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Stability and Well-Posedness of Fractional Navier–Stokes with Directional Fractional Diffusion

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Abstract

We investigate the three-dimensional incompressible fractional Navier–Stokes system with *directional fractional diffusion*: a vertical dissipative operator of order $2\alpha \in (0,2]$ acting on the full velocity field together with a horizontal fractional operator of order $2\beta \in (0,2]$ applied to the vertical average of one horizontal component. This anisotropic, nonlocal structure captures media in which smoothing acts with unequal strength by direction. For small, divergence-free initial data in an anisotropic energy class, we establish *global well-posedness and stability* of the zero state, including uniqueness and continuous dependence on the data. The analysis crucially relies on an average oscillation decomposition in the vertical variable, a fractional Poincaré inequality aligned with the vertical direction, and sharp product/commutator bounds compatible with the anisotropic splitting. We provide explicit estimates for direction-dependent smoothing and algebraic decay governed by (α, β), and we show that the conclusions persist under small perturbation of the dissipation parameters.

Keywords: fractional Navier–Stokes; anisotropy; directional fractional diffusion; stability; well-posedness

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Academic Editor: Jordan Hristov

Received: 26 September 2025
Revised: 30 October 2025
Accepted: 30 October 2025
Published: 31 October 2025

Citation: Abidin, M.Z.; Khan, A. Stability and Well-Posedness of Fractional Navier–Stokes with Directional Fractional Diffusion. *Fractal Fract.* **2025**, *9*, 708. https://doi.org/10.3390/fractalfract9110708

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1. Introduction

We consider a three-dimensional incompressible *fractional Navier–Stokes* model in which dissipation acts anisotropically: a vertical fractional operator of order 2α is applied to the full velocity field, while a horizontal fractional operator of order 2β is applied only to the vertical average of the flow. The system is as follows:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u = -\nabla p \;+\; \nu_v\,|\partial_3|^{2\alpha}u \;+\; \nu_h\,|\partial_1|^{2\beta}\,\bar{u}, & x \in \mathbb{T}^3,\ t>0, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1}$$

where $\mathbb{T}^3 = [0,1]^3$ is the periodic box, $u = u(x,t) \in \mathbb{R}^3$ is the velocity, and $p = p(x,t)$ is the scalar pressure. The constants $\nu_v, \nu_h > 0$ measure the strengths of the vertical and horizontal dissipative mechanisms. The vertical average $\bar{u}$ is defined by the following:

$$\bar{u}(x_1,x_2) \;:=\; \int_0^1 u(x_1,x_2,x_3)\,dx_3,$$

and the fractional directional operators are taken in the Fourier sense: $|\partial_j|^s$ has symbol $|\xi_j|^s$ for $j \in \{1,2,3\}$ and $s>0$. For brevity, we write ∂_i for $\partial/\partial x_i$ throughout.

The anisotropic dissipative architecture we impose is motivated by physical situations where mixing is directionally unbalanced. In stably stratified geophysical flows (oceans, atmosphere), vertical stirring is strongly inhibited, while horizontal transport persists on larger spatial scales and, in particular, acts effectively on vertically averaged fields; see, for instance [1,2]. In this setting, the exponent $\alpha \in (0,1]$ indexes the strength of *vertical* subgrid mixing inhibited by stable stratification (with $\alpha = 1$ recovering Laplacian viscosity and smaller α modeling weaker, intermittent smoothing), whereas $\beta \in (0,1]$ measures *horizontal* mesoscale/eddy mixing that primarily acts on depth-integrated, slowly varying modes. Applying the dissipative operator to the *vertical average* of a horizontal velocity component is consistent with barotropic (depth-averaged) dynamics, which dominate large-scale motions in stratified, rotating flows where vertical shear is suppressed; see [3,4] for background on anisotropic/eddy parameterizations and barotropic–baroclinic separation in geophysical fluids. In turbulence parameterizations, viscosity is often modeled as *anisotropic*: stratification amplifies the vertical component, while horizontal transport acts on larger length scales [5–7]. Within this regime, we study the core stability issue for small divergence-free data: does the system (1) admit a unique global solution that stays uniformly controlled for all $t > 0$? In the fractional, directionally applied setting considered here, there is vertical dissipation on the full velocity together with a horizontal fractional operator acting only on the vertical average. The damping sits strictly between one-directional and two-directional mechanisms. Moreover, the operators are nonlocal, so the smoothing is both anisotropic and fractional. These features place (1) outside the reach of tools tailored to fully isotropic viscosity and require estimates that track the interplay between anisotropy and fractional orders.

