



Article

The Tantawy Technique for Modeling Fractional Kinetic Alfvén Solitary Waves in an Oxygen–Hydrogen Plasma in Earth’s Upper Ionosphere

Shaukat Ali Shan ¹ , Wedad Albalawi ², Rania A. Alharbey ³ and Samir A. El-Tantawy ^{4,5,*}

¹ Theoretical Physics Division (TPD) PINSTECH, Nilore, Islamabad 45650, Pakistan; shaukatshan@gmail.com

² Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia; wsalbalawi@pnu.edu.sa

³ Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80081, Jeddah 21589, Saudi Arabia; rallehabi@kau.edu.sa

⁴ Department of Physics, Faculty of Science, Al-Baha University, P.O. Box 1988, Al-Baha 65779, Saudi Arabia

⁵ Department of Physics, Faculty of Science, Port Said University, Port Said 42521, Egypt

* Correspondence: tantawy@sci.psu.edu.eg or samireltantawy@yahoo.com

Abstract

Kinetic Alfvén waves (KAWs) are investigated in an Oxygen–Hydrogen plasma with electrons following the behavior of rq -distribution in an upper ionosphere. We aim to study low-frequency and long wavelengths at 1700 kms in the upper ionosphere of Earth as detected by Freja satellite. The fluid model and reductive perturbation method are combined to obtain the evolutionary wave equations that can be used to describe both fractional and non-fractional KAWs in an Oxygen–Hydrogen ion plasma. This procedure is used to obtain the integer-order Korteweg–de Vries (KdV) equation and then analyze its solitary wave solution. In addition, this study is carried out to evaluate the fractional KdV (FKdV) equation using a new approach called the “Tantawy technique” in order to generate more stable and highly accurate approximations that will be utilized to accurately depict physical events. This investigation also helps locate the existence regions of the solitary waves (SWs), and in turn displays that the characteristics of KAWs are affected by a number of physical factors, such as the nonthermal parameters/spectral indices “ r ”, “ q ”, and obliqueness (characterized by l_z). Depending on the parameter governing the distribution, especially the nonthermality of inertialess electrons, the rq -distribution of electrons has a major impact on the properties of KAWs.

Keywords: Pure kinetic Alfvén waves (KAWs); Oxygen–Hydrogen plasma in earth’s upper ionosphere; Fractional KdV for kinetic Alfvén wave; The Tantawy technique for FKdV; rq -distributed electron



Academic Editor: Kyungsik Kim

Received: 4 September 2025

Revised: 9 October 2025

Accepted: 16 October 2025

Published: 31 October 2025

Citation: Shan, S.A.; Albalawi, W.; Alharbey, R.A.; El-Tantawy, S.A. The Tantawy Technique for Modeling Fractional Kinetic Alfvén Solitary Waves in an Oxygen–Hydrogen Plasma in Earth’s Upper Ionosphere. *Fractal Fract.* **2025**, *9*, 705. <https://doi.org/10.3390/fractalfract9110705>

Copyright: © 2025 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

Research on Alfvén waves (AWs) has a rich history, beginning with the groundbreaking work of Hannes Alfvén himself in the 20th century [1]. Since then, AWs have gained increasing attention from the scientific community. These waves have important applications in space [2,3], astrophysics [4–6], and laboratory plasmas [7,8]. Nonlinear research in plasma physics spans a broad range of phenomena, and it provides insight into the fundamental processes occurring in both laboratory and space plasmas. Nonlinear plasma science also explains the creation of stationary electrostatic and electromagnetic structures in plasmas, such as single-pulse solitons, envelope solitons, shock waves, and vortices. AWs

have attracted attention due to their unique characteristics and their potential relevance to various plasma phenomena.

The fundamental AWs appear as a perturbation of the external magnetic field (MF) in the perpendicular direction, which propagates along the field lines, and the linear dispersion relation of these waves can be obtained using magnetohydrodynamic (MHD) equations [1]. These are low-frequency electromagnetic waves that propagate in an ideal plasma with frozen-in field lines. If the ambient MF is assumed to be along the z -axis, i.e., $B_0 = B_0 \hat{z}$, then it propagates along the z -direction with constant Alfvén velocity $v_A = (B_0^2 / 4\pi n_0 m_i)^{1/2}$, where n_0 indicates the equilibrium density of an electron-ion (e-i) plasma and m_i refers to the ion mass. The linear dispersion relation of this incompressible wave in the MHD framework turns out to be $\omega^2 = v_A^2 k_z^2$, whereas the electric field (EF) and MF associated with this wave lie in the xy -plane mutually perpendicular to each other. However, if the oblique propagation of this electromagnetic perturbation is considered, then the wave develops a component of the EF parallel to the external MF giving rise to the compressibility. If the effects of both electron inertia and temperature are taken into account, then the two fluid equations yield the linear dispersion relation in the following form:

$$\omega^2 = \frac{v_A^2 k_z^2 (1 + \rho_s^2 k_\perp^2)}{(1 + \lambda_e^2 k_\perp^2)}, \quad (1)$$

where $\rho_s = c_s / \Omega_i$ is the ion gyro-radius at electron temperature and $\lambda_e = c / \omega_{pe}$ is the electron skin depth. Here, $c_s = (T_e / m_i)^{1/2}$, $\Omega_i = eB_0 / cm_i$, $\omega_{pe} = (4\pi n_{i0} e^2 / m_e)^{1/2}$, m_e is the electron mass and c is the speed of light. The frequency of the wave is smaller than the ion gyrofrequency $\omega < \Omega_i$, $k_z \ll k_\perp$, $\rho_s^2 k_\perp^2 < 1$, $\lambda_e^2 k_\perp^2 < 1$ and it exists in a low beta plasma, i.e., $\beta = \text{plasma (thermal) pressure to magnetic pressure} = 2c_s^2 / v_A^2 < 1$ with $v_A = B_0 / (4\pi n_{i0} m_i)^{1/2}$ is the Alfvén wave speed. Equation (1) contains two branches of AWs; the fast mode in the limit $m_e \rightarrow 0$ and the slow mode in the limit $T_e \rightarrow 0$. The fast mode is also called the kinetic Alfvén wave (KAW) and exists for $m_e / m_i < \beta < 1$ while the slow mode exists for $\beta < m_e / m_i$. We focus our attention on the nonlinear dynamics of small-amplitude KAW.

The nonlinear solitary structures of kinetic AWs (KAWs) were investigated long ago by Hasegawa and Mima [9] in an ordinary e-i plasma following the Sagdeev potential approach. They used the Maxwellian distribution for hot electrons and obtained compressible (hump) solitons. Yu and Shukla [10] included the parallel motion of the ions in the study of nonlinear AWs, and again the hump solitons were obtained. Compressive (or hump) solitary structures were found to form when investigating nonlinear AWs with parallel motion of ions [10]. In a subsequent study [11], the full ion dynamics was taken into account along with the AW dynamics, and this resulted in the formation of both hump and dip solitons. The Freja satellite observations of the auroral plasma were compared with the theoretical results obtained through the comprehensive analysis of nonlinear AWs [12].

Although the Maxwellian distribution has long been used to characterize plasma species, it is evident that superthermal particles make this distribution fall short in expressing the true nature of space plasmas. The presence of nonthermal particles leads to velocity-space diffusion and the emergence of power-law distributions such as the kappa distribution, which better represents the behavior of space plasmas [13]. Several research works are present in the literature on the existence of nonthermal distributions of plasma charged particles [13–15]. The κ -distribution function can be reduced to the Maxwellian distribution as a special case of $\kappa \rightarrow \infty$. The κ -distribution function was modified, assuming that the electrons are at a constant temperature ($T_e = \text{constant}$) in the form of a double-spectral-index rq -distribution function [16,17].

Later, several investigations examined how nonlinear waves were affected by the presence of non-Maxwellian charged particle species in various plasma environments [18–20]. In 1982, the authors of [11] obtained the exact stationary solution of the nonlinear slow Alfvén mode and showed the existence of dip Alfvén solitons. Most of the nonlinear studies of AWs were performed using the Sagdeev potential approach for arbitrary amplitude waves. The conditions were explored and illustrated [21] for when current-driven inertial and KAWs become unstable due to the field-aligned shear flow of electrons and ions along with heavy stationary dust. The instability conditions and the growth rates of both inertial and KAWs have been presented.

Bains et al. [22] have rightly claimed that they are the first to investigate nonlinear solitary structures of KAWs in the small amplitude limit employing the reductive perturbation method (RPM). The periodic kinetic AWs have recently been studied in NPTE plasma, where the electrons obey the double spectral rq -distribution [23]. These authors also employed the RPM and obtained a Korteweg–de Vries (KdV)-type equation, but did not discuss the original KdV equation under the framework of RPM. It seems important to mention here that RPM was developed for the first time by Washimi and Tanuti in 1966 [24] to study the nonlinear propagation of ion-acoustic waves (IAWs) in unmagnetized $B_0 \neq 0$ plasma.

Recently [25], it has been pointed out that the RPM cannot be used to investigate the nonlinear dynamics of the IAWs in a magnetized $B_0 \neq 0$ electron-ion plasma because the dispersion term in the linear dispersion relation is treated as higher order. Therefore, the RPM equations in the lowest order do not produce the correct linear dispersion relation in the magnetized plasma. In the case of IAWs, the magnetic field's contribution to the linear dispersion relation appears through the polarization drift of ions, and it is killed due to the ordering of stretched coordinates. Hence, the dispersion relation becomes the same as it appears in an unmagnetized plasma. Since in a magnetized plasma $k_z < k_\perp$ is the requirement for oblique propagation of the wave, the nonlinear coefficient of the KdV equation turns out to be higher, and the normalized wave amplitude becomes greater than one, which is a contradiction to the initial assumption of a small amplitude limit. For more details, the reader can see Refs. [25]. Alfvén solitary structures have been observed in the upper ionosphere by satellites [2,26]. At an altitude of 1700 km a small percentage of hydrogen ions have also been detected in the oxygen plasma of the ionosphere. Several space and astrophysical systems contain multi-ion species [27].

Modeling real-world phenomena, especially complex and nonlinear systems, requires the use of differential equations (DEs). They explain the spatial and temporal evolution of physical systems. Fractional DEs (FDEs) are a generalized variant of classical equations that permit derivatives of non-integer order. Fractional derivatives include memory and heredity effects, which means that the current state depends on both the system's history and current conditions, in contrast to ordinary derivatives that solely reflect local behavior. Because of this, they are handy for simulating processes that involve spatial or long-term interactions, where traditional models are ineffective. Control theory, biology, physics, engineering, signal processing, and system identification are among the fields with applications [28–30]. As a potent extension of classical calculus, fractional calculus (FC) enables derivatives and integrals of non-integer order to capture long-range relationships, memory, and hereditary FC effects in complex systems [31–34]. FDEs are more accurate than traditional models in describing phenomena such as nonlinear plasma dynamics, anomalous diffusion, viscoelasticity, and wave propagation. FC has attracted a lot of scholarly interest since it is a potent tool for explaining nonlinear events and provides insights beyond classical integer calculus [35–37]. These days, fractional differential equations (FDEs) are used in many different domains, such as sophisticated control signal

processing, physics, and applied mathematics [38–40]. To gain a better understanding of solitons and wave patterns in nonlinear systems, recent research has focused on developing analytical and numerical techniques for solving fractional partial differential equations (FPDEs), particularly fractional KdV-type models. It is well known that analyzing integer differential equations using a variety of analytical and numerical methods is more flexible than dealing with fractional differential equations due to the inherent fractional nature of differentiation. Therefore, many researchers have sought to harness the many analytical methods available to analyze various FDEs, such as the Haar wavelet [41], finite difference method (FDM), Chebyshev collocation method, variational homotopy perturbation method (VHPM) [42,43], discontinuous Galerkin method [44] and many others. On the other hand, many researchers have devised different methods for analyzing FDEs, such as the Tantawy technique (TT) [45–51]. Additionally, a wide range of mathematical problems, such as linear and nonlinear algebraic equations, integral and integro-DEs, ordinary/partial DEs (both integer and fractional order), and coupled systems, have been successfully solved using the iterative approach [52,53].

Much research has focused on analyzing various integer evolutionary wave equations (EWEs) to model solitary waves and other nonlinear structures in different physical media, both numerically and analytically. However, after the success of various fractional EWEs in capturing properties that integer EWEs could not capture, many researchers began to consider fractional EWEs to unravel the mystery surrounding certain behaviors that accompany the propagation of nonlinear waves in many physical systems, including nonlinear waves in plasma physics [54–56]. For instance, Jaradat et al. [57] applied a bilinear method with fractional transforms to obtain multiple soliton solutions for fractional mKdV systems with variable coefficients. Similarly, Sahoo and Ray [58] presented 2D and 3D simulations of solitary wave solutions to the time-fractional third-order mKdV equation, highlighting the emergence of complex waves. El-Ajou et al. [59] used the residual power series method (RPSM) to analyze the nonlinear fractional KdV equations (FKdV), fractional modified KdV (FmKdV) and fractional KdV–Burgers (FKdVB), thereby deriving highly accurate approximations for various nonlinear structures described by these equations. Recently, El-Tantawy and his groups presented a novel and straightforward technique, namely “the Tantawy technique (TT)” to study various fractional EWEs and model various nonlinear structures in plasma and other multidisciplinary fields [45–51]. This technique is a novel analytical approach that retains exceptional precision and ease of use. There is a way to solve various FDEs without utilizing sophisticated mathematical procedures. Therefore, incorporating fractional calculus into the integer DEs that govern different physical phenomena is essential to unravel the mystery surrounding some of the behaviors that accompany these phenomena and cannot be explained by the analysis of integer DEs. Moreover, fractional calculus also reveals the time-dependent behavior (memory-dependent behavior) of many nonlinear phenomena, such as the various nonlinear structures that arise and propagate in plasma. This feature enables the precise modeling of difficult physical phenomena, such as marine shock waves and plasma, which have complex properties. For instance, The TT succeeded in modeling both FKdV-ion-acoustic solitary waves (IASWs) [46] and FmKdV-IASWs [47] in electronegative plasmas comprising inertial positive and negative ions, together with inertialess electrons following the Cairns distribution. In an unmagnetized homogeneous plasma, fractional and non-fractional electron-acoustic (EA) cnoidal waves (CWs) were examined in which electrons were assumed to follow the Cairns distribution [48]. Additionally, the FKdV-IASWs in an unmagnetized plasma containing dynamical ions, an electron beam, and inertialess Cairns-distributed electrons were investigated utilizing the TT [49]. Additionally, the TT was employed to analyze several physical (non-)homogeneous linear fractional KdV-type equations and derive highly accurate analyt-

ical approximations for these fractional models [50]. Based on the promising results of the TT and its comparison with more commonly utilized methods, this study aims to employ the novel TT to examine the propagation characteristics of (non)fractional nonlinear KAWs in a magnetized Oxygen–Hydrogen (O–H) plasma.

