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Hermite–Hadamard Framework for (h, m)-Convexity

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Abstract

This work presents generalizations and extensions of previous results by incorporating weighted integrals and a refined class of second-type (h, m)-convex functions. By utilizing classical inequalities, such as those of Hölder and Young and the Power Mean, we establish new Hermite–Hadamard-type inequalities. The findings offer a broader and more flexible analytical framework, enhancing existing results in the literature. Potential applications of the developed inequalities are also explored.

Keywords: weighted integrals; integral inequalities; Hermite–Hadamard-type inequalities; fractional derivatives

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1. Introduction

Convex functions play a fundamental role in various areas of mathematical sciences today, primarily due to their properties that guarantee existence, uniqueness and the ease of finding solutions in optimization problems. For example, in optimization (Mathematical Programming), convex functions are probably most crucial. In analysis and geometry, convexity is a property that connects concepts of analysis and geometry. Epigraph: A function is convex if and only if its epigraph (the set of points on or above its graph) is a convex set. This provides a powerful geometric interpretation. Derivatives and Criteria: For doubly differentiable functions, convexity is characterized by a nonnegative second derivative (or Hessian matrix in multiple dimensions). This facilitates its identification and analytical handling. Classical Inequalities: Convex functions are the basis of important inequalities, such as Jensen's inequality, which relates the value of a function to the expectation of a random variable and is fundamental in probability and information theory. Moreover, the uses of convex functions have become widespread in interdisciplinary applications: Data Science and Machine Learning, Economics and Finance, Engineering and Signal Processing, among others, are very fertile fields where the different notions of convexity have proven their worth. In short, convexity is a structural property that, when present, transforms mathematical problems that could be intractable into well-defined and efficiently solvable problems, making it an indispensable tool in modern applied mathematics. We add to the above the Hermite–Hadamard inequality, one of the most fundamental and elegant integral inequalities in the field of analysis, since it provides an upper and lower bound on the integral mean value of a function, based solely on the property of the convexity of that function. Today, this inequality is the focus of the attention of numerous researchers, both pure and applied, for four main reasons: using new notions of convexity; using different points on the interval, not just the endpoints; using new integral operators; and defining functionals that allow establishing new bounds.

Thus, this work focuses on two of the most dynamic topics in mathematical research today: convexity and Hermite–Hadamard inequalities.

2. Preliminaries

In [1], the following definitions were introduced.

Definition 1. *Let $h : [0,1] \to [0,+\infty)$ be a non-negative function, such that $h \neq 0$, and let $g : I \subseteq [0,+\infty) \to [0,+\infty)$. The function g is called modified and (h,m)-convex of the first type on I if it satisfies*

$$g(\gamma\mu_1 + m(1-\gamma)\mu_2) \leq h^s(\gamma)g(\mu_1) + m(1-h^s(\gamma))g(\mu_2), \tag{1}$$

for all $\mu_1, \mu_2 \in I$ and $\gamma \in [0,1]$, where $m \in [0,1]$ and $s \in [-1,1]$.

Definition 2. *Let $h : [0,1] \to [0,+\infty)$ be a non-negative function, such that $h \neq 0$, and let $g : I \subseteq [0,+\infty) \to [0,+\infty)$. The function g is called modified and (h,m)-convex of the second type on I if it satisfies*

$$g(\gamma\mu_1 + m(1-\gamma)\mu_2) \leq h^s(\gamma)g(\mu_1) + m(1-h(\gamma))^s g(\mu_2), \tag{2}$$

for all $\mu_1, \mu_2 \in I$ and $\gamma \in [0,1]$, where $m \in [0,1]$ and $s \in [-1,1]$.

Remark 1. *Definitions 1 and 2 enable us to define the set $N^s_{h,m}[\mu_1,\mu_2]$, where $\mu_1, \mu_2 \in I$, as the set of modified (h,m)-convex functions. Here are some convexity classes—special cases described by the triple $(h(\gamma), m, s)$:*

1. *$(h(\gamma),0,0)$, $(\gamma,0,1)$, $(\gamma,1,1)$ and $(\gamma,0,s)$; we have, respectively, the increasing starshaped classic convex on I and s-starshaped functions [2].*
2. *$(\gamma,1,s)$ $s \in (0,1]$; then ψ is s-convex (see [3,4]), and for $s \in [-1,1]$, it is extended and s-convex on I (see [5]).*
3. *(γ^α, m, s) with $\alpha \in (0,1]$; then ψ is an s-(α,m)-convex function on I [6]. If $\alpha = 1$, we have an (s,m)-convex function on I [7], but if $m = 1$, we have an (α,s)-convex function on I [8,9], and lastly, if $s = 1$, we have an (α,m)-convex function on I [10].*
4. *$(h(\gamma), m, 1)$; then ψ is a variant of an (h,m)-convex function on I [11].*

The weighted integral operators, which underpin our analysis, are presented next [1,12].

Adding a particular weight function to the definition of an integral operator is a new and general way to define an integral operator and start the process of generalizing a known result. This may be performed as follows:

Definition 3. *Let $g \in L[\mu_1, \mu_2]$ and let $w : I \to \mathbb{R}$ be a continuous, positive function, whose first derivative is integrable in I°. The weighted fractional integral operators are introduced as follows (right and left, respectively):*

$$\mathbf{J}^{w}_{\mu_1^+} g(\chi) = \int_{\mu_1}^{\chi} w'\left(\frac{\mu_2 - z}{\mu_2 - \mu_1}\right) g(z)\, dz, \quad \chi > \mu_1, \tag{3}$$

$$\mathbf{J}^{w}_{\mu_2^-} g(\chi) = \int_{\chi}^{\mu_2} w'\left(\frac{z - \mu_1}{\mu_2 - \mu_1}\right) g(z)\, dz, \quad \chi < \mu_2. \tag{4}$$

Remark 2. *The inclusion of the first derivative of the weight function w arises from the inherent nature of the problem. Alternatively, the second derivative or a higher order derivative, can also be considered.*

Remark 3. *We examine particular examples of the weight function w' to better demonstrate the scope of Definition 3:*

(a) *Setting $w'(z) \equiv 1$ recovers the classical Riemann integral.*
(b) *Choosing $w'(z) = \frac{z^{\alpha-1}}{\Gamma(\alpha)}$ leads to the Riemann–Liouville fractional integral.*
(c) *By selecting appropriate weight functions, w', various fractional integral operators can be derived, such as the k-Riemann–Liouville integrals [13]; right-sided fractional integrals of a function, g, relative to another function, h, on $[\mu_1, \mu_2]$ [14]; and integral operators introduced in [15–18].*
(d) *Additional well-known integral operators, fractional or otherwise, can be retrieved as particular cases of the above formulation. Interested readers may consult [19,20].*

The Caputo–Fabrizio definition's main basic feature can be explained (cf. [21]) with $0 < \alpha < 1$:

$$\left({}^{CF}_{\mu_1}\mathbf{I}^{\alpha} g\right)(\chi) = \frac{1-\alpha}{M(\alpha)} g(\chi) + \frac{\alpha}{M(\alpha)} \int_{\mu_1}^{\chi} g(z)\, dz, \tag{5}$$

$$\left({}^{CF}\mathbf{I}^{\alpha}_{\mu_2} g\right)(\chi) = \frac{1-\alpha}{M(\alpha)} g(\chi) + \frac{\alpha}{M(\alpha)} \int_{\chi}^{\mu_2} g(z)\, dz, \tag{6}$$

where $M(\alpha)$ is a normalization function, such that $M(0) = M(1) = 1$.

Caputo's fractional derivative is well known, given by the following expression [22]:

$$\left({}^{C}_{0}\mathbf{D}^{\alpha}_{\chi} g\right)(\chi) = \frac{1}{\Gamma(1-\alpha)} \int_0^{\chi} (\chi - z)^{-\alpha} g'(z)\, dz. \tag{7}$$

The idea comes from replacing the singular kernel $(\chi - z)^{-\alpha}$ in the Caputo fractional derivative, given in Formula (7), with the kernel $\exp\left[-\frac{\alpha(\chi - z)}{1-\alpha}\right]$.

In the paper [23], the same authors proposed a more complete study of the operator (7) by presenting the definition of the adapted fractional integral operator ${}^{CF}_{0}\mathbf{I}^{\alpha}_{\chi}$, when $M(\alpha) = 1$.

$$\left({}^{CF}_{0}\mathbf{I}^{\alpha}_{\chi} g\right)(\chi) = \frac{1}{\alpha} \int_0^{\chi} \exp\left[-\frac{(1-\alpha)}{\alpha}(\chi - z)\right] g(z)\, dz. \tag{8}$$

As one can notice, this definition shows a significant resemblance to the classical Riemann–Liouville fractional integral, as given by

$$\left({}^{RL}_{0}\mathbf{I}^{\alpha}_{\chi} g\right)(\chi) = \frac{1}{\Gamma(\alpha)} \int_0^{\chi} (\chi - z)^{\alpha - 1} g(z)\, dz. \tag{9}$$

In this work, we present some variants of the well-known Hermite–Hadamard inequality in the context of (h, m)-convex functions of the second kind using weighted integral operators. Our results include several well-known cases from the literature.

Definition 4. *Let $g \in L_1[\mu_1, \mu_2]$. The Riemann–Liouville integrals ${}^{RL}\mathbf{I}^{\alpha}_{\mu_1^+}g$ and ${}^{RL}\mathbf{I}^{\alpha}_{\mu_2^-}g$ of the order $\alpha > 0$ are defined as*

$$ {}^{RL}\mathbf{I}^{\alpha}_{\mu_1^+}g(\chi) = \frac{1}{\Gamma(\alpha)} \int_{\mu_1}^{\chi} (\chi - z)^{\alpha-1} g(z)\, dz, $$

$$ {}^{RL}\mathbf{I}^{\alpha}_{\mu_2^-}g(\chi) = \frac{1}{\Gamma(\alpha)} \int_{\chi}^{\mu_2} (z - \chi)^{\alpha-1} g(z)\, dz, $$

where $\Gamma(\alpha)$ is the Gamma function.

Throughout this work, $\mathbb{N}$ will be understood as the set of natural numbers $(0, 1, 2, \dots)$ and $\mathbb{R}$ will denote the set of real numbers.

