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Zeroing Operators for Differential Initial Values Applied to Fractional Operators in the Self-Congruent Physical Space

Gang Peng [1,*], Zhimo Jian [2], Meilin Li [1], Yu Wu [1], Meiling Yang [1] and Yajun Yin [2]

[1] Department of Civil Engineering, Chengdu Technological University, Chengdu 611730, China; 2202041120@cdtu.edu.cn (M.L.); 2202042133@cdtu.edu.cn (Y.W.); 2202041109@cdtu.edu.cn (M.Y.)
[2] Department of Engineering Mechanics, Tsinghua University, Beijing 100084, China; jzm22@mails.tsinghua.edu.cn (Z.J.); yinyj@tsinghua.edu.cn (Y.Y.)
* Correspondence: pgang1@cdtu.edu.cn

Abstract

Non-zero differential initial values hinder the application of fractional operator theory in practical systems. This paper proposes a differential initial values zeroing method, decomposing functions with non-zero differential initial values into a compensation function (Taylor polynomial) and a zeroing function (with all differential initial values being zero). A differential initial values "zeroing operator" is defined, with properties such as initial value annihilation and linearity, and operational rules compatible with unilateral Laplace transforms and Mikusinski calculus operators. Based on the zeroing operator, the "zeroing differential operator" is defined to extract the zero-initial-value differential intrinsic properties of the functions with non-zero differential initial values. Using the zeroing operator, fractional constitutive equations are reconstructed in both time and complex Laplace domains in the self-congruent physical space, introducing complex fractional operators and generalized fractional operators. Validated by the complex fractional constitutive model of bone, this method breaks the bottleneck of zero-initial-value assumption in fractional operator theory in the self-congruent physical space, providing a rigorous mathematical foundation and a standardized tool for modeling sophisticated fractional systems with non-zero differential initial values.

Keywords: zeroing operators; non-zero differential initial values; fractional operators; self-congruent physical space; complex constitutive relations

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1. Introduction

Non-zero differential initial values have always been a key factor influencing the regulatory mechanisms of systems such as chaos suppression [1], differential equations [2], control theory [3], and biomechanics [4], especially in fractional-order calculus systems [5–7]. The existence of non-zero initial values conditions render the solution of differential equations [8], circuit analysis [9], and flow analysis [10] extremely sophisticated. Chaos theory indicates that a tiny difference in the non-zero initial values can lead to significant changes in the behavior of the chaotic system [11]. According to the Lorenz model, the non-zero derivative of the initial temperature gradient can trigger the "butterfly effect" [12]. It can be seen that non-zero differential initial values have an extremely significant impact on the system response, especially for complicated systems such as chaotic systems and fractional-order differential systems.

However, for the field of fractional operators theory in the self-congruent physical space, the most recent and popular research of fractional-order biomechanical regulatory

systems [13] (such as viscoelasticity of skeletal muscles [14], bone mechanics [15], and hemodynamics [16]) typically assume zero differential initial conditions to analyze the inherent regulatory characteristics. This limitation restricts the application of the fractional operator theory in systems with non-zero differential initial input functions.

Note that the "fractional operators" considered in this paper are the calculus operators governing the fractional-order properties of self-congruent physical models which are composed of an infinite order of identical physical components stacked together [13]. Self-congruent physical space is also called the "physical fractal space" in the fractional rock permeability analysis [17]. Here, in order to describe the physical structural features more accurately, we refer to this type of physical fractals abstracted from biological fibers [14,15] and arterial hemodynamics [16] as "self-congruent physics".

Furthermore, fractional operators $\mathcal{F}(p)$, as key tools that describe the inherent fractional-order regulation of the self-congruent structure in the overall response of the system $\sigma(t) = \mathcal{F}(p)\epsilon(t)$ [13], are the nonlinear function of the differential operator $p = \mathrm{d}/\mathrm{d}t$, where p is the Heaviside differential operator restricted to act only on functions of zero differential initial values [13,15,16].

However, this has resulted in the existing fractional operator theory being inapplicable to practical non-zero differential initial values systems such as biomechanics of bone [18,19], fractional-order control systems [20,21] and signal recognition [22,23]. Even they cannot be addressed effectively by existing traditional fractional-order methods [5–7].

Thus, converting functions with non-zero differential initial values to zero differential initial values becomes a critical bottleneck for applying fractional operator theory in the self-congruent physical space to practical systems.

Existing research indicates that Podlubny [24], Angstmann [5], and Bai [25] employed a Taylor series decomposition algorithm to solve non-zero initial value problems of fractional-order differential equations. However, they did not refine this method nor analyze its intrinsic connotation. Moreover, they only examined the application of this method in fractional analysis, without addressing the implications of the method for fractional operators in the self-congruent physical space.

Therefore, this paper refines a non-zero differential initial values zeroing operator, which is intended to resolve the application limitations of the zeroing assumption in the theory of fractional operators. To address the aforementioned challenges, this paper is structured as follows: Section 2 elaborates on the differential initial values zeroing method, including the zeroing decomposition of univariate and multivariate continuous functions. Moreover, the uniqueness of decomposition, smoothness of the zeroing function, and its consistency with jet theory are rigorously derived. Section 3 defines the differential initial values zeroing operator (DIVZO) and zeroing differential operator, systematically analyzes their properties and operational rules, and explores their compatibility with unilateral Laplace transforms. Section 4 reconstructs fractional constitutive equations using DIVZO both in the time domain and complex Laplace domain with a specific focus on modeling the complex fractional constitutive relations of bone. Section 5 discusses the core value of the proposed method, identifies its current limitations, and outlines future research directions. Finally, Section 6 summarizes the key findings and contributions of this work.

2. Differential Initial Values Zeroing Method

2.1. Univariate Continuous Function Zeroing Decomposition

Let $f \in C^k([0,T])$ (k-times continuously differentiable and k is finite) has non-zero differential initial values set at $t_0 = 0$:

$$\mathcal{V} = \left\{ f(0) = a_0, f'(0) = a_1, \ldots, f^{(k)}(0) = a_k \right\}, \quad (a_i \neq 0). \tag{1}$$

The target decomposition is:

$$f(t) = \mathcal{I}(t) + \mathcal{N}(t). \tag{2}$$

(1). Compensation function $\mathcal{I}(t)$:
$\mathcal{I}(t)$ is a k-th order Taylor polynomial precisely matching $\mathcal{V}$ [25]:

$$\mathcal{I}(t) = \sum_{i=0}^{k} \frac{f^{(i)}(0)}{i!} t^i. \tag{3}$$

(2). Zeroing function $\mathcal{N}(t)$:

$$\mathcal{N}(t) = f(t) - \mathcal{I}(t), \tag{4}$$

which satisfies the zeroing property of differential initial values:

$$\mathcal{N}(0) = \mathcal{N}'(0) = \cdots = \mathcal{N}^{(k)}(0) = 0. \tag{5}$$

Note that almost everywhere, zeroing function is characterized, calling for the objective function $f(t)$.

2.2. Multivariate Continuous Function (Manifold Case) Zeroing Decomposition

Let n-variable function $f \in C^k(\Omega)$ ($\Omega \subseteq \mathbb{R}^n$) with differential initial point $\boldsymbol{x}_0 = \mathbf{0}$ and non-zero differential initial values as mixed partial derivatives be as follows:

$$\mathcal{V}_{\boldsymbol{x}_0} = \{\partial^\alpha f(\mathbf{0}) = a_\alpha \mid |\alpha| = \alpha_1 + \cdots + \alpha_n \leq k\}, \tag{6}$$

which is consistent with the k-th order jet $j^k f(x_0)$ at point $x_0 \in M$ for a smooth function $f : M \to \mathbb{R}$ on an n-dimensional manifold M [26]:

$$j^k f(x_0) = \big(f(x_0), \partial_i f(x_0), \partial_{i_1 i_2} f(x_0), \ldots, \partial_{i_1 \cdots i_k} f(x_0)\big), \tag{7}$$

where $\partial_{i_1 \cdots i_m}$ denotes the mixed partial derivative of the m-th order with indices $i_1, \ldots, i_m \in \{1, \ldots, n\}$. This structure fully captures derivative information of order 0 to k in x_0, inspiring the extraction of differential initial values from input signals.

