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Sandwich-Type Results and Existence Results of Analytic Functions Associated with the Fractional q-Calculus Operator

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Abstract: In the present investigation, we present certain subordination and superordination results for the q-integral operator of a fractional order associated with analytic functions in the open unit disk **U**. Using this q-integral operator, we obtain sandwich-type results. Furthermore, we employ the existence of univalent solutions to a q-differential equation connected with a fractional q-integral operator of fractional order. We use these results to demonstrate the significance of our findings for particular functions. We also derive some examples and corollaries that are pertinent to our results.

Keywords: analytic functions; fractional q-calculus operators; differential subordination; differential superordination; sandwich-type results; univalent solution

MSC: 30C45; 30C80

1. Introduction and Preliminaries

Many areas of science rely heavily on fundamental and fractional quantum calculus, also known as basic q-calculus and fractional q-calculus. We focus on the fascinating relations between geometric function theory and quantum calculus, which has attracted many scientific applications due to calculus, except the approach of limits. In 1783, Euler introduced the concept of quantum calculus for the pentagonal number theorem. In the recent century, it was further developed by Jackson [1] for the q-approach of differentiation and definite integrals. Nowadays, this advanced mathematical tool is essential due to its high accuracy and remarkably accurate results of complex geometric functions. Normally, fractional q-calculus behaves as a q-extension of ordinary fractional calculus. In prominent areas, like optimal control problems, engineering, statistics, combinatorics, economics, number theory, solutions of the q-differential and q-integral equations [2], physics (quantum mechanics, electronics, etc. see [3]), quantum calculus is used. For historical narratives of q-calculus, see the cluster of paper [4] and book [5]. Srivastava [6] presented a thorough examination of advancements in the domain of q-calculus in their review article. In 2011, Purohit and Raina [7] studied fractional q-calculus operators for specific subclasses of analytic functions. Many researchers studied analytic and multivalent functions associated with fractional q-calculus operators [8–10] (see also [11]), for convexity and majorization properties of analytic functions with q-integral operators [12,13]. In [14,15], the authors studied the existence and uniqueness of solutions to q-fractional difference equations.

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Mainly explore the realm of complex analysis, known as differential subordination. Historically, it can be traced back to S. S. Miller and P. T. Mocanu in 1978, which generalized the differential inequality on the real line. In this domain, researchers study geometric function theory and various classes of analytic functions. In 2003 [16], the authors examined the dual concept of subordination, i.e., superordination. However, it began as a mathematical tool in geometric function theory for many applications. Researchers can refer to the book for more information in this realm [17]. Moreover, the studies referenced in [18] highlight the initial progression of this realm. Various authors studied the sandwich-type findings for the class of analytic functions in [19,20]; for more information, we refer the reader to the paper [21].

The mentioned works motivated us to introduce some subordination and superordination results involving the technique of fractional q-calculus operators. This technique also allows us to find the existence of a univalent solution for the fractional q-differential equation:

$$D^{l}_{q,\varsigma}\varphi(\varsigma) = \mathfrak{f}(\varsigma, \varphi(\varsigma)), \tag{1}$$

subject to the initial condition $\varphi(0) = 0$, where $\varphi : \mathbf{U} \to \mathbb{C}$ is an analytic function for all $\varsigma \in \mathbf{U}$. Therefore, the univalent solution corresponds to a unique fractional differential equation.

Recall some requisite notations, definitions, and concept details of the q-calculus used in this paper.

We begin by denoting with $\mathcal{H} = \mathcal{H}(\mathbf{U})$ the class of functions which are analytic in the open unit disk $\mathbf{U} = \{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$. Also, let $\mathcal{A}$ denote a subclass of analytic function $\mathfrak{f}$ in $\mathcal{H}$, satisfying the following normalization condition $\mathfrak{f}(0) = \mathfrak{f}'(0) - 1 = 0$. In other words, a function $\mathfrak{f} \in \mathcal{A}$ has Taylor–Maclaurin series expansion of the following form:

$$\mathfrak{f}(\varsigma) := \varsigma + \sum_{j=2}^{\infty} a_j \varsigma^j \quad (\varsigma \in \mathbf{U}). \tag{2}$$

We next recall the principle of differential subordination between analytic functions (for details, see [17]). The function $\mathfrak{f}$ is subordinate to $\mathfrak{g}$, or equivalently, the function $\mathfrak{g}$ is superordinate to $\mathfrak{f}$, written as $\mathfrak{f} \prec \mathfrak{g}$ or $\mathfrak{f}(\varsigma) \prec \mathfrak{g}(\varsigma)$, if there exists a Schwarz function w analytic in $\mathbf{U}$ with $w(0) = 0$ and $|w(\varsigma)| < 1$ such that $\mathfrak{f}(\varsigma) = \mathfrak{g}(w(\varsigma))$. If $\mathfrak{g}$ is univalent in $\mathbf{U}$, then $\mathfrak{f}(\varsigma) \prec \mathfrak{g}(\varsigma)$ if and only if $\mathfrak{f}(0) = \mathfrak{g}(0)$ and $\mathfrak{f}(\mathbf{U}) \subset \mathfrak{g}(\mathbf{U})$.

Suppose that $\hbar, \in \mathcal{H}$. Let $\chi : \mathbb{C}^2 \to \mathbb{C}$ and (ς) be a univalent function in $\mathbf{U}$. If $\hbar$ satisfies the differential subordination, as follows:

$$\chi\big(\hbar(\varsigma), \varsigma\hbar'(\varsigma)\big) \prec (\varsigma), \tag{3}$$

then $\hbar(\varsigma)$ is called the solution of the differential subordination (3). A univalent function $\varkappa \in \mathcal{H}$ is called a dominant of the solutions of differential subordination if $\hbar(\varsigma) \prec \varkappa(\varsigma)$ for all $\hbar(\varsigma)$ satisfying (3). A dominant $\widetilde{\varkappa}$ that satisfies $\widetilde{\varkappa} \prec \varkappa$ for all the dominants $\varkappa$ of (3) is called the best dominant.

Recently, Miller and Mocanu [16] obtained conditions on , $\varkappa$, and χ, for which the following implication holds:

$$(\varsigma) \prec \chi\big(\hbar(\varsigma), \varsigma\hbar'(\varsigma)\big) \Longrightarrow \varkappa(\varsigma) \prec \hbar(\varsigma). \tag{4}$$

Using the results (4), Bulboacă [22] considered certain classes of first-order differential superordinations as well as superordination-preserving integral operators [23]. Ali et al. [24]

have used the results of Bulboacă and obtained sufficient conditions for certain normalized analytic functions $\mathfrak{f}(\varsigma)$ to satisfy the following:

$$\varkappa_1(\varsigma) \prec \frac{\varsigma \mathfrak{f}'(\varsigma)}{\mathfrak{f}(\varsigma)} \prec \varkappa_2(\varsigma),$$

where $\varkappa_1$ and $\varkappa_2$ are given univalent functions in $\mathbf{U}$ with $\varkappa_1(0) = 1$ and $\varkappa_2(0) = 1$.

We denote by $\mathbb{Q}$ the set of all functions $\mathfrak{f}$ that are analytic and injective on $\overline{\mathbf{U}} \setminus E(\mathfrak{f})$, where:

$$\mathbf{E}(\mathfrak{f}) = \left\{ \zeta \in \partial\mathbf{U} : \lim_{\varsigma \to \zeta} \mathfrak{f}(\varsigma) = \infty \right\},$$

and are such that $\mathfrak{f}'(\zeta) \neq 0$ for $\zeta \in \partial\mathbf{U} \setminus \mathbf{E}(\mathfrak{f})$ (see [17]).

Jackson ([25,26]) defined the q-derivative (or the q-difference) of a function $\mathfrak{f}(\varsigma) \in \mathcal{A}$ is denoted by: $\mathcal{D}_{q,\varsigma}\mathfrak{f}(\varsigma)$ and defined in a given subset of $\mathbb{C}$ by

$$\mathcal{D}_{q,\varsigma}\mathfrak{f}(\varsigma) := \begin{cases} \frac{\mathfrak{f}(q(\varsigma)) - \mathfrak{f}(\varsigma)}{(q-1)\varsigma} & (\varsigma \neq 0), \\ \mathfrak{f}'(0) & (\varsigma = 0). \end{cases}$$

The q-integral of a function $\mathfrak{f}(\varsigma) \in \mathcal{A}$ is defined in a given subset of $\mathbb{C}$ by:

$$\int_0^{\varsigma} \mathfrak{f}(\tau) d_q\tau = \varsigma(1-q) \sum_{k=0}^{\infty} q^k \mathfrak{f}\left(\varsigma q^k\right),$$

providing that the series converges, and:

$$\lim_{q \to 1^-} \int_0^{\varsigma} \mathfrak{f}(\tau) d_q\tau = \int_0^{\varsigma} \mathfrak{f}(\tau) d\tau,$$

provided that the series converges. The q-gamma function is defined by:

$$\Gamma_q(b) = \frac{(q;q)_\infty}{\left(q^b;q\right)_\infty} (1-q)^{1-b}, \quad for \quad b \in \mathbb{C} \setminus \{-n : n \in \mathbb{N}_0\}.$$

The $q-$shifted factorial for every $n \in \mathbb{N}, a \in \mathbb{C}$ is defined by:

$$(a;q)_\infty = \prod_{k=0}^{\infty} \left(1 - aq^k\right), \quad (a;q)_n = \frac{(a;q)_\infty}{\left(aq^n;q\right)_\infty}, \quad \text{and} \quad \left(q^a;q\right)_n = \frac{\Gamma_q(a+n)(1-q^n)}{\Gamma_q(a)}.$$

Purohit and Raina (see [27,28]) introduced the following fractional q-integral and derivative operator.

