu, Tv) \leq (K \cdot \psi(u, v))\|u - v\|_\infty \cdot \psi(u, v).$$

Notice that by condition (21) and our choice of λ, we have $K \cdot \psi(u, v) \leq K \cdot \sup \psi < \lambda$. Therefore,

$$K \cdot \psi(u, v)\|u - v\|_\infty \cdot \psi(u, v) \leq \lambda \cdot \|u - v\|_\infty \cdot \psi(u, v) = \lambda\|u - v\|_\infty \cdot a(u, v).$$

Thus, we have established the key inequality:

$$\|Tu - Tv\|_\infty \cdot a(Tu, Tv) \leq \lambda\|u - v\|_\infty \cdot a(u, v),$$

which is exactly (24). Therefore, the operator T is a fuzzy exponential contraction (13).

The fuzzy continuity of T follows directly from the continuity of f and G. Since $(X, M, \cdot)$ is M-complete, all conditions of Theorem 6 are satisfied. Consequently, T has a unique fixed point $u^* \in X$, which is the unique solution to the fractional boundary value problem (15). □

Remark 8. *The power of Theorem 6 lies in the use of the T-invariant function $\psi(u, v)$. This allows the Lipschitz constant to depend on the pair of functions (u, v) in a controlled way, generalizing the standard, uniform Lipschitz condition. The T-invariance property (20) is crucial for the iterative process in the proof and is a natural condition in many integral equations.*

Corollary 3. *If f satisfies the standard uniform Lipschitz condition*

$$|f(s, u) - f(s, v)| \leq L|u - v| \quad \textit{for all } s \in [0, 1], u, vs. \in \mathbb{R},$$

with $LK < 1$; then, the fractional boundary value problem (15) has a unique solution.

Proof. Take $\psi(u, v) \equiv L$ in Theorem 6. Conditions (19)–(21) are trivially satisfied. □

5. Conclusions

This paper successfully introduced and formalized the concept of fuzzy exponential contractions, a significant generalization of standard fuzzy contractions as defined in Definition 8. The key innovation lies in the incorporation of point-dependent exponential terms, which provides a more flexible and powerful framework for analyzing a broader class of mappings.

Our main contributions are supported by rigorous theoretical results. We established fixed point theorems for these new contractions under two different continuity assumptions: the standard fuzzy continuity (Theorem 3) and the weaker, newly introduced condition of fuzzy Picard continuity (Theorem 5). The robustness and expanded scope of our theory were demonstrated through examples. For instance, Example 1 showcased mappings on finite set that are fuzzy exponential contractions but fail to be standard fuzzy contractions, thereby confirming that our framework is a genuine extension of existing principles.

Furthermore, we demonstrated the practical utility of our findings by applying them to a nonlinear fractional boundary value problem (Theorem 6). This application showed how the flexibility of fuzzy exponential contractions can handle generalized, point-dependent Lipschitz conditions. In summary, this work not only broadens the theoretical landscape of fixed point theory in fuzzy metric spaces but also provides a versatile tool for future research in nonlinear analysis.

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