3. Generalizations

Theorem 1. *Let $g : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $\mu_1, \mu_2 \in I^\circ$ with $\mu_1 < \mu_2$. Let $w : [0, 1] \to \mathbb{R}$ be a continuous and positive function with first derivative integrable on $(0, 1)$. Suppose that g is modified and (h, m)-convex of the second type and $\frac{\mu_1}{m}, \frac{\mu_2}{m} \in \mathrm{Dom}(g)$; then it is true that*

$$ \begin{aligned} g\left(\frac{\mu_1 + \mu_2}{2}\right)(w(1) - w(0)) &\le h^s\left(\frac{1}{2}\right) \frac{r+1}{\mu_2 - \mu_1} \mathbf{J}^{w}_{\mu_1^+} g\left(\frac{r\mu_1 + \mu_2}{r+1}\right) \\ &\quad + m\left(1 - h\left(\frac{1}{2}\right)\right)^s \frac{m(r+1)}{\mu_2 - \mu_1} \mathbf{J}^{w}_{\mu_2^-} g\left(\frac{r\mu_2 + \mu_1}{m(r+1)}\right) \\ &\le h^s\left(\frac{1}{2}\right)\left[g(\mu_1)\mathbf{N}_1 + mg\left(\frac{\mu_2}{m}\right)\mathbf{N}_2\right] \\ &\quad + m\left(1 - h\left(\frac{1}{2}\right)\right)^s \left[g(\mu_2)\mathbf{N}_3 + mg\left(\frac{\mu_1}{m^2}\right)\mathbf{N}_4\right], \end{aligned} \tag{10} $$

where $r \in \mathbb{N}$, $\mathbf{N}_1 = \int_0^1 w'(\gamma)\, h^s\left(\frac{r+\gamma}{r+1}\right) d\gamma$, $\mathbf{N}_2 = \int_0^1 w'(\gamma) \left(1 - h\left(\frac{r+\gamma}{r+1}\right)\right)^s d\gamma$, $\mathbf{N}_3 = \int_0^1 w'(\gamma)\, h^s\left(\frac{r+\gamma}{m(r+1)}\right) d\gamma$ and $\mathbf{N}_4 = \int_0^1 w'(\gamma) \left(1 - h\left(\frac{r+\gamma}{m(r+1)}\right)\right)^s d\gamma$.

Proof. By means of the (h, m)-convexity of g with $\gamma = \frac{1}{2}$, we have

$$ g\left(\frac{x+y}{2}\right) \le h^s\left(\frac{1}{2}\right) g(x) + m\left(1 - h\left(\frac{1}{2}\right)\right)^s g\left(\frac{y}{m}\right), \tag{11} $$

for $x, y \in I$.

Substituting $x = \frac{r+\gamma}{r+1}\mu_1 + \frac{1-\gamma}{r+1}\mu_2$ and $y = \frac{r+\gamma}{r+1}\mu_2 + \frac{1-\gamma}{r+1}\mu_1$ in (11), we get

$$ \begin{aligned} g\left(\frac{\mu_1 + \mu_2}{2}\right) &\le h^s\left(\frac{1}{2}\right) g\left(\frac{r+\gamma}{r+1}\mu_1 + \frac{1-\gamma}{r+1}\mu_2\right) \\ &\quad + m\left(1 - h\left(\frac{1}{2}\right)\right)^s g\left(\frac{r+\gamma}{m(r+1)}\mu_2 + \frac{1-\gamma}{m(r+1)}\mu_1\right). \end{aligned} \tag{12} $$

Multiplying both sides of (12) by $w'(\gamma)$ and integrating over $[0, 1]$, we obtain

$$g\left(\frac{\mu_1+\mu_2}{2}\right)(w(1)-w(0)) \le h^s\left(\frac{1}{2}\right)\int_0^1 w'(\gamma)\,g\left(\frac{r+\gamma}{r+1}\mu_1+\frac{1-\gamma}{r+1}\mu_2\right)d\gamma$$
$$+m\left(1-h\left(\frac{1}{2}\right)\right)^s\int_0^1 w'(\gamma)\,g\left(\frac{r+\gamma}{m(r+1)}\mu_2+\frac{1-\gamma}{m(r+1)}\mu_1\right)d\gamma$$
$$=h^s\left(\frac{1}{2}\right)\mathbf{L}_1+m\left(1-h\left(\frac{1}{2}\right)\right)^s\mathbf{L}_2. \tag{13}$$

Rewriting the integrals, we find that

$$\mathbf{L}_1=-\left(\frac{r+1}{\mu_2-\mu_1}\right)\int_{\frac{r\mu_1+\mu_2}{r+1}}^{\mu_1} w'\left(\frac{\frac{r\mu_1+\mu_2}{r+1}-x}{\frac{\mu_2-\mu_1}{r+1}}\right)g(x)\,dx$$
$$=\left(\frac{r+1}{\mu_2-\mu_1}\right)J^w_{\mu_1^+}g\left(\frac{r\mu_1+\mu_2}{r+1}\right). \tag{14}$$

$$\mathbf{L}_2=\left(\frac{m(r+1)}{\mu_2-\mu_1}\right)\int_{\frac{r\mu_2+\mu_1}{m(r+1)}}^{\frac{\mu_2}{m}} w'\left(\frac{y-\frac{r\mu_2+\mu_1}{m(r+1)}}{\frac{\mu_2-\mu_1}{m(r+1)}}\right)g(y)\,dy$$
$$=\left(\frac{m(r+1)}{\mu_2-\mu_1}\right)J^w_{\frac{\mu_2}{m}^-}g\left(\frac{r\mu_2+\mu_1}{m(r+1)}\right). \tag{15}$$

From (13), (14) and (15), it follows that

$$g\left(\frac{\mu_1+\mu_2}{2}\right)(w(1)-w(0)) \le h^s\left(\frac{1}{2}\right)\frac{r+1}{\mu_2-\mu_1}\mathbf{J}^w_{\mu_1^+}g\left(\frac{r\mu_1+\mu_2}{r+1}\right)$$
$$+m\left(1-h\left(\frac{1}{2}\right)\right)^s\frac{m(r+1)}{\mu_2-\mu_1}\mathbf{J}^w_{\mu_2^-}g\left(\frac{r\mu_2+\mu_1}{m(r+1)}\right). \tag{16}$$

Again employing the (h,m)-convexity of g, we obtain

$$\int_0^1 w'(\gamma)\,g\left(\frac{r+\gamma}{r+1}\mu_1+\frac{1-\gamma}{r+1}\mu_2\right)d\gamma \le g(\mu_1)\mathbf{N}_1+mg\left(\frac{\mu_2}{m}\right)\mathbf{N}_2. \tag{17}$$

$$\int_0^1 w'(\gamma)\,g\left(\frac{r+\gamma}{m(r+1)}\mu_2+\frac{1-\gamma}{m(r+1)}\mu_1\right)d\gamma \le g(\mu_2)\mathbf{N}_3+mg\left(\frac{\mu_1}{m^2}\right)\mathbf{N}_4. \tag{18}$$

By combining (13)–(18), we arrive at (10). □

Remark 4. *Setting $s=m=1$, $r=0$, $h(z)=z$ and $w'(z)=1$, we recover the classical Hermite–Hadamard inequality.*

Remark 5. *Considering s, m, r and $h(z)$ as in Remark 4, but with $w'(z)=\frac{z^{\alpha-1}}{\Gamma(\alpha)}$, we obtain Theorem 2 of [24].*

Remark 6. *Letting $m=1$, $n=0$, $h(z)=z$ and $w'(z)=\frac{z^{\alpha-1}}{\Gamma(\alpha)}$, we have*

$$\begin{aligned}
\frac{g\left(\frac{\mu_1+\mu_2}{2}\right)}{\Gamma(\alpha+1)} &\leq \frac{1}{2^s(\mu_2-\mu_1)\Gamma(\alpha)} \int_{\mu_1}^{\mu_2} \left(\frac{\mu_2-z}{\mu_2-\mu_1}\right)^{\alpha-1} g(z)\,dz \\
&\quad + \frac{1}{2^s(\mu_2-\mu_1)\Gamma(\alpha)} \int_a^b \left(\frac{z-\mu_1}{\mu_2-\mu_1}\right)^{\alpha-1} g(z)\,dz \\
&\leq \frac{1}{2^s}\left[\frac{g(\mu_1)}{(\alpha+s)\Gamma(\alpha)} + \frac{g(\mu_2)}{\Gamma(\alpha)} \int_0^1 z^{\alpha-1}(1-z)^s\,dz\right] \\
&\quad + \frac{1}{2^s}\left[\frac{g(\mu_2)}{(\alpha+s)\Gamma(\alpha)} + \frac{g(\mu_1)}{\Gamma(\alpha)} \int_0^1 z^{\alpha-1}(1-z)^s\,dz\right].
\end{aligned} \tag{19}$$

Utilizing Definition 4 in (19), we find

$$\begin{aligned}
\frac{g\left(\frac{\mu_1+\mu_2}{2}\right)}{\Gamma(\alpha+1)} &\leq \frac{1}{2^s(\mu_2-\mu_1)^\alpha}\left[{}^{RL}\mathbf{I}^{\alpha}_{\mu_1^+} g(\mu_2) + {}^{RL}\mathbf{I}^{\alpha}_{\mu_2^-} g(\mu_1)\right] \\
&\leq \frac{\alpha}{2^{s-1}\Gamma(\alpha+1)}\left(\frac{g(\mu_1)+g(\mu_2)}{2}\right)\left[\frac{1}{\alpha+s} + \frac{2}{\alpha+s}\left(1-\frac{1}{2^{\alpha+s}}\right)\right].
\end{aligned} \tag{20}$$

Multiplying the three terms by $2^{s-1}\Gamma(\alpha+1)$ in (20), we complete (1) of [25].