Definition 1. *Let $l > 0$, and the fractional q-integral operator $I^l_{q,\varsigma}\mathfrak{f}(\varsigma)$ of order l is defined by:*

$$I^l_{q,\varsigma}\mathfrak{f}(\varsigma) := D^{-l}_{q,\varsigma}\mathfrak{f}(\varsigma) = \frac{1}{\Gamma_q(l)} \int_0^{\varsigma} \mathfrak{f}(\tau)(\varsigma - \tau q)_{l-1} d_q\tau, \tag{5}$$

where the function $\mathfrak{f}(\varsigma)$ is analytic in a simply connected region of the complex ς-plane containing the origin. The multiplicity of q-binomial coefficient function $(\varsigma - \tau)_{l-1}$ is defined by:

$$\begin{aligned} (\varsigma - \tau q)_{l-1} &= \varsigma^{l-1} \prod_{j=0}^{\infty} \left[\frac{\left(1 - \tau q \varsigma^{-1}\right) q^j}{1 - \left(\tau q \varsigma^{-1}\right) q^{l+j-1}} \right] \\ &= \varsigma^l {}_1\Phi_0\left(q^{1-l}; -; q, \tau q^l \varsigma^{-1}\right) \end{aligned}$$

The q-hypergeometric series ${}_1\Phi_0(l;-;q,\varsigma)$ *is known to be single-valued in open unit disk when* $|arg\varsigma|<\pi$ *([29]). Therefore, the q-binomial* $(\varsigma-\tau q)_{l-1}$ *is single-valued when* $|arg\left(-\tau q^l\varsigma^{-1}\right)|<\pi$, $\left|\frac{\tau q^l}{\varsigma}\right|<1$ *and* $|arg(\varsigma)|<\pi$.

Definition 2. *Let* $0\leq l<1$, *the fractional q-integral operator* $D^l_{q,\varsigma}\mathfrak{f}(\varsigma)$ *of order l is defined by:*

$$D^l_{q,\varsigma}\mathfrak{f}(\varsigma):=D_{q,\varsigma}I^{1-l}_{q,\varsigma}\mathfrak{f}(\varsigma)=\frac{1}{\Gamma_q(1-l)}D_{q,\varsigma}\int_0^{\varsigma}\mathfrak{f}(\tau)(\varsigma-\tau q)_{-l}d_q\tau,$$

where $\mathfrak{f}(\varsigma)$ *is suitably constrained and the multiplicity of* $(\varsigma-\tau)_{-l}$ *is removed by requiring as in Definition 1.*

As we proceed, we define the function $\mathfrak{f}_{q,l}(\varsigma)$:

$$\mathfrak{f}_{q,l}(\varsigma)=\begin{cases}\frac{(\varsigma)_{l-1}}{\Gamma_q(l)}, & \varsigma>0\\ 0, & otherwise\end{cases}$$

and come to the conclusion that the integral (5) can be expressed as a convolution as follows:

$$I^l_{q,\varsigma}\mathfrak{f}(\varsigma)=\mathfrak{f}_{q,l}(\varsigma)*\mathfrak{f}(\varsigma)=\int_0^{\varsigma}\mathfrak{f}_{q,l}(\varsigma-\tau q)\mathfrak{f}(\tau)d_q\tau.$$

Now, we define:

$$I^l_{q,\varsigma}\mathfrak{f}(\varsigma)=\begin{cases}\mathfrak{f}(\varsigma)\times & \frac{(\varsigma)_{l-1}}{\Gamma_q(l)}, \quad \varsigma>0\\ & 0, \quad otherwise.\end{cases}$$

Lemma 1 ([17]). *Let* $\mathfrak{g}(\varsigma)$ *be univalent in* $\boldsymbol{U}$ *and let* Θ *and* Φ *be analytic in a domain D containing* $\mathfrak{G}(\boldsymbol{U})$, *with* $\Phi(w)\neq 0$, *when* $w\in\mathfrak{G}(\boldsymbol{U})$. *Set* $\xi(\varsigma)=\varsigma\mathfrak{G}'(\varsigma)\cdot\Phi[\mathfrak{g}(\varsigma)]$, $(\varsigma)=\Theta[\mathfrak{g}(\varsigma)]+\xi(\varsigma)$. *Suppose that:*

(1) $\xi(\varsigma)$ *is starlike univalent in* $\boldsymbol{U}$;

(2) $\Re\left(\frac{\varsigma'(\varsigma)}{\xi(\varsigma)}\right)>0$ *for* $\varsigma\in\boldsymbol{U}$.

If $\mathfrak{F}$ *is analytic in* $\boldsymbol{U}$ *and:*

$$\Theta[\mathfrak{f}(\varsigma)]+\varsigma\mathfrak{F}'(\varsigma)\cdot\Phi[\mathfrak{f}(\varsigma)]\prec\Theta[\mathfrak{g}(\varsigma)]+\varsigma\mathfrak{G}'(\varsigma)\cdot\Phi[\mathfrak{g}(\varsigma)]=(\varsigma),$$

then $\mathfrak{f}(\varsigma)\prec\mathfrak{g}(\varsigma)$, *and* $\mathfrak{G}$ *is the best dominant.*

Lemma 2 ([30]). *Let* $\mathfrak{G}$ *be a convex univalent in* $\boldsymbol{U}$ *and let* $\delta,\ \eta\in\mathbb{C}$ *with* $\Re\left(1+\frac{\varsigma\mathfrak{G}''(\varsigma)}{\mathfrak{G}'(\varsigma)}+\frac{\delta}{\eta}\right)>0$. *If* $\mathfrak{F}$ *is analytic in* $\boldsymbol{U}$ *and:*

$$\delta\mathfrak{f}(\varsigma)+\eta\varsigma\mathfrak{F}'(\varsigma)\prec\delta\mathfrak{g}(\varsigma)+\eta\varsigma\mathfrak{G}'(\varsigma),$$

then $\mathfrak{f}(\varsigma)\prec\mathfrak{g}(\varsigma)$ *and* $\mathfrak{G}$ *is the best dominant.*

Lemma 3 ([22]). *Let* $\mathfrak{g}(\varsigma)$ *be convex univalent in* $\boldsymbol{U}$ *and let* $\alpha,\beta\in\mathcal{H}(D)$ *be analytic in a domain D containing* $\mathfrak{G}(\boldsymbol{U})$. *Suppose that:*

(1) $\Re\left(\frac{\alpha'(\mathfrak{f}(\varsigma))}{\beta(\mathfrak{g}(\varsigma))}\right)>0$ *for all* $\varsigma\in\boldsymbol{U}$;

(2) $\xi(\varsigma)=\varsigma\mathfrak{G}'(\varsigma)\beta(\mathfrak{g}(\varsigma))$ *is a starlike univalent function in* $\boldsymbol{U}$.

If $\mathfrak{F}\in\mathcal{H}[\mathfrak{G}(0),1]\cap\mathbb{Q}$, *with* $\mathfrak{F}(\boldsymbol{U})\subset D$, *and* $\alpha(\mathfrak{f}(\varsigma))+\varsigma\mathfrak{F}'(\varsigma)\beta(\mathfrak{f}(\varsigma))$ *is univalent in* $\boldsymbol{U}$, *and:*

$$\alpha(\mathfrak{g}(\varsigma))+\varsigma\mathfrak{G}'(\varsigma)\beta(\mathfrak{f}(\varsigma))\prec\alpha(\mathfrak{f}(\varsigma))+\varsigma\mathfrak{F}'(\varsigma)\beta(p(\varsigma)),$$

then $\mathfrak{g}(\varsigma) \prec \mathfrak{f}(\varsigma)$ *and* $\mathfrak{G}$ *is the best subordinant.*

Lemma 4 ([16]). *Let* $\mathfrak{G}$ *be convex univalent in* $\boldsymbol{U}$ *and* $\widetilde{\eta} \in \mathbb{C}$. *Assume that* $\Re(\widetilde{\eta}) > 0$. *If* $\mathfrak{f}(\varsigma) \in \mathcal{H}[q(0), 1] \cap \mathbb{Q}$, *and:* $\mathfrak{f}(\varsigma) + \eta\varsigma\mathfrak{F}'(\varsigma)$ *is univalent in* $\boldsymbol{U}$, *and*

$$\mathfrak{g}(\varsigma) + \eta\varsigma\mathfrak{G}'(\varsigma) \prec \mathfrak{f}(\varsigma) + \eta\varsigma\mathfrak{F}'(\varsigma),$$

then $\mathfrak{g}(\varsigma) \prec \mathfrak{f}(\varsigma)$ *and* $\mathfrak{G}$ *is the best subordinant.*

Lemma 5 ([31]). *Let* T *be a compact metric space and* $C(T)$ *be the Banach space of real or complex valued continuous functions normed by*

$$\| \mathfrak{f} \| := \sup_{t \in \mathrm{T}} | \mathfrak{f}(t) | .$$

If $B = \mathfrak{f}_n$ *is a sequence in* $C(T)$ *such that* $\mathfrak{f}_n$ *is uniformly bounded and equi-continuous, then* $\widetilde{B}$ *is compact.*

Lemma 6 ([32]). *Let* Υ *be a Banach space and subspace* $Y_1 \subset Y$ *be a nonempty closed bounded convex subset, while* $\mathcal{M} : Y_1 \to Y_1$ *is compact. Then,* $\mathcal{M}$ *has a fixed point.*

Lemma 7 ([32]). *If* Υ *is a Banach space and* $\mathcal{M} : Y \to Y$ *is a contraction map, then* $\mathcal{M}$ *has a unique fixed point.*

Let us outline this paper. In the current section, Section 1, we have demonstrated the connection between the fractional q-integral and q-derivative. In Section 2, we derive various subordination results involving the $I^{l}_{q,\varsigma}$ operator using the notion of differential subordination. Section 3 covers the results of differential superordination and merges with the second section to produce sandwich-style results. In Section 4, we examine the existence, univalence, and uniqueness of fractional q-differential equations. Section 5 presents a few examples and corollaries of the findings. Finally, Section 6 contains the paper's conclusions.