To place our model in context, we first recall the baseline case in which viscosity acts along *both* horizontal directions. On $\mathbb{T}^3$ the equations are as follows:

$$\begin{cases} \partial_t u + u\cdot\nabla u - \nabla p = \nu\,\Delta_h u, \\ \nabla\cdot u = 0, \end{cases} \qquad \Delta_h := \partial_1^2 + \partial_2^2. \tag{2}$$

For this setting, global well-posedness with respect to small data, is well developed in several functional frameworks; see [8–14]. Informally, damping in two horizontal directions, together with incompressibility, gives enough control to balance the quadratic convection term.

More recently, several works sharpened the description of the large-time behavior of such global solutions [10,15]; see also the anisotropic model of vertical dissipation combined with a horizontal operator acting on the vertical average, analyzed in [16], which highlights mechanisms relevant to stratified flows. Classical decay tools designed for fully isotropic viscosity, such as Fourier splitting, do not transfer directly to this anisotropic setting.

By contrast, if the dissipation acts in only *one* direction, e.g., the following:

$$\begin{cases} \partial_t u + u\cdot\nabla u - \nabla p = \nu\,\partial_3^2 u, \\ \nabla\cdot u = 0, \end{cases} \tag{3}$$

the globally well-posed question for small data on $\mathbb{T}^3$ or $\mathbb{R}^3$ remains subtle: a single directional Laplacian does not by itself provide sufficient damping to control the nonlinearity. The fractional, directionally applied dissipation considered in our work lies strictly between (3) and (2) and introduces additional nonlocal smoothing that we exploit in the analysis below.

There is also focused literature on *fractional* Navier–Stokes dynamics with diffusion applied by direction. For the 3D case with *fractional horizontal* dissipation $(-\Delta_h)^\alpha$, Ji, Luo and Jiang proved global small-data stability and time-decay, showing that non-local, direction-

wise smoothing controls the convection in the anisotropic setting [9]. Sun established the uniqueness of weak solutions for an anisotropic fractional model with only horizontal dissipation, clarifying the admissible exponents and function spaces for well-posedness [17]. Li and Yuan obtained global solutions for a generalized system with *fractional partial* dissipation, indicating that directional fractional operators can replace full isotropic viscosity when paired with sharp commutator estimates [18]. Yang, Jiu, and Wu studied *partial hyperdissipation* with component-wise fractional derivatives, providing a blueprint for balancing anisotropy, nonlocality, and the divergence constraint [19]. In a complementary direction, Lou, Yang, He, and He derived uniform analyticity for fractional Navier–Stokes in critical Fourier–Herz spaces, which is consistent with the enhanced smoothing rates produced by fractional operators [20]. Within this landscape, our model introduces vertical fractional dissipation on the full velocity together with a horizontal fractional operator acting only on the *vertical average*, and our analysis is tailored to quantify how this mixed, directionally applied nonlocal damping stabilizes the flow. For additional results on fractional Navier–Stokes and related rotating MHD models in critical frameworks, see also [21–23].

We address small-data global well-posedness and nonlinear stability for the fractional, directionally dissipative system (1) posed on the three-dimensional torus $\mathbb{T}^3 = [0,1]^3$. Our aim is to construct global solutions from small divergence-free data and to prove stability of the zero state with explicit decay governed by the fractional orders. As above, for any scalar or vector field f, we denote by $\bar{f}$ its vertical average and by $\tilde{f}$ the oscillation as follows:

$$\bar{f}(x_1, x_2) \;=\; \int_0^1 f(x_1, x_2, x_3)\, dx_3, \qquad \tilde{f} \;=\; f - \bar{f}. \tag{4}$$

With the conventions above and the operators in (1), we now state our main result on small-data global well-posedness and stability.

Theorem 1. *Consider the anisotropic fractional system* (1) *with* $\nu_v, \nu_h > 0$ *and exponents*

$$0 < \alpha \le 1, \qquad 0 < \beta \le 1.$$

Assume $u_0 \in H^3(\mathbb{T}^3)$ *with* $\nabla \cdot u_0 = 0$. *Then there exists a constant* $C_0 = C_0(\alpha, \beta) > 0$ *such that, if*

$$\|u_0\|_{H^3(\mathbb{T}^3)} \;\le\; C_0 \,\min\{\nu_v, \nu_h\},$$

the problem (1) *admits a unique global solution*

$$u \in L^\infty\big([0,\infty); H^3(\mathbb{T}^3)\big)$$

which depends continuously on u_0, *satisfies*

$$\|u(\cdot, t)\|_{H^2(\mathbb{T}^3)} \;\le\; C_0 \,\min\{\nu_v, \nu_h\},$$

and the solution satisfies the energy–dissipation control

$$\int_0^\infty \Big(\nu_v \,\big\| |\partial_3|^\alpha \nabla u(s) \big\|_{L^2}^2 \;+\; \nu_h \,\big\| |\partial_1|^\beta \,\overline{\nabla u}(s) \big\|_{L^2}^2 \Big)\, ds \;\lesssim_{\alpha,\beta}\; \|u_0\|_{H^2}^2.$$

Our contribution establishes global-in-time well-posedness and stability for small initial data, together with uniform regularity for the fractional, directionally dissipative model (1) on $\mathbb{T}^3$ under a minimal anisotropic damping compatible with the geometry: vertical fractional smoothing on the full velocity and a single horizontal fractional operator acting only on the vertical average. By comparison, many periodic small-data results require dissipation in (at least) two directions.