Remark 7. *Under the same assumptions as before, but with $w'(z) = 1$, we complete Theorem 2.1 of [26].*

Remark 8. *Maintaining the previous assumptions, but considering $w'(z) = \frac{z^{\alpha-1}}{\Gamma(\alpha)}$, we derive Theorem 3 of [27].*

Remark 9. *Under the conditions of Remark 4, but with $w'(z) = \frac{\exp(-\varsigma z)}{\alpha}$, where $\varsigma = \frac{1-\alpha}{\alpha}$, we retrieve Theorem 3.1 of [28].*

Remark 10. *Substituting $w(z) = \frac{z^\alpha}{\alpha}$, $r = 0$, $m = s = 1$ and $h(z) = z$ in the previous result leads to the following inequality for the Riemann–Liouville fractional integral (this refers to Theorem 2 in [24]):*

$$g\left(\frac{\mu_1+\mu_2}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(\mu_2-\mu_1)^\alpha}\left[{}^{RL}\mathbf{I}^{\alpha}_{\mu_2-} g(\mu_1) + {}^{RL}\mathbf{I}^{\alpha}_{\mu_1+} g(\mu_2)\right] \leq \frac{g(\mu_1)+g(\mu_2)}{2}.$$

Remark 11. *Theorem 5 in [29] (also see Theorem 1 in [30]), which is based on k-Riemann–Liouville fractional integrals, can be obtained from Theorem 1 by setting $w(z) = z^{\frac{\alpha}{k}}$, $r = 0$, $m = s = 1$ and $h(z) = z$.*

The above results form the foundation for deriving other inequalities by using different types of integral operators, as demonstrated in the following remark.

Remark 12. *We consider s-convex functions ($0 < \alpha < 1$; $m = 1$; $h(z) = z$); by putting $r = 0$ in (10) and choosing $w'(z) = 1$, we obtain*

$$2^{s-1} g\left(\frac{\mu_1+\mu_2}{2}\right) \leq \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} g(z)\,dz \leq \frac{g(\mu_1)+g(\mu_2)}{s+1},$$

taking into account

$$\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} g(z)\,dz = \frac{M(\alpha)}{\alpha(\mu_2-\mu_1)}\left[\frac{1-\alpha}{M(\alpha)}g(\chi)+\frac{\alpha}{M(\alpha)}\int_{\mu_1}^{\chi} g(z)\,dz \right.$$
$$\left. +\frac{1-\alpha}{M(\alpha)}g(\chi)+\frac{\alpha}{M(\alpha)}\int_{\chi}^{\mu_2} g(z)\,dz-\frac{2(1-\alpha)}{M(\alpha)}g(\chi)\right].$$

Using the last two results, we can easily derive Theorem 2.1 *of* [31]*. If, additionally, $s=1$, from the above we can derive Theorem* 2 *of* [32].

Theorem 2. *Let us have g, w, r, μ_1 and μ_2 as in Theorem 1. If $g'\in L[\mu_1,\mu_2]$, then*

$$\frac{\mu_2-\mu_1}{2}\int_0^1[w(1-\gamma)-w(\gamma)]g'\left(\frac{r+\gamma}{r+1}\mu_1+\frac{(1-\gamma)}{r+1}\mu_2\right)d\gamma$$
$$=(r+1)\left\{(w(0)-w(1))\left[g(\mu_1)+g\left(\frac{r\mu_1+\mu_2}{r+1}\right)\right]-\left[\frac{\mathbf{J}^{w}_{\mu_1^+}g\left(\frac{r\mu_1+\mu_2}{r+1}\right)+\mathbf{J}^{w}_{\left(\frac{r\mu_1+\mu_2}{r+1}\right)^-}g(\mu_1)}{2}\right]\right\}. \quad (21)$$

Proof. Let us consider

$$\int_0^1[w(1-\gamma)-w(\gamma)]g'\left(\frac{r+\gamma}{r+1}\mu_1+\frac{(1-\gamma)}{r+1}\mu_2\right)d\gamma$$
$$=\int_0^1 w(1-\gamma)g'\left(\frac{r+\gamma}{r+1}\mu_1+\frac{(1-\gamma)}{r+1}\mu_2\right)d\gamma-\int_0^1 w(\gamma)g'\left(\frac{r+\gamma}{r+1}\mu_1+\frac{(1-\gamma)}{r+1}\mu_2\right)d\gamma$$
$$=\mathcal{I}_1-\mathcal{I}_2. \quad (22)$$

Integrating $\mathcal{I}_1$ by parts, we get

$$\mathcal{I}_1=\frac{r+1}{\mu_1-\mu_2}\left[w(0)g(\mu_1)-w(1)g\left(\frac{r\mu_1+\mu_2}{r+1}\right)\right]$$
$$-\frac{r+1}{\mu_2-\mu_1}\int_0^1 w'(1-\gamma)\,g\left(\frac{r+\gamma}{r+1}\mu_1+\frac{1-\gamma}{r+1}\mu_2\right)d\gamma. \quad (23)$$

Making a change in the variables $x=\frac{r+\gamma}{r+1}\mu_1+\frac{(1-\gamma)}{r+1}\mu_2$ in (23), we find that

$$\mathcal{I}_1=\frac{r+1}{\mu_1-\mu_2}\left[w(0)g(\mu_1)-w(1)g\left(\frac{r\mu_1+\mu_2}{r+1}\right)\right]$$
$$-\frac{r+1}{\mu_2-\mu_1}\int_{\mu_1}^{\frac{r\mu_1+\mu_2}{r+1}} w'\left(\frac{x-\mu_1}{\frac{\mu_2-\mu_1}{r+1}}\right)g\left(\frac{r+\gamma}{r+1}\mu_1+\frac{1-\gamma}{r+1}\mu_2\right)d\gamma$$
$$=\frac{r+1}{\mu_1-\mu_2}\left[w(0)g(\mu_1)-w(1)g\left(\frac{r\mu_1+\mu_2}{r+1}\right)\right]-\frac{r+1}{\mu_2-\mu_1}\mathbf{J}^{w}_{\mu_1^+}g\left(\frac{r\mu_1+\mu_2}{r+1}\right). \quad (24)$$

Analogously for $\mathcal{I}_2$, we can prove

$$\mathcal{I}_2=\frac{r+1}{\mu_1-\mu_2}\left[w(1)g(\mu_1)-w(0)g\left(\frac{r\mu_1+\mu_2}{r+1}\right)\right]-\frac{r+1}{\mu_2-\mu_1}\mathbf{J}^{w}_{\left(\frac{r\mu_1+\mu_2}{r+1}\right)^-}g(\mu_1). \quad (25)$$

From (22), (24) and (25), we have

$$\int_0^1 [w(1-\gamma) - w(\gamma)] g'\left(\frac{r+\gamma}{r+1}\mu_1 + \frac{1-\gamma}{r+1}\mu_2\right) d\gamma$$
$$= \frac{2(r+1)}{\mu_1 - \mu_2}\left\{ (w(0) - w(1))\left[g(\mu_1) + g\left(\frac{r\mu_1 + \mu_2}{r+1}\right)\right] - \left[\frac{\mathbf{J}^w_{\mu_1^+} g\left(\frac{r\mu_1+\mu_2}{r+1}\right) + \mathbf{J}^w_{\left(\frac{r\mu_1+\mu_2}{r+1}\right)^-} g(\mu_1)}{2}\right]\right\}. \tag{26}$$

By multiplying both sides of (26) by $\frac{\mu_2-\mu_1}{2}$, we obtain the desired result. □

Remark 13. *Using convex functions, $r = 0$ and $w(z) = z$, in this way, Theorem 2 becomes the following lemma:*

Lemma 1. *Let g be a real-valued function defined on $[\mu_1, \mu_2]$ and differentiable on (μ_1, μ_2). If $g' \in L_1[\mu_1, \mu_2]$, then the following equality holds:*

$$\frac{g(\mu_1) + g(\mu_2)}{2} - \frac{1}{\mu_2 - \mu_1}\int_a^b g(u)\, du = \frac{\mu_2 - \mu_1}{2}\int_0^1 (1-2\gamma) g'(\gamma\mu_1 + (1-\gamma)\mu_2)\, d\gamma,$$

which is Lemma 2.1 of [33], one of the most important results in the Theory of Integral Inequalities.

Remark 14. *Establishing $r = 0$ and $w(z) = z^{\frac{\lambda}{k}}$, Lemma 2.1 of [29] is derived for $\lambda, k > 0$.*

Theorem 3. *Let g, w, r, μ_1 and μ_2 be defined as before. Suppose that $|g'|$ is modified and (h, m)-convex of the second type; the following inequality holds:*

$$\left|(r+1)\left\{ w(1) - w(0) - \left[\frac{\mathbf{J}^w_{\mu_1^+} g\left(\frac{r\mu_1+\mu_2}{r+1}\right) + \mathbf{J}^w_{\left(\frac{r\mu_1+\mu_2}{r+1}\right)^-} g(\mu_1)}{2}\right]\right\}\right|$$
$$\leq \frac{\mu_2 - \mu_1}{2}\left[|g'(\mu_1)|\mathbf{W}_1 + |g'(\mu_2)|\mathbf{W}_2\right], \tag{27}$$

where

$$\mathbf{W}_1 = \int_0^1 |w(1-\gamma) - w(\gamma)|\, h^s\left(\frac{r+\gamma}{r+1}\right) d\gamma,$$
$$\mathbf{W}_2 = \int_0^1 |w(1-\gamma) - w(\gamma)|\left(1 - h\left(\frac{r+\gamma}{r+1}\right)\right)^s d\gamma.$$

Proof. By using Lemma (2) and the (h, m)-convexity of g, we have

$$\left|(r+1)\left\{w(1)-w(0)-\left[\frac{\mathbf{J}^{w}_{\mu_1^+}g\left(\frac{r\mu_1+\mu_2}{r+1}\right)+\mathbf{J}^{w}_{\left(\frac{r\mu_1+\mu_2}{r+1}\right)^-}g(\mu_1)}{2}\right]\right\}\right|$$
$$\leq \frac{\mu_2-\mu_1}{2}\int_0^1 |w(1-\gamma)-w(\gamma)|\left|g'\left(\frac{r+\gamma}{r+1}\mu_1+\frac{1-\gamma}{r+1}\mu_2\right)\right|d\gamma$$
$$\leq \frac{\mu_2-\mu_1}{2}\left[|g'(\mu_1)|\int_0^1 |w(1-\gamma)-w(\gamma)|h^s\left(\frac{r+\gamma}{r+1}\right)d\gamma\right.$$
$$\left.+|g'(\mu_2)|\int_0^1 |w(1-\gamma)-w(\gamma)|\left(1-h\left(\frac{r+\gamma}{r+1}\right)\right)^s d\gamma\right]$$
$$=\frac{\mu_2-\mu_1}{2}\left[|g'(\mu_1)|\mathbf{W}_1+|g'(\mu_2)|\mathbf{W}_2\right].$$

The proof is finished. □

Remark 15. *Assuming the same conditions as in Remark 13 and invoking Lemma 1, we recover Theorem* 2.2 *of* [33].