The organization of our presentation of the findings of this research is as described below:

- A basic set of normalized equations is presented.
- The KdV equation for AWs is then derived using the two-potential approach.
- Then this simple KdV is turned into its fractional form.
- The results of the findings are compiled and illustrated with the help of parameters contained in Refs. [2,60].

2. Physical Model and Fluid Equations

The Maxwellian velocity distribution function (MVDF) is a very likely distribution in thermodynamic equilibrium. However, the data analysis of numerous satellite missions revealed in a wealth of literature that the existence of thermal energy that is higher than average results in distribution functions that deviate significantly from Maxwellian behavior, have longer tails, and behave according to a power law. Astrophysical and space plasmas have been known to have such plasma environments, which exhibit a nonthermal velocity distribution [13,61]. The magnetosheath plasma [62–64] verified the findings of flat top distributions, while the data collected for the solar wind and Earth’s magnetosphere [65–67] demonstrate that spiky distributions exist. The nonthermal velocity distribution is derived from the collected data [68] since these deviations from the Maxwellian velocity distribution function (MVDF) might be caused by a number of acceleration processes.

When the Kappa distribution exhibits a divergence from the Maxwellian in both low- and high-energy areas, its double spectral indices “ r ” and “ q ” make it the most suitable option; in contrast, the Kappa distribution [69] has a single spectral index. The Maxwellian, Kappa, and Cairns distributions are not the same as the rq -distribution function. The data in the Earth’s magnetosphere [64] and its impact in various plasma environments [70,71] are displayed using the rq -distribution.

The generalized form of the rq -function is described as follows [72,73]:

$$f_e(r, q; v_e) = \frac{3\Gamma(q)(q-1)^{-\frac{3}{2r+2}}}{4\pi\alpha^{3/2}(2K_B T_e/m_e)^{3/2}\Gamma[q-\frac{3}{2r+2}]\Gamma[1+\frac{3}{2r+2}]} \times \left[1 + \frac{1}{q-1} \left\{ \frac{v_e^2 - 2e\phi/m_e}{\alpha(2K_B T_e/m_e)} \right\}\right]^{-q}, \quad (2)$$

with

$$\alpha = \frac{3(q-1)^{-\frac{1}{1+4}}\Gamma[q-\frac{3}{2r+2}]\Gamma[\frac{3}{2r+2}]}{2\Gamma[q-\frac{5}{2r+2}]\Gamma[\frac{5}{2r+2}]}.$$

Here, the electron temperature is denoted by T_e , Γ is a gamma function, $(2K_B T_e/m_e)^{1/2}$ is the thermal velocity, “ e ” is the electron charge, “ ϕ ” is the electrostatic potential, K_B is the Boltzmann constant and m_e is the electron mass. However, the requirements $q > 1$ and $q(r+1) > 5/2$ must be met to obtain physically true results from this distribution function. The following limits can be used to extract the findings for the Maxwellian and Kappa distribution cases, respectively: (i) $r = 0$, $q \rightarrow \infty$ and (ii) $r = 0$, $q \rightarrow (\kappa + 1)$ [73].

We assume a magnetized O–H plasma (both positive ions) together with rq -distributed electrons and small but limited β such that $\beta \ll 1$. A set of fluid moment equations describes ions, whereas an rq -distributed velocity distribution models electrons. The direction of the ambient magnetic field (MV) B_0 is assumed parallel to the z -axis. Accordingly, the quasi-

neutrality requirement $\mu_{ei} = 1 + \mu_{bi}$ holds where $\mu_{ei} = n_e^{(0)}/n_i^{(0)}$ and $\mu_{bi} = n_b^{(0)}/n_i^{(0)}$. Parallel ψ and perpendicular φ potentials, which are justified for low- β plasmas, are handled by the two-potential theory by defining $E_x = -\partial\varphi/\partial x$ and $E_z = -\partial\psi/\partial z$ and (where $\varphi_{\perp} = \varphi$ and $\varphi_{\parallel} = \psi$ are the electrostatic and electromagnetic potentials, respectively). We suppose that the x - z plane is where the propagation takes place.

However, using the same normalization process as Hasegawa and Mima [9] employed to explore KAWs by using the Sagdeev potential approach, we can use the multi-scale RPM to obtain the KdV equation for O-H plasmas. Let $k = (k_x, 0, k_z)$ be the propagation vector and the ambient MF be constant parallel to the z -axis $B = B_0\hat{z}$ [74]. The fundamental set of equations governing the kinetic Alfvén wave in an O-H plasma is written as follows:

$$\partial_t n_i + \partial_x(n_i v_{ix}) = 0, \quad (3)$$

$$\partial_t n_b + \partial_x(n_b v_{bx}) = 0, \quad (4)$$

$$v_{ix} = -\frac{c}{\Omega_i B_0} \partial_t \partial_x \phi, \quad (5)$$

$$v_{bx} = \frac{c}{\Omega_b B_0} \partial_t \partial_x \phi, \quad (6)$$

and

$$\partial_x^2 \partial_z^2 (\phi - \psi) = \frac{4\pi e}{c^2} \partial_t^2 n_e. \quad (7)$$

All dependent and independent physical quantities in Equations (3)–(6) are rescaled as follows:

$$\begin{aligned} t' &= t\Omega_i, \quad x' = x/\rho_s, \quad z' = z/\lambda_i \\ v_j' &= v_j/v_A, \quad n_j' = n_j/n_0, \quad \forall j = e, i, b, \\ \Phi' &= e\phi/K_B T_e, \quad \Psi' = e\psi/K_B T_e, \end{aligned}$$

and plasma beta is given by $\beta = 2c_s^2/v_A^2$ for the isothermal case.

In the above equations, the quantity $c_s = \sqrt{T_e/m_i}$ indicates the ion-acoustic speed, $\Omega_i = eB_0/cm_i$ denotes the ion-cyclotron frequency, $\rho_s = c_s/\Omega_i$, $v_A = B_0/\sqrt{4\pi n_{i0}m_i}$ represents the Alfvén wave speed, $\omega_{pi} = \sqrt{4\pi n_{i0}e^2/m_i}$ is the ion plasma frequency, and $\lambda_i = c/\omega_{pi}$ gives the ion-skin depth. Following the previously indicated normalization, the aforementioned set of equations can be expressed as follows:

$$\partial_t n_i + \sqrt{\frac{2}{\beta}} \partial_x(n_i v_{ix}) = 0, \quad (8)$$

$$v_{ix} = -\sqrt{\frac{\beta}{2}} \partial_t \partial_x \Phi. \quad (9)$$

Substituting the value of v_{ix} given in Equation (9) into Equation (8), for positive ions “ i ”, we get

$$\partial_t n_i - \partial_x(n_i \partial_{x,t}^2 \Phi) = 0. \quad (10)$$

Similarly, the continuity equation for positive ions “ b ” reads

$$\partial_t n_b + \sqrt{\frac{2}{\beta}} \partial_x(n_b v_{bx}) = 0, \quad (11)$$

$$v_{b1x} = \sqrt{\frac{\beta}{2}} Q \partial_{x,t}^2 \Phi, \quad (12)$$

where $Q = m_b/m_i$ is the mass ratio of two ions of the same polarity.

Now, using the value of v_{b1x} given in Equation (11), we get

$$\partial_t n_b - Q \partial_x (n_b \partial_{x,t}^2 \Phi) = 0, \quad (13)$$

and

$$\partial_x^2 \partial_z^2 (\Phi - \Psi) = \partial_t^2 \mu_{ei} n_e. \quad (14)$$

When Equation (2) is integrated over v_e , it results in the following form [75]:

$$n_e = n_{e0} (1 + A_{rq} (e\psi/k_B T_e) + B_{rq} (e\psi/k_B T_e)^2 + \dots), \quad (15)$$

with

$$A_{rq} = \frac{(q-1)^{\frac{-1}{1+r}} \Gamma(\frac{1}{2+2r}) \Gamma(q - \frac{1}{2+2r})}{2\alpha \Gamma(\frac{3}{2+2r}) \Gamma(q - \frac{1}{2+2r})},$$

$$B_{rq} = -\frac{(q-1)^{\frac{-2}{1+r}} (1+4r) \Gamma(\frac{-1}{2+2r}) \Gamma(q + \frac{1}{2+2r})}{8\alpha^2 \Gamma(\frac{3}{2+2r}) \Gamma(q - \frac{3}{2+2r})}.$$

For the sake of convenience, the normalized quantities will not have superscript primes onward. The electron density normalized equation is as follows:

$$n_e = (1 + A_{rq} \Psi + B_{rq} \Psi^2 + \dots). \quad (16)$$

3. Linear Analysis Using Fourier Transform for Oxygen–Hydrogen Plasma

Linearizing the set of Equations (10) and (13) utilizing the sinusoidal approximations for both types of ions “ i ” and “ b ”, we get

$$\begin{cases} n_i^{(1)} = -k_x^2 \Phi^{(1)}, \\ n_b^{(1)} = -Q k_x^2 \Phi^{(1)}. \end{cases} \quad (17)$$

Perturbed number density expression for electrons is obtained as

$$n_e^{(1)} = A_{rq} \Psi^{(1)}. \quad (18)$$

Ampere’s and Faraday’s laws yield

$$\Phi^{(1)} = \Psi^{(1)} - \frac{\omega^2 \mu_{ei} A_{rq}}{k_x^2 k_z^2} \Psi^{(1)}. \quad (19)$$

Accordingly, we get

$$\frac{\omega^2}{k_z^2} = \frac{1}{(1 + \mu_{bi} Q)} + \frac{k_x^2}{\mu_{ei} A_{rq}}. \quad (20)$$

Equation (20) can be expressed as follows:

$$\frac{\omega^2 / \Omega_i^2}{k_z^2 \lambda_i^2} = \frac{1}{(1 + \mu_{bi} Q)} + \frac{k_x^2 \rho_s^2}{\mu_{ei} A_{rq}}. \quad (21)$$

With the utilization of the relations, $\lambda_i \Omega_i = c \Omega_i / \omega_{pi} = v_A$, the linear dispersion relation (LDR) for the plasma under consideration in Fourier space turns into the following form:

$$\omega^2 = k_z^2 v_A^2 \left[\frac{1}{(1 + \mu_{bi} Q)} + \frac{k_x^2 \rho_s^2}{\mu_{ei} A_{rq}} \right]. \quad (22)$$

4. Evolutionary Wave Equation for KAWs in O-H Plasma

Now, for deriving the evolutionary wave equation for the current plasma model, the RPM is employed. The multi-scale reductive perturbation method defines the stretching of independent variables as follows:

$$\eta = \epsilon^{1/2}(l_x x + l_z z - Mt), \text{ and } \tau = \epsilon^{3/2}t, \quad (23)$$

where l_x and l_z represent the cosines in the direction along the directions x - and z -directions, and the wave propagation angle, which B_0 can be expressed as $\theta = \cos^{-1}(l_z)$ with $l_x^2 + l_z^2 = 1$, while the quantity $M = u/v_A$ denotes the normalized phase speed of the wave. Various independent physical quantities are expanded as follows:

$$\begin{aligned} n_j &= 1 + \epsilon n_j^{(1)} + \epsilon^2 n_j^{(2)} + \dots, \\ v_{jx} &= \epsilon v_{jx}^{(1)} + \epsilon^2 v_{jx}^{(2)} + \dots, \\ \Phi &= \Phi^{(1)} + \epsilon \Phi^{(2)} + \dots, \\ \Psi &= \epsilon \Psi^{(1)} + \epsilon^2 \Psi^{(2)} + \dots. \end{aligned} \quad (24)$$