The proof crucially relies on two complementary decomposition and estimate layers. First, we decompose each quantity into its vertical mean and its x_3-fluctuation. A one-dimensional fractional Poincaré estimate along the vertical line yields control of the relevant H^s norms of the fluctuation via the vertical operator Λ_3^α. Next, we use direction-dependent Kato–Ponce product and commutator estimates to reassign derivatives to the dissipative pieces while preserving incompressibility. To take advantage of the x_1–directional fractional diffusion acting on the vertical average, we further decompose the following:

$$\bar{u} \;=\; \overline{\bar{u}} \;+\; \widetilde{\bar{u}}, \qquad \overline{\bar{u}}(x_2) \;:=\; \int_0^1 \bar{u}(x_1, x_2)\, dx_1,$$

so that $\widetilde{\bar{u}}$ has zero mean in x_1. This splitting isolates the component directly damped by $|\partial_1|^{2\beta}$ and permits sharp coercive estimates for the averaged flow, while the remaining pieces are controlled through the vertical fractional smoothing and the divergence constraint. Technical details of these tools are developed in Section 2.

The paper proceeds as follows. In Section 2, we gather the auxiliary estimates used to establish Theorem 1. We formalize the mean–fluctuation decomposition (1)–(4), prove a strengthened fractional Poincaré inequality for the oscillatory part $\tilde{f}$, and develop anisotropic Kato–Ponce product/commutator estimates tailored to the operators $|\partial_3|^\alpha$ and $|\partial_1|^\beta$, including sharp upper bounds for the triple products that occur in the energy method. In Section 3 we carry out the global *a priori* analysis: derive the decisive estimates, run a bootstrap argument, and close by continuity; together these steps prove Theorem 1.

2. Technical Lemmas

Before proving Theorem 1 we assemble the needed tools. We begin with the vertical mean–fluctuation decomposition: for any (scalar or vector) field f on $\mathbb{T}^3$, set the following:

$$\bar{f}(x_1, x_2) \;=\; \int_0^1 f(x_1, x_2, x_3)\, dx_3, \qquad \tilde{f} \;=\; f - \bar{f}. \tag{5}$$

so that

$$f \;=\; \bar{f} \;+\; \tilde{f},. \tag{6}$$

By construction,

$$\int_0^1 \tilde{f}(x_1, x_2, x_3)\, dx_3 \;=\; 0 \quad \text{for all } (x_1, x_2) \in \mathbb{T}^2, \qquad \int_{\mathbb{T}^3} \bar{f}\, \tilde{f}\, dx \;=\; 0, \tag{7}$$

so that the decomposition (6) is L^2-orthogonal.

A key advantage of (6) and (7) is that the oscillatory component $\tilde{f}$ holds a *fractional* Poincaré inequality in the vertical variable: for $0 < \alpha \leq 1$ and any $s \in \mathbb{R}$,

$$\|\tilde{f}\|_{H^s(\mathbb{T}^3)} \;\lesssim_\alpha\; \big\| |\partial_3|^\alpha \tilde{f} \big\|_{H^{s-\alpha}(\mathbb{T}^3)}, \tag{8}$$

which will be proved below. Inequality (8) allows us to estimate Sobolev's norms of $\tilde{f}$ through its vertical fractional derivative, a feature that dovetails with the vertical diffusion in (1).

When handling the nonlinear terms, vertical averages appear naturally. Since the x_1–directed dissipation acts only on $\bar{f}$, we *refine* the average by splitting it in the x_1 variable, as follows:

$$\bar{f} = \overline{\bar{f}} + \widetilde{\bar{f}}, \qquad \overline{\bar{f}}(x_2) = \int_0^1 \bar{f}(x_1, x_2)\, dx_1 = \int_0^1 \int_0^1 f(x_1, x_2, x_3)\, dx_3\, dx_1. \tag{9}$$

The fluctuation has zero mean in x_1 as follows:

$$\int_0^1 \tilde{\bar{f}}(x_1, x_2)\, dx_1 = 0 \quad \text{for all } x_2 \in \mathbb{T}. \tag{10}$$

The decomposition (9) and (10) isolates the component directly damped by $|\partial_1|^{2\beta}$ and will be combined with anisotropic Kato–Ponce product/commutator bounds to control triple products in the energy estimates. The precise statements are given in the lemmas that follow.