Then, the decomposition form is:

$$f(\boldsymbol{x}) = \mathcal{I}(\boldsymbol{x}) + \mathcal{N}(\boldsymbol{x}). \tag{8}$$

(1). Compensation function $\mathcal{I}(\boldsymbol{x})$:
A multivariate Taylor polynomial carrying jet information:

$$\mathcal{I}(\boldsymbol{x}) = \sum_{|\alpha|=1}^{k} \frac{f^{(\alpha)}(\mathbf{0})}{\alpha!} \boldsymbol{x}^\alpha, \quad (\alpha! = \alpha_1! \cdots \alpha_n!, \boldsymbol{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}); \tag{9}$$

(2). Zeroing function $\mathcal{N}(\boldsymbol{x})$:

$$\mathcal{N}(\boldsymbol{x}) = f(\boldsymbol{x}) - \mathcal{I}(\boldsymbol{x}), \tag{10}$$

satisfies $j^k \mathcal{N}(0) = 0$, that is, all partial derivatives $|\alpha| \leq k$ are zero at $x_0 = 0$:

$$\partial^\alpha \mathcal{N}(0) = 0, \quad (\forall |\alpha| \leq k). \tag{11}$$

2.3. Theoretical Derivation

(1). Uniqueness of Decomposition

For $f \in C^k([0,T])$, the decomposition $f(t) = \mathcal{I}(t) + \mathcal{N}(t)$ (Equation (2)) is unique [24]. Assume two decompositions $f = \mathcal{I}_1 + \mathcal{N}_1 = \mathcal{I}_2 + \mathcal{N}_2$, then $\mathcal{I}_1 - \mathcal{I}_2 = \mathcal{N}_2 - \mathcal{N}_1$. The left side is a polynomial of degree $\leq k$, and the right side satisfies $j^k(\mathcal{N}_2 - \mathcal{N}_1)(0) = 0$. For polynomial $P = \mathcal{I}_1 - \mathcal{I}_2$, its k-th order jet $j^k P(0) = 0$, i.e., $P(0) = P'(0) = \cdots = P^{(k)}(0) = 0$. So $P(t) \equiv 0$. Hence, $\mathcal{I}_1 = \mathcal{I}_2$ and $\mathcal{N}_1 = \mathcal{N}_2$.

(2). Smoothness of Zeroing Functions

If $f \in C^\infty([0,T])$, then $\mathcal{N} \in C^\infty([0,T])$ and $j^\infty \mathcal{N}(0) = 0$ (all-order derivatives are zero at 0). The infinite-order Taylor polynomial $\mathcal{I}(t) = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} t^m$ satisfies $j^\infty \mathcal{I}(0) = j^\infty f(0)$, so all-order derivatives of $\mathcal{N} = f - \mathcal{I}$ satisfy $\mathcal{N}^{(m)}(0) = f^{(m)}(0) - \mathcal{I}^{(m)}(0) = 0$. Since f and h are infinitely differentiable, $\mathcal{N} \in C^\infty$.

(3). Equivalence with Jet Theory

The compensation function $\mathcal{I}(t)$ is the unique k-th order polynomial satisfying $j^k \mathcal{I}(x_0) = j^k f(x_0)$, and the zeroing function $\mathcal{N}(t)$ is the unique C^k function satisfying $j^k \mathcal{N}(x_0) = 0$.

a. Jet uniqueness: By uniqueness of Taylor polynomials, $\mathcal{I}(t)$ is the unique k-th order polynomial with coefficients $j^k f(x_0)$, so $j^k \mathcal{I}(x_0) = j^k f(x_0)$ is unique.

b. Zeroing uniqueness: If $\mathcal{N}_1, \mathcal{N}_2$ both satisfy $j^k \mathcal{N}_i(x_0) = 0$ and $f = \mathcal{I} + \mathcal{N}_i$, then $\mathcal{N}_1 - \mathcal{N}_2 = 0$. Hence, $\mathcal{N}$ is unique.

(4). Derivation from Jet Theory to Decomposition Method

In local coordinates of manifold M, the fiber coordinates of the k-th order jet bundle $J^k(M,\mathbb{R})$ at x_0 are given by Equation (7). For any $f \in C^k(M)$, the jet of its k-th order Taylor polynomial $\mathcal{I}$ is $j^k \mathcal{I}(x_0) = j^k f(x_0)$, i.e., $\mathcal{I}$ is a polynomial section of $J^k(M,\mathbb{R})$ passing through $j^k f(x_0)$. Defining $\mathcal{N} = f - \mathcal{I}$ gives $j^k \mathcal{N}(x_0) = 0$, i.e., $\mathcal{N}$ is a section of the jet zero section $J_0^k(M,\mathbb{R}) = \{j^k \mathcal{N}(x_0) = 0\}$. Thus, decompositions Equations (2) and (8) are equivalent to the direct sum decomposition of jet bundles [26]:

$$J^k(M,\mathbb{R}) = \mathrm{Pol}^k(M,\mathbb{R}) \oplus J_0^k(M,\mathbb{R}), \tag{12}$$

where $\mathrm{Pol}^k(M,\mathbb{R})$ is the space of k-th order polynomial sections. This derivation shows that the proposed method is a specific implementation of jet bundle direct sum decomposition in fractional operator modeling.

3. Differential Initial Values Zeroing Operator (DIVZO)

3.1. Definition of the Zeroing Operator (DIVZO)

Based on the above differential initial value zeroing method, a differential initial value zeroing operator "$\mathcal{Z}$" can be defined as:

$$\boxed{\mathcal{Z} f(t) = f(t) - \mathcal{I}(t) = \mathcal{N}(t)}, \tag{13}$$

where, $\mathcal{I}(t) = \sum_{i=0}^{k} \frac{f^{(i)}(0)}{i!} t^i$; $\mathcal{N}(t)$ is the zeroing function.

For example, if $f(t) = t\sqrt{t} + \sin t$ (with non-zero differential initial values), the zeroing function $\mathcal{N}(t)$ (with zero differential initial values) obtained by the operator $\mathcal{Z}$ can be seen in Figure 1. It demonstrates the effect of the zeroing operator on the zero differential initial values of non-zero differential initial value functions.

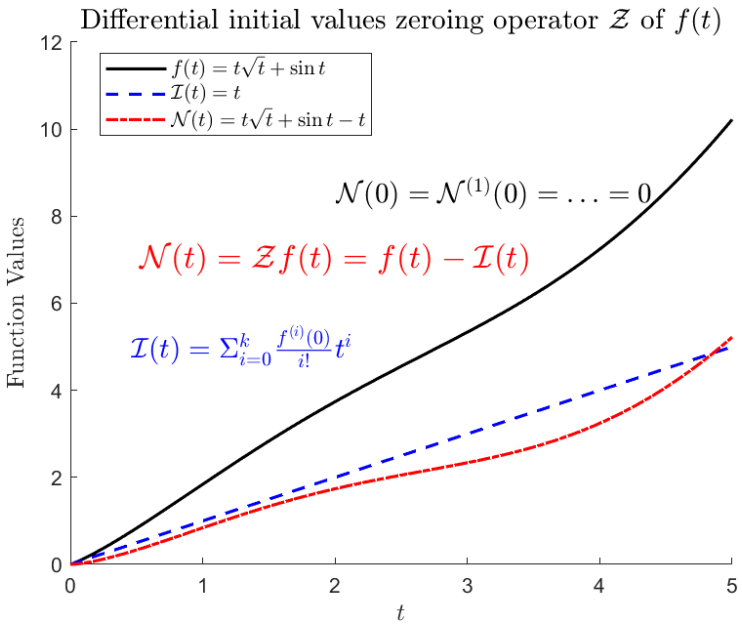


Figure 1. An example of a differential initial values zeroing operator $\mathcal{Z}$ dealing with non-zero initial values that are zeroed out as initial values. The black solid line represents the function with non-zero differential initial value, while the red thin dotted line represents the zeroing function with zero differential initial value after being processed by the zeroing operator.