Remark 16. *Under the same assumptions of Remark 14, we retrieve Theorem* 6 *of* [29].

Theorem 4. *Let g, w, n, μ_1 and μ_2 be defined as before. Suppose that g is modified and (h,m)-convex of the second type and $\frac{\mu_1}{m}, \frac{\mu_2}{m} \in \mathrm{Dom}(g)$; then it is true that*

$$g\left(\frac{\mu_1+\mu_2}{2}\right)(w(1)-w(0)) \leq \frac{n}{\mu_2-\mu_1}\left[h^s\left(\frac{1}{2}\right)\mathbf{J}^{w}_{\left(\frac{n-1}{n}\mu_2+\frac{\mu_1}{n}\right)^+}g(\mu_2)\right.$$
$$\left.+m^2\left(1-h\left(\frac{1}{2}\right)\right)^s\mathbf{J}^{w}_{\left(\frac{n-1}{n}\mu_1+\frac{\mu_2}{n}\right)^-}g\left(\frac{\mu_1}{m}\right)\right]$$
$$\leq h^s\left(\frac{1}{2}\right)\left[g(\mu_1)\mathbf{N}_5+mg\left(\frac{\mu_2}{m}\right)\mathbf{N}_6\right]$$
$$+m\left(1-h\left(\frac{1}{2}\right)\right)^s\left[g(\mu_2)\mathbf{N}_7+mg\left(\frac{\mu_1}{m^2}\right)\mathbf{N}_8\right], \tag{28}$$

where

$$\mathbf{N}_1=\int_0^1 w'(\gamma)\,h^s\left(\frac{\gamma}{n}\right)d\gamma, \quad \mathbf{N}_2=\int_0^1 w'(\gamma)\left(1-h\left(\frac{\gamma}{n}\right)\right)^s d\gamma,$$
$$\mathbf{N}_3=\int_0^1 w'(\gamma)\,h^s\left(\frac{n-\gamma}{mn}\right)d\gamma, \quad \mathbf{N}_4=\int_0^1 w'(\gamma)\left(1-h\left(\frac{n-\gamma}{mn}\right)\right)^s d\gamma.$$

Proof. By means of the (h,m)-convexity of g with $\gamma=\frac{1}{2}$, we have

$$g\left(\frac{x+y}{2}\right)\leq h^s\left(\frac{1}{2}\right)g(x)+m\left(1-h\left(\frac{1}{2}\right)\right)^s g\left(\frac{y}{m}\right), \tag{29}$$

for $x,y\in I$.

Substituting $x=\frac{\gamma}{n}\mu_1+\frac{n-\gamma}{n}\mu_2$ and $y=\frac{\gamma}{n}\mu_2+\frac{n-\gamma}{n}\mu_1$ in (27), we get

$$g\left(\frac{\mu_1+\mu_2}{2}\right) \le h^s\left(\frac{1}{2}\right)g\left(\frac{\gamma}{n}\mu_1+\frac{n-\gamma}{n}\mu_2\right) + m\left(1-h\left(\frac{1}{2}\right)\right)^s g\left(\frac{\gamma}{mn}\mu_2+\frac{n-\gamma}{mn}\mu_1\right). \tag{30}$$

Multiplying both sides of (12) by $w'(\gamma)$ and integrating over $[0,1]$, we obtain

$$\begin{aligned} g\left(\frac{\mu_1+\mu_2}{2}\right)(w(1)-w(0)) &\le h^s\left(\frac{1}{2}\right)\int_0^1 w'(\gamma)\,g\left(\frac{\gamma}{n}\mu_1+\frac{n-\gamma}{n}\mu_2\right)d\gamma \\ &\quad + m\left(1-h\left(\frac{1}{2}\right)\right)^s \int_0^1 w'(\gamma)\,g\left(\frac{\gamma}{mn}\mu_2+\frac{n-\gamma}{mn}\mu_1\right)d\gamma \\ &= h^s\left(\frac{1}{2}\right)\mathbf{L}_3 + m\left(1-h\left(\frac{1}{2}\right)\right)^s\mathbf{L}_4. \end{aligned} \tag{31}$$

Rewriting the integrals, we find

$$\begin{aligned} \mathbf{L}_3 &= -\left(\frac{n}{\mu_2-\mu_1}\right)\int_{\mu_2}^{\frac{(n-1)\mu_2+\mu_1}{n}} w'\left(\frac{x-\mu_2}{\frac{\mu_1-\mu_2}{n}}\right)g(x)\,dx \\ &= \left(\frac{n}{\mu_2-\mu_1}\right)\int_{\frac{(n-1)\mu_2+\mu_1}{n}}^{\mu_2} w'\left(\frac{\mu_2-x}{\frac{\mu_2-\mu_1}{n}}\right)g(x)\,dx \\ &= \left(\frac{n}{\mu_2-\mu_1}\right)\mathbf{J}^w_{\left(\frac{(n-1)\mu_2+\mu_1}{n}\right)^+}g(\mu_2), \end{aligned} \tag{32}$$

$$\begin{aligned} \mathbf{L}_4 &= \left(\frac{m\,n}{\mu_2-\mu_1}\right)\int_{\frac{\mu_1}{m}}^{\frac{\mu_2+(n-1)\mu_1}{m\,n}} w'\left(\frac{y-\frac{\mu_1}{m}}{\frac{\mu_2-\mu_1}{m\,n}}\right)g(y)\,dy \\ &= \left(\frac{m\,n}{\mu_2-\mu_1}\right)\mathbf{J}^w_{\left(\frac{\mu_2+(n-1)\mu_1}{m\,n}\right)^-}g\left(\frac{\mu_1}{m}\right). \end{aligned} \tag{33}$$

From (13), (14) and (15), it follows that

$$\begin{aligned} g\left(\frac{\mu_1+\mu_2}{2}\right)(w(1)-w(0)) \le \frac{n}{\mu_2-\mu_1}\Bigg[& h^s\left(\frac{1}{2}\right)\mathbf{J}^w_{\left(\frac{(n-1)\mu_2+\mu_1}{n}\right)^+}g(\mu_2) \\ & + m\left(1-h\left(\frac{1}{2}\right)\right)^s\mathbf{J}^w_{\left(\frac{\mu_2+(n-1)\mu_1}{m\,n}\right)^-}g\left(\frac{\mu_1}{m}\right)\Bigg]. \end{aligned} \tag{34}$$

Again employing again the (h,m)-convexity of g, we get

$$\int_0^1 w'(\gamma)\,g\left(\frac{\gamma}{n}\mu_1+\frac{n-\gamma}{n}\mu_2\right)d\gamma \le g(\mu_1)\mathbf{N}_1 + m\,g\left(\frac{\mu_2}{m}\right)\mathbf{N}_2, \tag{35}$$

$$\int_0^1 w'(\gamma)\,g\left(\frac{\gamma}{m\,n}\mu_2+\frac{n-\gamma}{m\,n}\mu_1\right)d\gamma \le g\left(\frac{\mu_2}{m}\right)\mathbf{N}_3 + m\,g\left(\frac{\mu_1}{m^2}\right)\mathbf{N}_4. \tag{36}$$

By combining (29)–(36), we arrive at (28). □

Remark 17. *Specializing to the case where g is convex, $w(z)=z$ and $n=1$, we yield the celebrated Hermite–Hadamard inequality.*

Remark 18. *Considering $n = 1$, we obtain a new result for modified (h, m)-convex functions of the second type.*

Remark 19. *If g is a convex function and $n = 1$, by setting $w'(z) = z^{\alpha-1}$ with $\alpha > 0$, we derive Expression* (2.1) *of Theorem* 2 *(see* [24]*).*

Indeed, applying Theorem 4, we obtain

$$\frac{1}{\alpha} g\left(\frac{\mu_1 + \mu_2}{2}\right) \leq \frac{1}{2(\mu_2 - \mu_1)} \left[\mathbf{J}^{w}_{\mu_1^+} g(\mu_2) + \mathbf{J}^{w}_{\mu_2^-} g(\mu_1)\right] \leq \frac{g(\mu_1) + g(\mu_2)}{2\alpha}.$$

According to Definition 3, we have

$$\begin{aligned}\frac{1}{\alpha} g\left(\frac{\mu_1 + \mu_2}{2}\right) &\leq \frac{1}{2(\mu_2 - \mu_1)^{\alpha}} \left[\int_{\mu_1}^{\mu_2} (z - \mu_1)^{\alpha-1} g(z)\, dz + \int_{\mu_1}^{\mu_2} (\mu_2 - z)^{\alpha-1} g(z)\, dz\right] \\ &\leq \frac{g(\mu_1) + g(\mu_2)}{2\alpha}.\end{aligned}$$

Given that $\Gamma(\alpha)$ is well defined for $\alpha > 0$, it follows that

$$\begin{aligned}g\left(\frac{\mu_1 + \mu_2}{2}\right) &\leq \frac{\alpha\Gamma(\alpha)}{2(\mu_2 - \mu_1)^{\alpha}} \left[\frac{1}{\Gamma(\alpha)} \int_{\mu_1}^{\mu_2} (z - \mu_1)^{\alpha-1} g(z)\, dz + \frac{1}{\Gamma(\alpha)} \int_{\mu_1}^{\mu_2} (\mu_2 - z)^{\alpha-1} g(z)\, dz\right] \\ &\leq \frac{g(\mu_1) + g(\mu_2)}{2}.\end{aligned}$$

From Definition 4, we conclude that

$$g\left(\frac{\mu_1 + \mu_2}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(\mu_2 - \mu_1)^{\alpha}} \left[{}^{RL}\mathbf{I}^{\alpha}_{\mu_2^-} g(\mu_1) + {}^{RL}\mathbf{I}^{\alpha}_{\mu_1^+} g(\mu_2)\right] \leq \frac{g(\mu_1) + g(\mu_2)}{2}.$$

Remark 20. *With $w(z) = \frac{z^{\alpha}}{\alpha}$, $m = s = 1$, $r = 2$ and $h(z) = z$, the previous result simplifies to Theorem* 4 *in* [34].