Lemma 1 (Properties of the average–oscillation splitting)**.** *Define $\bar{f}$ and $\tilde{f}$ as in (6) and (5). The following statements hold.*

(i) ***Derivatives commute with averaging.*** *For $i = 1, 2, 3$ one has*

$$\partial_i \bar{f} \;=\; \overline{\partial_i f}, \qquad \partial_i \tilde{f} \;=\; \widetilde{\partial_i f}.$$

In particular, if $\nabla \cdot u = 0$, then $\nabla \cdot \bar{u} = 0$ and $\nabla \cdot \tilde{u} = 0$. (For $i = 3$, periodicity yields $\partial_3 \bar{f} = 0 = \overline{\partial_3 f}$.)

(ii) ***Orthogonality in H^k.*** *For any non-negative integer k and any multi–index α with $|\alpha| \le k$,*

$$\int_{\mathbb{T}^3} \partial^\alpha \bar{f}(x)\, \partial^\alpha \tilde{f}(x)\, dx \;=\; 0.$$

Consequently,

$$\|f\|_{H^k}^2 \;=\; \|\bar{f}\|_{H^k}^2 + \|\tilde{f}\|_{H^k}^2, \qquad \|\bar{f}\|_{H^k} \le \|f\|_{H^k}, \quad \|\tilde{f}\|_{H^k} \le \|f\|_{H^k}.$$

(iii) ***Sharpened vertical fractional Poincaré estimate.*** *The oscillatory part satisfies the vertical Poincaré estimate*

$$\|\tilde{f}\|_{L^2(\mathbb{T}^3)} \;\lesssim\; \|\partial_3 \tilde{f}\|_{L^2(\mathbb{T}^3)}, \tag{11}$$

and, more generally, for any $\sigma > 0$,

$$\|\tilde{f}\|_{L^2(\mathbb{T}^3)} \;\lesssim_\sigma\; \|\Lambda_3^\sigma \tilde{f}\|_{L^2(\mathbb{T}^3)}, \tag{12}$$

where Λ_3^σ is the Fourier multiplier with symbol $|\xi_3|^\sigma$.

Remark 1. *The orthogonality follows from $\int_0^1 \tilde{f}\, dx_3 = 0$ and integration by parts on the torus; (15) and (16) are the one-dimensional (fractional) Poincaré inequalities along x_3 applied at each fixed (x_1, x_2). Standard proofs can be found, for example, in [24,25].*

Anisotropic mixed norms. Throughout, we use mixed Lebesgue notation that records the direction in which each norm is taken. For $1 \le p, q, r \le \infty$ and functions on $\mathbb{T}^3$, set as follows:

$$\|f\|_{L^p_{x_1} L^q_{x_2} L^r_{x_3}} \;:=\; \Big\| \, \| \, \|f(\cdot,\cdot,\cdot)\|_{L^p_{x_1}(\mathbb{T})} \|_{L^q_{x_2}(\mathbb{T})} \, \Big\|_{L^r_{x_3}(\mathbb{T})}. \tag{13}$$

When two directions are grouped, we write, for example,

$$\|f\|_{L^p_{x_1 x_2} L^q_{x_3}} \;:=\; \big\| \, \|f(\cdot,\cdot,\cdot)\|_{L^p_{x_1 x_2}(\mathbb{T}^2)} \big\|_{L^q_{x_3}(\mathbb{T})}. \tag{14}$$

Anisotropic Sobolev's norms are defined similarly by applying derivatives (fractional or integer) in the prescribed directions before taking the mixed norms.

Anisotropic triple–product control. The next lemma furnishes an upper bound for integrals of triple products in mixed norms adapted to the directional operators $|\partial_3|^\alpha$ and $|\partial_1|^\beta$. This

estimate is a key tool for handling the nonlinear term in anisotropic, fractionally dissipative systems such as (1). Variants for different spatial settings appear in [26,27].