3.2. Properties of the Zeroing Operator (DIVZO)

The properties of the zeroing operator $\mathcal{Z}$ (DIVZO) are as follows:

(1). Original Function Dependence

The regulatory effect of zeroing operator $\mathcal{Z}$ is dependent on the original function. On the one hand, the specific differential initial values $f^{(i)}(0)$ of the original continuous function have inevitably taken effect; on the other hand, it depends on the zeroing differential order required for addressing the problem, that is, the highest non-zero differential order k of the original function.

(2). Initial Value Annihilation

By construction of the Taylor polynomial, $\mathcal{I}^{(i)}(0) = f^{(i)}(0)$ for $i = 0, 1, \ldots, k$. Thus, $\mathcal{N}^{(i)}(0) = f^{(i)}(0) - \mathcal{I}^{(i)}(0) = 0$, for all $i = 0, 1, \ldots, k$.

This means that $\mathcal{Z}$ removes the initial values (function value and 0-th to k-th derivatives) of $f(t)$ at $t = 0$.

(3). Linearity for Functions

The zeroing operator $\mathcal{Z}$ is a linear operator, satisfying:

$$\mathcal{Z}(\alpha f + \beta g) = \alpha(\mathcal{Z}f) + \beta(\mathcal{Z}g), \tag{14}$$

for constants α, β.

3.3. Operation Rules of the Zeroing Operator (DIVZO)

The k-th order differential initial value zeroing operator removes the k-th order Taylor polynomial of $f(t)$ at $t = 0$ and redefines as:

$$\mathcal{Z}f(t) = f(t) - \mathcal{I}(t) = \mathcal{Z}_I f(t) - \mathcal{P}_k f(t), \tag{15}$$

where, $\mathcal{P}_k$ is the k-th order Taylor projection operator ($\mathcal{P}_k f(t) = \mathcal{I}(t) = \sum_{i=0}^{k} \frac{f^{(i)}(0)}{i!} t^i$) which is idempotent ($\mathcal{P}_k \circ \mathcal{P}_k = \mathcal{P}_k$), and $\mathcal{Z}_I$ denotes the identity operator ($\mathcal{Z}_I f = f$).

Thus, the zeroing operator $\mathcal{Z}$ was rewritten as:

$$\boxed{\mathcal{Z} = \mathcal{Z}_I - \mathcal{P}_k}. \tag{16}$$

Then, the operation rules for the zeroing operator $\mathcal{Z}$ are as follows:

(1) Comply with the Subtraction Decomposition Operations (Equation (16))

The operator $\mathcal{Z}$ can handle zeroing operations for all differential initial value cases from order 0 to order k (including cases where initial values are zero). However, if the exponential form ($\mathcal{N}(t) = \mathcal{Z}_{exp} f(t) = \ln \frac{f(t)}{f(0)}$) is used to obtain the differential zeroing function, it is necessary to restrict the 0th-order initial value of the original function to be non-zero and the differential initial values of all orders to be zero; that is, it is a constant function.

(2) Operator Addition ($\mathcal{Z}_1 + \mathcal{Z}_2$)

For two identical k-th order zeroing operators, there is $\mathcal{Z}_1 = \mathcal{Z}_2 = \mathcal{Z} = \mathcal{Z}_I - \mathcal{P}_k$ and

$$(\mathcal{Z}_1 + \mathcal{Z}_2)f = \mathcal{Z}_1 f + \mathcal{Z}_2 f = (f - \mathcal{P}_k f) + (f - \mathcal{P}_k f) = 2(f - \mathcal{P}_k f) = 2\mathcal{Z}f. \tag{17}$$

The sum of two identical k-th order zeroing operators is a scalar multiple (by 2) of the original operator.

(3) Operator Subtraction ($\mathcal{Z}_1 - \mathcal{Z}_2$)

For two identical k-th order zeroing operators, there is $\mathcal{Z}_1 = \mathcal{Z}_2 = \mathcal{Z}$ and

$$(\mathcal{Z}_1 - \mathcal{Z}_2)f = \mathcal{Z}_1 f - \mathcal{Z}_2 f = (f - \mathcal{P}_k f) - (f - \mathcal{P}_k f) = 0. \tag{18}$$

The difference of two identical k-th order zeroing operators is the zero operator ($\mathbf{0}$), which maps any function to the zero function.

(4) Scalar Multiplication ($a\mathcal{Z}$, $a \in \mathbb{R}$)

Multiplying $\mathcal{Z}$ by a scalar a scales its output pointwise:

$$(a\mathcal{Z})f = a \cdot (\mathcal{Z}f) = a(f - \mathcal{P}_k f) = af - a\mathcal{P}_k f, \tag{19}$$

the scalar multiplication preserves the order of $\mathcal{Z}$ and only scales the function values.

(5) Operator Multiplication ($\mathcal{Z}_1 \circ \mathcal{Z}_2$, for the same order k)

The product of two identical k-th order zeroing operators is the operator itself satisfying the commutativity and the idempotency:

$$\mathcal{Z}_1 \circ \mathcal{Z}_2 f = \mathcal{Z}_2 \circ \mathcal{Z}_1 f = \mathcal{Z} \circ \mathcal{Z} f = \mathcal{Z}f. \tag{20}$$

Compose $\mathcal{Z}_2$ first, then $\mathcal{Z}_1$: $(\mathcal{Z}_1 \circ \mathcal{Z}_2)f = \mathcal{Z}_1(\mathcal{Z}_2 f) = \mathcal{Z}_1(f - \mathcal{P}_k f)$. Since $f - \mathcal{P}_k f$ has all derivatives up to order k vanishing at $t = 0$ (by construction of $\mathcal{P}_k$), we have $\mathcal{P}_k(f - \mathcal{P}_k f) = 0$. Substituting this, there is $\mathcal{Z}_1(f - \mathcal{P}_k f) = (f - \mathcal{P}_k f) - \mathcal{P}_k(f - \mathcal{P}_k f) = f - \mathcal{P}_k f = \mathcal{Z}f$. If the sequence is exchanged, the same applies.

(6) Identity Operator ($\mathcal{Z}_I$)

The identity operator $\mathcal{Z}_I$ satisfies $\mathcal{Z}_I f = f$ for all f. For composition:

$$\mathcal{Z}_I \circ \mathcal{Z}(f) = \mathcal{Z}_I(\mathcal{Z}f) = \mathcal{Z}f, \quad \mathcal{Z} \circ \mathcal{Z}_I(f) = \mathcal{Z}(\mathcal{Z}_I f) = \mathcal{Z}f, \tag{21}$$

$\mathcal{Z}_I$ acts as the unit element, satisfying $\mathcal{Z}_I \circ \mathcal{Z} = \mathcal{Z} \circ \mathcal{Z}_I = \mathcal{Z}$.

(7) Zero Operator ($\mathbf{0}$)

The zero operator $\mathbf{0}$ satisfies $\mathbf{0}f = 0$ (the zero function) for all f. For composition:

$$\mathbf{0} \circ \mathcal{Z}(f) = \mathbf{0}(\mathcal{Z}f) = 0, \quad \mathcal{Z} \circ \mathbf{0}(f) = \mathcal{Z}(\mathbf{0}f) = \mathcal{Z}(0) = 0 - \mathcal{P}_k(0) = 0, \tag{22}$$

the $\mathbf{0}$ acts as the zero element, satisfying $\mathbf{0} \circ \mathcal{Z} = \mathcal{Z} \circ \mathbf{0} = \mathbf{0}$. Additionally, $\mathcal{Z} - \mathcal{Z} = \mathbf{0}$.

In conclusion, the operational rules for a k-th order differential initial value zeroing operator $\mathcal{Z}$:

a. Addition (same order): $\mathcal{Z}_1 + \mathcal{Z}_2 = 2\mathcal{Z}$;
b. Subtraction (same order): $\mathcal{Z}_1 - \mathcal{Z}_2 = \mathbf{0}$;
c. Scalar Multiplication: $a\mathcal{Z}$ scales outputs while preserving order;
d. Operator Multiplication (same order): $\mathcal{Z}_1 \circ \mathcal{Z}_2 = \mathcal{Z}$ (idempotent);
e. Identity Operator: $\mathcal{Z}_I \circ \mathcal{Z} = \mathcal{Z} \circ \mathcal{Z}_I = \mathcal{Z}$;
f. Zero Operator: $\mathbf{0} \circ \mathcal{Z} = \mathcal{Z} \circ \mathbf{0} = \mathbf{0}$, and $\mathcal{Z} - \mathcal{Z} = \mathbf{0}$.