For s-convex functions, using $w(z) = \frac{z^{\alpha}}{\alpha}$ and $n = 1$, we recover Theorem 2.1 *from* [25]. *Additionally, Theorem* 3 *in* [27]*, for $w(z) = z^{\alpha}$, provides further results. In this work, Theorem* 5 *for m-convex functions is also established under similar conditions and can be easily derived.*

Remark 21. *By assigning $n = m = s = 1$ and $h(z) = z$ in* (28)*, which corresponds to working with convex functions and choosing $w'(z) = \frac{\alpha z^{\frac{\alpha}{k}-1}}{kB(\alpha)\Gamma_k(\alpha)}$, the left-hand side yields*

$$\begin{aligned}g\left(\frac{\mu_1 + \mu_2}{2}\right) \frac{1}{B(\alpha)\Gamma_k(\alpha)} \leq \frac{1}{2(\mu_2 - \mu_1)^{\frac{\alpha}{k}}} &\left[\frac{\alpha}{kB(\alpha)\Gamma_k(\alpha)} \int_{\mu_1}^{\mu_2} \frac{g(z)}{(z - \mu_1)^{1-\frac{\alpha}{k}}}\, dz\right. \\ &\left.+ \frac{\alpha}{kB(\alpha)\Gamma_k(\alpha)} \int_{\mu_1}^{\mu_2} \frac{g(z)}{(\mu_2 - z)^{1-\frac{\alpha}{k}}}\, dz\right],\end{aligned}$$

$$\begin{aligned}g\left(\frac{\mu_1 + \mu_2}{2}\right) \frac{2(\mu_2 - \mu_1)^{\frac{\alpha}{k}}}{B(\alpha)\Gamma_k(\alpha)} \leq &\frac{\alpha}{kB(\alpha)\Gamma_k(\alpha)} \int_{\mu_1}^{\mu_2} \frac{g(z)}{(z - \mu_1)^{1-\frac{\alpha}{k}}}\, dz \\ &+ \frac{\alpha}{kB(\alpha)\Gamma_k(\alpha)} \int_{\mu_1}^{\mu_2} \frac{g(z)}{(\mu_2 - z)^{1-\frac{\alpha}{k}}}\, dz. \qquad (37)\end{aligned}$$

Adding the term $\frac{1-\alpha}{B(\alpha)}(g(\mu_1) + g(\mu_2))$ on both sides of (37) *and considering that $g\left(\frac{\mu_1+\mu_2}{2}\right) \leq \frac{g(\mu_1)+g(\mu_2)}{2}$, we obtain*

$$g\left(\frac{\mu_1+\mu_2}{2}\right)\left[\frac{2(\mu_2-\mu_1)^{\frac{\alpha}{k}}}{B(\alpha)\Gamma_k(\alpha)}+\frac{(1-\alpha)}{B(\alpha)}\right]\leq^{AB} I^{\alpha}_{\mu_1+}g(\mu_2)+^{AB} I^{\alpha}_{\mu_2-}g(\mu_1). \tag{38}$$

A similar approach applied to the right-hand side of (28) *gives*

$$\frac{1}{2(\mu_2-\mu_1)^{\frac{\alpha}{k}}}\left[\frac{\alpha}{kB(\alpha)\Gamma_k(\alpha)}\int_{\mu_1}^{\mu_2}\frac{g(z)}{(z-\mu_1)^{1-\frac{\alpha}{k}}}\,dz+\frac{\alpha}{kB(\alpha)\Gamma_k(\alpha)}\int_{\mu_1}^{\mu_2}\frac{g(z)}{(\mu_2-z)^{1-\frac{\alpha}{k}}}\,dz\right]$$
$$\leq\frac{1}{B(\alpha)\Gamma_k(\alpha)}\left(\frac{g(\mu_1)+g(\mu_2)}{2}\right).$$

Multiplying both sides by $2(\mu_2-\mu_1)^{\frac{\alpha}{k}}$, *adding* $\frac{(1-\alpha)}{B(\alpha)}(g(\mu_1)+g(\mu_2))$ *and rearranging terms, we arrive at*

$${}^{AB}\mathbf{I}^{\alpha}_{\mu_1+}g(\mu_2)+^{AB}\mathbf{I}^{\alpha}_{\mu_2-}g(\mu_1)\leq\left[\frac{2(\mu_2-\mu_1)^{\frac{\alpha}{k}}}{B(\alpha)\Gamma_k(\alpha)}+\frac{(1-\alpha)}{B(\alpha)}\right]\left(\frac{g(\mu_1)+g(\mu_2)}{2}\right). \tag{39}$$

By combining (38) *and* (39), *we obtain a relation that closely resembles Theorem 6 in* [35]. *Moreover, setting* $k=1$ *in this expression yields a result comparable to Proposition 2.1 in* [36].

Remark 22. *Theorem 7 of* [29] *can be established by taking* $m=s=1$, $n=2$ *and* $w'(z)=z^{\frac{\lambda}{k}-1}$.

Lemma 2. *Let* g, w, n, μ_1 *and* μ_2 *be defined as before. If* $g'\in L[\mu_1,\mu_2]$, *then*

$$\int_0^1 w(\gamma)\left[g'\left(\frac{\gamma}{n}\mu_1+\frac{(n-\gamma)}{n}\mu_2\right)-g'\left(\frac{\gamma}{n}\mu_2+\frac{(n-\gamma)}{n}\mu_1\right)\right]d\gamma$$
$$=\frac{n}{\mu_2-\mu_1}\left[w(0)(g(\mu_1)+g(\mu_2))-w(1)\left(\left(\frac{\gamma}{n}\mu_1+\frac{(n-\gamma)}{n}\mu_2\right)+g\left(\frac{\gamma}{n}\mu_2+\frac{(n-\gamma)}{n}\mu_1\right)\right)\right]$$
$$+\left(\frac{n}{\mu_2-\mu_1}\right)^2\left[\mathbf{J}^{w}_{\left(\frac{\gamma}{n}\mu_1+\frac{(n-\gamma)}{n}\mu_2\right)^+}g(\mu_2)+\mathbf{J}^{w}_{\left(\frac{\gamma}{n}\mu_2+\frac{(n-\gamma)}{n}\mu_1\right)^-}g(\mu_1)\right]. \tag{40}$$

Proof. Let

$$\int_0^1 w(\gamma)g'\left(\frac{\gamma}{n}\mu_1+\frac{(n-\gamma)}{n}\mu_2\right)d\gamma-\int_0^1 w(\gamma)g'\left(\frac{\gamma}{n}\mu_2+\frac{(n-\gamma)}{n}\mu_1\right)d\gamma=\mathcal{I}_3-\mathcal{I}_4 \tag{41}$$

By integrating $\mathcal{I}_3$ by parts and making a change in the variables $x=\frac{\gamma}{n}\mu_1+\frac{n-\gamma}{n}\mu_2$, we have, after some computations,

$$\mathcal{I}_3=\frac{n}{\mu_2-\mu_1}\left[w(0)g(\mu_2)-w(1)g\left(\frac{\gamma}{n}\mu_1+\frac{n-\gamma}{n}\mu_2\right)\right]+\left(\frac{n}{\mu_2-\mu_1}\right)^2\mathbf{J}^{w}_{\left(\frac{\gamma}{n}\mu_1+\frac{n-\gamma}{n}\mu_2\right)^+}g(\mu_2). \tag{42}$$

Analogously for $\mathcal{I}_4$, we get

$$\mathcal{I}_4=\frac{n}{\mu_2-\mu_1}\left[w(1)g\left(\frac{\gamma}{n}\mu_2+\frac{n-\gamma}{n}\mu_1\right)-w(0)g(\mu_2)\right]+\left(\frac{n}{\mu_2-\mu_1}\right)^2\mathbf{J}^{w}_{\left(\frac{\gamma}{n}\mu_1+\frac{n-\gamma}{n}\mu_2\right)^+}g(\mu_1). \tag{43}$$

From (42) and (43), (2) follows. □

Remark 23. *By setting* $n=2$ *and* $w(z)=z^{\alpha}$ *with* $\alpha>0$, *Lemma 3 is derived from* [34].

Remark 24. *Lemma* 3.1 *in* [29] *may be derived by setting* $m = s = 1$, $n = 2$ *and* $w'(z) = z^{\frac{\lambda}{k}}$.

Remark 25. *By adopting a strategy similar to that utilized in Lemma* 2*, we establish a comparable result concerning the midpoint of the interval.*

Lemma 3. *Let* g *be a real-valued function defined on a closed real interval,* $[\mu_1, \mu_2]$*, differentiable on* (μ_1, μ_2)*, and* w' *is an integrable function on* $[\mu_1, \mu_2]$*. If* $g' \in L_1[\mu_1, \mu_2]$*, then the following equality holds:*

$$\begin{aligned}
&\frac{1}{2(r+1)}\left[2w(1)g\left(\frac{\mu_1+\mu_2}{2}\right) - w(0)\left[g\left(\frac{(r+2)\mu_1 + r\mu_2}{2(r+1)}\right) + g\left(\frac{r\mu_1 + (r+2)\mu_2}{2(r+1)}\right)\right]\right] \\
&- \frac{1}{\mu_2-\mu_1}\left[\mathbf{J}^w_{\left(\frac{\mu_1+\mu_2}{2}\right)^-} g\left(\frac{(r+2)\mu_1 + r\mu_2}{2(r+1)}\right) + \mathbf{J}^w_{\left(\frac{\mu_1+\mu_2}{2}\right)^+} g\left(\frac{r\mu_1 + (r+2)\mu_2}{2(r+1)}\right)\right] \\
&= \frac{\mu_2-\mu_1}{4(r+1)^2}\int_0^1 w(t)\left[g'\left(\frac{r+\gamma}{r+1}\frac{\mu_1+\mu_2}{2} + \frac{1-\gamma}{r+1}\mu_1\right) - g'\left(\frac{r+\gamma}{r+1}\frac{\mu_1+\mu_2}{2} + \frac{1-\gamma}{r+1}\mu_2\right)\right] d\gamma,
\end{aligned} \tag{44}$$

for $r \in \mathbb{N} \cup \{0\}$.