Lemma 2 (Anisotropic triple–product bound on $\mathbb{T}^3$)**.** *Let $f, g, h : \mathbb{T}^3 \to \mathbb{R}^m$ (or $\mathbb{R}$) be given, and assume f, $\partial_1 f$, g, $\partial_2 g$, h, $\partial_3 h \in L^2(\mathbb{T}^3)$. Then there exists a constant $C > 0$, depending only on the domain, such that*

$$\begin{aligned}\Big| \int_{\mathbb{T}^3} f(x)\, g(x)\, h(x)\, dx \Big| \;\le\; \\ C\, \|f\|_{L^2}^{\frac{1}{2}} \big(\|f\|_{L^2} + \|\partial_1 f\|_{L^2}\big)^{\frac{1}{2}} \|g\|_{L^2}^{\frac{1}{2}} \big(\|g\|_{L^2} + \|\partial_2 g\|_{L^2}\big)^{\frac{1}{2}} \|h\|_{L^2}^{\frac{1}{2}} \big(\|h\|_{L^2} + \|\partial_3 h\|_{L^2}\big)^{\frac{1}{2}}.\end{aligned} \tag{15}$$

If, in addition, h has zero vertical mean (i.e., $h = \tilde{h}$), then

$$\Big| \int_{\mathbb{T}^3} f(x)\, g(x)\, \tilde{h}(x)\, dx \Big| \;\le\; C\, \|f\|_{L^2}^{\frac{1}{2}} \big(\|f\|_{L^2} + \|\partial_1 f\|_{L^2}\big)^{\frac{1}{2}} \|g\|_{L^2}^{\frac{1}{2}} \big(\|g\|_{L^2} + \|\partial_2 g\|_{L^2}\big)^{\frac{1}{2}} \|\tilde{h}\|_{L^2}^{\frac{1}{2}}\, \|\partial_3 \tilde{h}\|_{L^2}^{\frac{1}{2}}. \tag{16}$$

Proof. Apply Hölder's inequality successively in x_1, x_2, x_3 and use the one–dimensional Sobolev embedding $\|\phi\|_{L^\infty(\mathbb{T})} \le C\|\phi\|_{H^1(\mathbb{T})}$ in each corresponding direction. For (16), use the vertical Poincaré estimate $\|\tilde{h}\|_{L^2} \lesssim \|\partial_3 \tilde{h}\|_{L^2}$. □

Lemma 3 (Two–dimensional version on $\mathbb{T}^2$)**.** *Assume $f, \partial_1 f, g, \partial_2 g, h \in L^2(\mathbb{T}^2)$. Then there exists a constant $C > 0$, depending only on the domain, such that*

$$\begin{aligned}\Big| \int_{\mathbb{T}^2} f(x_1, x_2)\, g(x_1, x_2)\, h(x_1, x_2)\, dx_1 dx_2 \Big| \;\le\; \\ C\, \|f\|_{L^2}^{\frac{1}{2}} \big(\|f\|_{L^2} + \|\partial_1 f\|_{L^2}\big)^{\frac{1}{2}} \|g\|_{L^2}^{\frac{1}{2}} \big(\|g\|_{L^2} + \|\partial_2 g\|_{L^2}\big)^{\frac{1}{2}} \|h\|_{L^2}.\end{aligned} \tag{17}$$

Remark 2 (fractional variant)**.** *By replacing $\partial_1, \partial_2,$ and ∂_3 with the directional fractional operators $\Lambda_1^{\sigma_1}, \Lambda_2^{\sigma_2}, \Lambda_3^{\sigma_3}$ (symbols $|\xi_j|^{\sigma_j}$, and $0 < \sigma_j \le 1$) and using the 1D fractional embeddings, one obtains versions of (15)–(17) with $\|f\|_{L^2} + \|\Lambda_1^{\sigma_1} f\|_{L^2}$, etc., which we will use for the fractional dissipation in (1).*

The next lemma provides the nonlinear bound needed in our vorticity-based energy method. It shows that each trilinear contribution can be written as a product of a low-order size $\|u\|_{H^2}\|\Delta\omega\|_2$ and a dissipative factor, $\|\Lambda_3^\alpha \Delta\omega\|_2$ or $\|\Lambda_1^\beta \Delta\bar{\omega}\|_2$; see (18). This structure enables the dissipation to absorb the non-linearity under a small-data assumption.

Lemma 4 (Fractional anisotropic trilinear estimate)**.** *Let f, g, h be components of the gradient tensor ∇u. Then there exists a constant $C > 0$ (independent of f, g, h) such that*

$$\Big| \int_{\mathbb{T}^3} f\, \partial_j g\, \partial_{ik} h\, dx \Big| \;+\; \Big| \int_{\mathbb{T}^3} \partial_\ell f\, \partial_j g\, \partial_{ik} h\, dx \Big| \;\le\; C\, \|\omega\|_{H^2} \Big(\big\|\Lambda_3^\alpha \Delta\omega\big\|_2 + \big\|\Lambda_1^\beta \Delta\bar{\omega}\big\|_2 \Big), \tag{18}$$

where $\omega = \nabla \times u$, Λ_3^α and Λ_1^β denote the Fourier multipliers with symbols $|\xi_3|^\alpha$ and $|\xi_1|^\beta$, respectively, and $\bar{\omega}$ is the vertical average of ω.