2. Subordination Results

In this segment, the subordination results involve the operator $I^{l}_{q,\varsigma}$, defined in Definition 1. Also, the following Theorems give way to the best dominant by Lemmas 1 and 2.

Theorem 1. *Suppose* $\mathfrak{f}(\varsigma), \mathfrak{g}(\varsigma) \in \mathcal{A}$. *Let* $\vartheta \geq 1, \mathfrak{g}(\varsigma) \neq 0, \mathfrak{g}'(\varsigma) \neq 0, \left(I^{l}_{q,\varsigma}\mathfrak{g}(\varsigma)\right)^{\vartheta}$ *be convex univalent in* $\boldsymbol{U}$, *satisfying:*

$$Re\left(1 + \frac{\varsigma\mathfrak{g}''(\varsigma)}{\mathfrak{g}'(\varsigma)} + \frac{(\vartheta-1)\varsigma\mathfrak{g}'(\varsigma)}{\mathfrak{g}(\varsigma)} + \frac{\varsigma}{\mathfrak{g}'(\varsigma)} + \frac{1}{\eta}\right) > 0, \varsigma \in \mathbf{U}.$$

Let $\vartheta \geq 1, \eta \in \mathbb{C}, \left(I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)\right)^{\vartheta} \in \mathcal{A}$, *satisfying the differential subordination:*

$$\left(\frac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\vartheta}(\mathfrak{f}(\varsigma))^{\vartheta}\left[1 + \vartheta\eta\frac{\varsigma\mathfrak{f}'(\varsigma)}{\mathfrak{f}(\varsigma)}\right] \prec \left(\frac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\vartheta}(\mathfrak{g}(\varsigma))^{\vartheta}\left[1 + \vartheta\eta\frac{\varsigma\mathfrak{g}'(\varsigma)}{\mathfrak{g}(\varsigma)}\right], \tag{6}$$

then $\left(I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)\right)^{\vartheta} \prec \left(I^{l}_{q,\varsigma}\mathfrak{g}(\varsigma)\right)^{\vartheta}$ *and* $\left(I^{l}_{q,\varsigma}\mathfrak{g}(\varsigma)\right)^{\vartheta}$ *is the best dominant.*

Proof. We suppose the functions $\mathfrak{F}(\varsigma)$ and $\mathfrak{G}(\varsigma)$ by:

$$\mathfrak{F}(\varsigma)=\left(I_{q,\varsigma}^{l}\mathfrak{f}(\varsigma)\right)^{\vartheta}=\left(\frac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\vartheta}(\mathfrak{f}(\varsigma))^{\vartheta}\ \text{ and }\ \mathfrak{G}(\varsigma)=\left(I_{q,\varsigma}^{l}\mathfrak{g}(\varsigma)\right)^{\vartheta}=\left(\frac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\vartheta}(\mathfrak{g}(\varsigma))^{\vartheta}. \tag{7}$$

Then, a computation shows that:

$$\varsigma\mathfrak{F}'(\varsigma)=\varsigma\left(\left(I_{q,\varsigma}^{l}\mathfrak{f}(\varsigma)\right)^{\vartheta}\right)'=\vartheta\left(\tfrac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\vartheta}(\mathfrak{f}(\varsigma))^{\vartheta-1}\varsigma\mathfrak{f}'(\varsigma)$$

and

$$\varsigma\mathfrak{G}'(\varsigma)=\varsigma\left(\left(I_{q,\varsigma}^{l}\mathfrak{g}(\varsigma)\right)^{\vartheta}\right)'=\vartheta\left(\tfrac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\vartheta}(\mathfrak{g}(\varsigma))^{\vartheta-1}\varsigma\mathfrak{g}'(\varsigma).$$

Since $\mathfrak{g}(\varsigma)$ is convex and univalent in $\mathbf{U}$. Thus, in view of Lemma 2, by setting $\psi=1,\ Re\left(\frac{1}{\eta}\right)>0$, we obtain:

$$Re\left(1+\tfrac{\varsigma\mathfrak{G}''(\varsigma)}{\mathfrak{G}'(\varsigma)}+\tfrac{1}{\eta}\right)=Re\left(1+\tfrac{(\vartheta-1)\varsigma\mathfrak{g}'(\varsigma)}{\mathfrak{g}(\varsigma)}+\tfrac{\varsigma\mathfrak{g}''(\varsigma)}{\mathfrak{g}'(\varsigma)}+\tfrac{1}{\eta}\right)>0.$$

By simple computation in hypothesis (6) of Theorem 1 we get:

$$\mathfrak{F}(\varsigma)+\eta\varsigma\mathfrak{F}'(\varsigma)\prec\mathfrak{G}(\varsigma)+\eta\varsigma\mathfrak{G}'(\varsigma).$$

Since $\mathfrak{F}$ and $\mathfrak{G}$ satisfies the hypothesis of Theorem 1. Thus, by applying Lemma 2, our proof of Theorem 1 is completed. □

By using of Lemma 1, we have:

Theorem 2. *Suppose* $\mathfrak{f}(\varsigma),\mathfrak{g}(\varsigma)\in\mathcal{A}$. *Let* $\mathcal{T}_1(\varsigma)$ *be defined by:*

$$\mathcal{T}_1(\varsigma)=\tfrac{\mathfrak{g}'(\varsigma)}{\varsigma}-\tfrac{\mathfrak{g}(\varsigma)}{\varsigma^2}\ \ \textit{with}\ \ \Re\left(2+\tfrac{\varsigma\mathcal{T}_1'(\varsigma)}{\mathcal{T}_1(\varsigma)}+\tfrac{(\mathcal{E}-1)\varsigma^2\mathcal{T}_1(\varsigma)}{\mathfrak{g}(\varsigma)}\right)>0.$$

Assume that the function $\varsigma\left(\frac{\left(I_{q,\varsigma}^{l}\mathfrak{g}(\varsigma)\right)^{\mathcal{E}}}{\varsigma}\right)'$ *is starlike univalent in* $\mathbf{U}$. *Let* $\mathcal{E}\geq1$, $\left(\frac{I_{q,\varsigma}^{l}\mathfrak{f}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}\in\mathcal{A}$, *satisfying the following subordination:*

$$\left(\tfrac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\mathcal{E}}\left(\tfrac{\mathfrak{f}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}\left[1+\mathcal{E}\left(\tfrac{\varsigma\mathfrak{f}'(\varsigma)}{\mathfrak{f}(\varsigma)}-1\right)\right]\prec\left(\tfrac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\mathcal{E}}\left(\tfrac{\mathfrak{g}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}\left[1+\mathcal{E}\left(\tfrac{\varsigma\mathfrak{g}'(\varsigma)}{\mathfrak{g}(\varsigma)}-1\right)\right], \tag{8}$$

then

$$\left(\frac{I_{q,\varsigma}^{l}\mathfrak{f}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}\prec\left(\frac{I_{q,\varsigma}^{l}\mathfrak{g}(\varsigma)}{\varsigma}\right)^{\mathcal{E}},$$

and $\left(\frac{I_{q,\varsigma}^{l}\mathfrak{g}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}$ *is the best dominant.*

Proof. We suppose the functions $\mathfrak{F}(\varsigma)$ and $\mathfrak{G}(\varsigma)$ by:

$$\mathfrak{F}(\varsigma)=\left(\frac{I_{q,\varsigma}^{l}\mathfrak{f}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}=\left(\frac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\mathcal{E}}\left(\frac{\mathfrak{f}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}\ \text{ and }\ \mathfrak{G}(\varsigma)=\left(\frac{I_{q,\varsigma}^{l}\mathfrak{g}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}=\left(\frac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\mathcal{E}}\left(\frac{\mathfrak{g}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}. \tag{9}$$