Note: For operators of different orders, extend to mixed Taylor projections, but the above assumes identical orders.

3.4. Unilateral Laplace Transform Properties of the Zeroing Operator (DIVZO)

Let the unilateral Laplace transform of $f(t)$ be

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\mathrm{d}t = \hat{f}(s), \tag{23}$$

and let the initial values of $f(t)$ and its derivatives of all orders at $t = 0^+$ be $f(0^+)$, $f'(0^+)$, $\cdots$, $f^{(k-1)}(0^+)$, $f^{(k)}(0^+)$ (all non-zero). Then, the unilateral Laplace transform of the k-th derivative of $f(t)$ is:

$$\mathcal{L}\{f^{(k)}(t)\} = s^k\hat{f}(s) - s^{k-1}f(0^+) - \sum_{i=1}^{k-1} s^{k-1-i}f^{(i)}(0^+), k > 0. \tag{24}$$

(1). The unilateral Laplace transform of the compensation function $\mathcal{I}(t)$ calculated by Equation (3) is:

$$\hat{\mathcal{I}}(s) = \mathcal{L}\{\mathcal{I}(t)\} = \sum_{i=0}^{k} \frac{f^{(i)}(0)}{s^{i+1}}, \tag{25}$$

where, k represents the highest-order derivative of the function $f(t)$ that is non-zero.

Furthermore, the unilateral Laplace transform of $\mathcal{I}^{(n)}(t)$ satisfies:

$$\hat{\mathcal{I}}^{(n)}(s) = \mathcal{L}\{\mathcal{I}^{(n)}(t)\} = \sum_{i=0}^{k-n} \frac{f^{(n+i)}(0)}{s^{i+1}}, \quad (0 \le n \le k), \tag{26}$$

and the relations between $\hat{\mathcal{I}}^{(n)}(s)$ and $\hat{\mathcal{I}}(s)$ are:

$$\hat{\mathcal{I}}^{(n)}(s) = \mathcal{L}\{\mathcal{I}^{(n)}(t)\} = s^n\hat{\mathcal{I}}(s) - \sum_{i=0}^{n-1} \frac{f^{(i)}(0)}{s^{i+1-n}}, \quad (0 \le n \le k). \tag{27}$$

(2). The unilateral Laplace transform of the zeroing function ($\mathcal{N}(t) = \mathcal{Z}f(t)$) satisfies:

$$\hat{\mathcal{N}}(s) = \mathcal{L}\{\mathcal{N}(t)\} = \mathcal{L}\{f(t) - \mathcal{I}(t)\} = \hat{f}(s) - \sum_{i=0}^{k} \frac{f^{(i)}(0)}{s^{i+1}}. \tag{28}$$

Then, the unilateral Laplace transform of the n-th order derivative $\hat{\mathcal{N}}^{(n)}(t)$ is:

$$\boxed{\hat{\mathcal{N}}^{(n)}(s) = \mathcal{L}\{\mathcal{N}^{(n)}(t)\} = s^n\hat{\mathcal{N}}(s), \quad (0 \le n \le k)}, \tag{29}$$

which exhibits extremely beautiful symmetry and form invariance. Compared with the Equation (24), it indicates that the unilateral Laplace transform of the $\mathcal{N}^{(n)}(t)$ (the n-th

derivative of $\mathcal{N}(t)$) can completely eliminate the interference terms caused by the initial values of the function's differential.

Further, if we extend Equation (29) to arbitrary real number order ($\alpha \in \mathbb{R}$) [27,28], then,

$$\hat{\mathcal{N}}^{(\alpha)}(s) = \mathcal{L}\{\mathcal{N}^{(\alpha)}(t)\} = s^{\alpha}\hat{\mathcal{N}}(s), \quad (0 \le \alpha \le k, \alpha \in \mathbb{R}), \tag{30}$$

where the derivatives of $\mathcal{N}(t)$ of arbitrary real number order are all processed to be zero after the initial value zeroing:

$$\mathcal{N}^{(\alpha)}(0) = 0, \quad (0 \le \alpha \le k, \alpha \in \mathbb{R}). \tag{31}$$

(3). The unilateral Laplace transform of zeroing operator $\mathcal{Z}$

Let $\mathcal{L}\{\mathcal{Z}f(t)\} = \mathcal{L}\{\mathcal{N}(t)\} = \hat{\mathcal{N}}(s) = \hat{\mathcal{Z}}(s)\hat{f}(s)$, combine with Equations (16) and (28), the unilateral Laplace transform of zeroing operator $\mathcal{Z}$ satisfies:

$$\boxed{\hat{\mathcal{Z}}(s) = \hat{\mathcal{Z}}_I(s) - \hat{\mathcal{P}}_k(s) = 1 - \frac{\sum_{i=0}^{k} \frac{f^{(i)}(0)}{s^{i+1}}}{\hat{f}(s)}}. \tag{32}$$

3.5. Compatibility with Mikusinski Calculus Operators

In Mikusinski calculus operator theory [29], the differential operator p_M satisfies the core theorem: For a function $a = \{a(t)\}$ with continuous derivative $a' = \{a'(t)\}$ on $0 \le t < \infty$, there is:

$$p_M a = a' + a(0), \tag{33}$$

where $a(0)$ is the differential initial value of the function, revealing the relationship between differential operator action, function derivative, and differential initial value [29]. However, the differential initial value term $a(0)$ causes additional interference in operator operations, particularly disrupting the analytical structure properties of the fractional operators [29].

Using the proposed differential initial value zeroing method, the function $a(t)$ can be transformed into the zero initial value function ($\mathcal{N}(t) = \mathcal{Z}a(t)$). Substituting it into Mikusinski's operator formula (33), for zeroing function $\mathcal{N}(t)$, there is:

$$p_{\mathcal{N}}\mathcal{N}(t) = p_{\mathcal{N}}\mathcal{Z}a(t) = \mathcal{N}'(t), p_{\mathcal{N}}^2\mathcal{N}(t) = \mathcal{N}''(t), \ldots, p_{\mathcal{N}}^n\mathcal{N}(t) = \mathcal{N}^{(n)}(t), \tag{34}$$

where, $p_{\mathcal{N}}^n$ is defined as the power operator of the $p_{\mathcal{N}}$ known as Heaviside time differential operator p (namely, $p = p_{\mathcal{N}} = \mathrm{d}/\mathrm{d}t$) [14] when the Mikusinski's operator p_M acts on the zero differential initial value function $\mathcal{N}(t)$; $n = 1, 2, \ldots, \infty$.

That is, the differential initial value term is completely eliminated due to $\mathcal{N}(0) = \mathcal{N}'(0) = \mathcal{N}''(0) = \cdots = 0$, simplifying the action of differential operator $p_{\mathcal{N}}^n$ on the zeroing function to relate only to the derivative $\mathcal{N}^{(n)}$.

According to Equation (34), the zeroing differential operator "$p_{\mathcal{Z}}$" acting on the function $f(t)$ with non-zero differential initial values is defined as:

$$\boxed{p_{\mathcal{Z}} = p \cdot \mathcal{Z}}, \tag{35}$$

where, p is the Heaviside time differential operator [14]; $\mathcal{Z} = \mathcal{Z}_I - \mathcal{P}_k$ (Equation (16)). This zeroing differential operator $p_{\mathcal{Z}}$ is an eigen-extracting operator that strips off the influence of the differential initial value part and is used to explore the physical intrinsic properties. It can also be named the intrinsic differential operator. Its operation follows:

$$\boxed{p_{\mathcal{Z}}f(t) = p \cdot \mathcal{Z}f(t) = \mathcal{N}_f'(t) = \frac{\mathrm{d}\mathcal{N}_f(t)}{\mathrm{d}t}}, \tag{36}$$

where $\mathcal{N}_f(t)$ is the zeroing function of $f(t)$.