Proof. Write $f = \bar{f} + \tilde{f}$, $g = \bar{g} + \tilde{g}$, $h = \bar{h} + \tilde{h}$, cf. (6) and (7). Further split the average in x_1, $\bar{h} = \overline{\bar{h}} + \widetilde{\bar{h}}$, cf. (9) and (10). Expanding the product yields finitely many terms; representative contributions are treated below and the rest follow identically.

(a) Terms with $\tilde{h}$. Apply Lemma 2 (anisotropic triple product) with the choices $(f, g, h) = (\partial_a u^b, \partial_c u^d, \tilde{h})$ and use the fractional Poincaré inequality $\|\tilde{h}\|_2 \lesssim \|\Lambda_3^\alpha \tilde{h}\|_2$ (Lemma 1(iii)) as follows:

$$\Big| \int f\, \partial_j g\, \partial_{ik} \tilde{h}\, dx \Big| \;\lesssim\; \|\nabla u\|_{H^1}\, \|\nabla^2 u\|_{H^0}\, \|\Lambda_3^\alpha \partial_{ik} \tilde{h}\|_2 \;\lesssim\; \|\omega\|_{H^2}\, \|\Lambda_3^\alpha \Delta\omega\|_2.$$

(b) Terms with $\bar{\bar{h}}$. Since $\bar{\bar{h}}$ depends only on x_2, any x_1–derivative on it vanishes. Using Lemma 3 on each horizontal slice and the identities $\|\Delta u\|_2 \le C\|\nabla\omega\|_2$ and $\|\nabla\Delta u\|_2 \le C\|\Delta\omega\|_2$ produces the following:

$$\left|\int f\,\partial_j g\,\partial_{ik}\bar{\bar{h}}\,dx\right| \lesssim \|\nabla u\|_{H^1}\,\|\nabla^2 u\|_{H^0}\,\left\|\Lambda_1^{\beta}\Delta\bar{\omega}\right\|_2 \lesssim \|\omega\|_{H^2}\,\left\|\Lambda_1^{\beta}\Delta\bar{\omega}\right\|_2.$$

(c) Terms with $\widetilde{\bar{h}}$. Use Lemma 2 and the fact that $\widetilde{\bar{h}}$ has zero mean in x_1 so that $\|\widetilde{\bar{h}}\|_2 \lesssim \|\Lambda_1^{\beta}\widetilde{\bar{h}}\|_2$ by the 1D fractional Poincaré inequality in x_1. This yields the same bound as in (b).

The terms where a derivative falls on f are handled the same way, with the H^2-control of ω absorbing the extra derivative. Collecting all contributions proves (18). □

3. Main Proof

Proof of Theorem 1. The local well-posedness in $H^3(\mathbb{T}^3)$ for (1) follows from standard Picard iteration with the fractional semigroups and will not be repeated. We concentrate on deriving a *a priori*, time-uniform bounds.

Step 1: L^2–energy. Forming the L^2–inner product of the system (1) with u and noting that $\int_{\mathbb{T}^3}(u\cdot\nabla)u\cdot u\,dx = 0$ and $\int_{\mathbb{T}^3}\nabla p\cdot u\,dx = 0$, we obtain the following:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \nu_v\left\||\partial_3|^{\alpha}u(t)\right\|_{L^2}^2 + \nu_h\left\||\partial_1|^{\beta}\bar{u}(t)\right\|_{L^2}^2 = 0. \tag{19}$$

For the horizontal term we used the orthogonal splitting $u = \bar{u} + \tilde{u}$ (see (6) and (7)) to note $\int(|\partial_1|^{2\beta}\bar{u})\cdot u\,dx = \int(|\partial_1|^{2\beta}\bar{u})\cdot\bar{u}\,dx = \||\partial_1|^{\beta}\bar{u}\|_2^2$.

Step 2: Vorticity equation. Let $\omega = \nabla\times u$. Applying curl to (1), using Lemma 1(i) (average commutes with derivatives and thus with curl) and the commutation of $|\partial_1|^{\beta}$ with derivatives in x_1, x_2, we obtain the following:

$$\partial_t\omega + u\cdot\nabla\omega - \omega\cdot\nabla u + \nu_v\,|\partial_3|^{2\alpha}\omega + \nu_h\,|\partial_1|^{2\beta}\,\bar{\omega} = 0. \tag{20}$$

