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Non-Additivity and Additivity in General Fractional Calculus and Its Physical Interpretations †

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† This article is dedicated to the 140-th anniversary of the first publication on general fractional calculus, written by Nikolai Ya. Sonin.

Abstract: In this work, some properties of the general convolutional operators of general fractional calculus (GFC), which satisfy analogues of the fundamental theorems of calculus, are described. Two types of general fractional (GF) operators on a finite interval exist in GFC that are conventionally called the L-type and T-type operators. The main difference between these operators is that the additivity property holds for T-type operators and is violated for L-type operators. This property is very important for the application of GFC in physics and other sciences. The presence or violation of the additivity property can be associated with qualitative differences in the behavior of physical processes and systems. In this paper, we define L-type line GF integrals and L-type line GF gradients. For these L-type operators, the gradient theorem is proved in this paper. In general, the L-type line GF integral over a simple line is not equal to the sum of the L-type line GF integrals over lines that make up the entire line. In this work, it is shown that there exist two cases when the additivity property holds for the L-type line GF integrals. In the first case, the L-type line GF integral along the line is equal to the sum of the L-type line GF integrals along parts of this line only if the processes, which are described by these lines, are independent. Processes are called independent if the history of changes in the subsequent process does not depend on the history of the previous process. In the second case, we prove the additivity property holds for the L-type line GF integrals, if the conditions of the GF gradient theorems are satisfied.

Keywords: fractional calculus; general fractional calculus; additivity; non-additivity; line general fractional integral; line general fractional gradient

MSC: 26A33; 34A08

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1. Introduction

Fractional calculus is a mathematical theory of integral and integro-differential operators of arbitrary (integer and non-integer) orders. This theory is called calculus because these operators satisfy some analogs of the fundamental theorems of standard calculus. Operators that satisfy the fundamental theorems of fractional calculus are called fractional integrals (FIs) and fractional derivatives (FDs). These theorems connect the integral and differential operators of non-integer orders [1–7]. Fractional calculus has a rich history, which was first described in 1868 [8] and then in [1,9–16]. Fractional calculus has been actively used in recent decades to describe non-standard properties of systems and processes with nonlocality in space and time in various subjects of physics [17–26], biology [27], economics [28,29], and other sciences. It is important that the fractional derivatives and integrals have various non-standard properties and rules [30–40]. For example, the standard product rule, the standard chain rule, and the standard semi-group rule are violated for FDs of non-integer order [31–33,36].

Fractional calculus is usually used with FDs and FIs with kernels of the power-law type and some other types [1–7]. In physics and other sciences, it is important to formulate mathematical models of nonlocal processes and systems of a general form of nonlocality without any specification. To describe a general form of nonlocality, one can use the integral and integro-differential operators with kernels belonging to a wider class of functions. In order to have a mathematically self-consistent theory, these operators should satisfy some general analogs of the fundamental theorems of fractional calculus. This approach is based on the concepts of kernel pairs and operators, which were proposed by Sonin (1849–1915) in 1884 [41,42]. At the present time, such a theory is called general fractional calculus (GFC). The terms "general fractional calculus", "general fractional integrals", and "general fractional derivatives" were suggested by Kochubei in 2011 [43].

Although the first published work (the Sonin article [41]) on general fractional calculus (GFC) appeared exactly one hundred and forty years ago (in 1884), it can be said that GFC and its application is a new young area of research in mathematics, physics, and other sciences. Note that currently the number of published works on GFC does not exceed one hundred, and more precisely there are about seventy such publications. In the proposed paper, almost all works on general fractional calculus and its applications are presented in the references as [41–110].

At the present time, there exist three basic types of GFC [45], which are based on three different sets of Sonin kernels: (K) the Kochubei GFC [43] (see also [95,96]); (H) the Hanyga GFC [91]; and (L) the Luchko GFC [45]. For integral and differential operators of GFC, the general fundamental theorems are proved [43,45,91]. The Luchko GFC [45–60] is very convenient in applications. The Luchko GFC is actively developed and applied [65–83]. Other forms of GFC are considered and used for applications in [43,84–110]. In addition to the three basic types of GFC, the following types of GFC, which are generated by the Luchko approach, can be noted: GF vector calculus [66], the Riesz form of GFC [70], scale-invariant GFC based Mellin convolution operators, GFC of operators of distributed order suggested by Al-Refai and Luchko [56], GFC with construction, proposed by Al-Refai and Fernandez [61,62,64], parametric GFC [62,72], and some others types. Let us also note the discrete GFC that is proposed in 2024 papers by Ferreira and Rocha [108], and by Antoniouk and Kochubei [109].