Below we present some remarks that show the breadth and generality of (44).

Remark 26. *By setting* $w(z) = z^\alpha$ *and* $r = 0$*, we recover Lemma 2.1 of* [37]*. A similar result can be obtained very easily for the k-Riemann–Liouville integral of* [13].

Remark 27. *Letting* $w(z) = z$ *and* $r = 0$*, we find a new result in the framework of the Riemann integral:*

$$\begin{aligned}
&g\left(\frac{\mu_1+\mu_2}{2}\right) - \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} g(z)\,dz \\
&= \frac{\mu_2-\mu_1}{4}\int_0^1 w(\gamma)\left[g'\left(\gamma\frac{\mu_1+\mu_2}{2} + (1-\gamma)\mu_1\right) - g'\left(\gamma\frac{\mu_1+\mu_2}{2} + (1-\gamma)\mu_2\right)\right] d\gamma.
\end{aligned}$$

Remark 28. *Considering* $w(z)$ *to be a linear function, but different for* $\mathbb{I}_1$ *and* $\mathbb{I}_2$*, and* $r = 0$*, we get*

$$\mathbb{I} = \int_0^1 (\gamma - \lambda_1) g'\left(\gamma\frac{\mu_1+\mu_2}{2} + (1-\gamma)\mu_1\right) d\gamma - \int_0^1 (\gamma - \lambda_2) g'\left(\gamma\frac{\mu_1+\mu_2}{2} + (1-\gamma)\mu_2\right) d\gamma,$$

where

$$\begin{aligned}
&\lambda_1, \lambda_2 \in \mathbb{R}, \\
\mathbb{I}_1 &= \int_0^1 w(\gamma) g'\left(\frac{r+\gamma}{r+1}\frac{\mu_1+\mu_2}{2} + \frac{1-\gamma}{r+1}\mu_1\right) d\gamma, \\
\mathbb{I}_2 &= \int_0^1 w(\gamma) g'\left(\frac{r+\gamma}{r+1}\frac{\mu_1+\mu_2}{2} + \frac{1-\gamma}{r+1}\mu_2\right) d\gamma.
\end{aligned}$$

From here we obtain

$$\frac{2-\lambda_1-\lambda_2}{2} g\left(\frac{\mu_1+\mu_2}{2}\right) + \frac{\lambda g(\mu_1) + \mu g(\mu_2)}{2} - \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} g(u)\,du = \frac{\mu_2-\mu_1}{4}\mathbb{I}.$$

Given that

$$\int_0^1 (\gamma-\lambda_1)g'\left(\gamma\frac{\mu_1+\mu_2}{2}+(1-\gamma)\mu_1\right)d\gamma=\int_0^1 (1-\gamma-\lambda_1)g'\left(\gamma\mu_1+(1-\gamma)\frac{\mu_1+\mu_2}{2}\right)d\gamma,$$

we retrieve Lemma 2.1 *of* [38].

Remark 29. *Readers will have no difficulty in proving, in a similar manner, the following result.*

Lemma 4. *Let g be a real function defined on some closed real interval $[\mu_1,\mu_2]$, differentiable on (μ_1,μ_2), and w' is an integrable function on $[\mu_1,\mu_2]$. If $g'\in L_1[\mu_1,\mu_2]$, then we find the following equality:*

$$\begin{aligned}&\frac{1}{n+1}\left[-w(1)\frac{g(\mu_1)+g(\mu_2)}{2}+w(0)\frac{g\left(\frac{n\mu_1+\frac{\mu_1+\mu_2}{2}}{n+1}\right)+g\left(\frac{n\mu_2+\frac{\mu_1+\mu_2}{2}}{n+1}\right)}{2}\right]\\&+\frac{1}{\mu_2-\mu_1}\left[\mathbf{J}^{w}_{\mu_1+}g\left(\frac{n\mu_1+\frac{\mu_1+\mu_2}{2}}{n+1}\right)+\mathbf{J}^{w}_{\mu_2-}g\left(\frac{n\mu_2+\frac{\mu_1+\mu_2}{2}}{n+1}\right)\right]\\&=\frac{\mu_2-\mu_1}{4(n+1)^2}\int_0^1 w(\gamma)\left[g'\left(\frac{n+\gamma}{n+1}\frac{\mu_1+\mu_2}{2}+\frac{1-\gamma}{n+1}\mu_1\right)-g'\left(\frac{n+\gamma}{n+1}\frac{\mu_1+\mu_2}{2}+\frac{1-\gamma}{n+1}\mu_2\right)\right]d\gamma,\end{aligned}$$

for $n\in\mathbb{N}$.

This result completes Lemma 2.1 *of* [37]*. Of course, remarks, similar to those presented above, can be derived.*

Theorem 5. *Let g, w, n, μ_1 and μ_2 be defined as before. If $|g'|$ is modified and (h,m)-convex of the second type, then it is true that*

$$|\mathbf{L}|\le\left(|g'(\mu_1)|+|g'(\mu_2)|\right)\mathbf{W}_3+m\left(\left|g'\left(\frac{\mu_1}{m}\right)\right|+\left|g'\left(\frac{\mu_2}{m}\right)\right|\right)\mathbf{W}_4, \tag{45}$$

where $\mathbf{L}$ is the left-hand side of (2)*, $\mathbf{W}_3=\int_0^1 w(\gamma)h^s\left(\frac{\gamma}{n}\right)d\gamma$ and $\mathbf{W}_4=\int_0^1 w(\gamma)\left(1-h\left(\frac{\gamma}{n}\right)\right)^s d\gamma$.*

Proof. From Lemma 2, by employing the properties of the modulus, we obtain

$$|\mathbf{L}|\le\int_0^1|w(\gamma)|\left[\left|g'\left(\frac{\gamma}{n}\mu_1+\frac{n-\gamma}{n}\mu_2\right)\right|+\left|g'\left(\frac{n-\gamma}{n}\mu_1+\frac{\gamma}{n}\mu_2\right)\right|\right]d\gamma.$$

Utilizing the convexity property of $|g'|$, we get

$$\left|g'\left(\frac{\gamma}{n}\mu_1+\frac{n-\gamma}{n}\mu_2\right)\right|\le h^s\left(\frac{\gamma}{n}\right)|g'(\mu_1)|+m\left(1-h\left(\frac{\gamma}{n}\right)\right)^s\left|g'\left(\frac{\mu_2}{m}\right)\right|,$$

and

$$\left|g'\left(\frac{n-\gamma}{n}\mu_1+\frac{\gamma}{n}\mu_2\right)\right|\le m\left(1-h\left(\frac{\gamma}{n}\right)\right)^s\left|g'\left(\frac{\mu_1}{m}\right)\right|+h^s\left(\frac{\gamma}{n}\right)|g'(\mu_2)|.$$

Summing the last two inequalities, we have

$$\begin{aligned}&\left|g'\left(\frac{\gamma}{n}\mu_1+\frac{n-\gamma}{n}\mu_2\right)\right|+\left|g'\left(\frac{n-\gamma}{n}\mu_1+\frac{\gamma}{n}\mu_2\right)\right|\\&\le h^s\left(\frac{\gamma}{n}\right)\left(|g'(\mu_1)|+|g'(\mu_2)|\right)+m\left(1-h\left(\frac{\gamma}{n}\right)\right)^s\left(\left|g'\left(\frac{\mu_1}{m}\right)\right|+\left|g'\left(\frac{\mu_2}{m}\right)\right|\right).\end{aligned}$$

Taking into account the accepted notations, we obtain (5). The proof is completed. □

Remark 30. *If we consider the usual class of convex functions and $n = 2$, then from Theorem 5, we obtain*

$$|\mathbf{L}| \le \left(|g'(\mu_1)| + |g'(\mu_2)|\right) \int_0^1 w(\gamma)\, d\gamma.$$

Here, if we take $w(z) = z$, then we obtain Theorem 2.2 from [39] and Theorem 5 from [34]. If we choose $w(z) = (1 - z)$, then we have Theorem 2.2 in [33], and if $w(z) = z^\alpha$, then we obtain the inequality from [40] (remark of Theorem 1, for $w(z) = (1 - z)^\alpha$).

Remark 31. *By adopting a strategy similar to that utilized in Theorem 5 and by employing Lemma 3, we establish a comparable result concerning the midpoint of the interval.*

Theorem 6. *Let $g : [\mu_1, \mu_2] \to \mathbb{R}$ be a differentiable function on (μ_1, μ_2), such that $g' \in L_1[\mu_1, \mu_2]$. If $|g'|$ is modified and (h, m)-convex of the second type and $\frac{\mu_1}{m}, \frac{\mu_2}{m} \in \mathrm{Dom}(|g'|)$, then the following inequality holds:*

$$|\mathcal{L}(w, g, \mu_1, \mu_2, n)| \le 2\left|g'\left(\frac{\mu_1 + \mu_2}{2}\right)\right|\mathcal{H}_1 + m\left(\left|g'\left(\frac{\mu_1}{m}\right)\right| + \left|g'\left(\frac{\mu_2}{m}\right)\right|\right)\mathcal{H}_2,$$

where

$$\begin{aligned} &\mathcal{L}(w, g, \mu_1, \mu_2, r) = \\ = \;& \frac{1}{2(r+1)}\left\{2w(1)g\left(\frac{\mu_1+\mu_2}{2}\right) - w(0)\left[g\left(\frac{(r+2)\mu_1 + r\mu_2}{2(r+1)}\right) + g\left(\frac{r\mu_1 + (r+2)\mu_2}{2(r+1)}\right)\right]\right\} \\ - \;& \frac{1}{\mu_2 - \mu_1}\left[J^w_{\left(\frac{\mu_1+\mu_2}{2}\right)^-} g\left(\frac{(r+2)\mu_1 + r\mu_2}{2(r+1)}\right) + J^w_{\left(\frac{\mu_1+\mu_2}{2}\right)^+} g\left(\frac{r\mu_1 + (r+2)\mu_2}{2(r+1)}\right)\right], \end{aligned}$$

$$\begin{aligned} \mathcal{H}_1 &= \int_0^1 w(\gamma) h^s\left(\frac{r+\gamma}{r+1}\right) d\gamma, \\ \mathcal{H}_2 &= \int_0^1 w(\gamma)\left[1 - h\left(\frac{r+\gamma}{r+1}\right)\right]^s d\gamma. \end{aligned}$$

Corollary 1. *Under the assumptions of Theorem 6, we have the following:*

1. *If we choose $m = 1$, then we derive the following inequality:*

$$|\mathcal{L}(w, g, \mu_1, \mu_2, r)| \le 2\left|g'\left(\frac{\mu_1 + \mu_2}{2}\right)\right|\mathcal{H}_1 + \left(|g'(\mu_1)| + |g'(\mu_2)|\right)\mathcal{H}_2,$$

   $\mathcal{H}_1$ and $\mathcal{H}_2$ are as before.