For positive ions “ i ” and “ b ” in stretched coordinates (η, τ) , the ion continuity equation is written as follows:

$$\begin{aligned} &-M\epsilon^{\frac{3}{2}}\partial_\eta n_i^{(1)} - M\epsilon^{\frac{5}{2}}\partial_\eta n_i^{(2)} + \epsilon^{\frac{5}{2}}\partial_\tau n_i^{(1)} + \epsilon^{\frac{3}{2}}Ml_x^2\partial_\eta^3\Phi^{(1)} \\ &- \epsilon^{\frac{5}{2}}l_x^2\partial_\tau[\partial_\eta^2\Phi^{(1)}] + \epsilon^{\frac{5}{2}}Ml_x^2\partial_\eta n_i^{(1)}[\partial_\eta^2\Phi^{(1)}] + \epsilon^{\frac{5}{2}}Ml_x^2\partial_\eta^3\Phi^{(2)} = 0, \end{aligned} \quad (25)$$

and

$$\begin{aligned} &-M\epsilon^{\frac{3}{2}}\partial_\eta n_b^{(1)} - M\epsilon^{\frac{5}{2}}\partial_\eta n_b^{(2)} + \epsilon^{\frac{5}{2}}\partial_\tau n_b^{(1)} + \epsilon^{\frac{3}{2}}Ql_x^2\partial_\eta^3\Phi^{(1)} \\ &- \epsilon^{\frac{5}{2}}Ql_x^2\partial_\tau[\partial_\eta^2\Phi^{(1)}] + \epsilon^{\frac{5}{2}}Ql_x^2\partial_\eta n_b^{(1)}[\partial_\eta^2\Phi^{(1)}] + \epsilon^{\frac{5}{2}}Ql_x^2\partial_\eta^3\Phi^{(2)} = 0. \end{aligned} \quad (26)$$

When stretched coordinates are used, Equations (14) and (16) turn into the following form, respectively:

$$\begin{aligned} &l_x^2 l_z^2 \epsilon^2 \partial_\eta^4 \{ (\Phi^{(1)} + \epsilon \Phi^{(2)}) - (\epsilon \Psi^{(1)} + \epsilon^2 \Psi^{(2)}) \} \\ &= [M^2 \epsilon \partial_\eta^2 - 2M\epsilon^2 \partial_{\eta,\tau}^2 + \epsilon^3 \partial_\tau^2] \mu_{ei} (\epsilon n_e^{(1)} + \epsilon^2 n_e^{(2)}), \end{aligned} \quad (27)$$

and

$$\begin{aligned} &\mu_{ei} [1 + \epsilon A_{rq} \Psi^{(1)} + \epsilon^2 A_{rq} \Psi^{(2)} + \epsilon^2 B_{rq} (\Psi^{(1)})^2] = \\ &1 + \epsilon n_i^{(1)} + \epsilon^2 n_i^{(2)} + \mu_{bi} (1 + \epsilon n_b^{(1)} + \epsilon^2 n_b^{(2)}), \end{aligned} \quad (28)$$

The RPM framework is used to obtain the LDR of KAW in an O-H plasma.

Proceeding towards getting LDR in the nonlinear case, for singly charged positive ions “ i ”, the continuity equation in its lowest ϵ -order, yields

$$-M\partial_{\eta}n_i^{(1)} + Ml_x^2\partial_{\eta}^3\Phi^{(1)} = 0. \quad (29)$$

Similarly for singly charged positive ions “ b ” the continuity equation in its lowest ϵ -order yields

$$-M\partial_{\eta}(n_b^{(1)}) + QMl_x^2\partial_{\eta}^3\Phi^{(1)} = 0. \quad (30)$$

Faraday’s and Ampere’s laws take the following form:

$$l_x^2l_z^2\partial_{\eta}^4\Phi^{(1)} = M^2[\partial_{\eta}^2(n_i^{(1)} + \mu_{bi}n_b^{(1)})]. \quad (31)$$

Using Equations (29) and (30) along with the quasi-neutrality condition, we obtain

$$[l_x^2(1 + \mu_{bi}Q)]\partial_{\eta}^2\Phi^{(1)} = [\mu_{ei}A_{rq}]\Psi^{(1)}. \quad (32)$$

In this case, the LDR turns out to be

$$M^2 = \frac{l_z^2}{(1 + \mu_{bi}Q)}. \quad (33)$$

For $n_{b0} = 0$, Equation (33) becomes $M^2 = l_z^2$ which is equal to the result obtained in the case of a single (Oxygen ion) electron-ion plasma [76].

Comparing the subsequent higher-order terms, in which $\sim\epsilon$ provides relations between first- and second-order perturbed quantities, the KdV equation for KAW in an O-H plasma is derived. Thus, for both positive ions “ i ” and “ b ”, Equations (10) and (13) yield the following

$$M\partial_{\eta}n_i^{(2)} - Ml_x^2\partial_{\eta}^3\Phi^{(2)} = f_3, \quad (34)$$

and

$$M\partial_{\eta}n_b^{(2)} - QMl_x^2\partial_{\eta}^3\Phi^{(2)} = f_4, \quad (35)$$

and Ampere’s law gives

$$l_x^2l_z^2\partial_{\eta}^3\Phi^{(2)} - M^2\partial_{\eta}(\mu_{ei}n_e^{(2)}) = f_5, \quad (36)$$

with

$$\begin{aligned} f_3 &= \partial_{\tau}n_i^{(1)} - l_x^2\partial_{\tau}[\partial_{\eta}^2\Phi^{(1)}] + Ml_x^2\partial_{\eta}n_i^{(1)}[\partial_{\eta}^2\Phi^{(1)}], \\ f_4 &= \partial_{\tau}n_b^{(1)} - Ql_x^2\partial_{\tau}[\partial_{\eta}^2\Phi^{(1)}] + QMl_x^2\partial_{\eta}n_b^{(1)}[\partial_{\eta}^2\Phi^{(1)}], \\ f_5 &= l_x^2l_z^2\partial_{\eta}^3\Psi^{(1)} + 2M\partial_{\tau}(\mu_{ei}n_e^{(1)}). \end{aligned} \quad (37)$$

Combining Equations (34)–(37) yields

$$Mf_3 + \mu_{bi}Mf_4 + f_5 = 0. \quad (38)$$

The following expressions describe the relationships between first-order terms,

$$n_i^{(1)} = l_x^2\partial_{\eta}^2\Phi^{(1)}, n_b^{(1)} = Ql_x^2\partial_{\eta}^2\Phi^{(1)} \text{ \& } l_x^2\partial_{\eta}^2\Phi^{(1)} = \frac{\mu_{ei}n_e^{(1)}}{(1 + \mu_{bi}Q)}, \quad (39)$$

and

$$n_e^{(1)} = A_{rq}\Psi^{(1)} \text{ \& } M = l_z/\sqrt{(1 + \mu_{bi}Q)}. \quad (40)$$

Equation (38) is converted into the standard form of the non-fractional planar KdV equation with the use of the aforementioned relations

$$\partial_\tau \Phi + A \Phi \partial_\eta \Phi + B \partial_\eta^3 \Phi = 0, \quad (41)$$

with

$$A = -\frac{\mu_{ei}(1 + \mu_{bi}Q^2)I_z A_{rq}}{(1 + \mu_{bi}Q)^{5/2}} \& B = -\frac{\sqrt{(1 + \mu_{bi}Q)}I_x^2 I_z}{2\mu_{ei}A_{rq}}, \quad (42)$$

where $\Phi \equiv \Psi^{(1)}$.

The steady-state solution to the KdV Equation (41) can be obtained by transforming the independent variables (η, τ) into $\xi = \eta + m_0 \tau$, where m_0 is a constant velocity normalized by v_{Ai} , and imposing appropriate boundary conditions for localized perturbations, namely $\Phi \rightarrow 0$, $\partial_\xi \Phi \rightarrow 0$, $\partial_\xi^2 \Phi \rightarrow 0$ when $\xi \rightarrow \pm\infty$. Accordingly, the following soliton solution is obtained:

$$\Phi = \Phi_m \operatorname{sech}^2(\xi/\Delta), \quad (43)$$

where $\Psi_m = 3m_0/A$ and $\Delta = \sqrt{4B/m_0}$ are, respectively, the amplitude and width of the soliton.

5. Kinetic Alfvén Solitary Waves (KASWs) in an O-H Plasma of Upper Ionosphere

Nonthermal distribution in velocity space is frequently followed by the charged species in space plasmas. To investigate the linear and nonlinear characteristics of KAWs in the upper ionosphere, we have used the rq -distribution function for electrons. The change in the behavior of the rq -distribution function with variations given to the spectral indices “ r ” and “ q ” is illustrated in Figure 1 of Ref. [77].

The higher ionosphere’s O-H plasmas will be taken into consideration to prepare the theoretical findings and their illustrations [2,60]. At a height of roughly 1700 km (upper ionosphere), the Freja satellite detected AWs, where $B_0 = 0.4$ G, the electron temperature $T_{e0} = (1\text{--}2)$ eV, and the ion temperature is found as $T_i < (0.3)T_e$. Oxygen and Hydrogen ions have linear ion plasma frequencies of $f_{pi} = (7\text{--}14) \times 10^3$ Hz and $f_{pb} = (1.7\text{--}3.5) \times 10^3$ Hz, respectively [2]. For preparing the illustrations in this investigation, the estimates of the plasma populations are made by selecting the upper bounds of these frequencies reported in Refs. [2,60] as $n_{b0} \approx 2.79 \times 10^2 \text{ cm}^{-3}$ and $n_{i0} \approx 7.01 \times 10^4 \text{ cm}^{-3}$, respectively, which are mentioned in detail in Ref. [77]. According to these values, $n_{b0} \approx 0.004 n_{i0}$, and the electron density is $n_{e0} = 7.17 \times 10^4 \text{ cm}^{-3}$. These plasma parameters give us $\omega_{pb} = 2.2 \times 10^4 \text{ s}^{-1}$, $\omega_{pi} = 8.8 \times 10^4 \text{ s}^{-1}$, $\Omega_b = 3.9 \times 10^3 \text{ s}^{-1}$, $\Omega_i = 2.44 \times 10^2 \text{ s}^{-1}$, $m_e/m_i = 3.47 \times 10^{-4}$, $\beta_b = 8\pi n_{b0} T_e / B_0^2 = 2.27 \times 10^{-3}$, and $\beta_i = 8\pi n_{i0} T_e / B_0^2 = 2.3 \times 10^{-3}$. For Hydrogen and Oxygen ions, the Alfvén speeds are $v_{Ab} = 3.30 \times 10^8 \text{ cm/s}$ and $v_{Ai} = 1.32 \times 10^9 \text{ cm/s}$, respectively.

The LDR as given in Equation (22) is solved for different values of “ r ” and “ q ” to plot the frequency “ ω ” vs. “ k_z ” in Figure 1. This graph shows that variations incremented in the values of the nonthermal parameters, such as “ r ” and “ q ” in case of rq -distribution, modify linear KAW frequency. While plotting all parts in the panel of Figure 1, the perpendicular component of the wave vector, $k_\perp = k_x$, is set while taking into account the conditions $k_z \ll k_x$, $k_x^2 \rho_{si}^2 \ll 1$, and $k_x^2 \rho_{sb}^2 \ll 1$. The plasma’s superthermal particle population is increased when “ q ” is reduced. We now examine the profiles (both amplitude and width) of nonlinear KAW structures that might be present in the higher Earth ionosphere’s Oxygen–Hydrogen bi-ion plasmas.

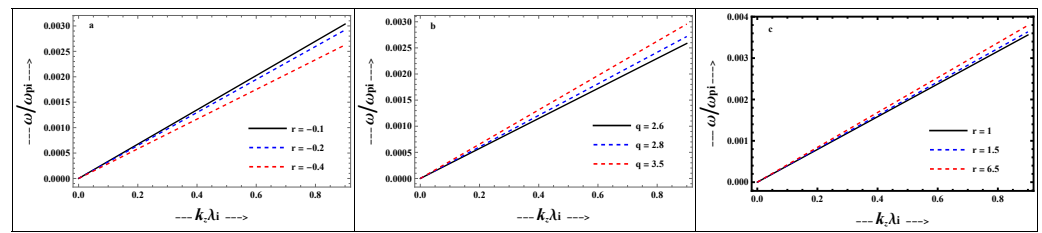


Figure 1. Frequency “ ω ” of kinetic Alfvén wave versus k_z shown O-H bi-ion plasma for different values for: (a) negative “ r ” (b) “ q ” and (c) positive “ r ” with fixed $k_x \rho_s = 0.9$.

In solving Equation (41) against changes introduced into the negative values of “ r ” which, when calculated, correspond to $A_{rq} = 1.4, 1.67, 2.18$, and $A_{rq} = 7.5$, while $q = 5$ and $l_z = 0.86$ are kept the same, the normalized potential “ Ψ ” of nonlinear kinetic Alfvén solitons is displayed vs. “ ξ ” in Figure 1. It is observed that the amplitude of the isolated structure of the nonlinear kinetic Alfvén wave diminishes against the changed negative value of “ r ”. The values of $\Delta = 49.3, 45.0, 39.4$, and 21.6 m are the widths of the solitons that correspond to the various variations in $r = -0.1, -0.2, -0.3$, and -0.4 , respectively.

The variation in amplitude of the solitary structures corresponding to various values of “ q ” with fixed $r = 0$ and obliqueness $l_z = 0.86$ is shown in Figure 2a. This graphic makes it clear that as “ q ” increases, so does the height of the kinetic Alfvén isolated structure. Using $A_{rq} = 11.0, 4.3, 3.0$, and $A_{rq} = 2.1$ in this figure, the widths are $24.5, 39.1, 47.0$, and 56.0 , respectively, compared to $q = 2.6, 2.8, 3.0$ and $q = 3.4$.

