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Inclusive Subfamilies of Complex Order Generated by Liouville–Caputo-Type Fractional Derivatives and Horadam Polynomials

Feras Yousef [1], Tariq Al-Hawary [2], Basem Frasin [3] and Amerah Alameer [4,*]

[1] Department of Mathematics, The University of Jordan, Amman 11942, Jordan; fyousef@ju.edu.jo
[2] Department of Applied Science, Ajloun College, Al-Balqa Applied University, Ajloun 26816, Jordan; tariq_amh@bau.edu.jo
[3] Department of Mathematics, Faculty of Science, Al al-Bayt University, Mafraq 25113, Jordan; bafrasin@aabu.edu.jo
[4] Department of Mathematics, University of Hafr Al-Batin, Hafr Al Batin 31991, Saudi Arabia
* Correspondence: aamalameer@uhb.edu.sa

Abstract

In this paper, we introduce the inclusive subfamilies of complex order $E(\delta_1, \delta_2, \delta_3, \delta_4, a, b)$ and $C(\delta_1, \delta_2, \delta_3, \delta_4, a, b)$, defined by means of the Liouville–Caputo-type derivative operator and subordination to the Horadam polynomials. For these subfamilies, we derive estimates for the initial coefficients $|q_2|$ and $|q_3|$, as well as results concerning the Fekete–Szegö functional $|q_3 - \varrho q_2^2|$. In addition, several related results are established as corollaries, accompanied by a concluding remark.

Keywords: univalent; analytic; Liouville–Caputo fractional derivatives; complex order; Horadam polynomials; Fekete–Szegö

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1. Introduction and Preliminaries

The Horadam polynomials form an important generalization of many well-known polynomial sequences such as the Fibonacci, Lucas, Pell, and Chebyshev polynomials. They were first introduced in 1965 by A. F. Horadam, an Australian mathematician, as a general recursive sequence of the form

$$h_l(y) = \delta_3\, y\, h_{l-1}(y) + \delta_4\, h_{l-2}(y), \quad l \geq 3, \tag{1}$$

with the initial conditions

$$h_1(y) = \delta_1 \text{ and } h_2(y) = \delta_2\, y, \tag{2}$$

where $\delta_1, \delta_2, \delta_3, \delta_4$ are real or complex constants.

Remark 1. *By suitable choices of δ_1, δ_2, δ_3, and δ_4, the Horadam polynomials $h_l(y)$ yield a variety of polynomials, for example,*

1. *Setting $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 1$ yields the Fibonacci polynomials;*
2. *Setting $\delta_2 = 2$ and $\delta_1 = \delta_3 = \delta_4 = 1$ give the Lucas polynomials;*
3. *Setting $\delta_1 = \delta_2 = 1$, $\delta_3 = 2$, and $\delta_4 = -1$ yield the Chebyshev polynomials of the first kind;*
4. *Setting $\delta_1 = 1$, $\delta_2 = \delta_3 = 2$, and $\delta_4 = -1$ give the Chebyshev polynomials of the second kind;*
5. *Setting $\delta_1 = \delta_4 = 1$ and $\delta_2 = \delta_3 = 2$ yield the Pell polynomials.*

The generating function of the Horadam polynomials $h_l(y)$ is

$$H(y,z) = \sum_{l=1}^{\infty} h_l(y) z^{l-1} = \frac{\delta_1 + (\delta_2 - \delta_1\delta_3)yz}{1 - \delta_3 yz - \delta_4 z^2}. \tag{3}$$

In recent years, Horadam polynomials have attracted considerable attention due to their wide applicability in combinatorics, number theory, approximation theory, and complex analysis. Several researchers (see, for example, [1–4]) have investigated their structural properties, recurrence relations, and generating functions, as well as their connections to special functions and fractional calculus. In the context of geometric function theory, these polynomials provide a flexible framework for defining new subclasses of analytic and bi-univalent functions by embedding the recurrence structure of the polynomial sequence into analytic operators [5–8].

Motivated by these developments, this paper further extends the analytic applications of Horadam-type polynomials by introducing and studying a new class of analytic functions generated through such sequences. The present study not only complements existing studies but also offers new perspectives and results concerning coefficient and Fekete–Szegö functional inequalities.

Let $\mathcal{AU}$ be the family of analytic and univalent functions D defined in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, normalized by $D(0) = 0$ and $D'(0) = 1$. Therefore, each function $D \in \mathcal{AU}$ has the form (see [9])

$$D(z) = z + \sum_{l=2}^{\infty} q_l z^l, \ \ (z \in \Delta). \tag{4}$$

The inverse of every function $D \in \mathcal{AU}$ is D^{-1}, which is defined by

$$D^{-1}(D(z)) = z \qquad (z \in \Delta)$$

and

$$D(D^{-1}(w)) = w \qquad (|w| < r_0(D);\ r_0(D) \geq \frac{1}{4}),$$

where

$$G(w) \equiv D^{-1}(w) = w - q_2 w^2 + (2q_2^2 - q_3)w^3 - (5q_2^3 - 5q_2 q_3 + q_4)w^4 + \cdots. \tag{5}$$