2. *If $s = m = 1$, then*

$$|\mathcal{L}(w, g, \mu_1, \mu_2, r)| \le 2\left|g'\left(\frac{\mu_1 + \mu_2}{2}\right)\right|\mathcal{H}_3(\gamma) + \left(|g'(\mu_1)| + |g'(\mu_2)|\right)\mathcal{H}_4(\gamma),$$

   where

$$\mathcal{H}_3 = \int_0^1 w(\gamma) h\left(\frac{r+\gamma}{r+1}\right) d\gamma, \quad \mathcal{H}_4 = \int_0^1 w(\gamma)\left[1 - h\left(\frac{r+\gamma}{r+1}\right)\right] d\gamma.$$

3. *If we take $w(z) = z$, $r = 0$ and $s = m = 1$, we obtain the following inequality, new for the Riemann integral:*

$$\left|g\left(\frac{\mu_1+\mu_2}{2}\right)-\int_{\mu_1}^{\mu_2}g(z)\,dz\right|\leq 2\left|g'\left(\frac{\mu_1+\mu_2}{2}\right)\right|\int_0^1\gamma\,h(\gamma)\,d\gamma$$
$$+\left(|g'(\mu_1)|+|g'(\mu_2)|\right)\int_0^1\gamma(1-h(\gamma))d\gamma.$$

4. *Putting* $w'(z)=\dfrac{z^\alpha}{\Gamma(\alpha+1)}$, $r=0$, *readers will have no difficulty in obtaining a new inequality for the Riemann–Liouville integral.*

Remark 32. *The generality of this result can be easily verified since, for different notions of convexity contained in our Definition 2, with different values of r and for different kernels,* w', *new results can be derived under the conditions from Theorem 6.*

Theorem 7. *Let us have* g, g', w, μ_1, μ_2 *and* n *as in Theorem 9. Suppose that* $|g'|^q$ *is modified and* (h,m)*-convex of the second type and* $\frac{\mu_1}{m}, \frac{\mu_2}{m}\in Dom(|g'|^q)$*; then the inequality below is satisfied:*

$$|\mathbf{U}|\leq\frac{\mu_2-\mu_1}{n}\,\mathbf{W}_6\left[\left(|g'(\mu_1)|^q\,\mathbf{H}_1+m\left|g'\left(\frac{\mu_2}{m}\right)\right|^q\mathbf{H}_2\right)^{\frac{1}{q}}+\left(|g'(\mu_2)|^q\,\mathbf{H}_1+m\left|g'\left(\frac{\mu_1}{m}\right)\right|^q\mathbf{H}_2\right)^{\frac{1}{q}}\right],$$

where $p,q>1$, $\mathbf{U}$ *is the right-hand side of Equation (2),* $\mathbf{W}_5=\left(\int_0^1 w^p(\gamma)\,d\gamma\right)^{\frac{1}{p}}$, $\mathbf{H}_1=\int_0^1 h^s\left(\frac{\gamma}{n}\right)d\gamma$ *and* $\mathbf{H}_2=\int_0^1\left(1-h\left(\frac{\gamma}{n}\right)\right)^s d\gamma$.

Proof. By adapting the approach used in Theorem 9 but by employing Hölder's inequality instead, we arrive at

$$|\mathbf{U}|\leq\frac{\mu_2-\mu_1}{n}\left[\int_0^1 w(\gamma)\left|g'\left(\frac{\gamma}{n}\mu_1+\frac{(n-\gamma)}{n}\mu_2\right)\right|d\gamma+\int_0^1 w(\gamma)\left|g'\left(\frac{\gamma}{n}\mu_2+\frac{(n-\gamma)}{n}\mu_1\right)\right|d\gamma\right]$$
$$\leq\frac{\mu_2-\mu_1}{n}\left(\int_0^1 w^p(\gamma)\,d\gamma\right)^{\frac{1}{p}}\left[\left(\int_0^1\left|g'\left(\frac{\gamma}{n}\mu_1+\frac{(n-\gamma)}{n}\mu_2\right)\right|^q d\gamma\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left|g'\left(\frac{\gamma}{n}\mu_2+\frac{(n-\gamma)}{n}\mu_1\right)\right|^q d\gamma\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{\mu_2-\mu_1}{n}\,\mathbf{W}_5\left[\left(|g'(\mu_1)|^q\,\mathbf{H}_1+m\left|g'\left(\frac{\mu_2}{m}\right)\right|^q\mathbf{H}_2\right)^{\frac{1}{q}}+\left(|g'(\mu_2)|^q\,\mathbf{H}_1+m\left|g'\left(\frac{\mu_1}{m}\right)\right|^q\mathbf{H}_2\right)^{\frac{1}{q}}\right].$$

Therefore, the desired result has been established. □

Remark 33. *If* $w(z)=z^\alpha$ *and* g *is convex, we obtain the inequality of Theorem 6 presented in* [34]:

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mu_2-\mu_1)^\alpha}\left[{}^{RL}\mathbf{I}^\alpha_{\left(\frac{\mu_1+\mu_2}{2}\right)^-}g(\mu_1)+{}^{RL}\mathbf{I}^\alpha_{\left(\frac{\mu_1+\mu_2}{2}\right)^+}g(\mu_2)\right]-g\left(\frac{\mu_1+\mu_2}{2}\right)\right|$$
$$\leq\frac{\mu_2-\mu_1}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|g'(\mu_1)|^q}{4}+\frac{3|g'(\mu_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|g'(\mu_2)|^q}{4}+\frac{3|g'(\mu_1)|^q}{4}\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{\mu_2-\mu_1}{4}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}\left[|g'(\mu_1)|+|g'(\mu_2)|\right].$$

Remark 34. *If* $w(z)=4z$, $q=\frac{p-1}{p}$ *and* g *is convex, we obtain an inequality similar to Theorem* 2.3 *presented in* [39]:

$$\left|\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} g(z)\,dz - g\left(\frac{\mu_1+\mu_2}{2}\right)\right| \le \frac{\mu_2-\mu_1}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|g'(\mu_1)|^{\frac{p-1}{p}}}{4}+\frac{3|g'(\mu_2)|^{\frac{p-1}{p}}}{4}\right)^{\frac{p}{p-1}} + \left(\frac{|g'(\mu_2)|^{\frac{p-1}{p}}}{4}+\frac{3|g'(\mu_1)|^{\frac{p-1}{p}}}{4}\right)^{\frac{p}{p-1}}\right].$$

Remark 35. *Utilizing a procedure parallel to that applied in Theorem 7 and invoking Lemma 3, we obtain an equivalent statement pertaining to the midpoint of the interval:*

Theorem 8. *Let $g:[\mu_1,\mu_2]\to\mathbb{R}$ be a differentiable function on (μ_1,μ_2) such that $g'\in L_1[\mu_1,\mu_2]$. If $|g'|^q$ is modified and (h,m)-convex of the second type and $\frac{\mu_1}{m},\frac{\mu_2}{m}\in \mathrm{Dom}(|g'|^q)$, then it is true that*

$$\begin{aligned}
&|\mathcal{L}(w,g,\mu_1,\mu_2,r)| \\
&\le \frac{\mu_2-\mu_1}{4(r+1)^2}\mathbf{W}_5\left\{\left[\left|g'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q\mathcal{H}_5 + m\left(\left|g'\left(\frac{\mu_1}{m}\right)\right|^q\right)\mathcal{H}_6\right]^{\frac{1}{q}}\right. \\
&\quad\left. + \left[\left|g'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q\mathcal{H}_5 + m\left(\left|g'\left(\frac{\mu_2}{m}\right)\right|^q\right)\mathcal{H}_6\right]^{\frac{1}{q}}\right\}.
\end{aligned} \tag{46}$$

with $\frac{1}{p}+\frac{1}{q}=1$, $\mathcal{H}_5=\int_0^1 h^s\left(\frac{r+t}{r+1}\right)dt$, $\mathcal{H}_6=\int_0^1\left(1-h\left(\frac{r+t}{r+1}\right)\right)^s dt$ and $\mathbf{W}_5$ defined as before.

Corollary 2. *Under the assumptions of Theorem 8, we have the following:*

1. *Choosing $m=1$, then we obtain the following inequality:*

$$\begin{aligned}
&|\mathcal{L}(w,g,\mu_1,\mu_2,r)| \\
&\le \frac{\mu_2-\mu_1}{4(r+1)^2}\left(\int_0^1 w^p(t)dt\right)^{\frac{1}{p}}\left\{\left[\left|g'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q\mathcal{H}_5 + \left(|g'(\mu_1)|^q\right)\mathcal{H}_6\right]^{\frac{1}{q}}\right. \\
&\quad\left. + \left[\left|g'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q\mathcal{H}_5 + \left(|g'(\mu_2)|^q\right)\mathcal{H}_6\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

2. *If $s=m=1$, then*

$$\begin{aligned}
|\mathcal{L}(w,g,\mu_1,\mu_2,r)| &\le \frac{\mu_2-\mu_1}{4(r+1)^2}\left(\int_0^1 w^p(t)dt\right)^{\frac{1}{p}} \\
&\times\left\{\left[\left|g'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q\int_0^1 h\left(\frac{r+t}{r+1}\right)dt + |g'(\mu_1)|^q\int_0^1\left(1-h\left(\frac{r+t}{r+1}\right)\right)dt\right]^{\frac{1}{q}}\right. \\
&\left. + \left[\left|g'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q\int_0^1 h\left(\frac{r+t}{r+1}\right)dt + |g'(\mu_2)|^q\int_0^1\left(1-h\left(\frac{r+t}{r+1}\right)\right)dt\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

3. *Bearing in mind Corollary 1, items 3 and 4, we can derive new inequalities for Riemann and Riemann–Liouville integrals, respectively.*

Theorem 9. *Let $g, g', w, \mu_1, \mu_2, n$, $\mathbf{W}_3$ and $\mathbf{W}_4$ be as in Lemma 2. Suppose that $|g'|^q$ is modified and (h,m)-convex of the second type and $\frac{\mu_1}{m}, \frac{\mu_2}{m} \in Dom(|g'|^q)$; then the following result emerges:*

$$|\mathbf{U}| \le \frac{\mu_2-\mu_1}{n}\mathbf{W}_6\left[\left(|g'(\mu_1)|^q\mathbf{W}_3 + m\left|g'\left(\frac{\mu_2}{m}\right)\right|^q\mathbf{W}_4\right)^{\frac{1}{q}} + \left(|g'(\mu_2)|^q\mathbf{W}_3 + m\left|g'\left(\frac{\mu_1}{m}\right)\right|^q\mathbf{W}_4\right)^{\frac{1}{q}}\right],$$

where $q \ge 1$, $\mathbf{W}_6 = \left(\int_0^1 w(\gamma)\,d\gamma\right)^{1-\frac{1}{q}}$ and $\mathbf{U}$ is defined as before.