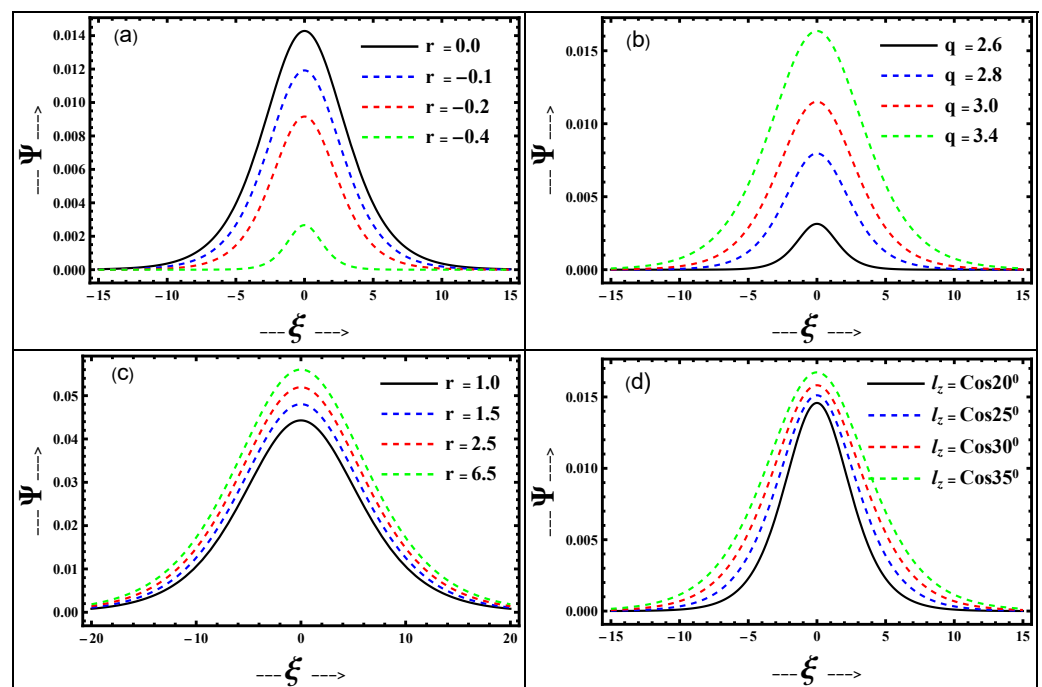


Figure 2. The profile of the non-fractional potential Ψ of KAW solitons in O-H bi-ion plasma is investigated against various related physical parameters, such as (a) negative values of the spectral index “ r ” at $(l_z, q, u_0) = (0.86, 5, -0.01)$, (b) the spectral index q at $(l_z, r, u_0) = (0.86, 0, -0.01)$, (c) positive values of the spectral index “ r ” at $(l_z, q, u_0) = (0.86, 5, -0.01)$, and (d) l_z at $(r, q, u_0) = (-0.2, 5, -0.01)$.

While keeping the values of $q = 5$ and obliqueness $l_z = 0.86$ unchanged, Figure 2b shows the impact of flatness present in the top region of the rq -distribution of electrons on the amplitudes and widths of solitary structures by adjusting the values of “ r ”. This graphic shows that $r = 4$ corresponds to the maximum soliton amplitude that is created.

The widths that correspond to the flatness parameters $r = 1, 1.5, 2.5$ and $r = 6.5$ in this case are 92.2, 96.0, 99.9, and 103.7 m.

Figure 2c shows how the overall profiles (which include both amplitude and widths) of solitary structures are affected by flatness in the upper part of the electrons' rq -distribution by adjusting the values of " r " with an unchanged nonthermal spectral index $q = 5$ and $l_z = 0.86$ (corresponding to obliqueness of the propagation angle). This graphic shows that $r = 4$ corresponds to the maximum soliton amplitude that is created. The widths that correspond to the flatness parameters $r = 1, 1.5, 2.5$ and $r = 6.5$ in this case are 92.2, 96.0, 99.9, and 103.7 m.

Propagation angle of the kinetic Alfvén wave also affects the profiles of isolated nonlinear solitary structures. For $l_z = 0.94, 0.90, 0.86$, and $l_z = 0.82$, the trend shown by the kinetic Alfvén solitary structures is depicted in Figure 2d. The solitary structure with the maximum amplitude is created according to $l_z = 0.82$, which is in contrast to the height (or amplitude) of the nonlinear kinetic Alfvén wave solitary structures that are formed related to $l_z = 0.93$. The nonlinear constructions have respective widths of 9.3, 47.6, 55.1, and 61.5 m. The variables $r = -0.2$ and $q = 5$ do not change.

6. Fractional KASWs (FKASWs)

This section will examine the influence of memory on the dynamical behavior of KAWs by converting the integer KdV (41) into its fractional form. Employing the identical methods utilized in Refs. [55,56], we ultimately get the subsequent FKdV equation:

$$D_\tau^\alpha \Phi + A\Phi\partial_\eta\Phi + B\partial_\eta^3\Phi = 0, \quad \forall 0 < \alpha \leq 1, \quad (44)$$

where $D_\tau^\alpha \equiv \partial^\alpha / \partial \tau^\alpha$ indicates the fractional derivative operator, which can be treated from the point of view of the Caputo sense. Consequently, the general Caputo fractional derivative operator $D_\tau^\alpha \Phi$ for $\tau > 0$ is defined by [78,79]

$$D_\tau^\alpha \Phi(\eta, \tau) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^\tau (\tau-t)^{m-\alpha-1} \Phi^{(m)}(\eta, t) dt, & \forall m-1 < \alpha \leq m \text{ \& } m \in \mathbb{N}, \\ \partial_\tau^m \psi(\zeta, \tau), & \forall \alpha = m \text{ \& } m \in \mathbb{N}. \end{cases} \quad (45)$$

The following relation is fulfilled according to the definition (45)

$$D_\tau^\alpha \tau^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} \tau^{\beta-\alpha}. \quad (46)$$

According to the current problem, the definition (45) can be written in the form ($m = 1$)

$$D_\tau^\alpha \Phi(\eta, \tau) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^\tau (\tau-t)^{-\alpha} \Phi^{(1)}(\eta, t) dt, & \forall 0 < \alpha \leq 1, \\ \partial_\tau \psi(\eta, \tau). \end{cases} \quad (47)$$

To simulate the propagation of FKdV-KASWs in the current plasma model, the TT is utilized to analyze the planar FKdV Equation (44) and generate an analytical approximation. This approach is presented in the subsequent concise steps [45–51]:

Step-1: The TT supposes that the solution to any fractional PDE is expressed as a convergent series solution as follows:

$$\Phi(\eta, \tau) \equiv H_0 + \sum_{i=1}^{\infty} \tau^{i\alpha} H_i. \quad (48)$$

where $\Phi \equiv \Phi(\eta, \tau)$, $H_i \equiv H_i(\eta) \forall i = 1, 2, 3, \dots$, are unknown functions, and $H_0 \equiv H_0(\eta)$ indicates any IC for the problem under consideration; in our problem, we can use the following IC for the KASWs:

$$H_0 = \Phi_m \operatorname{sech}^2\left(\frac{\eta}{\Delta}\right). \quad (49)$$

Step-2: By plugging Equation (48) into Equation (44), we get

$$\begin{aligned} D_\tau^\alpha \left(H_0 + \sum_{i=1}^{\infty} \tau^{i\alpha} H_i \right) + A \left(H_0 + \sum_{i=1}^{\infty} \tau^{i\alpha} H_i \right) \partial_\zeta \left(H_0 + \sum_{i=1}^{\infty} \tau^{i\alpha} H_i \right) \\ + B \partial_\zeta^3 \left(H_0 + \sum_{i=1}^{\infty} \tau^{i\alpha} H_i \right) = 0. \end{aligned} \quad (50)$$

Step-3: To simplify, we can take only the first three approximations from Ansatz (48), and by rewriting Equation (50), we have

$$\begin{aligned} H_1 (D_\tau^\alpha \tau^\alpha) + H_2 (D_\tau^\alpha \tau^{2\alpha}) + H_3 (D_\tau^\alpha \tau^{3\alpha}) \\ + A (H_0 + \tau^\alpha H_1 + \tau^{2\alpha} H_2 + \tau^{3\alpha} H_3) \times \\ \partial_\eta (H_0 + \tau^\alpha H_1 + \tau^{2\alpha} H_2 + \tau^{3\alpha} H_3) \\ + B \partial_\eta^3 (H_0 + \tau^\alpha H_1 + \tau^{2\alpha} H_2 + \tau^{3\alpha} H_3) = 0. \end{aligned} \quad (51)$$

Step-4: By substituting the definition (46) into Equation (51), we finally obtain

$$\begin{aligned} R_1 H_1 + R_2 H_2 \tau^\alpha + R_3 H_3 \tau^{2\alpha} \\ + A (H_0 + \tau^\alpha H_1 + \tau^{2\alpha} H_2 + \tau^{3\alpha} H_3) \times \\ \partial_\eta (H_0 + \tau^\alpha H_1 + \tau^{2\alpha} H_2 + \tau^{3\alpha} H_3) \\ + B \partial_\eta^3 (H_0 + \tau^\alpha H_1 + \tau^{2\alpha} H_2 + \tau^{3\alpha} H_3) = 0. \end{aligned} \quad (52)$$

with

$$R_i = \frac{\Gamma(i\alpha + 1)}{\Gamma((i-1)\alpha + 1)} \forall i = 1, 2, 3, \dots$$

Step-5: If we rearrange Equation (52) and collect the coefficients of the same power of $\tau^{i\alpha}$, we finally get

$$C_0 + C_1 \tau^\alpha + C_2 \tau^{2\alpha} + C_3 \tau^{3\alpha} + \dots = 0, \quad (53)$$

with

$$\begin{aligned} C_0 &= R_1 H_1 + A H_0 H_0' + B H_0^{(3)}, \\ C_1 &= R_2 H_2 + A H_1 H_0' + A H_0 H_1' + B H_1^{(3)}, \\ C_2 &= R_3 H_3 + A H_2 H_0' + A H_1 H_1' + A H_0 H_2' + B H_2^{(3)}, \\ &\vdots \end{aligned}$$

Step-6: Equating to zero the coefficients $C_0, C_1, C_2, \dots$, we get

$$\begin{aligned} R_1 H_1 + A H_0 H_0' + B H_0^{(3)} &= 0, \\ R_2 H_2 + A H_1 H_0' + A H_0 H_1' + B H_1^{(3)} &= 0, \\ R_3 H_3 + A H_2 H_0' + A H_1 H_1' + A H_0 H_2' + B H_2^{(3)} &= 0, \\ &\vdots \end{aligned} \quad (54)$$

Step-7: Using the built-in MATHEMATICA command "Solve[]" we can solve the system (54) and obtain the values of H_1, H_2 , and H_3 as functions of H_0 and its derivatives

$$\begin{aligned} H_1 &= \frac{-1}{\Gamma_1} (A H_0 H_0' + B H_0^{(3)}), \\ H_2 &= \frac{K_0}{\Gamma_2}, \\ H_3 &= \frac{-1}{\Gamma_1^2 \Gamma_3} (K_1 + K_2 + K_3), \end{aligned} \quad (55)$$

with

$$\begin{aligned} K_0 &= \left(\begin{aligned} &2A^2 H_0 (H_0')^2 + A^2 H_0^2 H_0'' + 3AB (H_0'')^2 \\ &+ 5AB H_0' H_0^{(3)} + 2AB H_0 H_0^{(4)} + B^2 H_0^{(6)} \end{aligned} \right), \\ K_1 &= A^3 \Gamma_1^2 H_0^3 H_0^{(3)} + A^2 H_0^2 [A H_0' H_0'' (7\Gamma_1^2 + \Gamma_2) + 3B \Gamma_1^2 H_0^{(5)}], \\ K_2 &= A H_0 \left\{ \begin{aligned} &A^2 (H_0')^3 (4\Gamma_1^2 + \Gamma_2) + AB H_0' H_0^{(4)} (19\Gamma_1^2 + \Gamma_2) \\ &+ B [A H_0'' H_0^{(3)} (31\Gamma_1^2 + \Gamma_2) + 3B \Gamma_1^2 H_0^{(7)}] \end{aligned} \right\}, \\ K_3 &= B \left\{ \begin{aligned} &A^2 (H_0')^2 H_0^{(3)} (25\Gamma_1^2 + \Gamma_2) + AB H_0^{(3)} H_0^{(4)} (40\Gamma_1^2 + \Gamma_2) \\ &+ 3A \Gamma_1^2 H_0' (11A (H_0'')^2 + 4B H_0^{(6)}) + B \Gamma_1^2 (27A H_0'' H_0^{(5)} + B H_0^{(9)}) \end{aligned} \right\}, \end{aligned}$$

where $\Gamma_i = \Gamma(ip + 1) \forall i = 1, 2, 3, \dots$.