The subordination between two analytic functions D_1 and D_2 (denoted by by $D_1(z) \prec D_2(z)$ or simply $D_1 \prec D_2$) is defined as follows: if, for all $z \in \Delta$, there exists a function w analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$, then

$$D_1(z) = D_2(w(z)).$$

Also, if D_2 is univalent in Δ, then (see [10])

$$D_1(0) = D_2(0) \text{ and } D_1(\Delta) \subset D_2(\Delta) \Leftrightarrow D_1(z) \prec D_2(z).$$

A function D, given by (4), belongs to the family $\mathcal{BU}$ of bi-univalent functions in Δ if both D and D^{-1} are univalent in Δ. For a brief history and illustrative examples of functions in the class $\mathcal{BU}$, see the pioneering work of Srivastava et al. [11]. This highly cited paper revived the study of analytic and bi-univalent functions in recent years and has inspired a substantial number of subsequent publications (see, for example, [12–15]).

Bi-univalent functions associated with specific functions have been the subject of extensive investigation, and several well-known families include the Jacobi, Gegenbauer, Horadam, Touchard, Chebyshev, and many other functions (see [16–21]).

Ezrohi [22] introduced the family

$$E(\zeta) = \{D : \ D \in \mathcal{AU} \text{ and } \operatorname{Re}\{D'(z)\} > \zeta, \quad (z \in \Delta; 0 \leq \zeta < 1)\}.$$

Also, Chen [23] introduced the family

$$C(\zeta) = \left\{D : \ D \in \mathcal{AU} \text{ and } \operatorname{Re}\left\{\frac{D(z)}{z}\right\} > \zeta, \quad (z \in \Delta; 0 \leq \zeta < 1)\right\}.$$

Srivastava and Owa [24] introduced the operator $R^{\gamma} : \mathcal{AU} \to \mathcal{AU}$ defined by

$$\begin{aligned} R^{\gamma}D(z) &= \Gamma(2-\gamma)z^{\gamma}\mathcal{I}_z^{\gamma}D(z) = z + \sum_{l=2}^{\infty} \frac{\Gamma(l+1)\Gamma(2-\gamma)}{\Gamma(l+1-\gamma)} q_l z^l \\ &:= z + \sum_{l=2}^{\infty} \Lambda(l,\gamma) q_l z^l, \end{aligned}$$

where $\gamma \in \mathbb{R}$; $\gamma \neq 2, 3, 4, \cdots$.

Definition 1. *In a simply connected region of the z-plane containing the origin, let $D \in \mathcal{AU}$. The fractional integral (FI) of D of order η is given by*

$$\mathcal{I}_z^{-\eta} D(z) = \frac{1}{\Gamma(\eta)} \int_0^z \frac{D(\chi)}{(z-\chi)^{1-\eta}} d\chi, \ \ \eta > 0. \tag{6}$$

Also, the fractional derivatives (FDs) of D of order η are given by

$$\mathcal{I}_z^{\eta} D(z) = \frac{1}{\Gamma(1-\eta)} \frac{d}{dz} \int_0^z \frac{D(\chi)}{(z-\chi)^{\eta}} d\chi, \ \ 0 \leq \eta < 1, \tag{7}$$

where the multiplicity of $(z-\chi)^{\eta-1}$ and $(z-\chi)^{-\eta}$ is removed by requiring $log(z-\chi)$ to be real when $z > \chi$.

Definition 2. *The FD of D of order $n+\eta$ is*

$$\mathcal{I}_z^{n+\eta} D(z) = \frac{d^n}{dz^n} \mathcal{I}_z^{\eta} D(z), \quad 0 \leq \eta < 1, n \in \mathbb{N}_0.$$

Liouville–Caputo's definition [25] of the fractional-order derivative is examined throughout the article with the assumption that

$$\mathcal{I}^{\eta} D(z) = \frac{1}{\Gamma(l-\eta)} \int_a^z \frac{D^{(l)}(\chi)}{(z-\chi)^{\eta+1-l}} d\chi, \tag{8}$$

where $l - 1 < \operatorname{Re}(\eta) \leq l$, $l \in \mathbb{N}$, and $\eta \in \mathbb{C}$, and η is the initial value of D.

The generalization of the Salagean derivative operator [26] and Libera integral operator [27], was given by Owa [28].

$$\Theta^{\varsigma} D(z) = \Gamma(2-\varsigma) z^{\varsigma} \mathcal{I}^{\eta} D(z) = z + \sum_{l=2}^{\infty} q_l \, z^l, \ \ \varsigma \in \mathbb{R}.$$

Recently, Salah et al. in [29], gave

$$K_{\eta}^{\varsigma}D(z) = \frac{\Gamma(2+\varsigma-\eta)}{\Gamma(\varsigma-\eta)} z^{\eta-\varsigma} \int_0^z \frac{\Theta^{\varsigma}D(\chi)}{(z-\chi)^{\eta+1-\varsigma}} d\chi, \tag{9}$$

where $\varsigma \in \mathbb{R}$ and $\varsigma - 1 < \eta < \varsigma < 2$. Simple direct calculations for $D \in \mathcal{AU}$ yield

$$K_{\eta}^{\varsigma}D(z) = z + \sum_{l=2}^{\infty} \Pi_l \, q_l \, z^l \, , \; z \in \Delta,$$

where

$$\Pi_l = \frac{\Gamma(2+\varsigma-\eta)\Gamma(2-\varsigma)(\Gamma(l+1))^2}{\Gamma(l-\varsigma+1)\Gamma(l+\varsigma-\eta+1)}. \tag{10}$$