Proof. Employing Lemma 2, the triangle inequality, the Power Mean inequality and Definition 2 for $|g|^q$, we obtain

$$\begin{aligned}
|\mathbf{U}| &\le \left|\frac{\mu_2-\mu_1}{n}\int_0^1 w(\gamma)\left[g'\left(\frac{\gamma}{n}\mu_1 + \frac{(n-\gamma)}{n}\mu_2\right) - g'\left(\frac{\gamma}{n}\mu_2 + \frac{(n-\gamma)}{n}\mu_1\right)\right]d\gamma\right| \\
&\le \frac{\mu_2-\mu_1}{n}\left[\int_0^1 w(\gamma)\left|g'\left(\frac{\gamma}{n}\mu_1 + \frac{(n-\gamma)}{n}\mu_2\right)\right|d\gamma + \int_0^1 w(\gamma)\left|g'\left(\frac{\gamma}{n}\mu_2 + \frac{(n-\gamma)}{n}\mu_1\right)\right|d\gamma\right] \\
&\le \frac{\mu_2-\mu_1}{n}\left(\int_0^1 w(\gamma)\,d\gamma\right)^{1-\frac{1}{q}}\left[\left(\int_0^1 w(\gamma)\left|g'\left(\frac{\gamma}{n}\mu_1 + \frac{(n-\gamma)}{n}\mu_2\right)\right|^q d\gamma\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\int_0^1 w(\gamma)\left|g'\left(\frac{\gamma}{n}\mu_2 + \frac{(n-\gamma)}{n}\mu_1\right)\right|^q d\gamma\right)^{\frac{1}{q}}\right] \\
&\le \frac{\mu_2-\mu_1}{n}\mathbf{W}_6\left[\left(|g'(\mu_1)|^q\mathbf{W}_3 + m\left|g'\left(\frac{\mu_2}{m}\right)\right|^q\mathbf{W}_4\right)^{\frac{1}{q}} + \left(|g'(\mu_2)|^q\mathbf{W}_3 + m\left|g'\left(\frac{\mu_1}{m}\right)\right|^q\mathbf{W}_4\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Hence, the proof is finished. □

Remark 36. *Theorem 8 in [29] follows as a consequence when the parameters m, s and n and the function w' are selected as in Remark 24.*

Remark 37. *In light of Theorem 9 and Lemma 2, we similarly obtain a result for the midpoint of the interval:*

Theorem 10. *Let $g : [\mu_1, \mu_2] \to \mathbb{R}$ be a differentiable function on (μ_1, μ_2), such that $g' \in L_1[\mu_1, \mu_2]$. If $|g'|^q$ is modified and (h,m)-convex of the second type with $q \ge 1$ and $\frac{\mu_1}{m}, \frac{\mu_2}{m} \in$ $\mathrm{Dom}(|g'|^q)$, then it is true that*

$$\begin{aligned}
|\mathcal{L}(w, g, \mu_1, \mu_2, r)| \le \frac{\mu_2-\mu_1}{4(r+1)^2}&\left(\int_0^1 w(t)\,dt\right)^{1-\frac{1}{q}} \\
&\times\left\{\left[\left|g'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q\mathcal{H}_1 + m\left(\left|g'\left(\frac{\mu_1}{m}\right)\right|^q\right)\mathcal{H}_2\right]^{\frac{1}{q}}\right. \\
&\quad\left. + \left[\left|g'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q\mathcal{H}_1 + m\left(\left|g'\left(\frac{\mu_2}{m}\right)\right|^q\right)\mathcal{H}_2\right]^{\frac{1}{q}}\right\},
\end{aligned} \tag{47}$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are defined above in Theorem 6.

Theorem 11. *Let g, g', w, μ_1, μ_2, n, p, q, $\mathbf{U}$, $\mathbf{H}_1$ and $\mathbf{H}_2$ be as defined in the preceding result. Suppose that $|g'|^q$ is modified and (h,m)-convex of the second type and $\frac{\mu_1}{m}, \frac{\mu_2}{m} \in Dom(|g'|^q)$; then it is true that*

$$|\mathbf{U}| \le \frac{\mu_2-\mu_1}{n}\left[\mathbf{W}_7 + \left(|g'(\mu_1)|^q + |g'(\mu_2)|^q\right)\frac{\mathbf{H}_1}{q} + m\left(\left|g'\left(\frac{\mu_2}{m}\right)\right|^q + \left|g'\left(\frac{\mu_1}{m}\right)\right|^q\right)\frac{\mathbf{H}_2}{q}\right], \quad (48)$$

where $\mathbf{W}_7 = 2\int_0^1 \frac{w(\gamma)}{p}\,d\gamma$.

Proof. Following a similar line of reasoning as in Theorem 9 but replacing the key inequality with that of Young, we get

$$\begin{aligned}|\mathbf{U}| &\le \frac{\mu_2-\mu_1}{n}\left[\int_0^1 w(\gamma)\left|g'\left(\frac{\gamma}{n}\mu_1 + \frac{(n-\gamma)}{n}\mu_2\right)\right|d\gamma + \int_0^1 w(\gamma)\left|g'\left(\frac{\gamma}{n}\mu_2 + \frac{(n-\gamma)}{n}\mu_1\right)\right|d\gamma\right] \\ &\le \frac{\mu_2-\mu_1}{n}\left[2\int_0^1 \frac{w^p(\gamma)}{p}\,d\gamma + \int_0^1 \frac{\left|g'\left(\frac{\gamma}{n}\mu_1 + \frac{(n-\gamma)}{n}\mu_2\right)\right|^q}{q}\,d\gamma\right. \\ &\quad \left. + \int_0^1 \frac{\left|g'\left(\frac{\gamma}{n}\mu_2 + \frac{(n-\gamma)}{n}\mu_1\right)\right|^q}{q}\,d\gamma\right] \\ &\le \frac{\mu_2-\mu_1}{n}\left[\mathbf{W}_7 + \left(|g'(\mu_1)|^q + |g'(\mu_2)|^q\right)\frac{\mathbf{H}_1}{q} + m\left(\left|g'\left(\frac{\mu_2}{m}\right)\right|^q + \left|g'\left(\frac{\mu_1}{m}\right)\right|^q\right)\frac{\mathbf{H}_2}{q}\right].\end{aligned}$$

This concludes the proof. □

Remark 38. *If we consider the usual class of convex functions and* $n=2$*, then from* (48)*, we obtain*

$$|\mathbf{U}| \le \frac{2(\mathbf{W}_5)^p}{p} + \frac{|g'(\mu_1)|^q + |g'(\mu_2)|^q}{q}.$$

Here, if we take $w(z)=z$*, then we get*

$$\left|\int_{\mu_1}^{\mu_2} g(z)\,dz - g\left(\frac{\mu_1+\mu_2}{2}\right)\right| \le \frac{2}{p(p+1)} + \frac{|g'(\mu_1)|^q + |g'(\mu_2)|^q}{q}.$$

Remark 39. *By building upon the method employed in Theorem 7 and drawing on Lemma 2, we derive a parallel result concerning the midpoint of the interval.*

Theorem 12. *Let* $g:[\mu_1,\mu_2]\to\mathbb{R}$ *be a differentiable function on* (μ_1,μ_2) *such that* $g' \in L_1[\mu_1,\mu_2]$*. If* $|g'|^q$ *is modified and* (h,m)*-convex of the second type with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $\frac{\mu_1}{m}, \frac{\mu_2}{m} \in \mathrm{Dom}(|g'|^q)$*, then*

$$\begin{aligned}|\mathcal{L}(w,g,\mu_1,\mu_2,r)| \le \frac{\mu_2-\mu_1}{4(r+1)^2}&\left\{\frac{2}{p}\int_0^1 w^p(t)\,dt + \frac{2}{q}\left|g'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q \mathcal{H}_5\right. \\ &\left. + \frac{m}{q}\left(\left|g'\left(\frac{\mu_1}{m}\right)\right|^q + \left|g'\left(\frac{\mu_2}{m}\right)\right|^q\right)\mathcal{H}_6\right\}, \end{aligned} \quad (49)$$

holds, where $\mathcal{H}_5$ *and* $\mathcal{H}_6$ *are defined above in Theorem 8.*

Remark 40. *Remark 32 remains valid in these results.*

Remark 41. *Readers will have no difficulty in formulating the corresponding corollaries to Theorems 10 and 11.*

4. Conclusions

This work focuses on the generalization and extension of existing results related to integral inequalities. The main results and contributions are Theorem 1 and Theorem 2, which establish new inequalities for (h, m)-convex functions of second type using weighted integral operators. It also provides remarks showing how these new results generalize or connect with existing theorems in the literature by establishing specific parameters for s, m, and h and the weighting function w'.

In essence, we consider this work to contribute significantly to the theory of convex functions by providing a more generalized and flexible framework for Hermite–Hadamard-type inequalities through the introduction of weighted integrals and refined classes of (h, m)-convex functions.

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