Then, a computation shows that:

$$\varsigma\mathfrak{F}'(\varsigma)=\varsigma\left(\left(\frac{I_{q,\varsigma}^{l}\mathfrak{f}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}\right)'=\mathcal{E}\left(\tfrac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\mathcal{E}}\left(\tfrac{\mathfrak{f}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}\left(\tfrac{\varsigma\mathfrak{f}'(\varsigma)}{\mathfrak{f}(\varsigma)}-1\right)$$

and

$$\varsigma\mathfrak{G}'(\varsigma)=\varsigma\left(\left(\frac{I^{l}_{q,\varsigma}\mathfrak{g}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}\right)'=\mathcal{E}\left(\frac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\mathcal{E}}\left(\frac{\mathfrak{g}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}\left(\frac{\varsigma\mathfrak{g}'(\varsigma)}{\mathfrak{g}(\varsigma)}-1\right).$$

By setting $\theta(w)=w$ and $\phi(w)=1$, it can be easily noticed that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}^*$, and that $\phi(w)\neq 0$ $(w\in\mathbb{C}^*)$. From Lemma 1, $\xi(\varsigma)$ and $h(\varsigma)$ can be written as:

$$\xi(\varsigma)=\varsigma\mathfrak{G}'(\varsigma)\phi(\mathfrak{g}(\varsigma))=\varsigma\mathfrak{G}'(\varsigma)$$

and

$$h(\varsigma)=\theta(\mathfrak{g}(\varsigma))+\xi(\varsigma)=\mathfrak{g}(\varsigma)+\varsigma\mathfrak{G}'(\varsigma).$$

We find that $\xi(\varsigma)$ is starlike univalent in **U**, so that:

$$\Re\left(\frac{\varsigma h'(\varsigma)}{\xi(\varsigma)}\right)=\Re\left(2+\frac{\varsigma\mathfrak{G}''(\varsigma)}{\mathfrak{G}'(\varsigma)}\right)=\Re\left(2+\frac{\varsigma\mathcal{T}_1'(\varsigma)}{\mathcal{T}_1(\varsigma)}+\frac{(\mathcal{E}-1)\varsigma^2\Gamma_q(l)\mathcal{T}_1(\varsigma)}{\mathfrak{g}(\varsigma)}\right)>0.$$

Based on the hypothesis (8) of Theorem 2, we obtain:

$$\mathfrak{F}(\varsigma)+\varsigma\mathfrak{F}'(\varsigma)\prec\mathfrak{G}(\varsigma)+\varsigma\mathfrak{G}'(\varsigma)$$

Since $\mathfrak{F}$ and $\mathfrak{G}$ satisfies the hypothesis of Theorem 2. Thus, by applying Lemma 1, our proof of Theorem 2 is completed. □

Theorem 3. *Suppose* $\mathfrak{f}(\varsigma)\in\mathcal{A}$. *Assume that* $\mathfrak{g}(\varsigma)$ *is convex univalent in* ***U*** *and satisfies:*

$$\Re\left(1+\frac{\varsigma\mathfrak{G}''(\varsigma)}{\mathfrak{G}'(\varsigma)}+\frac{1}{\eta}\right)>0. \tag{10}$$

Let $\vartheta\geq 1,\eta\in\mathbb{C}$, $\left(I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)\right)^{\vartheta}\in\mathcal{A}$, *satisfying the differential subordination:*

$$\left(\frac{\varsigma\mathfrak{f}'(\varsigma)}{\mathfrak{f}(\varsigma)}\right)^{\vartheta}\left[1+\vartheta\eta\left(\frac{\varsigma\mathfrak{f}''(\varsigma)}{\mathfrak{f}'(\varsigma)}+\frac{\varsigma\mathfrak{f}'(\varsigma)}{\mathfrak{f}(\varsigma)}+1\right)\right]\prec\mathfrak{G}(\varsigma)+\eta\varsigma\mathfrak{G}'(\varsigma), \tag{11}$$

then $\left(\frac{\varsigma\left(I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)\right)'}{I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)}\right)^{\vartheta}\prec\mathfrak{G}(\varsigma)$ *and* $\mathfrak{G}(\varsigma)$ *is the best dominant.*

Proof. We suppose the function $\mathfrak{F}(\varsigma)$ by:

$$\mathfrak{F}(\varsigma)=\left(\frac{\varsigma\left(I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)\right)'}{I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)}\right)^{\vartheta}. \tag{12}$$

Then, a computation shows that:

$$\varsigma\mathfrak{F}'(\varsigma)=\varsigma\left(\left(\frac{\varsigma\left(I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)\right)'}{I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)}\right)^{\vartheta}\right)'=\vartheta\left(\frac{\varsigma\mathfrak{f}'(\varsigma)}{\mathfrak{f}(\varsigma)}\right)^{\vartheta}\left(\frac{\varsigma\mathfrak{f}''(\varsigma)}{\mathfrak{f}'(\varsigma)}+\frac{\varsigma\mathfrak{f}'(\varsigma)}{\mathfrak{f}(\varsigma)}+1\right).$$

Since $q(\varsigma)$ is convex and univalent in **U**. Thus, in view of Lemma 2, by setting $\delta=1$, $Re\left(\frac{1}{\eta}\right)>0$, we obtain the following:

$$Re\left(1+\frac{\varsigma\mathfrak{G}''(\varsigma)}{\mathfrak{G}'(\varsigma)}+\frac{1}{\eta}\right)>0.$$

By simple computation in hypothesis (11) of Theorem 3, we have:

$$\mathfrak{F}(\varsigma)+\eta\varsigma\mathfrak{F}'(\varsigma)\prec\mathfrak{G}(\varsigma)+\eta\varsigma\mathfrak{G}'(\varsigma).$$

Since $\mathfrak{F}$ satisfies the hypothesis of Theorem 3, so as an application of Lemma 2, we obtain the following:

$$\mathfrak{F}(\varsigma)\prec\mathfrak{G}(\varsigma)$$

and $\mathfrak{G}$ is the best dominant, which completes the proof of Theorem 3. □

3. Superordination and Sandwich-Type Results

This section examines the dual concept of the subordination theorem, i.e., the superordination results. Furthermore, the best subordinate is determined using the Lemmas 3 and 4. Lastly, we state a sandwich-type theorem that combines the findings of differential subordination and superordination.

Theorem 4. *Suppose* $\mathfrak{f}(\varsigma),\mathfrak{g}(\varsigma)\in\mathcal{A}$. *Assume that* $\left(I^{l}_{q,\varsigma}\mathfrak{g}(\varsigma)\right)^{\vartheta}$ *is convex univalent in* $\boldsymbol{U}$. *Let* $\vartheta\geq 1,\Re(\overline{\eta})>0,\left(I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)\right)^{\vartheta}\in\mathcal{A},\left(\frac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\vartheta}(\mathfrak{f}(\varsigma))^{\vartheta}\left[1+\vartheta\gamma\frac{\varsigma\mathfrak{f}'(\varsigma)}{\mathfrak{f}(\varsigma)}\right]$ *is univalent in* $\boldsymbol{U}$, $\left(I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)\right)^{\vartheta}\in\mathcal{H}[0,1]\cap\mathbb{Q}$ *and satisfies the following superordination:*

$$\left(\frac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\vartheta}(\mathfrak{g}(\varsigma))^{\vartheta}\left[1+\vartheta\eta\frac{\varsigma\mathfrak{g}'(\varsigma)}{\mathfrak{g}(\varsigma)}\right]\prec\left(\frac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\vartheta}(\mathfrak{f}(\varsigma))^{\vartheta}\left[1+\vartheta\eta\frac{\varsigma\mathfrak{f}'(\varsigma)}{\mathfrak{f}(\varsigma)}\right],$$

then $\left(I^{l}_{q,\varsigma}\mathfrak{g}(\varsigma)\right)^{\vartheta}\prec\left(I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)\right)^{\vartheta}$ *and* $\left(I^{l}_{q,\varsigma}\mathfrak{g}(\varsigma)\right)^{\vartheta}$ *is the best subordinant.*

Proof. Suppose the function $\mathfrak{F}(\varsigma)$ and $\mathfrak{G}(\varsigma)$ are defined in (7). We confine our proof comparable to that of Theorem 1, based on Lemma 4. Hence, the result follows from a hypothesis of Theorem 4 as:

$$\mathfrak{G}(\varsigma)+\eta\varsigma\mathfrak{G}'(\varsigma)\prec\mathfrak{F}(\varsigma)+\eta\varsigma\mathfrak{F}'(\varsigma)\Rightarrow\mathfrak{G}(\varsigma)\prec\mathfrak{F}(\varsigma)$$

and $\mathfrak{G}$ is the best subordinant. □

We can use Lemma 3 to derive the following theorem.