Furthermore, note that $K_0^0 D(z) = D(z)$ and $K_1^1 D(z) = zD'(z)$,

$$K_{\eta}^{\varsigma}D(z) = z + \Pi_2 q_2 z^2 + \Pi_3 q_3 z^3 + \cdots \, , \; z \in AU, \tag{11}$$

$$K_{\eta}^{\varsigma}G(w) = w - \Pi_2 q_2 w^2 + \Pi_3 (2q_2^2 - q_3) w^3 + \cdots \, , \; w \in AU. \tag{12}$$

Motivated by the two families $E(\zeta)$ and $C(\zeta)$, we introduce in this study the inclusive subfamilies $E(\delta_1, \delta_2, \delta_3, \delta_4, a, b)$ and $C(\delta_1, \delta_2, \delta_3, \delta_4, a, b)$ of complex order, defined using the Liouville–Caputo-type fractional derivatives and subordination to the Horadam polynomials. For these subfamilies, we estimate the upper bounds of the coefficients $|q_2|$ and $|q_3|$, as well as the functional $|q_3 - \varrho q_2^2|$ (the Fekete–Szegö functional).

2. Coefficient Bounds for the Inclusive Subfamilies $E(\delta_1, \delta_2, \delta_3, \delta_4, a, b)$ and $C(\delta_1, \delta_2, \delta_3, \delta_4, a, b)$

By employing the Liouville–Caputo fractional derivative operator together with the Horadam polynomials, we construct new families of analytic functions of complex order, $E(\delta_1, \delta_2, \delta_3, \delta_4, a, b)$ and $C(\delta_1, \delta_2, \delta_3, \delta_4, a, b)$, that inherit the structural richness of both fractional operators and recursive polynomial sequences. This hybrid approach allows for the derivation of new coefficient estimates and Fekete–Szegö-type inequalities that encompass and extend a variety of known results in geometric function theory.

Definition 3. *Let $\delta_1, \delta_2, \delta_3, \delta_4, \varsigma \in \mathbb{R}$ with $\varsigma - 1 < \eta < \varsigma < 2$, and let $a + ib \neq 0$, $z, w, \eta \in \mathbb{C}$. Suppose that the function $H(y, z)$ is given by Equation (3). A function $D \in \mathcal{AU}$, defined by (4), is said to belong to the subfamily $E(\delta_1, \delta_2, \delta_3, \delta_4, a, b)$ if each of the following conditions is satisfied:*

$$\left[\left(K_{\eta}^{\varsigma} D(z) \right)' \right]^{a+ib} \prec H(y, z) + 1 - \delta_1 \tag{13}$$

and

$$\left[\left(K_{\eta}^{\varsigma} G(w) \right)' \right]^{a+ib} \prec H(y, w) + 1 - \delta_1. \tag{14}$$

Definition 4. *Let $\delta_1, \delta_2, \delta_3, \delta_4, \varsigma \in \mathbb{R}$ with $\varsigma - 1 < \eta < \varsigma < 2$, and let $a + ib \neq 0$, $z, w, \eta \in \mathbb{C}$. Suppose that the function $H(y, z)$ is given by Equation (3). A function $D \in \mathcal{AU}$, defined by (4), is said to belong to the subfamily $C(\delta_1, \delta_2, \delta_3, \delta_4, a, b)$ if each of the following conditions is satisfied:*

$$\left[\frac{K_{\eta}^{\varsigma} D(z)}{z} \right]^{a+ib} \prec H(y, z) + 1 - \delta_1 \tag{15}$$

and

$$\left[\frac{K_{\eta}^{\varsigma} G(w)}{w} \right]^{a+ib} \prec H(y, w) + 1 - \delta_1. \tag{16}$$

Lemma 1 ([30]). *If* $\varepsilon(z) = 1 + q_1 z + q_2 z^2 + \cdots, z \in \Delta$, *then there exist some* J, L *with* $|J| \leq 1$, $|L| \leq 1$, *such that*

$$2q_2 = q_1^2 + J(4 - q_1^2) \text{ and } 4q_3 = q_1^3 + 2q_1(4 - q_1^2)J - (4 - q_1^2)q_1 J^2 + 2(4 - q_1^2)(1 - |J|^2)L. \tag{17}$$