Step-8: Using IC (49) in Equation (55), the following explicit values of H_1, H_2 and H_3 are obtained:

$$H_1 = \frac{2\Phi_m I_0 \operatorname{sech}^4(X) \tanh(X)}{\Delta^3 \Gamma_1}, \quad (56)$$

$$H_2 = \frac{2\Phi_m \operatorname{sech}^8(X)}{\Delta^6 \Gamma_2} (B^2 I_1 + A \Delta^2 \Phi_m I_2), \quad (57)$$

$$H_3 = \frac{4\Phi_m \operatorname{sech}^{10}(X) \tanh(X)}{\Delta^9 \Gamma_1^2 \Gamma_3} [B^3 \Gamma_1^2 I_3 + A \Delta^2 \Phi_m (\Gamma_2 I_4 + \Gamma_1^2 I_5)], \quad (58)$$

$\vdots$,

with

$$\begin{aligned}
 I_0 &= [2B \cosh(2X) - 10B + A\Delta^2\Phi_m], \\
 I_1 &= (-1208 + 1191 \cosh(2X) - 120 \cosh(4X) + \cosh(6X)), \\
 I_2 &= [2B(74 - 68 \cosh(2X) + 5 \cosh(4X)) + A\Delta^2\Phi_m(-4 + 3 \cosh(2X))], \\
 I_3 &= \begin{pmatrix} 450995 - 408364 \cosh(2X) + 46828 \cosh(4X) \\ -1012 \cosh[6X] + \cosh(8X) \end{pmatrix}, \\
 I_4 &= \left\{ \begin{array}{l} B^2(-356 - 327 \cosh(2X) - 36 \cosh(4X) + \cosh(6X)) \\ + A\Delta^2\Phi_m \left[\begin{array}{l} B(65 - 52 \cosh(2X) + 3 \cosh(4X)) \\ + A\Delta^2\Phi_m(-3 + 2 \cosh(2X)) \end{array} \right] \end{array} \right\}, \\
 I_5 &= \left\{ \begin{array}{l} 2B^2(-30445 + 26115 \cosh(2X) - 2619 \cosh(4X) + 41 \cosh(6X)) \\ + A\Delta^2\Phi_m \left[\begin{array}{l} B(2219 - 1660 \cosh(2X) + 111 \cosh(4X)) \\ + A\Delta^2\Phi_m(-23 + 12 \cosh(2X)) \end{array} \right] \end{array} \right\}.
 \end{aligned}$$

Step-9: Inserting the obtained values of H_1 , H_2 , and H_3 given in Equations (56)–(58), we ultimately get an analytical approximation for the FKdV Equation (44) up to the third order for modeling FKASWs as follows:

$$\begin{aligned}
 \varphi_T &= H_0 + H_1\tau^\alpha + H_2\tau^{2\alpha} + H_3\tau^{3\alpha} + \dots \\
 &= \Phi_m \operatorname{sech}^2(X) + \frac{2\Phi_m I_0 \operatorname{sech}^4(X) \tanh(X)}{\Delta^3 \Gamma_1} \tau^\alpha \\
 &\quad + \frac{2\Phi_m \operatorname{sech}^8(X)}{\Delta^6 \Gamma_2} (B^2 I_1 + A\Delta^2 \Phi_m I_2) \tau^{2\alpha} \\
 &\quad + \frac{4\Phi_m \operatorname{sech}^{10}(X) \tanh(X)}{\Delta^9 \Gamma_1^2 \Gamma_3} [B^3 \Gamma_1^2 I_3 + A\Delta^2 \Phi_m (\Gamma_2 I_4 + \Gamma_1^2 I_5)] \tau^{3\alpha} + \dots \quad (59)
 \end{aligned}$$

7. Discussion and Conclusions

A theoretical model for the formation of solitary kinetic AWs (SKAWs) in Oxygen–Hydrogen (O–H) bi-ion plasmas at 1700 km altitude at the top of the ionosphere has been presented. Interestingly, the occurrence of kinetic AWs in this area is facilitated by the extremely low percentage of protons in the ionosphere [76]. Figure 3 illustrates that the frequency of KAWs decreases in the case of O–H plasma. For example, selecting $\frac{\omega}{\omega_{pi}} = 0.01$ in Figure 1 results in $\omega = 88$ rad/s, which is consistent with the findings and corresponds to the linear frequency $\nu = \frac{\omega}{2\pi} \simeq 14$ Hz. The widths of the SKAWs range from a few hundred to a thousand meters, although our calculations indicate that they are closer to 100 m. As a result, the provided linear frequency and breadth estimates are more accurate than the satellite data.

It is crucial to remember that KAWs' spatial scales in directions parallel and perpendicular to the external magnetic field are always perpendicular to one another. Keeping this idea in mind, Hasegawa and Mima [9] employed suitable normalizations of spatial scales in directions parallel and perpendicular to the external magnetic field to obtain the exact solution of the nonlinear equations for KAWs. Proceeding on the same lines, we used the same normalization procedure to obtain small-amplitude linear waves and nonlinear structures by employing the RPM framework. The plasma of the higher ionosphere, where these waves have been reported by satellites, has been selected to obtain analytical results. Although a small population of protons is found in the Oxygen plasma of the upper ionosphere, it is still enough to modify the AW and its corresponding nonlinear structures [2,60].

Weakly nonlinear waves that are non-dispersive in the lowest order can be investigated using the RPM. If dispersion effects are not taken into account, higher-order kinetic AWs follow the straightforward linear dispersion relation in Fourier space: $\omega^2 = k_z^2 v_A^2$. The dispersion term can be regarded as an account of a higher-order term since it manifests itself through the ion polarization term. Consequently, the RPM can be used to study the nonlinear solitary structures of KAWs in O-H plasmas in the small amplitude limit. At the lowest order, the normalized dispersion relations are $l_z = M$, and the dispersion term is absent from RPM.

Since nonthermal charged entities are frequently seen in space plasmas, it has been believed that electrons follow the rq -distribution function. In O-H plasmas, the simulated results demonstrate that the presence of high-energy particles contributes negatively to the generation of electromagnetic solitons by nonlinear KAWs (Figure 2). However, as shown in Figure 1, when the contribution of nonthermal electrons is considered, the values of the Alfvén wave frequencies in the linear range vary substantially. Additionally, the effects of oblique propagation and flatness in the electron rq -distribution have been taken into account. As the flatness rises, so does the amplitude of the single forms. It is associated with higher values of the electron distribution function's parameter " r ". Any ideal bi-ion plasma can be used with the theoretical model that is described here.

Here, we examine how fractionality " α " affects the properties of FKASWs to better understand the dynamics of these waves and reveal certain atypical behaviors absent in the solutions of integer evolutionary wave equations. To this end, we examine the influence of fractionality " α " on the profile of the FKASWs by analyzing the generated approximation (59) at various physical parameters, in addition to the fractional order parameter " α ". At $(q, r) = (5, -0.4)$, we investigate the effect of fractionality " α " on the profile of FKASWs, as shown in Figure 3. Moreover, the impact of fractionality " α " on the profile of FKASWs $(q, r) = (3, 0)$ is investigated, as elucidated in Figure 4. Furthermore, at $(q, r) = (5, 1)$, we examined how the fractional order parameter α influences the behavior of the FKASWs, as illustrated in Figure 5. Thus, in Figures 3–5, we examined how fractionality affects the behavior of the FKASWs across the three cases of generalized distribution, i.e., for the spiky distribution $(q, r) = (5, -0.4)$, Kappa distribution $(q, r) = (3, 0)$, and (r, q) – distribution $(q, r) = (5, 1)$. These graphs show how the fractional parameter " α " affects the behavior of the propagation of FKASWs. They reveal that changing this parameter greatly impacts the wave profile, especially over long periods. It is also observed that the FKASWs split into two parts as the fractionality increases until it approaches unity over long time scales. Additionally, at a fixed value of the fractionality and with increasing time intervals, it is noted that the profile of the FKASWs splits into two parts during long time intervals, as seen in Figures 6–8 for the three mentioned cases, respectively. This effect is undetectable in the exact solutions to the integer KdV equation, which may be the most crucial explanation for some of the strange behaviors that occur in these phenomena during propagation. The results suggest that fractionality has a significant influence on soliton dynamics. As a result, this may explain some experimental observations that the exact solution of the integer evolutionary wave equations could not account for.

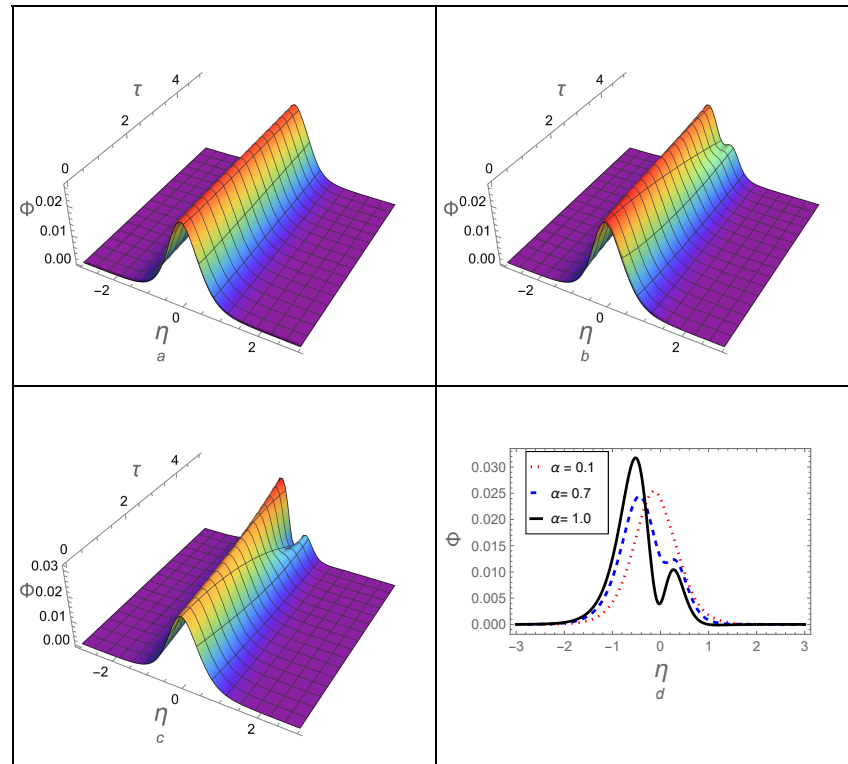


Figure 3. The fractional Kinetic Alfvén SWs approximation $\Phi(\eta, \tau)$ (59) is analyzed vs the fractional order parameter α : (a) 3D- $\Phi(\eta, \tau)$ at $\alpha = 0.1$, (b) 3D- $\Phi(\eta, \tau)$ at $\alpha = 0.7$, (c) 3D- $\Phi(\eta, \tau)$ at $\alpha = 1$, and (d) 2D- $\Phi(\eta, 5)$ at various values to α . Here, $q = 5$, $r = -0.4$, $l_z = \cos(30^\circ)$, $B_0 = 0.4$ which lead to $B = -0.00737426$ and $A = -11.2549$.

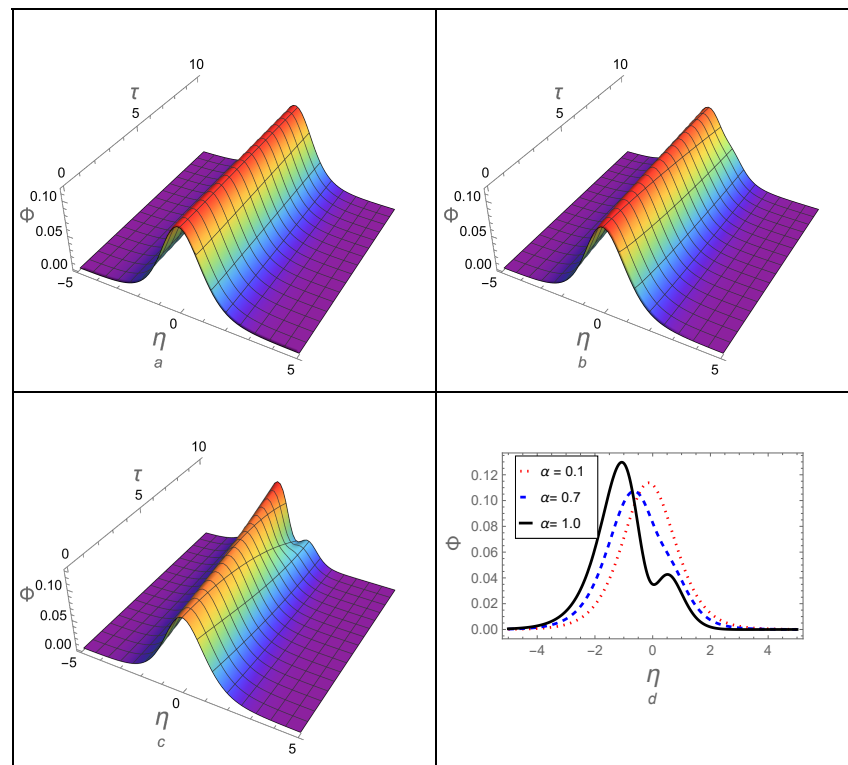


Figure 4. The fractional kinetic Alfvén SW approximation $\Phi(\eta, \tau)$ (59) is analyzed vs. the fractional order parameter α : (a) 3D- $\Phi(\eta, \tau)$ at $\alpha = 0.1$, (b) 3D- $\Phi(\eta, \tau)$ at $\alpha = 0.7$, (c) 3D- $\Phi(\eta, \tau)$ at $\alpha = 1$, and (d) 2D- $\Phi(\eta, 10)$ at various values to α . Here, $q = 3$, $r = 0$, $l_z = \cos(30^\circ)$, $B_0 = 0.4$ which lead to $B = -0.0359484$ and $A = -2.60667$.