Theorem 5. *Let* $\mathfrak{f}(\varsigma),\mathfrak{g}(\varsigma)\in\mathcal{A}$. *Assume* $\left(\frac{I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}$ *is convex univalent and* $\varsigma\left(\left(\frac{I^{l}_{q,\varsigma}\mathfrak{g}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}\right)'$ *is starlike univalent in* $\boldsymbol{U}$. *Suppose* $\mathcal{E}\geq 1,\left(\frac{I^{l}_{q,\varsigma}\mathfrak{g}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}\in\mathcal{A},\left(\frac{I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}\in\mathcal{H}[0,1]\cap\mathbb{Q},\left(\frac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\mathcal{E}}\left(\frac{\mathfrak{f}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}\left[1+\mathcal{E}\left(\frac{\varsigma\mathfrak{f}'(\varsigma)}{\mathfrak{f}(\varsigma)}-1\right)\right]$ *is univalent in* $\boldsymbol{U}$, *and satisfies the following superordination:*

$$\left(\frac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\mathcal{E}}\left(\frac{\mathfrak{g}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}\left[1+\mathcal{E}\left(\frac{\varsigma\mathfrak{g}'(\varsigma)}{\mathfrak{g}(\varsigma)}-1\right)\right]\prec\left(\frac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\mathcal{E}}\left(\frac{\mathfrak{f}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}\left[1+\mathcal{E}\left(\frac{\varsigma\mathfrak{f}'(\varsigma)}{\mathfrak{f}(\varsigma)}-1\right)\right],$$

then:

$$\left(\frac{I^{l}_{q,\varsigma}\mathfrak{g}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}\prec\left(\frac{I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}$$

and $\left(\frac{I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}$ *is the best subordinant.*

Proof. Suppose the function $\mathfrak{F}(\varsigma)$ and $\mathfrak{G}(\varsigma)$ are defined in (9). We build our proof on Lemma 3 and restrict it to that of Theorem 2. Thus, Theorem 5's hypothesis leads to the following:

$$\mathfrak{G}(\varsigma)+\varsigma\mathfrak{G}'(\varsigma)\prec\mathfrak{F}(\varsigma)+\varsigma\mathfrak{F}'(\varsigma)\Rightarrow\mathfrak{G}(\varsigma)\prec\mathfrak{F}(\varsigma)$$

and $\mathfrak{G}$ is the best subordinant. □

Theorem 6. *Suppose $\mathfrak{f}(\varsigma)\in\mathcal{A}$. Assume that $\mathfrak{G}$ is convex univalent in $\boldsymbol{U}$ and (10) holds. Let $\vartheta\geq 1,\Re(\overline{\eta})>0,\left(\frac{\varsigma\left(I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)\right)'}{I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)}\right)^{\vartheta}\in\mathcal{A},\left(\frac{\varsigma\mathfrak{f}'(\varsigma)}{\mathfrak{f}(\varsigma)}\right)^{\vartheta}\left[1+\eta\vartheta\left(\frac{\varsigma\mathfrak{f}''(\varsigma)}{\mathfrak{f}'(\varsigma)}+\frac{\varsigma\mathfrak{f}'(\varsigma)}{\mathfrak{f}(\varsigma)}+1\right)\right]$ is univalent in $\boldsymbol{U},\left(\frac{\varsigma\left(I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)\right)'}{I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)}\right)^{\vartheta}\in\mathcal{H}[0,1]\cap\mathbb{Q}$ and satisfies the following superordination:*

$$\mathfrak{G}(\varsigma)+\eta\varsigma\mathfrak{G}'(\varsigma)\prec\left(\frac{\varsigma\mathfrak{f}'(\varsigma)}{\mathfrak{f}(\varsigma)}\right)^{\vartheta}\left[1+\eta\vartheta\left(\frac{\varsigma\mathfrak{f}''(\varsigma)}{\mathfrak{f}'(\varsigma)}+\frac{\varsigma\mathfrak{f}'(\varsigma)}{f(\varsigma)}+1\right)\right],$$

then $\mathfrak{G}(\varsigma)\prec\left(\frac{\varsigma\left(I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)\right)'}{I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)}\right)^{\vartheta}$ and $\mathfrak{G}(\varsigma)$ is the best subordinant.

Proof. The function $\mathfrak{F}(\varsigma)$ was assumed to be defined by (12). Our proof is limited, akin to Theorem 3, and is predicated on Lemma 4. Thus, the outcome of Theorem 6's hypothesis is as follows:

$$\mathfrak{G}(\varsigma)+\eta\varsigma\mathfrak{G}'(\varsigma)\prec\mathfrak{F}(\varsigma)+\eta\varsigma\mathfrak{F}'(\varsigma),$$

then $\mathfrak{G}(\varsigma)\prec\mathfrak{F}(\varsigma)$ and $\mathfrak{G}$ is the best subordinant. □

Combining Theorems 1 and 4, we obtain the following sandwich-type theorem.

Theorem 7. *Suppose $\mathfrak{f}(\varsigma),\mathfrak{g}_1(\varsigma),\mathfrak{g}_2(\varsigma)\in\mathcal{A}$. Assume that $\left(I^{l}_{q,\varsigma}\mathfrak{g}_1(\varsigma)\right)^{\vartheta},\left(I^{l}_{q,\varsigma}\mathfrak{g}_2(\varsigma)\right)^{\vartheta}$ is convex univalent in $\boldsymbol{U}$, satisfying:*

$$Re\left(1+\frac{\varsigma\mathfrak{g}_2''(\varsigma)}{\mathfrak{g}_2'(\varsigma)}+(\vartheta-1)\left(\frac{\varsigma\mathfrak{g}_2'(\varsigma)}{\mathfrak{g}_2(\varsigma)}\right)+\left(\frac{\varsigma}{\mathfrak{g}_2'(\varsigma)}\right)+\frac{1}{\gamma}\right)>0.$$

Let $\vartheta\geq 1,\Re(\overline{\eta})>0,\gamma\in\mathbb{C},\left(I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)\right)^{\vartheta}\in\mathcal{H}[0,1]\cap\mathbb{Q},\left(\frac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\vartheta}(\mathfrak{f}(\varsigma))^{\vartheta}\left[1+\vartheta\gamma\frac{\varsigma\mathfrak{f}'(\varsigma)}{\mathfrak{f}(\varsigma)}\right]$ is univalent in $\boldsymbol{U}$ and satisfies:

$$\begin{aligned}\left(\frac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\vartheta}(\mathfrak{g}_1(\varsigma))^{\vartheta}\left[1+\eta\vartheta\frac{\varsigma\mathfrak{g}_1'(\varsigma)}{\mathfrak{g}_1(\varsigma)}\right]&\prec\left(\frac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\vartheta}(\mathfrak{f}(\varsigma))^{\vartheta}\left[1+\eta\vartheta\frac{\varsigma\mathfrak{f}'(\varsigma)}{\mathfrak{f}(\varsigma)}\right]\\&\prec\left(\frac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\eta}(\mathfrak{g}_2(\varsigma))^{\eta}\left[1+\eta\gamma\frac{\varsigma\mathfrak{g}_2'(\varsigma)}{\mathfrak{g}_2(\varsigma)}\right],\end{aligned}$$

then:

$$\left(I^{l}_{q,\varsigma}\mathfrak{g}_1(\varsigma)\right)^{\vartheta}\prec\left(I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)\right)^{\vartheta}\prec\left(I^{l}_{q,\varsigma}\mathfrak{g}_2(\varsigma)\right)^{\vartheta},$$

and $\left(I^{l}_{q,\varsigma}\mathfrak{g}_1(\varsigma)\right)^{\vartheta},\left(I^{l}_{q,\varsigma}\mathfrak{g}_2(\varsigma)\right)^{\vartheta}$ are the best subordinant and best dominant, respectively.

Combining Theorems 2 and 5, we obtain the following sandwich-type theorem.

Theorem 8. *Suppose* $\mathfrak{f}(\varsigma), \mathfrak{g}_1(\varsigma), \mathfrak{g}_2(\varsigma) \in \mathcal{A}$. *Assume that* $\left(\frac{I^l_{q,\varsigma}\mathfrak{f}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}$ *is convex univalent functions in* $\boldsymbol{U}$. *Also,* $\varsigma\left(\left(\frac{I^l_{q,\varsigma}\mathfrak{g}_1(\varsigma)}{\varsigma}\right)^{\mathcal{E}}\right)', \varsigma\left(\left(\frac{I^l_{q,\varsigma}\mathfrak{g}_2(\varsigma)}{\varsigma}\right)^{\mathcal{E}}\right)'$ *are starlike univalent functions in* $\boldsymbol{U}$. *Let* $\mathcal{T}_2(\varsigma)$ *be defined by:*

$$\mathcal{T}_2(\varsigma) = \frac{\mathfrak{g}_2'(\varsigma)}{\varsigma} - \frac{\mathfrak{g}_2(\varsigma)}{\varsigma^2} \quad \text{with} \quad \Re\left(2 + \frac{\varsigma\mathcal{T}_2'(\varsigma)}{\mathcal{T}_2(\varsigma)} + \frac{(\mathcal{E}-1)\varsigma^2\mathcal{T}_2(\varsigma)}{g_2(\varsigma)}\right) > 0.$$