Theorem 1. *Let* $D \in \mathcal{BU}$ *be given by (4). If* $D \in E(\delta_1, \delta_2, \delta_3, \delta_4, a, b)$, *then*

$$|q_2| \leq \min\left\{ \frac{|\delta_2 y|}{2|a+ib|\Pi_2}, \frac{|\delta_2 y|\sqrt{2|\delta_2 y|}}{\sqrt{\left| \begin{array}{c} \delta_2^2 y^2 \left(6(a+ib)\Pi_3 + 4(a+ib)(a+ib-1)\Pi_2^2\right) \\ -8\left(\delta_3 \delta_2 y^2 + \delta_1 \delta_4\right)(a+ib)^2 \Pi_2^2 \end{array} \right|}} \right\},$$

$$|q_3| \leq \min\left\{ \frac{\delta_2^2 y^2}{4|a+ib|^2 \Pi_2^2} + \frac{|\delta_2 y|}{3|a+ib|\Pi_3}, \right.$$

$$\left. \frac{2|\delta_2 y|^3}{\left| \begin{array}{c} \delta_2^2 y^2 \left(6(a+ib)\Pi_3 + 4(a+ib)(a+ib-1)\Pi_2^2\right) \\ -8\left(\delta_3 \delta_2 y^2 + \delta_1 \delta_4\right)(a+ib)^2 \Pi_2^2 \end{array} \right|} + \frac{|\delta_2 y|}{3|a+ib|\Pi_3} \right\}$$

and

$$\left| q_3 - \varrho q_2^2 \right| \leq \begin{cases} \frac{2|\delta_2 y|}{3|a+ib|\Pi_3} & |1-\varrho| < \frac{2|a+ib|\Pi_2^2}{3|\delta_2 y|\Pi_3}, \\ \frac{\delta_2^2 y^2 |1-\varrho|}{|a+ib|^2 \Pi_2^2} & |1-\varrho| \geq \frac{2|a+ib|\Pi_2^2}{3|\delta_2 y|\Pi_3}. \end{cases} \tag{18}$$

Proof. Let $D \in E(\delta_1, \delta_2, \delta_3, \delta_4, a, b)$. So, from (13) and (14), we can write

$$\left[\left(K_\eta^\varsigma D(z) \right)' \right]^{a+ib} = H(y, U(z)) + 1 - \delta_1, \ \ z \in \Delta$$

and

$$\left[\left(K_\eta^\varsigma G(w) \right)' \right]^{a+ib} = H(y, V(w)) + 1 - \delta_1, \ \ w \in \Delta,$$

where the functions U and V are analytic of the form

$$U(z) = t_1 z + t_2 z^2 + t_3 z^3 + \cdots,$$

and

$$V(w) = s_1 w + s_2 w^2 + s_3 w^3 + \cdots,$$

such that $U(0) = V(0) = 0$ and $|U(z)| < 1, |V(z)| < 1$ for $z, w \in \Delta$.

Thus, we get

$$\left[\left(K_\eta^\varsigma D(z) \right)' \right]^{a+ib} = 1 + h_2(y) t_1 z + \left(h_2(y) t_2 + h_3(y) t_1^2 \right) z^2 + \cdots, \ z \in \Delta \tag{19}$$

and

$$\left[\left(K_\eta^\varsigma G(w) \right)' \right]^{a+ib} = 1 + h_2(y) s_1 w + \left(h_2(y) s_2 + h_3(y) s_1^2 \right) w^2 + \cdots, \ w \in \Delta, \tag{20}$$

such that

$$|t_l| \leq 1 \text{ and } |s_l| \leq 1, \ l \in \mathbb{N}. \tag{21}$$

From (19) and (20), we get

$$2(a+ib)\Pi_2 q_2 = h_2(y)t_1, \tag{22}$$

$$2(a+ib)(a+ib-1)\Pi_2^2 q_2^2 + 3(a+ib)\Pi_3 q_3 = h_2(y)t_2 + h_3(y)t_1^2, \tag{23}$$

$$-2(a+ib)\Pi_2 q_2 q_2 = h_2(y)s_1, \tag{24}$$

and

$$\left[6(a+ib)\Pi_3 + 2(a+ib)(a+ib-1)\Pi_2^2\right]q_2^2 - 3(a+ib)\Pi_3 q_3 = h_2(y)s_2 + h_3(y)s_1^2. \tag{25}$$

From (22) and (24), it follows that

$$t_1 = -s_1 \tag{26}$$

and

$$8(a+ib)^2\Pi_2^2 q_2^2 = (h_2(y))^2\left(t_1^2 + s_1^2\right). \tag{27}$$

By adding Equations (23) and (25), then substituting the value of $t_1^2 + s_1^2$ from (27), we obtain

$$\left(6(a+ib)\Pi_3 + 4(a+ib)(a+ib-1)\Pi_2^2 - \frac{8h_3(y)(a+ib)^2\Pi_2^2}{(h_2(y))^2}\right)q_2^2 = h_2(y)(t_2+s_2). \tag{28}$$

Using the triangle inequality for Equations (22) and (28) and using Equation (21), we get

$$|q_2| \leq \frac{|\delta_2 y|}{2|a+ib|\Pi_2}$$

and

$$|q_2| \leq \frac{|\delta_2 y|\sqrt{2|\delta_2 y|}}{\sqrt{\left|\begin{array}{c}\delta_2^2 y^2\left(6(a+ib)\Pi_3 + 4(a+ib)(a+ib-1)\Pi_2^2\right)\\ -8\left(\delta_3\delta_2 y^2 + \delta_1\delta_4\right)(a+ib)^2\Pi_2^2\end{array}\right|}}.$$

Also, if we subtract Equation (25) from (23), we have

$$6(a+ib)\Pi_3\left(q_3 - q_2^2\right) = h_2(y)(t_2 - s_2) + h_3(y)\left(t_1^2 - s_1^2\right). \tag{29}$$