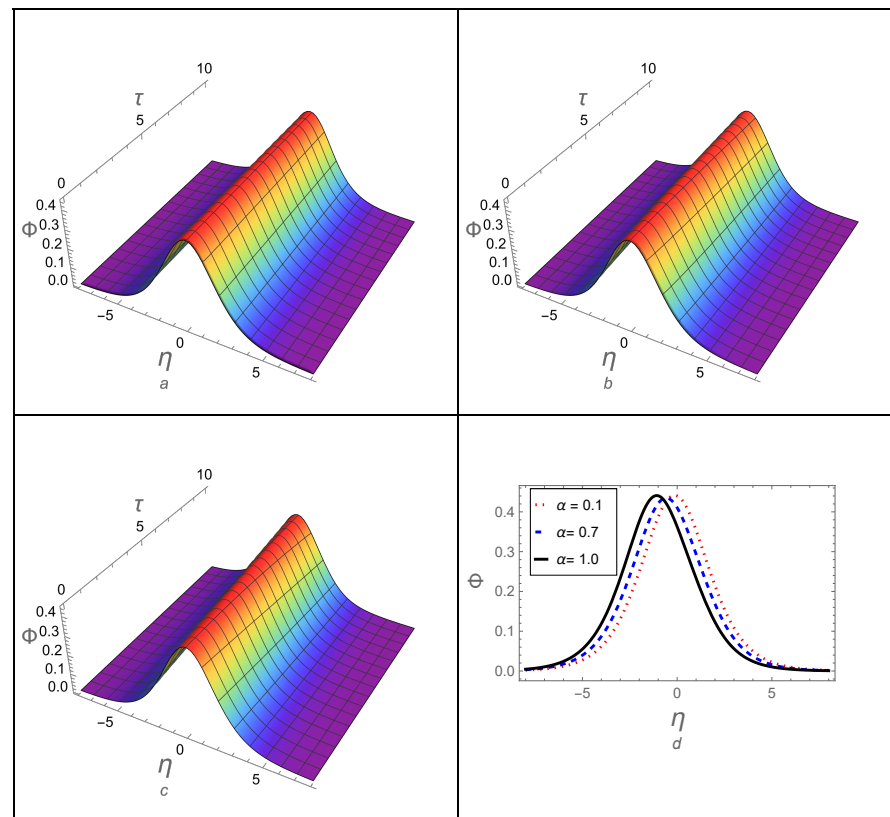


Figure 5. The fractional kinetic Alfvén SW approximation $\Phi(\eta, \tau)$ (59) is analyzed vs. the fractional order parameter α : (a) 3D- $\Phi(\eta, \tau)$ at $\alpha = 0.1$, (b) 3D- $\Phi(\eta, \tau)$ at $\alpha = 0.7$, (c) 3D- $\Phi(\eta, \tau)$ at $\alpha = 1$, and (d) 2D- $\Phi(\eta, 5)$ at various values to α . Here, $q = 5$, $r = 1$, $l_z = \cos(30^\circ)$, $B_0 = 0.4$ which lead to $B = -0.138331$ and $A = -0.677404$.

Moreover, to verify the high accuracy of the derived approximation (59), we also performed a graphical comparison between this approximation and the exact solution (43) at $\alpha = 1$, as illustrated in Figures 9–11 for the spiky distribution $(q, r) = (5, -0.4)$, Kappa distribution $(q, r) = (3, 0)$, and (r, q) – distribution $(q, r) = (5, 1)$, respectively. Additionally, the absolute errors of the generated approximation (59) according to the following relation are estimated for the spiky distribution $(q, r) = (5, -0.4)$, Kappa distribution $(q, r) = (3, 0)$, and (r, q) – distribution $(q, r) = (5, 1)$, as presented in Tables 1–3, respectively,

$$L_2 = |\text{Approx. (59)} - \text{Exact (43)}|_{H_{3,4}=0},$$

$$L_3 = |\text{Approx. (59)} - \text{Exact (43)}|_{H_4=0},$$

$$L_4 = |\text{Approx. (59)} - \text{Exact (43)}|_{H_4 \neq 0},$$

where L_2 , L_3 and L_4 are, respectively, the absolute errors of the second-, third-, and fourth-order approximations according to relation (59). As noted in the Tables 1–3, the absolute error decreases and the accuracy of the derived approximation increases with an increase in the order of the approximation. This, in turn, confirms the convergence of all approximations derived using the TT. We can conclude that the high accuracy and greater stability of the derived approximations enhance the efficiency of the used technique and its ability to analyze more complicated fractional evolutionary wave equations.

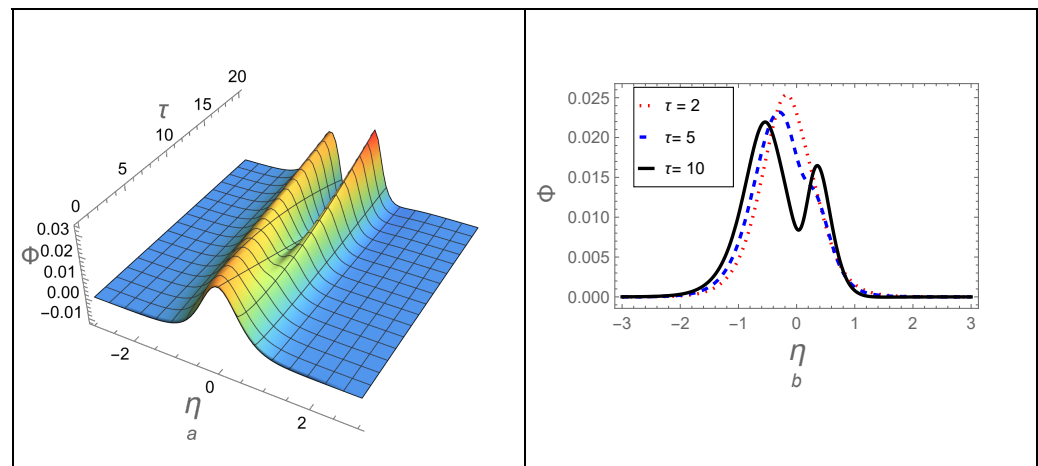


Figure 6. The fractional kinetic Alfvén SW approximation $\Phi(\eta, \tau)$ (59) at $\alpha = 0.5$ is examined: (a) 3D- $\Phi(\eta, \tau)$ and (b) 2D- $\Phi(\eta, \tau)$ at various values to τ . All other parameters are like Figure 3.

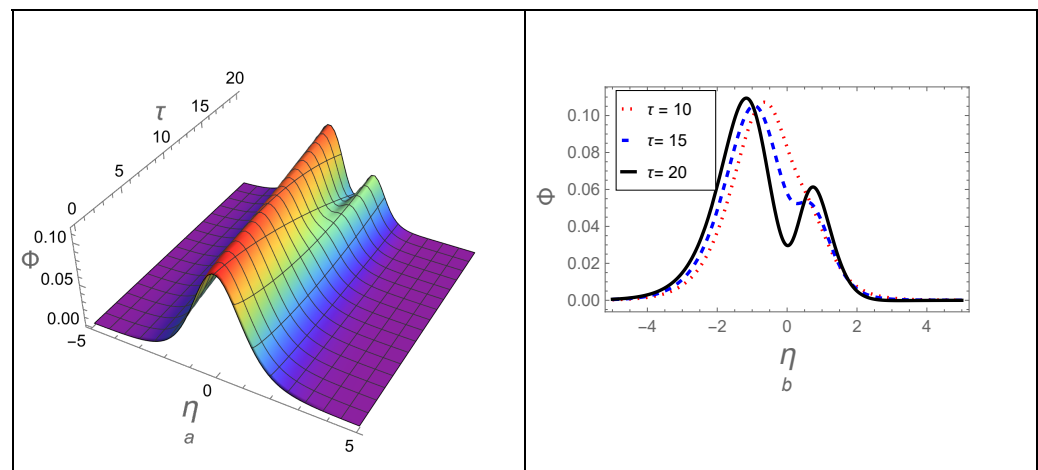


Figure 7. The fractional kinetic Alfvén SW approximation $\Phi(\eta, \tau)$ (59) at $\alpha = 0.7$ is examined: (a) 3D- $\Phi(\eta, \tau)$ and (b) 2D- $\Phi(\eta, \tau)$ at various values to τ . All other parameters are like Figure 4.

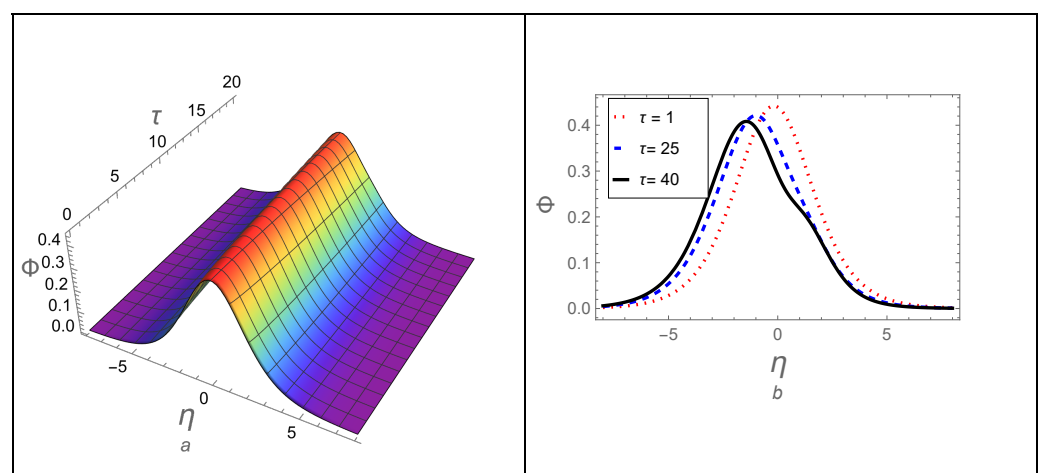


Figure 8. The fractional kinetic Alfvén SW approximation $\Phi(\eta, \tau)$ (59) at $\alpha = 0.7$ is examined: (a) 3D- $\Phi(\eta, \tau)$ and (b) 2D- $\Phi(\eta, \tau)$ at various values to τ . All other parameters are like Figure 5.

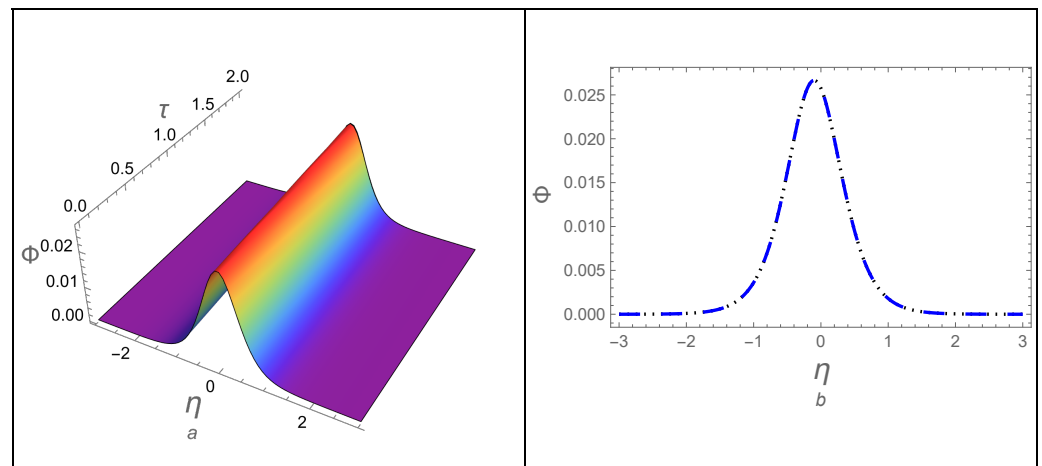


Figure 9. The exact solution (43) for the integer case is compared to the generated approximation (59) at $\alpha = 1$: (a) 3D- $\Phi(\eta, \tau)$ and (b) 2D- $\Phi(\eta, \tau)$ at $\tau = 1$. All other parameters like Figure 3.

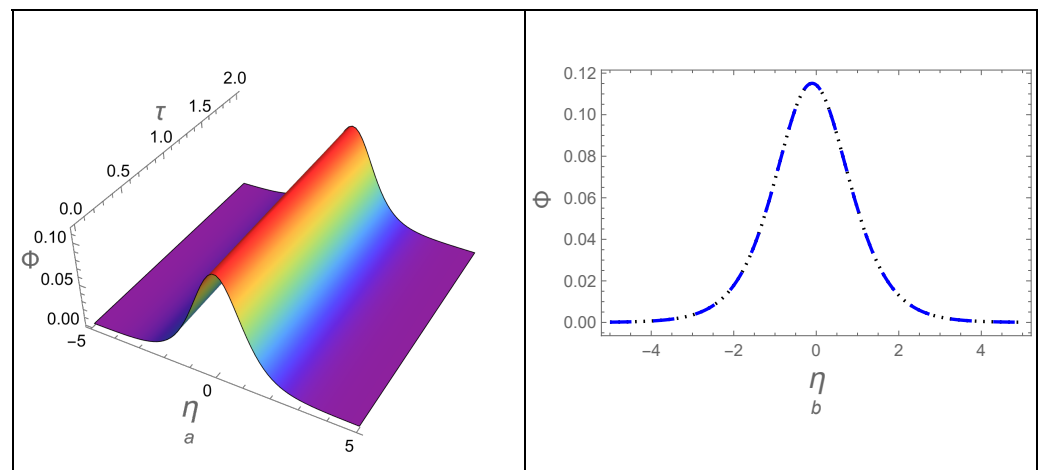


Figure 10. The exact solution (43) for the integer case is compared to the generated approximation (59) at $\alpha = 1$: (a) 3D- $\Phi(\eta, \tau)$ and (b) 2D- $\Phi(\eta, \tau)$ at $\tau = 1$. All other parameters are like Figure 4.

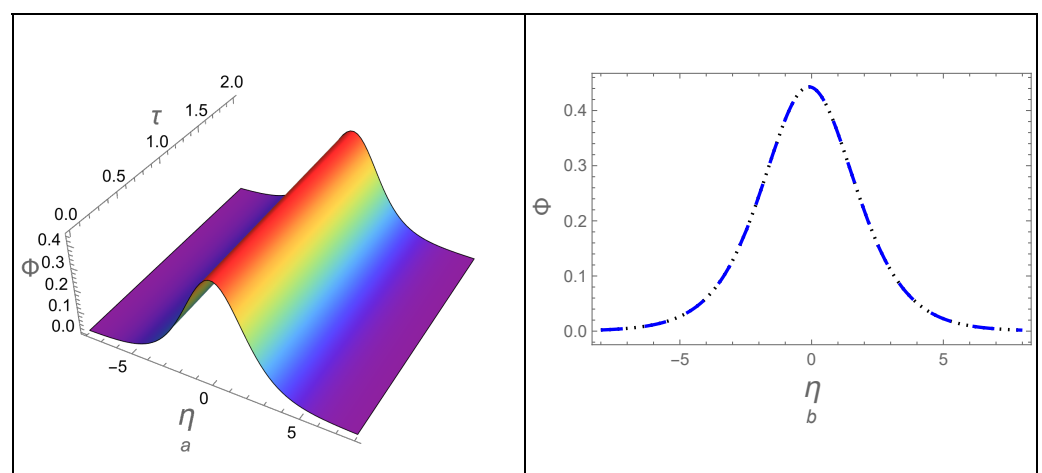


Figure 11. The exact solution (43) for the integer case is compared to the generated approximation (59) at $\alpha = 1$: (a) 3D- $\Phi(\eta, \tau)$ and (b) 2D- $\Phi(\eta, \tau)$ at $\tau = 1$. All other parameters are like Figure 5.