Let $\mathcal{E} \geq 1$, $\left(\frac{I^l_{q,\varsigma}\mathfrak{f}(\varsigma)}{\varsigma}\right)^{\mathcal{E}} \in \mathcal{A}$, $\left(\frac{I^l_{q,\varsigma}\mathfrak{f}(\varsigma)}{\varsigma}\right)^{\mathcal{E}} \in \mathcal{H}[0,1] \cap \mathbb{Q}$, $\left(\frac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\mathcal{E}}\left(\frac{\mathfrak{f}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}\left[1 + \mathcal{E}\left(\frac{\varsigma(\mathfrak{f}(\varsigma))'}{(\mathfrak{f}(\varsigma))} - 1\right)\right]$ *being univalent in* $\boldsymbol{U}$, *and satisfying:*

$$\begin{aligned}\left(\frac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\mathcal{E}}\left(\frac{\mathfrak{g}_1(\varsigma)}{\varsigma}\right)^{\mathcal{E}}\left[1 + \mathcal{E}\left(\frac{\varsigma\mathfrak{g}_1'(\varsigma)}{(\mathfrak{g}_1(\varsigma))} - 1\right)\right] &\prec \left(\frac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\mathcal{E}}\left(\frac{\mathfrak{f}(\varsigma)}{\varsigma}\right)^{\mathcal{E}}\left[1 + \mathcal{E}\left(\frac{\varsigma\mathfrak{f}'(\varsigma)}{(\mathfrak{f}(\varsigma))} - 1\right)\right] \\ &\prec \left(\frac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^{\mathcal{E}}\left(\frac{\mathfrak{g}_2(\varsigma)}{\varsigma}\right)^{\mathcal{E}}\left[1 + \mathcal{E}\left(\frac{\varsigma\mathfrak{g}_2'(\varsigma)}{(\mathfrak{g}_2(\varsigma))} - 1\right)\right],\end{aligned}$$

then:

$$\left(\frac{I^l_{q,\varsigma}\mathfrak{g}_1(\varsigma)}{\varsigma}\right)^{\mathcal{E}} \prec \left(\frac{I^l_{q,\varsigma}\mathfrak{f}(\varsigma)}{\varsigma}\right)^{\mathcal{E}} \prec \left(\frac{I^l_{q,\varsigma}\mathfrak{g}_2(\varsigma)}{\varsigma}\right)^{\mathcal{E}},$$

and $\left(\frac{I^l_{q,\varsigma}\mathfrak{g}_1(\varsigma)}{\varsigma}\right)^{\mathcal{E}}$, $\left(\frac{I^l_{q,\varsigma}\mathfrak{g}_2(\varsigma)}{\varsigma}\right)^{\mathcal{E}}$ *are the best dominant and the best subordinant, respectively.*

Combining Theorems 3 and 6, we obtain the following sandwich-type theorem.

Theorem 9. *Suppose* $\mathfrak{f}(z) \in \mathcal{A}$. *Assume that* $\mathfrak{G}_1, \mathfrak{G}_2$ *are convex univalent in* $\boldsymbol{U}$, *and (10) holds. Let* $\vartheta \geq 1, \Re(\overline{\eta}) > 0, \gamma \in \mathbb{C}$, $\left(\frac{z\left(I^l_{q,\varsigma}\mathfrak{f}(\varsigma)\right)'}{I^l_{q,\varsigma}\mathfrak{f}(\varsigma)}\right)^{\vartheta} \in \mathcal{H}[0,1] \cap \mathbb{Q}$, $\left(\frac{\varsigma\mathfrak{f}'(\varsigma)}{\mathfrak{f}(\varsigma)}\right)^{\vartheta}\left[1 + \vartheta\gamma\left(\frac{\varsigma\mathfrak{f}''(\varsigma)}{\mathfrak{f}'(\varsigma)} + \frac{\varsigma\mathfrak{f}'(\varsigma)}{\mathfrak{f}(\varsigma)} + 1\right)\right]$ *being univalent in* $\boldsymbol{U}$, *and satisfying:*

$$\mathfrak{G}_1(\varsigma) + \eta\varsigma\mathfrak{G}_1'(\varsigma) \prec \left(\frac{\varsigma\mathfrak{f}'(\varsigma)}{\mathfrak{f}(\varsigma)}\right)^{\vartheta}\left[1 + \vartheta\gamma\left(\frac{\varsigma\mathfrak{f}''(\varsigma)}{\mathfrak{f}'(\varsigma)} + \frac{\varsigma\mathfrak{f}'(\varsigma)}{\mathfrak{f}(\varsigma)} + 1\right)\right] \prec \mathfrak{G}_2(\varsigma) + \eta\varsigma\mathfrak{G}_2'(\varsigma),$$

then:

$$\mathfrak{G}_1(\varsigma) \prec \left(\frac{\varsigma\left(I^l_{q,\varsigma}\mathfrak{f}(\varsigma)\right)'}{I^l_{q,\varsigma}\mathfrak{f}(\varsigma)}\right)^{\vartheta} \prec \mathfrak{G}_2(\varsigma)$$

and $\mathfrak{G}_1(\varsigma), \mathfrak{G}_2(\varsigma)$ *are the best subordinant and best dominant, respectively.*

4. Existence and Uniqueness Results

This segment demonstrates the idea of fractional q-differential Equation (1) for an analytic function defined in an open unit disk. We establish the existence and uniqueness of the solution to the fractional q-differential equation of fractional order.

Let $\mathfrak{B} := \mathrm{C}[\mathbf{U}, \mathbb{C}]$ be a Banach space of all continuous functions $\phi : \mathbf{U} \to \mathbb{C}$ endowed with the sup norm, defined as:

$$\| \phi \| := \sup_{\varsigma \in \mathbf{U}} | \phi(\varsigma) | .$$

Theorem 10. *Let* $\mathcal{K} > 0$, $\mathfrak{f} : [\boldsymbol{U} \times \mathbb{C}] \to \mathbb{C}$ *be a continuous analytic function with* $||\mathfrak{f}|| \leq \mathcal{K}$. *Then, the initial value problem (1) has a solution* $\phi(\varsigma)$, *where* $\phi(\varsigma)$ *is univalent in* $\boldsymbol{U}$.

Proof. We suppose the operator $\mathcal{M} : \mathbb{C} \to \mathbb{C}$ by:

$$\mathcal{M}_\phi(\varsigma) = \frac{1}{\Gamma_q(l)} \int_0^\varsigma \mathfrak{f}(\tau, \phi(\tau))(\varsigma - \tau q)_{l-1} d_q\tau \quad (l \in (0,1)). \tag{13}$$

Taking the modulus in (13), we obtain:

$$\begin{aligned} \mid \mathcal{M}_\phi(\varsigma) \mid &\leq \int_0^\varsigma \left| \tfrac{(\varsigma-\tau q)_{l-1}}{\Gamma_q(l)} \right| \mid \mathfrak{f}(\tau, \phi(\tau)) \mid d_q\tau \\ &\leq \mathcal{K} \int_0^\varsigma \left| \tfrac{(\varsigma-\tau q)_{l-1}}{\Gamma_q(l)} \right| d_q\tau \leq \tfrac{\mathcal{K}}{\Gamma_q(l+1)} = \xi. \end{aligned} \tag{14}$$

Therefore, we can consider that $\mathcal{M}_\phi(\varsigma)$ is bounded operator, and:

$$\parallel \mathcal{M}_\phi \parallel_{\mathfrak{B}} = \sup_{\varsigma \in \mathbf{U}} \mid \mathcal{M}_\phi(\varsigma) \mid .$$

Consider $\mathcal{M}_\phi : F \to F$, where:

$$F := \{\phi \in \mathcal{B} : \parallel \phi \parallel \leq r, \quad r \geq 0\}.$$

According to the hypothesis of Theorem 10, $\mathfrak{f}(\varsigma)$ is a continuous function on $\mathbf{U} \times F$, implying that it is uniformly continuous on a compact set $\widetilde{\mathbf{U}} \times F$, where:

$$\widetilde{\mathbf{U}} := \{\varsigma \in \mathbf{U} : |\varsigma| \leq l, \quad 0 < l < 1\}.$$

Consider for $\epsilon > 0$, there exists $\delta > 0$ such that for all $\phi_1, \phi_2 \in F$, we have:

$$\parallel \mathfrak{f}(\varsigma, \phi_1) - \mathfrak{f}(\varsigma, \phi_2) \parallel < \frac{\Gamma_q(l+1)}{l^l} \epsilon \quad \text{for} \quad \parallel \phi_1 - \phi_2 \parallel < \delta.$$

Now:

$$\begin{aligned} \mid \mathcal{M}_{\phi_1}(\varsigma) - \mathcal{M}_{\phi_2}(\varsigma) \mid &= \left| \tfrac{1}{\Gamma_q(l)} \int_0^\varsigma \mathfrak{f}(\tau, \phi_1(\tau))(\varsigma - \tau q)_{l-1} d_q\tau - \tfrac{1}{\Gamma_q(l)} \int_0^\varsigma \mathfrak{f}(\tau, \phi_2(\tau))(\varsigma - \tau q)_{l-1} d_q\tau \right| \\ &\leq \int_0^\varsigma \tfrac{|(\varsigma-\tau q)_{l-1}|}{\Gamma_q(l)} \times |\mathfrak{f}(\tau, \phi_1(\tau)) - \mathfrak{f}(\tau, \phi_2(\tau))| d_q\tau \leq \epsilon. \end{aligned}$$