Step 3: H^3–level control via $\Delta\omega$. Recall the standard vector identity on the torus with $\nabla\cdot u = 0$: $\nabla\times\omega = -\Delta u$, so that $\|\nabla\Delta u\|_2 \lesssim \|\Delta\omega\|_2$ and $\|\Delta u\|_2 \lesssim \|\nabla\omega\|_2$. Applying Δ to (20), taking the L^2 inner product with $\Delta\omega$, and utilizing Lemma 1 again, we obtain the following:

$$\frac{1}{2}\frac{d}{dt}\|\Delta\omega\|_2^2 + \nu_v\left\||\partial_3|^{\alpha}\Delta\omega\right\|_2^2 + \nu_h\left\||\partial_1|^{\beta}\Delta\bar{\omega}\right\|_2^2 = I_1 + I_2, \tag{21}$$

where

$$I_1 = -\int_{\mathbb{T}^3}\big[\Delta,\, u\cdot\nabla\big]\omega\;\cdot\;\Delta\omega\,dx, \qquad I_2 = \int_{\mathbb{T}^3}\big[\Delta,\,\omega\cdot\nabla\big]u\;\cdot\;\Delta\omega\,dx.$$

Using $\nabla\cdot u = 0$ and the commutator identity, we split as follows:

$$\begin{aligned} I_1 &= I_{11} + I_{12},\\ I_{11} &:= -\int_{\mathbb{T}^3}\Delta u\cdot\nabla\omega\,\Delta\omega\,dx,\\ I_{12} &:= -2\int_{\mathbb{T}^3}\nabla u : \nabla(\nabla\omega)\,\Delta\omega\,dx, \end{aligned}$$

$$
\begin{aligned}
I_2 &= I_{21} + I_{22} + I_{23},\\
I_{21} &:= \int_{\mathbb{T}^3} \Delta\omega \cdot \nabla u \, \Delta\omega \, dx,\\
I_{22} &:= 2\int_{\mathbb{T}^3} \nabla\omega : \nabla(\nabla u) \, \Delta\omega \, dx,\\
I_{23} &:= \int_{\mathbb{T}^3} \omega \cdot \nabla(\Delta u) \, \Delta\omega \, dx.
\end{aligned}
$$

Step 4: Nonlinear bound and closure. We continue from (21) by estimating the nonlinear terms $I_{11}, I_{12}, I_{21}, I_{22}, I_{23}$. All of them are linear combinations of integrals having the generic structure as follows:

$$
\int_{\mathbb{T}^3} f(x)\, \partial_j g(x)\, \partial_{ik} h(x)\, dx \quad \text{or} \quad \int_{\mathbb{T}^3} \partial_\ell f(x)\, \partial_j g(x)\, \partial_{ik} h(x)\, dx,
$$

with f, g, h drawn from the entries of the matrix $(\partial_m u^n)_{3\times 3}$. Applying Lemma 4 to each of $I_{11}, I_{12}, I_{21}, I_{22}, and I_{23}$ in (21) we obtain

$$
\frac{1}{2}\frac{d}{dt}\|\Delta\omega\|_2^2 + \nu_v \left\|\Lambda_3^\alpha \Delta\omega\right\|_2^2 + \nu_h \left\|\Lambda_1^\beta \Delta\bar{\omega}\right\|_2^2 \;\le\; C\,\|\omega\|_{H^2}\left(\left\|\Lambda_3^\alpha \Delta\omega\right\|_2^2 + \left\|\Lambda_1^\beta \Delta\bar{\omega}\right\|_2^2\right). \tag{22}
$$

Finally, add the L^2-energy identity (19) to (22) and use the standard equivalence $\|u\|_{H^3} \simeq \|u\|_2 + \|\Delta\omega\|_2$ on $\mathbb{T}^3$ to obtain the following:

$$
\|u(t)\|_{H^3}^2 + \int_0^t \left(2\min\{\nu_v, \nu_h\} - C\|u(\tau)\|_{H^3}\right)\left(\|\Lambda_3^\alpha \Delta\omega(\tau)\|_2^2 + \|\Lambda_1^\beta \Delta\bar{\omega}(\tau)\|_2^2\right) d\tau \;\le\; \|u_0\|_{H^3}^2. \tag{23}
$$

If $\|u_0\|_{H^3}$ is small so that $2\min\{\nu_v, \nu_h\} - C\|u_0\|_{H^3} \le 0$, then by a continuity argument the same inequality holds with u_0 replaced by $u(t)$ for all $t \ge 0$, which yields a uniform bound on $\|u(t)\|_{H^3}$ and the integrability of the dissipative terms. Uniqueness and continuous dependence follow from the analogous energy estimate for the difference of two solutions. This completes the proof of Theorem 1. □

4. Conclusions

We established small-data global well-posedness and stability for the three-dimensional incompressible *fractional Navier–Stokes* system with *directional fractional diffusion*: vertical smoothing of order 2α acting on the full velocity and a horizontal operator of order 2β applied only to the vertical average. The analysis crucially relies on an average–oscillation decomposition, a fractional Poincaré inequality in the vertical variable, and anisotropic product/commutator estimates that control the nonlinear triple products. We provide explicit estimates that yield uniform H^2 bounds and coercive dissipation for $\Lambda_3^\alpha \Delta\omega$ and $\Lambda_1^\beta \Delta\bar{\omega}$, from which decay consistent with the underlying semigroups follows. The framework is flexible and can be extended to incorporate forcing, variable fractional orders, or different anisotropic placements of diffusion; treating rougher data in critical spaces and quantifying Gevrey-type smoothing are natural directions for future work.