Table 1. The absolute error for the second-order approximation L_2 , the third-order approximation L_3 , and the fourth-order approximation L_4 , as given in Equation (59) at $\tau = 0.1$ and $\alpha = 1$. All other parameters are like Figure 3.

J	$\varphi_T _{2^{nd}}$	$\varphi_T _{3^{rd}}$	$\varphi_T _{4^{th}}$	Exact	$10^{-10} \times L_2$	$10^{-11} \times L_3$	$10^{-13} \times L_4$
−3	1.76185×10^{-6}	1.76186×10^{-6}	1.76186×10^{-6}	1.76186×10^{-6}	0.142614	0.0131017	0.00963252
−2.5	0.0000111054	0.0000111055	0.0000111055	0.0000111055	0.897863	0.0823731	0.0603966
−2	0.000069935	0.0000699355	0.0000699355	0.0000699355	5.61187	0.510414	0.367728
−1.5	0.000437841	0.000437845	0.000437845	0.000437845	33.4862	2.87463	1.82291
−1	0.00264341	0.00264343	0.00264343	0.00264343	144.662	6.82378	3.25209
−0.5	0.0129378	0.0129378	0.0129378	0.0129378	326.685	84.4666	54.3366
0	0.026646	0.026646	0.026646	0.026646	20.4196	204.196	3.9229
0.5	0.012264	0.012264	0.012264	0.012264	309.901	83.3689	55.4304
1	0.00246465	0.00246463	0.00246463	0.00246463	143.291	6.88683	3.05355
1.5	0.000406998	0.000406995	0.000406995	0.000406995	32.9149	2.83833	1.80749
2	0.0000649765	0.000064976	0.000064976	0.000064976	5.51052	0.503102	0.363427
2.5	0.0000103172	0.0000103171	0.0000103171	0.0000103171	0.881509	0.0811725	0.0596647
3	1.63678×10^{-6}	1.63677×10^{-6}	1.63677×10^{-6}	1.63677×10^{-6}	0.140013	0.0129102	0.00951517

Table 2. The absolute error for the second-order approximation L_2 , the third-order approximation L_3 , and the fourth-order approximation L_4 , as given in Equation (59) at $\tau = 0.1$ and $\alpha = 1$. All other parameters are like Figure 4.

η	$\varphi_T _{2^{nd}}$	$\varphi_T _{3^{rd}}$	$\varphi_T _{4^{th}}$	Exact	$10^{-10} \times L_2$	$10^{-11} \times L_3$	$10^{-14} \times L_4$
−5	0.000111792	0.000111792	0.000111792	0.000111792	0.850848	0.0353118	0.116802
−4	0.000591353	0.000591353	0.000591353	0.000591353	4.43601	0.181029	0.578454
−3	0.00310026	0.00310027	0.00310027	0.00310027	21.4974	0.795057	2.00304
−2	0.0155158	0.0155158	0.0155158	0.0155158	66.2694	0.663699	9.16871
−1	0.0621687	0.0621687	0.0621687	0.0621687	197.11	17.195	26.3754
0	0.115081	0.115081	0.115081	0.115081	3.71063	37.1063	1.46549
1	0.0607692	0.0607692	0.0607692	0.0607692	193.676	17.1415	27.1581
2	0.0150413	0.0150413	0.0150413	0.0150413	66.1349	0.681898	9.03045
3	0.00299989	0.00299989	0.00299989	0.00299989	21.3388	0.791055	1.99871
4	0.000572001	0.000572001	0.000572001	0.000572001	4.39992	0.179875	0.575527
5	0.000108126	0.000108126	0.000108126	0.000108126	0.843809	0.0350788	0.116187

Table 3. The absolute error for the second-order approximation L_2 , the third-order approximation L_3 , and the fourth-order approximation L_4 , as given in Equation (59) at $\tau = 0.1$ and $\alpha = 1$. All other parameters are like Figure 5.

η	$\varphi_T _{2^{nd}}$	$\varphi_T _{3^{rd}}$	$\varphi_T _{4^{th}}$	Exact	$10^{-9} \times L_2$	$10^{-11} \times L_3$	$10^{-14} \times L_4$
−8	0.00198164	0.00198164	0.00198164	0.00198164	0.198579	0.0414353	0.0677843
−7	0.00462355	0.00462355	0.00462355	0.00462355	0.453624	0.0923006	0.143115
−6	0.0107446	0.0107446	0.0107446	0.0107446	1.00235	0.191402	0.255004
−5	0.0247381	0.0247381	0.0247381	0.0247381	2.03998	0.324767	0.210248

Table 3. Cont.

η	$\varphi_T _{2^{nd}}$	$\varphi_T _{3^{rd}}$	$\varphi_T _{4^{th}}$	Exact	$10^{-9} \times L_2$	$10^{-11} \times L_3$	$10^{-14} \times L_4$
−4	0.0557552	0.0557552	0.0557552	0.0557552	3.31669	0.2161	1.00614
−3	0.119804	0.119804	0.119804	0.119804	2.01254	1.23096	5.30409
−2	0.232649	0.232649	0.232649	0.232649	9.3254	4.39207	4.0995
−1	0.372835	0.372835	0.372835	0.372835	23.1696	0.0819567	23.509
0	0.442859	0.442859	0.442859	0.442859	0.0964315	9.64315	0.0999201
1	0.3703	0.3703	0.3703	0.3703	23.1675	0.128908	23.4479
2	0.229931	0.229931	0.229931	0.229931	9.23765	4.38382	4.14946
3	0.118074	0.118074	0.118074	0.118074	2.03705	1.22037	5.29021
4	0.0548753	0.0548753	0.0548753	0.0548753	3.31234	0.218103	0.997119
5	0.0243326	0.0243326	0.0243326	0.0243326	2.03349	0.324348	0.209208
6	0.0105656	0.0105656	0.0105656	0.0105656	0.998524	0.190893	0.253617
7	0.004546	0.004546	0.004546	0.004546	0.45178	0.0920149	0.142594
8	0.0019483	0.0019483	0.0019483	0.0019483	0.197751	0.0412998	0.0676976

Author Contributions: Methodology, S.A.S., W.A. and S.A.E.-T.; Software, S.A.E.-T.; Formal analysis, R.A.A. and S.A.E.-T.; Investigation, S.A.S., R.A.A. and S.A.E.-T.; Writing—original draft, S.A.S., W.A. and S.A.E.-T.; Writing—review & editing, R.A.A. and S.A.E.-T.; Supervision, R.A.A. All authors have read and agreed to the published version of the manuscript.

Funding: The project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia under grant No. (13-135-35-RG). The authors, therefore, acknowledge with thanks to DSR technical and financial support. The authors express their gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2025R157), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Data Availability Statement: The authors confirm that the data supporting the findings of this study are available within the article. Also, we confirm that this investigation does not involve any clinical trial. For more Mathematica codes about the Tantawy technique, you can contact El-Tantawy.

Acknowledgments: The project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia under grant No. (13-135-35-RG). The authors, therefore, acknowledge with thanks to DSR technical and financial support. The authors express their gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting Project Number (PNURSP2025R157), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Alfvén, H. Existence of electromagnetic-hydrodynamic waves. *Nature* **1942**, *150*, 405–406. [\[CrossRef\]](#)
2. Wahlund, J.E.; Louarn, P.; Chust, T.; De Feraudy, H.; Roux, A.; Holback, B.; Dovner, P.-O.; Holmgren, G. On ion acoustic turbulence and the nonlinear evolution of kinetic alfvén waves in aurora. *Geophys. Res. Lett.* **1994**, *21*, 1831–1834. [\[CrossRef\]](#)
3. Pokhotelov, O.; Onishchenko, O.; Shukla, P.; Stenflo, L. Drift-alfvén vortices in dusty plasmas. *J. Geophys. Res. Space Phys.* **1999**, *104*, 19797–19800. [\[CrossRef\]](#)
4. Hertzberg, M.; Cramer, N.; Vladimirov, S. Parametric instabilities of alfvén waves in a dusty plasma. *Phys. Plasmas* **2003**, *10*, 3160–3167. [\[CrossRef\]](#)
5. Ruderman, M.S.; Goossens, M.; Ballester, J.L.; Oliver, R. Resonant alfvén waves in coronal arcades driven by footpoint motions. *Astron. Astrophys.* **1997**, *328*, 361–370.
6. Scalisi, J.; Oxley, W.; Ruderman, M.S.; Erdélyi, R. Propagation of torsional alfvén pulses in zero-beta flux tubes. *Astrophys. J.* **2021**, *911*, 39. [\[CrossRef\]](#)
7. Weiland, J. *Collective Modes in Inhomogeneous Plasmas: Kinetic and Advanced Fluid Theory*; Series in Plasma Physics; CRC Press: Bristol, UK, 1999.
8. Wu, D.-J.; Chen, L. *Laboratory Experiments of KAWs*; Springer: Singapore, 2020; pp. 49–81.
9. Hasegawa, A.; Mima, K. Exact solitary alfvén wave. *Phys. Rev. Lett.* **1976**, *37*, 690. [\[CrossRef\]](#)
10. Yu, M.; Shukla, P. Finite amplitude solitary alfvén waves. *Phys. Fluids* **1978**, *21*, 1457. [\[CrossRef\]](#)

11. Shukla, P.; Rahman, H.; Sharma, R. Alfvén soliton in a low-beta plasma. *J. Plasma Phys.* **1982**, *28*, 125–131. [\[CrossRef\]](#)
12. Stasiewicz, K.; Bellan, P.; Chaston, C.; Kletzing, C.; Lysak, R.; Maggs, J.; Pokhotelov, O.; Seyler, C.; Shukla, P.; Stenflo, L.; et al. Small scale alfvénic structure in the aurora. *Space Sci. Rev.* **2000**, *92*, 423–533. [\[CrossRef\]](#)
13. Vasyliunas, V.M. A survey of low-energy electrons in the evening sector of the magnetosphere with ogo 1 and ogo 3. *J. Geophys. Res.* **1968**, *73*, 2839–2884. [\[CrossRef\]](#)
14. Cairns, R.; Mamun, A.; Bingham, R.; Boström, R.; Dendy, R.; Nairn, C.; Shukla, P. Electrostatic solitary structures in non-thermal plasmas. *Geophys. Res. Lett.* **1995**, *22*, 2709–2712. [\[CrossRef\]](#)
15. Gurevich, A.V. Nonlinear effects in the ionosphere. *Physics-Uspekh* **2007**, *50*, 1091. [\[CrossRef\]](#)
16. Qureshi, M.; Shah, H.; Murtaza, G.; Schwartz, S.; Mahmood, F. Parallel propagating electromagnetic modes with the generalized (r, q) distribution function. *Phys. Plasmas* **2004**, *11*, 3819–3829. [\[CrossRef\]](#)
17. Qureshi, M.; Shi, J.; Ma, S. Landau damping in space plasmas with generalized (r, q) distribution function. *Phys. Plasmas* **2005**, *12*, 122902. [\[CrossRef\]](#)
18. Christon, S.; Mitchell, D.; Williams, D.; Frank, L.; Huang, C.; Eastman, T. Energy spectra of plasma sheet ions and electrons from 50 eV/e to 1 MeV during plasma temperature transitions. *J. Geophys. Space Phys.* **1988**, *93*, 2562–2572. [\[CrossRef\]](#)
19. Leubner, M. Fundamental issues on kappa-distributions in space plasmas and interplanetary proton distributions. *Phys. Plasmas* **2004**, *11*, 1308–1316. [\[CrossRef\]](#)
20. Shan, S.A.; Khan, D.; Saleem, H. Kappa distributed trapped electrons, drift wave instability and nonlinear structures in O–H plasma of ionosphere. *Eur. Phys. J. Plus* **2020**, *135*, 1–12. [\[CrossRef\]](#)
21. Shan, S.A.; Haque, Q.; Saleem, H. Current-driven alfvén waves in dusty magnetospheric plasmas. *Astrophys. Space Sci.* **2014**, *349*, 285–291. [\[CrossRef\]](#)
22. Bains, A.; Li, B.; Xia, L.-D. Kinetic alfvén solitary and rogue waves in superthermal plasmas. *Phys. Plasmas* **2014**, *21*, 032123. [\[CrossRef\]](#)
23. El-Labany, S.; El-Taibany, W.; El-Tantawy, A.; Abdelghany, A. Quasi-periodic and chaotic structure of alfvén waves in a plasma containing double spectral distributed electrons. *Phys. Plasmas* **2022**, *29*, 122103. [\[CrossRef\]](#)
24. Washimi, H.; Taniuti, T. Propagation of ion-acoustic solitary waves of small amplitude. *Phys. Rev. Lett.* **1966**, *17*, 996. [\[CrossRef\]](#)
25. Saleem, H.; Shan, S.A.; Poedts, S. Reductive perturbation method in magnetized plasma and role of negative ions. *Phys. Plasmas* **2023**, *30*, 122111. [\[CrossRef\]](#)
26. Marklund, G. Auroral phenomena related to intense electric fields observed by the Freja satellite. *Plasma Phys. Control. Fusion* **1997**, *39*, 195. [\[CrossRef\]](#)
27. Weber, E. Multi-ion plasmas in astrophysics: I. general requirements for the existence of critical points. *Astrophys. Space Sci.* **1973**, *20*, 391–400. [\[CrossRef\]](#)
28. Jumarie, G. Modified riemann-liouville derivative and fractional Taylor series of nondifferentiable functions further results. *Comput. Math. Appl.* **2006**, *51*, 1367–1376. [\[CrossRef\]](#)
29. Cimpoeasu, R.; Constantinescu, R. The inverse symmetry problem for a 2d generalized second order evolutionary equation. *Nonlinear Anal. Theory Methods Appl.* **2010**, *73*, 147–154. [\[CrossRef\]](#)
30. Abdou, M.A. An analytical method for space–time fractional nonlinear differential equations arising in plasma physics. *J. Ocean. Eng. Sci.* **2017**, *2*, 288–292. [\[CrossRef\]](#)
31. Oldman, K.B.; Spanier, J. *The Fractional Calculus*; Academic Press: New York, NY, USA, 1974.
32. Hilfer, R. *Applications of Fractional Calculus in Physics*; World Scientific: Singapore, 2000.
33. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
34. Ortigueira, M.D. *Fractional Calculus for Scientists and Engineers*; Springer: Dordrecht, The Netherlands, 2011.
35. Yang, X.J. *General Fractional Derivatives: Theory, Methods and Applications*; CRC Press: New York, NY, USA, 2019.
36. Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*; Wiley: New York, NY, USA, 2000.
37. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives: Theory and Applications*; Gordon and Breach Science: Yverdon-les-Bains, Switzerland, 1993.
38. Hamamci, S.E. Stabilization using fractional-order PI and PID controllers. *Nonlinear Dyn.* **2008**, *51*, 329–343. [\[CrossRef\]](#)
39. Yang, X.J.; Gao, F.; Ju, Y. *General Fractional Derivatives with Applications in Viscoelasticity*; Academic Press: New York, NY, USA, 2019.
40. Dhayal, R.; Malik, M.; Abbas, S.; Debbouche, A. Optimal controls for second order stochastic differential equations driven by mixed-fractional brownian motion with impulses. *Math. Methods Appl. Sci.* **2020**, *43*, 4107–4124. [\[CrossRef\]](#)
41. Zada, L.; Aziz, I. The numerical solution of fractional Korteweg-de Vries and Burgers’ equations via Haar wavelet. *Math. Methods Appl. Sci.* **2021**, *44*, 10564–10577. [\[CrossRef\]](#)
42. Kelil, A.S.; Rao, A. On the Numerical Solution of 1D and 2D KdV Equations Using Variational Homotopy Perturbation and Finite Difference Methods. *Mathematics* **2022**, *10*, 4443. [\[CrossRef\]](#)