Then, define the n-th power of $p_{\mathcal{Z}}$ as $p_{\mathcal{Z}}^n$:

$$\boxed{p_{\mathcal{Z}}^2 = p \cdot p \cdot \mathcal{Z}, \quad p_{\mathcal{Z}}^3 = p^3 \cdot \mathcal{Z}, \ldots, \quad p_{\mathcal{Z}}^n = p^n \cdot \mathcal{Z}}, \tag{37}$$

where n is positive integer $(n = 1, 2, \ldots, \infty)$.

Further, define the arbitrary real number α-th power of $p_{\mathcal{Z}}$ as $p_{\mathcal{Z}}^{\alpha}$:

$$\boxed{p_{\mathcal{Z}}^{\alpha} = p^{\alpha} \cdot \mathcal{Z}, \quad \alpha \in \mathbb{R}}. \tag{38}$$

The properties of the zeroing differential operator "$p_{\mathcal{Z}}$":

(1). It can act directly on the function $f(t)$ with non-zero differential initial values and be directly applied to existing fractional complex circuits [30,31] and biomechanical fractional operators [13,15,16], thereby expanding the application scope of the existing fractional circuit theory and biomechanical fractional operators.

(2). It also represents the eigen-differential operation of the original function $f(t)$ that is independent of differential initial values.

Then, taking the unilateral Laplace transform on both sides of Equation (35) and combining the Equation (32), the Laplace domain expression $\hat{p}_{\mathcal{Z}}(s)$ of the zeroing differential operator $p_{\mathcal{Z}}$ is:

$$\boxed{\hat{p}_{\mathcal{Z}}(s) = \hat{p}(s)\hat{\mathcal{Z}}(s) = s - \frac{\sum_{i=0}^{k} \frac{f^{(i)}(0)}{s^i}}{\hat{f}(s)}}. \tag{39}$$

Furthermore, taking the unilateral Laplace transform on both sides of Equation (34), there is:

$$\mathcal{L}\{p^n\}\hat{\mathcal{N}}(s) = \mathcal{L}\{p^n \mathcal{N}(t)\} = \mathcal{L}\{\mathcal{N}^{(n)}(t)\} = s^n \hat{\mathcal{N}}(s). \tag{40}$$

Then,

$$[\mathcal{L}\{p^n\} - s^n]\hat{\mathcal{N}}(s) = [\mathcal{L}\{p^n\} - s^n]\hat{\mathcal{Z}}(s)\hat{f}(s) = 0. \tag{41}$$

Therefore, the Laplace transform of the n-th power of the differential operator p^n is as follows:

$$\boxed{\mathcal{L}\{p^n\} = s^n, \quad (n = 1, 2, \ldots, \infty, \text{when} \quad \hat{\mathcal{N}}(s) = \hat{\mathcal{Z}}(s)\hat{f}(s) \neq 0)}. \tag{42}$$

Further, if Equation (42) is extended to arbitrary real number order $n = \alpha \in \mathbb{R}$, then

$$\mathcal{L}\{p^{\alpha}\} = s^{\alpha}, \quad (\alpha \in \mathbb{R}, \text{when} \quad \hat{\mathcal{N}}(s) = \hat{\mathcal{Z}}(s)\hat{f}(s) \neq 0), \tag{43}$$

where, p^{α} is an arbitrary real number-order differential operator; s represents the complex variable in the Laplace domain.

Moreover, the unilateral Laplace transform $\hat{p}_{\mathcal{Z}}^{\alpha}(s)$ of the α-th power of the zeroing differential operator is:

$$\boxed{\hat{p}_{\mathcal{Z}}^{\alpha}(s) = \hat{p}^{\alpha}(s)\hat{\mathcal{Z}}(s) = s^{\alpha} - \frac{\sum_{i=0}^{k} \frac{f^{(i)}(0)}{s^{i+1-\alpha}}}{\hat{f}(s)}}, \tag{44}$$

which is equivalent to the unilateral Laplace transform of the α-th order derivative of the function $f(t)$ with non-zero differential initial values (denoted as $\mathcal{L}\{f^{(\alpha)}(t)\}$) divided by $\hat{f}(s)$.

The zeroing operator $\mathcal{Z}$ renders the differential operator p free from interference by initial value terms. Meanwhile, it can be found that the differential operator p can be composed with the zeroing operator $\mathcal{Z}$ (Equations (34) and (41)), providing a standardized tool for operator cascading and composition in fractional mechanics and fractional calculus [32].

In short, it is only necessary to replace the differential operator p (or the corresponding Laplacian variable s) in existing fractional circuit theory [30,31], biomechanical fractional operator theory [13,15,16], and so on (all derived under the assumption that the initial value is zero) with the zeroing differential operator $p_{\mathcal{Z}}$ (or $\hat{p}_{\mathcal{Z}}(s)$). And all these theories will be valid for problems with non-zero differential initial values.

4. Reconstructing Fractional Constitutive Equations by Zeroing Operator

4.1. Time-Domain Fractional Constitutive Relations Reconstructed by the Differential Initial Value Zeroing Operator

In the fields of viscoelastic mechanics, biomechanics, and hemodynamics [13,14,16], the existing fractional constitutive research paradigm in the self-congruent physical space is as follows.

Due to the self-equivalence of structures, the fractional constitutive model must satisfy the following relationship, that is the output response function $\sigma(t)$ is obtained directly through the action of fractional operator $\mathcal{F}(p_{\mathcal{N}})$ on input function $\epsilon(t)$:

$$\sigma(t) = \mathcal{F}(p_{\mathcal{N}})\epsilon(t). \tag{45}$$

Note that if $\epsilon(t)$ is the zero differential initial value function $\epsilon_{\mathcal{N}}(t)$, that is, $(\epsilon(t) = \epsilon_{\mathcal{N}}(t), \epsilon_{\mathcal{N}}(0) = \epsilon'_{\mathcal{N}}(0) = \cdots = 0)$, then relation (45) holds. However, if $\epsilon(t)$ is a non-zero differential initial value function, then Equation (45) does not hold.

For instance, in the fractional constitutive model of bone (whose physical elements only contain first-order differential operators $p_{\mathcal{N}}$, as shown in Figure 2a), it is required that the function acted upon by the differential operator $p_{\mathcal{N}}$ has zero initial value.

Then, $\epsilon_3(t)$ has zero initial value ($\epsilon_3(0) = 0$) (see Figure 2b). Moreover, in the fractional model, $\epsilon(t) = \epsilon_3(t)$, which means that $p_{\mathcal{N}}$ will act on $\epsilon(t)$, that is $p_{\mathcal{N}}\epsilon(t)$. Therefore, $\epsilon(t)$ must also be a function with zero initial value ($\epsilon(0) = 0$).

Then, the Equation (45) should be strictly rewritten as:

$$\sigma(t) = \mathcal{F}(p_{\mathcal{N}})\epsilon_{\mathcal{N}}(t), \quad (\epsilon_{\mathcal{N}}(t) = \epsilon(t), \epsilon_{\mathcal{N}}(0) = \epsilon'_{\mathcal{N}}(0) = \cdots = 0), \tag{46}$$

where $\epsilon_{\mathcal{N}}(t)$ is the function with zero differential initial value (see Figure 2b). If the physical elements have multi-order differential ($p^i_{\mathcal{N}}$) properties, $\epsilon_{\mathcal{N}}(t)$ must have the zero-differential initial values with the corresponding order.

In conclusion, the existing fractional constitutive models [13–15] are not applicable to systems with non-zero differential initial input signals.

However, for many physical scenarios where the initial strain ($\epsilon^{(i)}(0) \neq 0, i = 0, 1, 2, \ldots, k$) is not zero [5–7], this constitutive model (Equation (46)) fails.

Fortunately, the differential initial value zeroing method proposed in this paper can extract the zero initial value part $\epsilon_{\mathcal{N}}(t), (\epsilon_{\mathcal{N}}(t) = \epsilon(t) - \epsilon_{\mathcal{I}}(t), \epsilon^{(i)}_{\mathcal{N}}(0) = 0)$ from the function $\epsilon(t), (\epsilon^{(i)}(0) \neq 0)$ with non-zero differential initial values. If the signal is still regulated by the self-equivalence, the stress response can still be given by Equation (46):

$$\sigma(t) = \mathcal{F}(p_{\mathcal{N}})[\epsilon(t) - \epsilon_{\mathcal{I}}(t))], \tag{47}$$

where, $\epsilon_{\mathcal{I}}(t) = \sum_{i=0}^{k} \frac{\epsilon^{(i)}(0)}{i!} t^i; \epsilon^{(i)}(0) \neq 0, (i = 0, 1, 2, \ldots, k)$.