Author Contributions: Conceptualization, M.Z.A.; formal analysis, M.Z.A. and A.K.; funding acquisition, M.Z.A.; investigation, M.Z.A.; methodology, M.Z.A. and A.K.; resources, A.K.; software, M.Z.A.; validation, A.K.; visualization, A.K.; writing—original draft, M.Z.A.; writing—review and editing, A.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research is supported by Taizhou University Research Start-up Fund (Project No. T20250101221)

Data Availability Statement: This paper presents theoretical results, and therefore does not involve any data creation or data analysis.

Conflicts of Interest: The authors declare no conflicts of interest.

Nomenclature

Symbol	Meaning / Definition
$\mathbb{T} = [0,1]$	One-dimensional periodic torus; $\mathbb{T}^3 = \mathbb{T} \times \mathbb{T} \times \mathbb{T}$.
$x = (x_1, x_2, x_3)$, $t \geq 0$	Space and time variables.
$u(x,t) \in \mathbb{R}^3$, $p(x,t) \in \mathbb{R}$	Velocity and pressure.
$\nabla = (\partial_1, \partial_2, \partial_3)$, $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$	Gradient and Laplacian; $\nabla_h = (\partial_1, \partial_2)$, $\Delta_h = \partial_1^2 + \partial_2^2$.
$\omega = \nabla \times u$	Vorticity.
$\bar{f}(x_1, x_2) = \int_{\mathbb{T}} f(x_1, x_2, x_3)\, dx_3$	*Vertical average* (in x_3). Defined component-wise for vector/tensor fields.
$\tilde{f} = f - \bar{f}$	Oscillatory part with respect to x_3 (mean zero in x_3).
$\bar{f} = \overline{f} + \widetilde{\bar{f}}$, $\overline{f}(x_2) = \int_{\mathbb{T}} \bar{f}(x_1, x_2)\, dx_1$	Further splitting of $\bar{f}$ into its x_1–mean $\overline{f}$ and the x_1–mean–zero remainder $\widetilde{\bar{f}}$; $\int_{\mathbb{T}} \widetilde{\bar{f}}(x_1, x_2)\, dx_1 = 0$.
$\bar{\omega}$, $\overline{\omega}$	Vertical and secondary (x_1) averages applied to ω (component-wise).
$\Lambda_3^\alpha = \lvert\partial_3\rvert^\alpha$	One-dimensional fractional derivative in x_3 (Fourier multiplier $\lvert k_3\rvert^\alpha$).
$\Lambda_1^\beta = \lvert\partial_1\rvert^\beta$	One-dimensional fractional derivative in x_1 (Fourier multiplier $\lvert k_1\rvert^\beta$).
$\nu_v > 0$, $\nu_h > 0$	Vertical and horizontal dissipation coefficients.
$\alpha, \beta \in (0,1]$	Fractional exponents (orders 2α and 2β in dissipation).
$\langle f, g\rangle = \int_{\mathbb{T}^3} f\, g\, dx$	L^2 inner product; $\lVert f\rVert_2^2 = \langle f, f\rangle$.
$A : B = \sum_{i,j} A_{ij} B_{ij}$	Frobenius (tensor) product.
$[A,B]f = A(Bf) - B(Af)$	Commutator of operators A and B.
$\lVert f\rVert_{H^s}$, $\lVert f\rVert_{L^p}$	Sobolev/Lebesgue norms on $\mathbb{T}^3$.
$\lVert f\rVert_{L^p_{x_1} L^q_{x_2} L^r_{x_3}} = \big\lVert \lVert \lVert f(\cdot,\cdot,x_3)\rVert_{L^p_{x_1}} \rVert_{L^q_{x_2}} \big\rVert_{L^r_{x_3}}$	Anisotropic mixed-norm notation (Lebesgue).
$\lesssim$, $\approx$	Inequalities up to a harmless constant; equivalence up to two-sided constants.
C	Generic positive constant independent of the solution (may change from line to line). Subscripts indicate dependence, e.g., $C_{\alpha,\beta}$.

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