So, $\mathcal{M}_\phi(\varsigma)$ is continuous on F. Let $\varsigma_1, \varsigma_2 \in \widetilde{U}$ with $\varsigma_1 \neq \varsigma_2$. Then:

$$\begin{aligned} \mid \mathcal{M}_\phi(\varsigma_1) - \mathcal{M}_\phi(\varsigma_2) \mid &= \left| \frac{1}{\Gamma_q(l)} \int_0^{\varsigma_1} \mathfrak{f}(\tau, \phi(\tau))(\varsigma_1 - \tau q)_{l-1} d_q\tau - \frac{1}{\Gamma_q(l)} \int_0^{\varsigma_2} \mathfrak{f}(\tau, \phi(\tau))(\varsigma_2 - \tau q)_{l-1} d_q\tau \right| \\ &\leq \int_0^{\varsigma_1} \left| \frac{(\varsigma_1 - \tau q)_{l-1} \mathfrak{f}(\tau, \phi(\tau))}{\Gamma_q(l)} d_q\tau \right| + \int_0^{\varsigma_2} \left| \frac{(\varsigma_2 - \tau q)_{l-1} \mathfrak{f}(\tau, \phi(\tau))}{\Gamma_q(l)} d_q\tau \right| \\ &\leq \mathcal{K} \int_0^{\varsigma_1} \left| \frac{(\varsigma_1 - \tau q)_l}{\Gamma_q(l)} d_q\tau \right| + \mathcal{K} \int_0^{\varsigma_2} \left| \frac{(\varsigma_2 - \tau q)_l}{\Gamma_q(l)} d_q\tau \right| \\ &\leq \mathcal{K} \left[\frac{(|\varsigma_1|)^l}{\Gamma_q(l+1)} + \frac{(|\varsigma_2|)^l}{\Gamma_q(l+1)} \right] \leq \frac{2\mathcal{K} l^l}{\Gamma_q(l+1)}. \end{aligned}$$

Clearly, the above computation shows that it is independent from ϕ; thus, $\mathcal{M}_\phi(F)$ is equicontinuous. The kind of function $\mathcal{M}_\phi$, defined in the above proof, is univalent due to [33]. $\mathcal{M}_\phi(F)$ has a uniformly convergent subsequence of every sequence of functions. Hence, it is compact by Arzelà–Ascoli Lemma 5. Applying Lemma 6, $\mathcal{M}_\phi$ has a fixed point and is a univalent solution of (1). This completes the proof of Theorem 10. □

Theorem 11. *Suppose the function $\mathfrak{f}(\varsigma, \phi)$ satisfies the Lipschitz condition:*

$$\parallel \mathfrak{f}(\varsigma, \phi_1) - \mathfrak{f}(\varsigma, \phi_2) \parallel \leq \varkappa \parallel \phi_1 - \phi_2 \parallel, \quad (\phi_1, \phi_2 \in \mathbb{C})$$

for some $\varkappa > 0$. *If* $\varkappa < \Gamma_q(l+1)$, *then there exists at most one univalent function* $\phi : \boldsymbol{U} \to \mathbb{C}$ *solving the initial value problem (1).*

Proof. We are supposing the operator $\mathcal{M}_\phi : \mathbb{C} \to \mathbb{C}$ is defined in (13) as:

$$\mathcal{M}_\phi(\varsigma) = \frac{1}{\Gamma_q(l)} \int_0^\varsigma \mathfrak{f}(\tau, \phi(\tau))(\varsigma - \tau q)_{l-1} d_q\tau, \quad l \in (0,1).$$

For $\phi_1, \phi_2 \in \mathbb{C}$, we have:

$$\begin{aligned} \mid \mathcal{M}_{\phi_1}(\varsigma) - \mathcal{M}_{\phi_2}(\varsigma) \mid &= \left| \frac{1}{\Gamma_q(l)} \int_0^\varsigma \mathfrak{f}(\tau, \phi_1(\tau))(\varsigma - tq)_{l-1} d_q\tau - \frac{1}{\Gamma_q(l)} \int_0^\varsigma \mathfrak{f}(\tau, \phi_2(\tau))(\varsigma - tq)_{l-1} d_q\tau \right| \\ &\leq \| \mathfrak{f}(\varsigma, \phi_1) - \mathfrak{f}(\varsigma, \phi_2) \| \int_0^\varsigma \frac{\mid (\varsigma - tq)_{l-1} \mid}{\Gamma_q(l)} d_q\tau \\ &\leq \frac{\varkappa(\mid \varsigma \mid)^l}{\Gamma_q(l+1)} \| \phi_1 - \phi_2 \| < \frac{\varkappa}{\Gamma_q(l+1)} \| \phi_1 - \phi_2 \|. \end{aligned}$$

Since $\frac{\varkappa}{\Gamma_q(l+1)} < 1$, the map $\mathcal{M}_\phi$ is a contraction mapping. By making use of Theorem 10, we obtain that $\mathcal{M}_\phi$ is an equicontinuous mapping, and hence, it is a univalent function (see [33]). Moreover, this assertion includes a unique fixed point and is the solution of the initial value problem (1) based on Banach fixed point Lemma 7. This completes the proof of Theorem 11. □

5. Corollaries and Examples

This segment presents some corollaries and examples of our Theorems for particular parameters and functions.

Letting $\mathfrak{f}(\varsigma) = \varsigma \exp(\varsigma)$, in Theorem 1, we obtain the following result.

Corollary 1. *Suppose* $\mathfrak{g}(\varsigma) \in \mathcal{H}$ *and* $\mathfrak{g}(0) = 0, \mathfrak{g}'(0) = 1$. *Assume that* $\vartheta \geq 1, \mathfrak{g}(\varsigma) \neq 0, \mathfrak{g}'(\varsigma) \neq 0, \left(I^l_{q,\varsigma}\mathfrak{g}(\varsigma)\right)^\vartheta$ *is convex univalent in* $\boldsymbol{U}$ *and satisfies:*

$$\Re\left(1 + \frac{\varsigma\mathfrak{g}''(\varsigma)}{\mathfrak{g}'(\varsigma)} + (\vartheta - 1)\frac{\varsigma\mathfrak{g}'(\varsigma)}{\mathfrak{g}(\varsigma)} + \frac{\varsigma}{\mathfrak{g}'(\varsigma)} + \frac{1}{\eta}\right) > 0.$$

Let $\vartheta \geq 1, \eta \in \mathbb{C}, \left(I^l_{q,\varsigma}\varsigma\exp(\varsigma)\right)^\vartheta \in \mathcal{A}$, *satisfying the differential subordination:*

$$\left(\frac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^\vartheta (\varsigma\exp(\varsigma))^\vartheta [1 + \eta\vartheta(\varsigma + 1)] \prec \left(\frac{(\varsigma)_{l-1}}{\Gamma_q(l)}\right)^\vartheta (\mathfrak{g}(\varsigma))^\vartheta \left[1 + \eta\vartheta\frac{\varsigma\mathfrak{g}'(\varsigma)}{\mathfrak{g}(\varsigma)}\right],$$

then $\left(I^l_{q,\varsigma}\varsigma\exp(\varsigma)\right)^\vartheta \prec \left(I^l_{q,\varsigma}\mathfrak{g}(\varsigma)\right)^\vartheta$ *and* $\left(I^l_{q,\varsigma}\mathfrak{g}(\varsigma)\right)^\vartheta$ *is the best dominant.*

Letting $l = 1$, in Corollary 1, we obtain the following result.

Corollary 2. *Suppose* $\mathfrak{g}(\varsigma) \in \mathcal{H}$ *and* $\mathfrak{g}(0) = 0, \mathfrak{g}'(0) = 1$. *Assume that* $\vartheta \geq 1$, $\mathfrak{g}(\varsigma) \neq 0$, $\mathfrak{g}'(\varsigma) \neq 0$, $\left(I^l_{q,\varsigma}\mathfrak{g}(\varsigma)\right)^\vartheta$ *is convex univalent in* $\boldsymbol{U}$ *and satisfies:*

$$\Re\left(1 + \frac{\varsigma\mathfrak{g}''(\varsigma)}{\mathfrak{g}'(\varsigma)} + (\vartheta - 1)\frac{\varsigma\mathfrak{g}'(\varsigma)}{\mathfrak{g}(\varsigma)} + \frac{\varsigma}{\mathfrak{g}'(\varsigma)} + \frac{1}{\eta}\right) > 0.$$

Let $\vartheta \geq 1, \eta \in \mathbb{C}, \left(I_{q,\varsigma}\varsigma\exp(\varsigma)\right)^\vartheta \in \mathcal{A}$, *satisfying the differential subordination:*

$$(\varsigma\exp(\varsigma))^\vartheta [1 + \eta\vartheta(\varsigma + 1)] \prec (\mathfrak{g}(\varsigma))^\vartheta \left[1 + \eta\vartheta\frac{\varsigma\mathfrak{g}'(\varsigma)}{\mathfrak{g}(\varsigma)}\right],$$

then $\left(I_{q,\varsigma}\varsigma\exp(\varsigma)\right)^{\vartheta} \prec \left(I_{q,\varsigma}\mathfrak{g}(\varsigma)\right)^{\vartheta}$ *and* $\left(I_{q,\varsigma}\mathfrak{g}(\varsigma)\right)^{\vartheta}$ *is the best dominant.*

Letting $\vartheta = 1$, in Corollary 2, we obtain the following result.