43. Khader, M.M.; Saad, K.M. On the numerical evaluation for studying the fractional KdV, KdV-Burgers and Burgers equations. *Eur. Phys. J. Plus* **2018**, *133*, 335. [\[CrossRef\]](#)
44. Gu, Y. High-order numerical method for the fractional Korteweg-de Vries equation using the discontinuous Galerkin method. *Aims Math.* **2025**, *10*, 1367–1383. [\[CrossRef\]](#)
45. El-Tantawy, S.A.; Al-Johani, A.S.; Almuqrin, A.H.; Khan, A.; El-Sherif, L.S. Novel approximations to the fourth-order fractional Cahn–Hilliard equations: Application to the Tantawy Technique and other two techniques with Yang transform. *J. Low Freq. Noise Vib. Act. Control.* **2025**, *44*, 1374–1400. [\[CrossRef\]](#)
46. El-Tantawy, S.A.; Bacha, S.I.H.; Khalid, M.; Alhejaili, W. Application of the Tantawy technique for modeling fractional ion-acoustic waves in electronegative plasmas having Cairns distributed-electrons, Part (I): Fractional KdV Solitary Waves. *Braz. J. Phys.* **2025**, *55*, 123. [\[CrossRef\]](#)
47. El-Tantawy, S.A.; Alhejaili, W.; Khalid, M.; Al-Johani, A.S. Application of the Tantawy Technique for Modeling Fractional Ion-Acoustic Waves in Electronegative nonthermal Plasmas, Part (II): Fractional modified KdV-Solitary Waves. *Braz. J. Phys.* **2025**, *55*, 176. [\[CrossRef\]](#)
48. El-Tantawy, S.A.; Khan, D.; Khan, W.; Khalid, M.; Alhejaili, W. A Novel Approximation to the Fractional KdV Equation Using the Tantawy Technique and Modeling Fractional Electron-Acoustic Cnoidal Waves in a Nonthermal Plasma. *Braz. J. Phys.* **2025**, *55*, 163. [\[CrossRef\]](#)
49. El-Tantawy, S.A.; Khalid, M.; Bacha, S.I.H.; Alyousef, H.A.; El-Sherif, L.S. On the Tantawy Technique for Modeling Fractional Ion-Acoustic KdV Solitary Waves in a Nonthermal Plasma Having Electron Beams. *Braz. J. Phys.* **2025**, *55*, 191. [\[CrossRef\]](#)
50. El-Tantawy, S.A.; Alhejaili, W.; Al-Johani, A.S. On the Tantawy technique for analyzing (in)homogeneous fractional physical wave equations. *J. Supercomput.* **2025**, *81*, 1377. [\[CrossRef\]](#)
51. Alhejaili, W.; Khan, A.; Al-Johani, A.S.; El-Tantawy, S.A. Novel Approximations to the Multi-Dimensional Fractional Diffusion Models Using the Tantawy Technique and Two Other Transformed Methods. *Fractal Fract.* **2025**, *9*, 423. [\[CrossRef\]](#)
52. Awawdeh, F. On new iterative method for solving systems of nonlinear equations. *Numer. Algorithms* **2010**, *54*, 395–409. [\[CrossRef\]](#)
53. Bhalekar, S.; Daftardar-Gejji, V. New iterative method: Application to partial differential equations. *Appl. Math. Comput.* **2008**, *203*, 778–783. [\[CrossRef\]](#)
54. El-Wakil, S.; Abulwafa, E.M.; El-Shewy, E.; Mahmoud, A.A. Time-fractional kdv equation for plasma of two different temperature electrons and stationary ion. *Phys. Plasmas* **2011**, *18*, 092116. [\[CrossRef\]](#)
55. El-Wakil, S.A.; Abulwafa, E.M.; Zahran, M.A.; Mahmoud, A.A. Time-fractional kdv equation: Formulation and solution using variational methods. *Nonlinear Dyn.* **2011**, *65*, 55–63. [\[CrossRef\]](#)
56. El-Wakil, S.A.; Abulwafa, E.M.; El-Shewy, E.K.; Mahmoud, A.A. Time fractional kdv equation for electron-acoustic waves in plasma of cold electron and two different temperature isothermal ions. *Astrophys. Space Sci.* **2011**, *333*, 269–276. [\[CrossRef\]](#)
57. Jaradat, H. New solitary wave and multiple soliton solutions for the time-space fractional boussinesq equation. *Ital. J. Pure Appl. Math* **2016**, *36*, 367–376.
58. Sahoo, S.; Ray, S.S. Solitary wave solutions for time fractional third order modified kdv equation using two reliable techniques (g/g)-expansion method and improved (g/g)-expansion method. *Phys. Stat. Mech. Its Appl.* **2016**, *448*, 265–282. [\[CrossRef\]](#)
59. El-Ajou, A.; Arqub, O.A.; Momani, S. Approximate analytical solution of the nonlinear fractional KdV–Burgers equation: A new iterative algorithm. *J. Comput. Phys.* **2015**, *293*, 81–95. [\[CrossRef\]](#)
60. Wahlund, J.-E.; Louarn, P.; Chust, T.; De Feraudy, H.; Roux, A.; Holback, B.; Cabrit, B.; Eriksson, A.; Kintner, P.; Kelley, M.; et al. Observations of ion acoustic fluctuations in the auroral topside ionosphere by the freja s/c. *Geophys. Res. Lett.* **1994**, *21*, 1835–1838. [\[CrossRef\]](#)
61. Abid, A.; Khan, M.; Lu, Q.; Yap, S. A generalized az-non-maxwellian velocity distribution function for space plasmas. *Phys. Plasmas* **2017**, *24*, 033702. [\[CrossRef\]](#)
62. Montgomery, M.D.; Asbridge, J.; Bame, S. Vela 4 plasma observations near the earth’s bow shock. *J. Geophys. Res.* **1970**, *75*, 1217–1231. [\[CrossRef\]](#)
63. Feldman, W.; Bame, S.; Gary, S.; Gosling, J.; McComas, D.; Thomsen, M.; Paschmann, G.; Scokopke, N.; Hoppe, M.; Russell, C. Electron heating within the earth’s bow shock. *Phys. Rev. Lett.* **1982**, *49*, 199. [\[CrossRef\]](#)
64. Qureshi, M.; Nasir, W.; Bruno, R.; Masood, W. Whistler instability based on observed flat-top two-component electron distributions in the earth’s magnetosphere. *Mon. Not. R. Astron. Soc.* **2019**, *488*, 954–964. [\[CrossRef\]](#)
65. Qureshi, M.; Pallochia, G.; Bruno, R.; Cattaneo, M.; Formisano, V.; Reme, H.; Bosqued, J.; Dandouras, I.; Sauvaud, J.; Kistler, L.; et al. Solar wind particle distribution function fitted via the generalized kappa distribution function: Cluster observations. In Proceedings of the AIP Conference Proceedings, Anchorage, AK, USA, 22–26 September 2003; American Institute of Physics: College Park, MD, USA, 2003; Volume 679, pp. 489–492.
66. Parks, G.; Lee, E.; Lin, N.; Mozer, F.; Wilber, M.; Dandouras, I.; Reme, H.; Lucek, E.; Fazakerley, A.; Goldstein, M.; et al. Solitary electromagnetic pulses detected with super-alfvénic flows in earth’s geomagnetic tail. *Phys. Rev. Lett.* **2007**, *98*, 265001. [\[CrossRef\]](#)

67. Parks, G.; Lee, E.; Teste, A.; Wilber, M.; Lin, N.; Canu, P.; Dandouras, I.; Reme, H.; Fu, S.; Goldstein, M. Transport of transient solar wind particles in earth's cusps. *Phys. Plasmas* **2008**, *15*, 080702. [[CrossRef](#)]
68. Kourakis, I.; Sultana, S.; Hellberg, M. Dynamical characteristics of solitary waves, shocks and envelope modes in kappa-distributed non-thermal plasmas: An overview. *Plasma Phys. Control. Fusion* **2012**, *54*, 124001. [[CrossRef](#)]
69. Bhatti, M.; Qureshi, M.; Shah, K. Effect of ion temperature on ion acoustic shock structures in dissipative (r, q) distributed plasma. *Aip Adv.* **2022**, *12*, 045105. [[CrossRef](#)]
70. Naeem, I.; Masood, W.; Mirza, A.M. Dipolar and kelvin-stuart's cat's eyes vortices in magnetoplasmas with non-maxwellian electron distribution. *Astrophys. Space Sci.* **2020**, *365*, 52. [[CrossRef](#)]
71. El-Labany, S.; El-Taibany, W.; El-Tantawy, A.; Zedan, N. Nonplanar dust acoustic waves in a four-component dusty plasma with double spectral distributed electrons: Modulational instability and rogue waves. *Waves Random Complex Media* **2024**, *34*, 2128–2147. [[CrossRef](#)]
72. Shah, K.; Qureshi, M.; Masood, W.; Shah, H. An alternative explanation for the density depletions observed by freja and viking satellites. *Aip Adv.* **2018**, *8*, 085010. [[CrossRef](#)]
73. Shan, S.A.; Haque, Q. Coupled drift and dust ion acoustic wave driven double layers in magnetized plasma with (r, q)-distributed electrons. *Waves Random Complex Media* **2023**, *33*, 98–121. [[CrossRef](#)]
74. Hasegawa, A.; Mima, K. Pseudo-three-dimensional turbulence in magnetized nonuniform plasma. *Phys. Fluids* **1978**, *21*, 87–92. [[CrossRef](#)]
75. Qureshi, M.; Shah, K.; Shi, J.; Masood, W.; Shah, H. Investigation of cubic nonlinearity-driven electrostatic structures in the presence of double spectral index distribution function. *Contrib. Plasma Phys.* **2020**, *60*, 201900065. [[CrossRef](#)]
76. Shan, S.A.; Arooj, S.; Saleem, H. Nonlinear kinetic alfvén waves in upper ionosphere and reductive perturbation method. *Eur. Phys. J. Plus* **2024**, *139*, 1–15. [[CrossRef](#)]
77. Shan, S.A.; Saeed, U.; Saleem, H. Coupled nonlinear drift and iaws in streaming O–H plasma of upper ionosphere. *Phys. Plasmas* **2024**, *31*, 032104. [[CrossRef](#)]
78. Alawad, F.A.; Yousif, E.A.; Arbab, A.I. A new technique of laplace variational iteration method for solving space-time fractional telegraph equations. *Int. J. Differ. Equations* **2013**, *2013*, 256593. [[CrossRef](#)]
79. Jafari, H.; Nazari, M.; Baleanu, D.; Khalique, C.M. A new approach for solving a system of fractional partial differential equations. *Comput. Math. Appl.* **2013**, *66*, 838–843. [[CrossRef](#)]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.