Note that, $\mathcal{F}(p_{\mathcal{N}})$ is no longer a linear operator because $\mathcal{F}(p_{\mathcal{N}})[\epsilon(t) - \epsilon_{\mathcal{I}}(t))]$ does not satisfy the distributive law of multiplication.

Here, a definition is proposed: The differential initial value zeroing operation of the derivative of a function is regarded as an operator "$\mathcal{Z}$", which represents:

$$\epsilon_{\mathcal{N}}(t) = \mathcal{Z}\epsilon(t) = \epsilon(t) - \epsilon_{\mathcal{I}}(t), \tag{48}$$

where $\epsilon_{\mathcal{I}}(t) = \sum_{i=0}^{k} \frac{\epsilon^{(i)}(0)}{i!} t^i$. This is a filtering process for the initial values of the signal derivative.

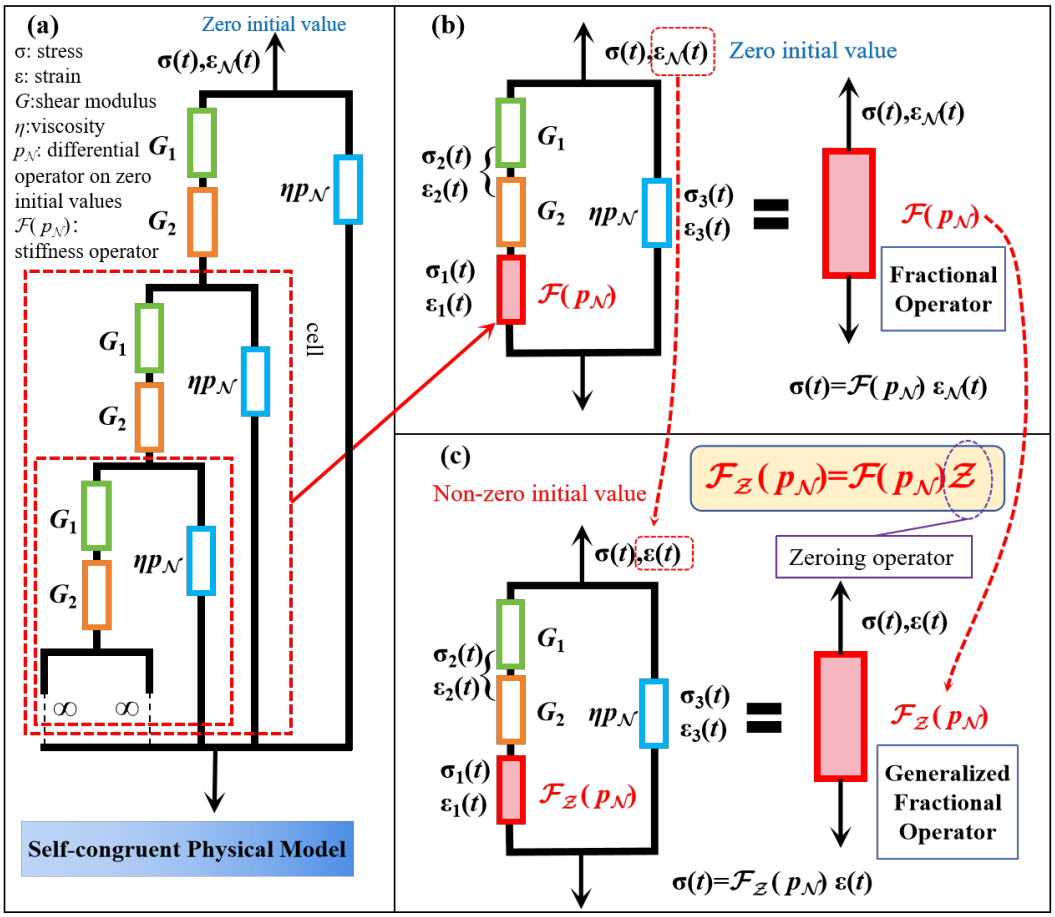


Figure 2. The time-domain fractional constitutive model of bone tissue. (**a**) The self-congruent physical constitutive model. It is derived from the physical phenomena including real biological fibers, arterial hemodynamics, and biomechanics. (**b**) Time-domain fractional equivalent constitutive model with zero initial values. Through self-equivalence, fractional operators can represent the overall physical constitutive relationship. (**c**) A time-domain generalized fractional equivalent constitutive model for non-zero initial value problems under the zeroing operator.

Then, Equation (47) can be written as:

$$\boxed{\sigma(t) = \mathcal{F}(p_{\mathcal{N}})\mathcal{Z}\epsilon(t) = \mathcal{F}_{\mathcal{Z}}(p_{\mathcal{N}})\epsilon(t)}, \tag{49}$$

where $\mathcal{F}_{\mathcal{Z}}(p_{\mathcal{N}})$ is called the generalized fractional operator, which satisfies:

$$\boxed{\mathcal{F}_{\mathcal{Z}}(p_{\mathcal{N}}) = \mathcal{F}(p_{\mathcal{N}})\mathcal{Z}}, \tag{50}$$

as shown in Figure 2c.

The remaining task is to solve $\mathcal{F}_{\mathcal{Z}}(p_{\mathcal{N}})$, which follows the existing fractional constitutive model processing method.

However, the emergence of $\mathcal{Z}$ has affected the specific calculations of the system's response. Furthermore, the specific operation of operator $\mathcal{F}(p_{\mathcal{N}})$ on functions depends on the properties of $\mathcal{F}(p_{\mathcal{N}})$ itself. One approach to obtain the stress response is through convolution $K(t) * \epsilon_{\mathcal{N}}(t)$, though the analytical form of $K(t)$ is generally difficult to determine. Usually, $K(t)$ needs to be solved by means of the Laplace transform [33].

4.2. Complex Fractional Constitutive Relations Reconstructed by the Differential Initial Values Zeroing Operator

Based on the analysis in the previous section, it may be advisable to express the fractional constitutive equation in the Laplace complex variable domain directly.

Using the convolution theorem [34]:

$$\mathcal{L}[f(t) * \mathcal{N}(t)] = \hat{f}(s) \cdot \hat{\mathcal{N}}(s), \tag{51}$$

the complex domain expression of fractional operators $\mathcal{F}(p_{\mathcal{N}})$ (which is the operator function of $p_{\mathcal{N}}$) is defined as the complex fractional operators $\hat{\mathcal{F}}(s)$, which satisfies:

$$\boxed{\hat{\mathcal{F}}(s)\hat{\mathcal{N}}(s) = \mathcal{L}\{\mathcal{F}(p_{\mathcal{N}})\mathcal{N}(t)\} = \mathcal{L}\{K(t) * \mathcal{N}(t)\} = \hat{K}(s)\hat{\mathcal{N}}(s)}, \tag{52}$$

where $\mathcal{F}(p_{\mathcal{N}})\mathcal{N}(t) = K(t) * \mathcal{N}(t) = \int_0^t K(t-\tau)\mathcal{N}(\tau)\mathrm{d}\tau$ [35].