Corollary 3. *Suppose* $\mathfrak{g}(\varsigma) \in \mathcal{H}$ *and* $\mathfrak{g}(0) = 0, \mathfrak{g}'(0) = 1$. *Assume that* $\mathfrak{g}(\varsigma) \neq 0$, $\mathfrak{g}'(\varsigma) \neq 0$, $I^{l}_{q,\varsigma}\mathfrak{g}(\varsigma)$ *is convex univalent in* $\boldsymbol{U}$ *and satisfies:*

$$\Re\left(1 + \frac{\varsigma\mathfrak{g}''(\varsigma)}{\mathfrak{g}'(\varsigma)} + \frac{\varsigma}{\mathfrak{g}'(\varsigma)} + \frac{1}{\eta}\right) > 0.$$

Let $\eta \in \mathbb{C}$, *satisfying the differential subordination:*

$$(\varsigma\exp(\varsigma))[1 + \eta(\varsigma + 1)] \prec \mathfrak{g}(\varsigma)\left(1 + \eta\frac{\varsigma\mathfrak{g}'(\varsigma)}{\mathfrak{g}(\varsigma)}\right),$$

then $\sum_{n=2}^{\infty} \frac{\varsigma^n}{[n]_q} \prec \left(I_{q,\varsigma}\mathfrak{g}(\varsigma)\right)$ *and* $\left(I_{q,\varsigma}\mathfrak{g}(\varsigma)\right)$ *is the best dominant.*

Letting $\mathfrak{g}(\varsigma) = \frac{1+A\varsigma}{1+B\varsigma}$, in Theorem 3, we obtain the following result.

Corollary 4. *Suppose* $\mathfrak{f}(\varsigma) \in \mathcal{H}$ *and* $\mathfrak{f}(0) = 0$. *Assume that* $q(\varsigma)$ *is convex univalent in* $\boldsymbol{U}$ *and satisfies:*

$$\Re\left(1 + \frac{\varsigma\mathfrak{G}''(\varsigma)}{\mathfrak{G}'(\varsigma)} + \frac{1}{\eta}\right) > 0.$$

Let $\vartheta \geq 1, \eta \in \mathbb{C}$, $\left(I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)\right)^{\vartheta} \in \mathcal{A}$, *satisfying the differential subordination:*

$$\left(\frac{\varsigma\mathfrak{f}'(\varsigma)}{\mathfrak{f}(\varsigma)}\right)^{\vartheta}\left[1 + \eta\vartheta\left(\frac{\varsigma\mathfrak{f}''(\varsigma)}{\mathfrak{f}'(\varsigma)} + \frac{\varsigma\mathfrak{f}'(\varsigma)}{\mathfrak{f}(\varsigma)} + 1\right)\right] \prec \frac{1+A\varsigma}{1+B\varsigma} + \eta\frac{(A-B)\varsigma}{(1+B\varsigma)^2},$$

then $\left(\frac{\varsigma\left(I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)\right)'}{I^{l}_{q,\varsigma}\mathfrak{f}(\varsigma)}\right)^{\vartheta} \prec \frac{1+A\varsigma}{1+B\varsigma}$ *and* $\frac{1+A\varsigma}{1+B\varsigma}$ *is the best dominant.*

Example 1. *Let* $l = 1$ *be fixed and* $\mathfrak{f}(\varsigma)$ *is defined in (2), then:*

$$I_{q,\varsigma}\mathfrak{f}(\varsigma) = \sum_{n=2}^{\infty} a_{n-1}\frac{\varsigma^n}{[n]_q} \tag{15}$$

Proof. Using (15), it follows that:

$$\begin{aligned}
I_{q,\varsigma}\mathfrak{f}(\varsigma) &= \int_0^{\varsigma} f(\tau)d_q\tau \\
&= \int_0^{\varsigma}\left(\tau + a_2\tau^2 + a_3\tau^3 + a_4\tau^4\cdots\right)d_q\tau \\
&= \int_0^{\varsigma}\tau d_q\tau + \int_0^{\varsigma} a_2\tau^2 d_q\tau + \int_0^{\varsigma} a_3\tau^3 d_q\tau + \cdots \\
&= \varsigma(1-q)\sum_{k=0}^{\infty} q^k\left(\varsigma q^k\right) + \varsigma(1-q)\sum_{k=0}^{\infty} a_2 q^k\left(\varsigma q^k\right)^2 + \cdots \\
&= \varsigma^2\frac{(1-q)}{1-q^2} + a_2\varsigma^3\frac{(1-q)}{1-q^3} + \cdots \\
&= \varsigma^2\left(\frac{1}{[2]_q}\right) + a_2\varsigma^3\left(\frac{1}{[3]_q}\right) + \cdots \\
&= \sum_{n=2}^{\infty} a_{n-1}\frac{\varsigma^n}{[n]_q}.
\end{aligned}$$

□

Example 2. *Suppose for a function* $\mathfrak{f}(\varsigma) = \left(\frac{\varsigma}{\tau}; q\right)_l$, *the Jackson q-derivative is defined by:*

$$\begin{aligned} d_{q,\varsigma}\left(\frac{\varsigma}{\tau}; q\right)_l &= \frac{1}{\tau}\left(\frac{1-q^l}{1-q}\right)\prod_{n=0}^{\infty}\frac{1+q\left(\frac{\varsigma}{\tau}q^n\right)}{1+q\left(\frac{\varsigma}{\tau}q^n\right)} \\ &= \left(\frac{1}{\tau}\right)\left(\frac{\Gamma_q(1+l)}{\Gamma_q(l)}\right)\left(\frac{q\varsigma}{\tau}; q\right)_{l-1} \\ &= \left(\frac{1}{\tau}\right)[l]_q\left(\frac{q\varsigma}{\tau}; q\right)_{l-1}. \end{aligned}$$

Example 3. *Let* $\mathfrak{g}(\varsigma) = \varsigma + \varsigma^2$, *with* $\mathfrak{g}(0) = 0$ *and* $\mathfrak{g}'(0) = 1$. *For* $l = 1$, $\vartheta = 1$, *and* $\eta = 1$, *we have:*

$$\Re\left(1 + \frac{\varsigma\mathfrak{g}''(\varsigma)}{\mathfrak{g}'(\varsigma)} + \frac{(\vartheta-1)\varsigma\mathfrak{g}'(\varsigma)}{\mathfrak{g}(\varsigma)} + \frac{\varsigma}{\mathfrak{g}'(\varsigma)} + \frac{1}{\eta}\right) = \Re\left(2 + \frac{3\varsigma}{1+2\varsigma}\right) = \left(\frac{2 + 11r\cos\theta + 14r^2}{1 + 4r^2 + 4r\cos\theta}\right) > 0.$$

Now, $\Re\left(\left(I^l_{q,\varsigma}\mathfrak{g}(\varsigma)\right)'\right) = \Re(1+2\varsigma) = \Re(1+2r\cos\theta) > 0$, $\left(0 \le \theta \le \frac{\pi}{2}\right)$. *Therefore,* $I^l_{q,\varsigma}\mathfrak{g}(\varsigma)$ *is convex univalent. Let* $\mathfrak{f}(\varsigma) = \varsigma - \varsigma^2$. *Thus, by using the Theorem 1, we get:*

$$2\varsigma - 3\varsigma^2 \prec 2\varsigma + 3\varsigma^2 \Rightarrow \varsigma - \varsigma^2 \prec \varsigma + \varsigma^2.$$

Example 4. *Consider the initial value problem:*

$$\begin{aligned} &D^{\frac{1}{2}}_{\frac{1}{4},\varsigma}\phi(\varsigma) = \tfrac{(\varsigma+3)\phi+1}{16}, \quad (\varsigma \in \mathbf{U}) \\ &\phi(0) = 0. \end{aligned} \tag{16}$$

where $q = \frac{1}{4}$, $l = \frac{1}{2}$, $\mathfrak{f}(\varsigma, \phi) = \frac{(\varsigma+3)\phi+1}{16}$. *For* $\mathfrak{f} : [\boldsymbol{U} \times \mathbb{C}] \to \mathbb{C}$, *we have:*

$$\begin{aligned} |\mathfrak{f}(\varsigma, \phi_1) - \mathfrak{f}(\varsigma, \phi_2)| &= \left|\tfrac{\varsigma+3}{16}\right||\phi_1 - \phi_2| \\ &\le \tfrac{1}{4}|\phi_1 - \phi_2|. \end{aligned}$$

A computation shows that:

$$\begin{aligned} \varkappa < \Gamma_q(l+1) = \tfrac{1}{4} &< [0.5]_{(0.25)}\Gamma_{(0.25)}(0.5) \\ &= \tfrac{0.26}{\Gamma_{0.25}(0.5)} < 1. \end{aligned}$$

In this way, the deduction of Theorem 5 is valid to problem (16).

6. Conclusions

In this paper, we obtained basic properties using fractional quantum operators and studied solutions of the existence, univalence, and uniqueness for fractional q-differential Equation (1). We examined theorems and corollaries of subordination and superordination outcomes associated with the fractional q-calculus operator $I^l_{q,\varsigma}$ and identified the best dominants and subordinants, applying the findings derived from the subordination and superordination that supports the formulation of sandwich-type theorems. We demonstrate the significance of our findings through specific examples and explore relevant corollaries that complement our results. This investigation can utilize other operators because the literature contains many differential operators, and by extending our basic results, we can derive integral operators, leading to numerous conclusions with possible applications in various mathematical and scientific contexts. The results of this paper may generalize for (p, q) analogs in further research.

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