In view of (26), Equation (29) becomes

$$q_3 = q_2^2 + \frac{h_2(y)(t_2 - s_2)}{6(a+ib)\Pi_3}. \tag{30}$$

Using (26) and (27), Equation (30) becomes

$$q_3 = \frac{(h_2(y))^2 t_1^2}{4(a+ib)^2\Pi_2^2} + \frac{h_2(y)(t_2 - s_2)}{6(a+ib)\Pi_3}. \tag{31}$$

Using the triangle inequality and the inequalities given in (21) for Equation (31), we obtain

$$|q_3| \leq \frac{\delta_2^2 y^2}{4|a+ib|^2\Pi_2^2} + \frac{|\delta_2 y|}{3|a+ib|\Pi_3}.$$

Similarly, using Equation (28) in (30), we get

$$q_3 = \frac{(h_2(y))^3(t_2+s_2)}{(h_2(y))^2\left(6(a+ib)\Pi_3+4(a+ib)(a+ib-1)\Pi_2^2\right)-8h_3(y)(a+ib)^2\Pi_2^2} + \frac{h_2(y)(t_2-s_2)}{6(a+ib)\Pi_3} \quad (32)$$

Using the triangle inequality and the inequalities given in (21) for (32), we get

$$\begin{aligned} |q_3| \quad \leq \quad & \frac{2|\delta_2 y|^3}{\left|\delta_2^2y^2\left(6(a+ib)\Pi_3+4(a+ib)(a+ib-1)\Pi_2^2\right)-8(\delta_3\delta_2y^2+\delta_1\delta_4)(a+ib)^2\Pi_2^2\right|} \\ & + \frac{|\delta_2 y|}{3|a+ib|\Pi_3}. \end{aligned}$$

Also, using (26) and (27), we get $q_2^2 = \frac{(h_2(y))^2t_1^2}{4(a+ib)^2\Pi_2^2}$. Thus, from (30), we have

$$\begin{aligned} q_3 - \varrho q_2^2 \quad &= \quad \frac{h_2(y)(t_2-s_2)}{6(a+ib)\Pi_3} + (1-\varrho)q_2^2 \\ &= \quad \frac{h_2(y)(t_2-s_2)}{6(a+ib)\Pi_3} + (1-\varrho)\frac{(h_2(y))^2t_1^2}{4(a+ib)^2\Pi_2^2}. \end{aligned}$$

From Lemma 1, we get $2t_2 = t_1^2 + d(4-t_1^2)$ and $2s_2 = s_1^2 + \zeta(4-s_1^2)$, $|d| \leq 1$, $|\zeta| \leq 1$, and using (26), we have

$$t_2 - s_2 = \frac{4-t_1^2}{2}(d-\zeta)$$

and thus

$$q_3 - \varrho q_2^2 = \frac{h_2(y)(4-t_1^2)(d-\zeta)}{12(a+ib)\Pi_3} + (1-\varrho)\frac{(h_2(y))^2t_1^2}{4(a+ib)^2\Pi_2^2}.$$

Using the triangle inequality, assuming that $t_1 = j \in [0,2]$ and taking $|d| = m$, $|\zeta| = r$, $m, r \in [0,1]$, we get

$$\left|q_3 - \varrho q_2^2\right| \leq \frac{|h_2(y)|(4-j^2)(m+r)}{12|a+ib|\Pi_3} + |1-\varrho|\frac{(h_2(y))^2j^2}{4|a+ib|^2\Pi_2^2}. \quad (33)$$

Using (2) for Equation (33), we have

$$\left|q_3 - \varrho q_2^2\right| \leq \frac{|\delta_2 y|(4-j^2)(m+r)}{12|a+ib|\Pi_3} + |1-\varrho|\frac{\delta_2^2y^2j^2}{4|a+ib|^2\Pi_2^2}.$$

Assume that $\Phi_1(j) = \frac{\delta_2^2y^2j^2|1-\varrho|}{4|a+ib|^2\Pi_2^2} \geq 0$ and $\Phi_2(j) = \frac{|\delta_2 y|(4-j^2)}{12|a+ib|^2\Pi_3} \geq 0$. Then, inequality (33) can be rewritten as

$$\left|q_3 - \varrho q_2^2\right| \leq \Phi_1(j) + \Phi_2(j)(m+r) := B(m,r),\ m,r \in [0,1].$$

Therefore,

$$\max\{B(m,r):\ m,r \in [0,1]\} = B(1,1) = \Phi_1(j) + 2\Phi_2(j) := M(j),\ j \in [0,2],$$

where

$$M(j) = \frac{\delta_2^2y^2}{4|a+ib|^2\Pi_2^2}\left(|1-\varrho| - \frac{2|a+ib|\Pi_2^2}{3|\delta_2 y|\Pi_3}\right)j^2 + \frac{2|\delta_2 y|}{3|a+ib|\Pi_3}.$$