Then,

$$\hat{\mathcal{F}}(s) = \hat{K}(s), \quad (\hat{\mathcal{N}}(s) \neq 0). \tag{53}$$

The kernel function of fractional operator $\hat{\mathcal{F}}(s)$ is:

$$K(t) = \mathcal{L}^{-1}\{\hat{K}(s)\} = \mathcal{L}^{-1}\{\hat{\mathcal{F}}(s)\}, \quad (\hat{\mathcal{N}}(s) \neq 0). \tag{54}$$

Therefore, using Laplace transforms and the complex fractional operators Equation (52), the fractional constitutive Equation (46) is reconstructed as a complex fractional constitutive equation:

$$\boxed{\hat{\sigma}(s) = \hat{\mathcal{F}}(s)\hat{\epsilon}_{\mathcal{N}}(s)}, \tag{55}$$

where, $\hat{\sigma}(s)$ is the unilateral Laplace transform of $\sigma(t)$, $\hat{\sigma}(s) = \int_0^{\infty} \sigma(t)e^{-st}\mathrm{d}t$; $\hat{\epsilon}_{\mathcal{N}}(s)$ is the unilateral Laplace transform ($\hat{\epsilon}_{\mathcal{N}}(s) = \int_0^{\infty} \epsilon_{\mathcal{N}}(t)e^{-st}\mathrm{d}t$) of ($\epsilon_{\mathcal{N}}(t) = \epsilon(t) - \epsilon_{\mathcal{I}}(t)$), which satisfies ($\epsilon_{\mathcal{N}}^{(i)}(0) = 0, \quad (i = 0, 1, 2, \ldots, k)$); $\epsilon_{\mathcal{I}}(t)$ is known that $\epsilon_{\mathcal{I}}(t) = \sum_{i=0}^{k} \frac{\epsilon^{(i)}(0)}{i!} t^i$;$\hat{\mathcal{F}}(s)$ is called the complex fractional operator (see Figure 3a).

By introducing the zeroing operator $\hat{\mathcal{Z}}(s)$, there is:

$$\boxed{\hat{\sigma}(s) = \hat{\mathcal{F}}(s)\hat{\mathcal{Z}}(s)\hat{\epsilon}(s) = \hat{\mathcal{F}}_{\mathcal{Z}}(s)\hat{\epsilon}(s)}, \tag{56}$$

where $\hat{\mathcal{F}}_{\mathcal{Z}}(s)$ is defined as the complex generalized fractional operator:

$$\boxed{\hat{\mathcal{F}}_{\mathcal{Z}}(s) = \hat{\mathcal{F}}(s)\hat{\mathcal{Z}}(s)}, \tag{57}$$

which can be seen in Figure 3b.

The complex generalized fractional operator $\hat{\mathcal{F}}_{\mathcal{Z}}(s)$ has the following properties:

(1) Commutativity. Although $\mathcal{F}(p_{\mathcal{N}})\mathcal{Z}$ does not possess commutativity because $\mathcal{Z}$ must act on the zeroing function in the time domain, $\hat{\mathcal{F}}_{\mathcal{Z}}(s)$ possess commutativity in the Laplace complex variable domain, that is, $\hat{\mathcal{F}}_{\mathcal{Z}}(s) = \hat{\mathcal{F}}(s)\hat{\mathcal{Z}}(s) = \hat{\mathcal{Z}}(s)\hat{\mathcal{F}}(s)$.

(2) Separability. $\hat{\mathcal{F}}_{\mathcal{Z}}(s)$ neatly separates the operator $\hat{\mathcal{F}}(s)$, which expresses the intrinsic property of the fractional structure, from the operator $\hat{\mathcal{Z}}(s)$, which expresses the influence of the initial value of the input signal on the response.

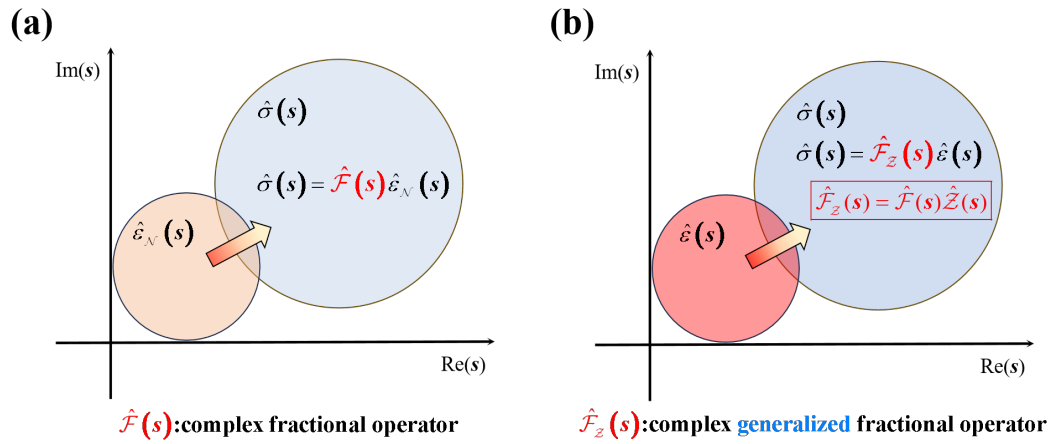


Figure 3. Complex stiffness operator and complex generalized fractional stiffness operator. (**a**) The complex operator acts on input function of zero-differential initial strain; (**b**) The complex generalized fractional operator can be applied to general input strain functions with non-zero differential initial values.

4.3. Complex Fractional Constitutive Relations of Bone with the Zeroing Operator

To characterize bone's viscoelastic-fractional behavior, a series-parallel network of four operators can be used the same as in the time-domain [15]. Further, transform them into the Laplace complex variable domain, as shown in Figure 4.

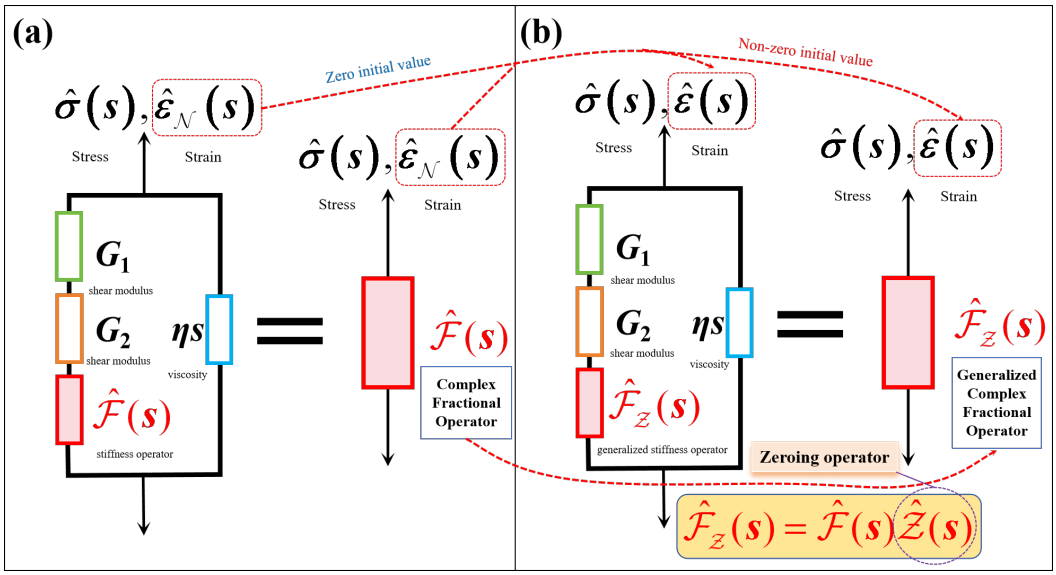


Figure 4. The complex fractional constitutive model of bone tissue. (**a**) The complex fractional constitutive model based on the zero-differential initial value strain function; (**b**) The complex generalized fractional constitutive model based on the non-zero differential initial value for the general input strain function.

Therefore, the total equivalent complex stiffness $\hat{\mathcal{F}}_{\text{total}}$ is:

$$\hat{\mathcal{F}}_{\text{total}} = \hat{\mathcal{F}}(s) = \frac{G_1 G_2 \hat{\mathcal{F}}(s)}{G_1 G_2 + \hat{\mathcal{F}}(s)(G_1 + G_2)} + \eta s. \tag{58}$$

Then, when $\hat{\epsilon}_{\mathcal{N}}(s) = \hat{\mathcal{Z}}(s)\hat{\epsilon}(s) \neq 0$, the complex fractional operator of bone is:

$$\boxed{\hat{\mathcal{F}}(s) = \frac{\eta\left(s \pm \sqrt{s^2 + 4G_0 s}\right)}{2}}, \tag{59}$$

where $G_0 = \frac{G_1 G_2}{\eta(G_1+G_2)}$.

Meanwhile, the total stress is:

$$\hat{\sigma}(s) = T_{\text{total}}\hat{\epsilon}_{\mathcal{N}}(s) = \hat{\mathcal{F}}(s)\hat{\epsilon}_{\mathcal{N}}(s) = \hat{\mathcal{F}}(s)\hat{\mathcal{Z}}(s)\hat{\epsilon}(s) = \hat{\mathcal{F}}_{\mathcal{Z}}(s)\hat{\epsilon}(s). \tag{60}$$

Combine Equations (32), (59), and (60), the complex generalized fractional operator of bone is:

$$\boxed{\hat{\mathcal{F}}_{\mathcal{Z}}(s) = \hat{\mathcal{F}}(s)\hat{\mathcal{Z}}(s) = \frac{\eta\left(s \pm \sqrt{s^2 + 4G_0 s}\right)}{2}\left[1 - \frac{\sum_{i=0}^{k} \frac{\epsilon^{(i)}(0)}{s^{i+1}}}{\hat{\epsilon}(s)}\right]}. \tag{61}$$

Therefore, the complex fractional operator and the complex generalized fractional operator possess the following characteristics:

(1). For complex fractional operator $\hat{\mathcal{F}}(s)$

It found that this result is formally consistent with the theory proposed by Jian et al. However, s is a complex variable here, while Jian's theory [15] uses the differential operator p. The above analysis confirms their equivalence after zeroing differential initial values from the input function:

$$\boxed{s \leftrightarrow p}, \tag{62}$$

which exact formula is given in Equation (42).

Furthermore, the complex fractional operator Equation (59) obtained via differential initial value zeroing directly and analytically demonstrates bone's fractional regulation mechanism.

Moreover, the term $\sqrt{s^2 + 4G_0 s}$ introduces a fractional (non-integer order) derivative effect, reflecting bone's hierarchical structure.

- For $s \gg 4G_0$ (high-frequency limit), $\sqrt{s^2 + 4G_0 s} \approx s + 2G_0$, so $\hat{\mathcal{F}}(s) \approx \frac{\eta}{2}(s \pm (s + 2G_0))$, recovering elastic/viscous behavior.
- For $s \ll 4G_0$ (low-frequency limit), $\sqrt{s^2 + 4G_0 s} \approx 2\sqrt{G_0 s}$, revealing a square-root (fractional order 1/2) dependency [27,28], characteristic of power-law relaxation in self-equivalence media. This simplification clarifies the operator's frequency-dependent behavior and its connection to bone's multi-scale mechanics.

Thus, $\hat{\mathcal{F}}(s)$ corresponds to a fractional convolution (reflecting non-local, power-law behavior). For a fractional order α, $\hat{\mathcal{F}}(s) \propto s^{\alpha}$ (encoding hierarchical scaling). It encodes bone's hierarchical self-equivalence structure (e.g., collagen branching [19]). It explains non-local stress propagation and power-law deformation, which traditional models miss.

If $\epsilon(t) = t\sqrt{t} + \sin t$ (with only 1st-order non-zero initial derivative), the decomposition gives $\epsilon_{\mathcal{I}}(t) = t$ and $\epsilon_{\mathcal{N}}(t) = t\sqrt{t} + \sin t - t$, where $\epsilon_{\mathcal{N}}(t) \neq 0$ and satisfies zero-initial conditions, making it applicable for solving $\hat{\mathcal{F}}(s)$.

(2). For complex generalized fractional operator $\hat{\mathcal{F}}_{\mathcal{Z}}(s)$

Based on the above examples ($\epsilon(t) = t\sqrt{t} + \sin t$), there is:

$$\hat{\epsilon}(s) = \mathcal{L}\{\epsilon(t)\} = \frac{3\sqrt{\pi}}{4s^{5/2}} + \frac{1}{s^2 + 1}. \tag{63}$$

Then, the complex zeroing operator $\hat{\mathcal{Z}}(s)$ is:

$$\boxed{\hat{\mathcal{Z}}(s) = 1 - \frac{1}{\left(\frac{3\sqrt{\pi}}{4s^{1/2}} + \frac{s^2}{s^2+1}\right)}}. \tag{64}$$

Then, the complex generalized fractional operator of bone $\hat{\mathcal{F}}_{\mathcal{Z}}(s)$ can be obtained:

$$\hat{\mathcal{F}}_{\mathcal{Z}}(s) = \hat{\mathcal{F}}(s)\hat{\mathcal{Z}}(s) = \frac{\eta\left(s \pm \sqrt{s^2 + 4G_0 s}\right)}{2}\left[1 - \frac{1}{\left(\frac{3\sqrt{\pi}}{4s^{1/2}} + \frac{s^2}{s^2+1}\right)}\right], \tag{65}$$

which represents the total regulatory effect operator of self-equivalence bone tissue when there is input strain with non-zero differential initial values, and it can be given by a specific analytical expression which is not realizable in the time domain.

Thus, the complex generalized fractional operator provides a new perspective for the analysis of bone mechanics regulation in bioengineering.

Of course, in addition to bone mechanics, other biomechanical factors (such as biological ligament fibers [14], hemodynamics [16], etc.) can also be described by using the zero-operator combined with the complex generalized fractional operator to represent the system response. Due to the length of the article, this will not be discussed here.

5. Discussion

5.1. Core Value

The differential initial values zeroing method and its associated zeroing operator (DIVZO) offer threefold core value:

First, they break the long-standing bottleneck of the zero-initial-value assumption in fractional operator theory, enabling the application of fractional operators to practical systems with non-zero differential initial values (e.g., biomechanics, fractional-order mechanics).

Secondly, the method is theoretically rigorous: Based on Taylor polynomial decomposition and manifold jet theory, it guarantees the uniqueness of decomposition, smoothness of the zeroing function, and consistency with jet bundle structures, providing a solid mathematical foundation.

Thirdly, its compatibility with Laplace transforms and Mikusinski operators simplifies the analysis of fractional-order derivatives and complex system responses. Validated by the complex fractional constitutive model of bone mechanics, this method successfully reconstructs the intrinsic fractional stiffness operator regulating bone stress, offering a standardized tool for modeling complex fractional operators across disciplines, and establishing a new analytical framework for non-zero differential initial value problems in sophisticated systems.

5.2. Limitations and Future Work

Despite its merits, the proposed method has limitations. Currently, it is primarily applicable to functions with finite-order continuous differentiability (C^k), and its extension to functions with lower smoothness (e.g., C^0 or non-smooth functions) remains underexplored. Additionally, the computational efficiency of DIVZO in large-scale systems (e.g., Kaplan self-similar power-law scaling impedance model [31], self-similar scaling viscoelastic biomechanical model [36], and even the fractional circuit model of the whole-body fractal structure [37] of arterial hemodynamics, etc.) needs clarification.

Future work will focus on the following: (1) Extending the method to handle functions with infinite or non-smooth differentiability, enhancing its adaptability; (2) Developing efficient numerical algorithms for DIVZO to facilitate practical engineering applications; (3) Investigating the application of zeroing method to self-similar and proportional scaling physical fractal models.

6. Conclusions

This paper tackles the inability of fractional operator theory in self-congruent physical space to handle systems with non-zero differential initial values by introducing a method to zero out differential initial values. This method decomposes functions with non-zero differential initial values into two parts: a compensation function, which is a Taylor polynomial, and a zeroing function where all differential initial values are zero. A specific zeroing operator for stripping differential initial values is defined, with properties such as eliminating initial values and linearity, and its operational rules are compatible with unilateral Laplace transforms and Mikusinski calculus operators. Compared with the fractional model with zero initial value assumption, by applying this zeroing operator, the zeroing differential operator has been defined and fractional constitutive equations are re-established both in the time domain and the complex Laplace domain, leading to the introduction of complex fractional operators and complex generalized fractional operators. Verified through the complex fractional constitutive model of bone, this approach overcomes the long-standing limitation of relying on the zero-initial-value assumption in fractional operator theory. It establishes a new analytical framework for problems involving non-zero differential initial values and provides a standardized tool for modeling complicated fractional systems across various fields, for instance, biological fiber mechanics and hemodynamics.

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Abbreviations

The following abbreviations are used in this manuscript:

DIVZ	Differential Initial Values Zeroing
DIVZO	Differential Initial Values Zeroing Operator

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