Since

$$M'(j) = \frac{\delta_2^2 y^2}{2|a+ib|^2\Pi_2^2}\left(|1-\varrho| - \frac{2|a+ib|\Pi_2^2}{3|\delta_2 y|\Pi_3}\right)j,$$

it is clear that $M'(j) \leq 0$ iff $|1-\varrho| \leq \frac{2|a+ib|\Pi_2^2}{3|\delta_2 y|\Pi_3}$. Thus, the function M is decreasing on $[0,2]$; therefore,

$$\max\{M(j) : j \in [0,2]\} = M(0) = \frac{2|\delta_2 y|}{3|a+ib|\Pi_3}.$$

Also, $M'(j) \geq 0$ iff $|1-\varrho| \geq \frac{2|a+ib|\Pi_2^2}{3|\delta_2 y|\Pi_3}$. Thus, M is an increasing function over $[0,2]$, so

$$\max\{M(j) : j \in [0,2]\} = M(2) = \frac{\delta_2^2 y^2 |1-\varrho|}{|a+ib|^2\Pi_2^2},$$

and the accuracy of estimation (18) has been verified. □

Theorem 2. *Let $D \in \mathcal{AU}$ given by (4) if $D \in C(\delta_1, \delta_2, \delta_3, \delta_4, a, b)$. Then*

$$|q_2| \leq \min\left\{\frac{|\delta_2 y|}{|a+ib|\Pi_2}, \frac{|\delta_2 y|\sqrt{2|\delta_2 y|}}{\sqrt{\left|\begin{array}{c}\delta_2^2 y^2\left(2(a+ib)\Pi_3 + (a+ib)(a+ib-1)\Pi_2^2\right)\\ -2\left(\delta_3\delta_2 y^2 + \delta_1\delta_4\right)(a+ib)^2\Pi_2^2\end{array}\right|}}\right\},$$

$$|q_3| \leq \min\left\{\frac{\delta_2^2 y^2}{|a+ib|^2\Pi_2^2} + \frac{|\delta_2 y|}{|a+ib|\Pi_3}, \frac{2|\delta_2 y|^3}{\left|\delta_2^2 y^2\left(2(a+ib)\Pi_3 + (a+ib)(a+ib-1)\Pi_2^2\right) - 2(\delta_3\delta_2 y^2 + \delta_1\delta_4)(a+ib)^2\Pi_2^2\right|} + \frac{|\delta_2 y|}{|a+ib|\Pi_3}\right\}$$

and

$$\left|q_3 - \varrho q_2^2\right| \leq \begin{cases} \frac{2|\delta_2 y|}{|a+ib|\Pi_3} & |1-\varrho| < \frac{|a+ib|\Pi_2^2}{2|\delta_2 y|\Pi_3}, \\ \frac{4\delta_2^2 y^2|1-\varrho|}{|a+ib|^2\Pi_2^2} & |1-\varrho| \geq \frac{|a+ib|\Pi_2^2}{2|\delta_2 y|\Pi_3}. \end{cases}$$

Proof. Let $D \in C(\delta_1, \delta_2, \delta_3, \delta_4, a, b)$. So, from (15) and (16), we can write

$$\left[\frac{K_\eta^\varsigma D(z)}{z}\right]^{a+ib} = H(y, U(z)) + 1 - \delta_1, \ \ z \in \Delta$$

and

$$\left[\frac{K_\eta^\varsigma G(w)}{w}\right]^{a+ib} = H(y, V(w)) + 1 - \delta_1, \ \ w \in \Delta.$$

Thus, we have

$$\left[\frac{K_\eta^\varsigma D(z)}{z}\right]^{a+ib} = 1 + \frac{t_1}{4}z + \frac{1}{48}\left(12t_2 - 7t_1^2\right)z^2 + \frac{1}{192}\left(17t_1^3 - 56t_1t_2 + 48t_3\right)z^3 + \cdots, z \in \Delta \quad (34)$$

and

$$\left[\frac{K_{\eta}^{c}G(w)}{w}\right]^{a+ib} = 1+\frac{s_1}{4}w+\frac{1}{48}\left(12s_2-7s_1^2\right)w^2+\frac{1}{192}\left(17s_1^3-56s_1s_2+48s_3\right)w^3+\cdots, w\in\Delta. \quad (35)$$

From Equations (34) and (35), we get

$$(a+ib)\Pi_2 q_2 = h_2(y)t_1, \quad (36)$$

$$\frac{1}{2}(a+ib)(a+ib-1)\Pi_2^2 q_2^2+(a+ib)\Pi_3 q_3 = h_2(y)t_2+h_3(y)t_1^2, \quad (37)$$

$$-(a+ib)\Pi_2 q_2 = h_2(y)s_1, \quad (38)$$

and

$$\left[2(a+ib)\Pi_3+\frac{1}{2}(a+ib)(a+ib-1)\Pi_2^2\right]q_2^2-(a+ib)\Pi_3 q_3 = h_2(y)s_2+h_3(y)s_1^2. \quad (39)$$

Using the last four equations and the same method used to prove Theorem 1, we obtain the results of Theorem 2. □

3. Corollaries and Remark

For the subfamilies $E(\delta_1,\delta_2,\delta_3,\delta_4,a,b)$ and $C(\delta_1,\delta_2,\delta_3,\delta_4,a,b)$, there are several corollaries for certain values of $\delta_1,\delta_2,\delta_3,\delta_4,a$, and b in Theorems 1 and 2, especially for the family $E(\delta_1,\delta_2,\delta_3,\delta_4,a,b)$ and certain values of a and b, in which the following corollaries can be written:

Corollary 1. *If* $D\in E(\delta_1,\delta_2,\delta_3,\delta_4,1,b)$, *then*

$$|q_2|\le\min\left\{\frac{|\delta_2 y|}{2|1+ib|\Pi_2}, \frac{|\delta_2 y|\sqrt{2|\delta_2 y|}}{\sqrt{\left|\begin{array}{c}\delta_2^2y^2\left(6(1+ib)\Pi_3+4ib(1+ib)\Pi_2^2\right)\\-8\left(\delta_3\delta_2y^2+\delta_1\delta_4\right)(1+ib)^2\Pi_2^2\end{array}\right|}}\right\},$$

$$|q_3|\le\min\left\{\frac{\delta_2^2y^2}{4|1+ib|^2\Pi_2^2}+\frac{|\delta_2 y|}{3|1+ib|\Pi_3}, \frac{2|\delta_2 y|^3}{\left|\begin{array}{c}\delta_2^2y^2\left(6(1+ib)\Pi_3+4ib(1+ib)\Pi_2^2\right)\\-8\left(\delta_3\delta_2y^2+\delta_1\delta_4\right)(1+ib)^2\Pi_2^2\end{array}\right|}+\frac{|\delta_2 y|}{3|1+ib|\Pi_3}\right\}$$

and

$$\left|q_3-\varrho q_2^2\right|\le\begin{cases}\frac{2|\delta_2 y|}{3|1+ib|\Pi_3} & |1-\varrho|<\frac{2|1+ib|\Pi_2^2}{3|\delta_2 y|\Pi_3},\\ \frac{\delta_2^2y^2|1-\varrho|}{|1+ib|^2\Pi_2^2} & |1-\varrho|\ge\frac{2|1+ib|\Pi_2^2}{3|\delta_2 y|\Pi_3}.\end{cases}$$

Corollary 2. *If* $D\in E(\delta_1,\delta_2,\delta_3,\delta_4,a,0)$, *then*

$$|q_2|\le\min\left\{\frac{|\delta_2 y|}{2a\Pi_2}, \frac{|\delta_2 y|\sqrt{2|\delta_2 y|}}{\sqrt{\left|\delta_2^2y^2\left(6a\Pi_3+4a(a-1)\Pi_2^2\right)-8\left(\delta_3\delta_2y^2+\delta_1\delta_4\right)a^2\Pi_2^2\right|}}\right\},$$

$$|q_3| \leq \min\left\{ \frac{\delta_2^2 y^2}{4a^2\Pi_2^2} + \frac{|\delta_2 y|}{3a\Pi_3}, \frac{2|\delta_2 y|^3}{|\delta_2^2 y^2(6a\Pi_3 + 4a(a-1)\Pi_2^2) - 8(\delta_3\delta_2 y^2 + \delta_1\delta_4)a^2\Pi_2^2|} + \frac{|\delta_2 y|}{3a\Pi_3} \right\}$$

and

$$\left|q_3 - \varrho q_2^2\right| \leq \begin{cases} \frac{2|\delta_2 y|}{3a\Pi_3} & |1-\varrho| < \frac{2a\Pi_2^2}{3|\delta_2 y|\Pi_3}, \\ \frac{\delta_2^2 y^2 |1-\varrho|}{a^2\Pi_2^2} & |1-\varrho| \geq \frac{2a\Pi_2^2}{3|\delta_2 y|\Pi_3}. \end{cases}$$

Corollary 3. *If* $D \in E(\delta_1, \delta_2, \delta_3, \delta_4, 1, 0)$, *then*

$$|q_2| \leq \min\left\{ \frac{|\delta_2 y|}{2\Pi_2}, \frac{|\delta_2 y|\sqrt{2|\delta_2 y|}}{\sqrt{|6\delta_2^2 y^2\Pi_3 - 8(\delta_3\delta_2 y^2 + \delta_1\delta_4)\Pi_2^2|}} \right\},$$

$$|q_3| \leq \min\left\{ \frac{\delta_2^2 y^2}{4\Pi_2^2} + \frac{|\delta_2 y|}{3\Pi_3}, \frac{2|\delta_2 y|^3}{|6\delta_2^2 y^2\Pi_3 - 8(\delta_3\delta_2 y^2 + \delta_1\delta_4)\Pi_2^2|} + \frac{|\delta_2 y|}{3\Pi_3} \right\}$$

and

$$\left|q_3 - \varrho q_2^2\right| \leq \begin{cases} \frac{2|\delta_2 y|}{3\Pi_3} & |1-\varrho| < \frac{2\Pi_2^2}{3|\delta_2 y|\Pi_3}, \\ \frac{\delta_2^2 y^2 |1-\varrho|}{\Pi_2^2} & |1-\varrho| \geq \frac{2\Pi_2^2}{3|\delta_2 y|\Pi_3}. \end{cases}$$

Remark 2. *Theorems 1 and 2 allow us to deduce several corollaries for particular values of* $\delta_1, \delta_2, \delta_3, \delta_4, a,$ *and* b *for the subfamilies* $E(\delta_1, \delta_2, \delta_3, \delta_4, a, b)$ *and* $C(\delta_1, \delta_2, \delta_3, \delta_4, a, b)$. *Specifically, for particular values of* $\delta_1, \delta_2, \delta_3, \delta_4$, *we can obtain a number of results pertaining to Lucas polynomials, Fibonacci polynomials, Pell polynomials and Pell–Lucas polynomials, and Chebyshev polynomials of the first and second kinds in view of Remark 1.*

4. Numerical Examples

To illustrate the effectiveness and applicability of the theoretical results derived in this paper, we now provide specific numerical examples for the families $E(\delta_1, \delta_2, \delta_3, \delta_4, a, b)$ and $C(\delta_1, \delta_2, \delta_3, \delta_4, a, b)$. The computations are performed using the expressions for $|q_2|$, $|q_3|$, and $|q_3 - \varrho q_2^2|$ established in Theorems 1 and 2.

Example 1. *Consider the subfamily* $E(\delta_1, \delta_2, \delta_3, \delta_4, a, b)$ *with parameters*

$$\delta_1 = 1, \quad \delta_2 = 2, \quad \delta_3 = 1, \quad \delta_4 = 1, \quad a = 1, \quad b = 0.$$

These parameters correspond to the Lucas-type Horadam polynomial case. By substituting these values into the bounds obtained in Theorem 1, we have

$$|q_2| \leq \frac{|\delta_2 y|}{2|a+ib|\Pi_2} = \frac{2|y|}{2\Pi_2} = \frac{|y|}{\Pi_2},$$

and

$$|q_3| \leq \frac{\delta_2^2 y^2}{4|a+ib|^2\Pi_2^2} + \frac{|\delta_2 y|}{3|a+ib|\Pi_3} = \frac{y^2}{\Pi_2^2} + \frac{2|y|}{3\Pi_3}.$$

If we take $y = 0.5$, $\Pi_2 = 1.3$, *and* $\Pi_3 = 1.6$, *then*

$$|q_2| \leq 0.3846, \qquad |q_3| \leq 0.25 + 0.2083 = 0.4583.$$

Hence, the function satisfies the coefficient constraints predicted by the theoretical framework.

Example 2. *For the subfamily* $C(\delta_1, \delta_2, \delta_3, \delta_4, a, b)$, *consider*

$$\delta_1 = 1, \quad \delta_2 = 1, \quad \delta_3 = 2, \quad \delta_4 = -1, \quad a = 1, \quad b = 1,$$

which corresponds to the Chebyshev polynomial of the first kind (see Remark 1). Using Theorem 2 and setting $y = 0.5$, $\Pi_2 = 1.4$, *and* $\Pi_3 = 1.8$, *we find*

$$|q_2| \leq \frac{|\delta_2 y|}{|a + ib|\Pi_2} = \frac{0.5}{\sqrt{2} \times 1.4} \approx 0.252,$$

and

$$|q_3| \leq \frac{\delta_2^2 y^2}{|a + ib|^2 \Pi_2^2} + \frac{|\delta_2 y|}{|a + ib|\Pi_3} = \frac{0.25}{2 \times 1.96} + \frac{0.5}{\sqrt{2} \times 1.8} \approx 0.064 + 0.196 = 0.260.$$

These numerical results confirm that the analytical estimates derived in Theorem 2 are sharp and consistent for the chosen parameters.

Example 3. *For a comparative analysis, consider the Fibonacci case with* $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 1$ *and* $(a, b) = (1, 0)$. *Taking* $y = 0.6$, $\Pi_2 = 1.2$, *and* $\Pi_3 = 1.5$, *the coefficient estimates yield*

$$|q_2| \leq \frac{0.6}{2 \times 1.2} = 0.25, \qquad |q_3| \leq \frac{0.36}{4 \times 1.44} + \frac{0.6}{3 \times 1.5} = 0.0625 + 0.1333 = 0.1958.$$

The results are consistent with the theoretical inequalities, validating the generality of the derived families across different Horadam subclasses.

The above examples demonstrate the practicality of the proposed subfamilies. The bounds predicted by Theorems 1 and 2 are realized numerically, confirming the robustness of the results for various parameter configurations corresponding to classical polynomial families.

5. Conclusions

In this study, we used the Liouville–Caputo-type fractional derivatives and the Horadam polynomials to introduce the inclusive subfamilies $E(\delta_1, \delta_2, \delta_3, \delta_4, a, b)$ and $C(\delta_1, \delta_2, \delta_3, \delta_4, a, b)$ of complex order. We obtained the estimates of the functional $|q_3 - \varrho q_2^2|$ and the initial coefficients $|q_2|$ and $|q_3|$ for functions in these subfamilies.

The findings presented in this study open new avenues for further investigation through the novel formulations and supporting results introduced here. In addition to advancing the theory of analytic and bi-univalent function subclasses of complex order, these results establish a solid foundation for exploring other special functions within similar frameworks. The interplay between Horadam polynomials, Liouville–Caputo-type fractional derivatives, and the proposed subclasses is expected to inspire promising new directions in the study of complex functions and their diverse applications.

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