In papers [55,66,69], GFC on finite intervals is proposed. General fractional integrals (GFIs) and general fractional derivatives (GFDs) form GFC, in which generalizations of the fundamental theorems of standard calculus are proved [55,66,69]. In paper [55], the following basic properties of GFDs and GFIs on the finite interval are proved: GFIs satisfy the semi-group and commutativity properties; equations expressing GFDs of the Riemann–Liouville type in terms of GFDs of the Caputo type; the fundamental theorems of GFC; GFDs satisfy rules of integration by parts; and GFIs satisfy rules of integration by parts.

One can state that two types of GF operators on a finite interval exist in GFC, which can be conventionally called the L-type proposed in [55,69], and the T-type proposed in [66,69]. The main difference between these two types of GF operators is that the additivity property is satisfied for the T-type operators, and the additivity property is violated for the L-type operators.

In this work, we will discuss only one of the most important properties of linear GF integrals, namely the property of additivity, which can be performed and violated for different types of operators. The fact that the focus of this paper is on the additivity property is due to the importance of this property for the application of GF calculus in physics and other sciences. The presence or violation of the additivity property leads to qualitative differences in the behavior of physical processes and systems. Because of this, without understanding the mathematical aspects and their physical interpretation, it is difficult to build self-consistent adequate mathematical models of various processes and systems with nonlocality in space and time.

The GF gradients are integro-differential operators that depend on the domain of integration in the space [66]. Therefore, in three-dimensional space there are line, surface, and region GF gradient operators [66]. In [66], it was proved that the GF gradient theorem is satisfied only for the line GF gradient. The GF gradient theorem does not hold for the surface and region GF gradient operators [66].

In this paper, we define the L-type line GF integrals and the L-type line GF gradients, and the gradient theorem is proved for these L-type GF operators. Note that the T-type line GF integrals, the T-type line GF gradients, and the gradient theorem if proved for these T-type GF operators have been suggested in [66]. Using these operators of the L-type and the T-type and theorems, the properties of non-additivity and additivity are considered in this paper.

In general, the L-type line GFI along a simple line $L = ,\bigcup_{k=1}^{n} L_k$ (and piecewise simple line) is not equal to the sum of the L-type line GFIs along the simple lines L_k. There exists two cases when the additivity property is satisfied for the L-type line GF integrals.

(1) The L-type line GFI along the line $L = ,\bigcup_{k}^{n} L_k$ is equal to the sum of the L-type line GFIs along lines L_k only on the condition that the processes, which are described by these lines, are independent. Processes are called independent if the history of changes in the subsequent process does not depend on the history of the previous process.

(2) In this paper, we prove the additivity property holds for the L-type line GF integrals, if the conditions of the GF gradient theorems for the L-type operators are satisfied.

In Section 2, we present several equations for a more initial introductory explanation of the property of additivity in GFC. The additive and non-additive properties of general fractional operators are discussed for simple cases. In Section 3, line GF integrals of the T-type and L-type are defined. The definition of the L-type line GF integrals is proposed first time. The additive and non-additive properties of line GF integrals of the T-type and L-type are discussed. Section 3 also has a preliminary part with definitions of GFIs and GFDs of the T-type and L-type. In Section 4, we prove the additivity property of the L-type line GF integrals if the conditions of the GF gradient theorems for the L-type operators are satisfied. This theorem for L-type line GF integrals and L-type line GF gradient operator is proposed first time. This theorem gives the mathematical condition of the additivity of L-type line GF integrals. In Section 5, nonlocality and additivity properties in applications are described. The conditions of additivity for L-type line GF integrals are discussed. In Section 6, a short conclusion is suggested.

2. Initial Introductory Explanations

2.1. Additivity and Non-Additivity of GF Integral on Finite Interval

Let us briefly explain why it is possible and necessary to define two types of general fractional integral operators, one of which will be non-additive and the other additive.

STEP 1. The additivity property of a standard first-order integral is well known. For example, if function $f(x) \in C[a,c]$, with $-\infty < a < b < c < \infty$, the additivity property leads the equality

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx \tag{1}$$

holds for all $b \in (a,c)$. Equation (1) can be writted as

$$\int_b^c f(x)\,dx = \int_a^c f(x)\,dx - \int_a^b f(x)\,dx. \tag{2}$$

To simplify the expressions, one can consider the case $a = 0$,

$$\int_b^c f(x)\,dx = \int_0^c f(x)\,dx - \int_0^b f(x)\,dx. \tag{3}$$

To generalize the standard integrated operators on the final interval, one can use both the left side of Equation (3) and the right side of the equation. As a result, we obtain two different types of general fractional operators that are not equal and do not coincide ($\mathcal{J}^{(M)}_{[a,x]} \neq I^{(M)}_{[a,x]}$).

STEP 2. Let us consider the integral operator

$$I^{x}_{(M)}[z]\, f(z) \;=\; \int_0^x M(x-z)\, f(z)\, dz, \tag{4}$$

where $f(x) \in C_{-1}(0,\infty)$, $M(x) \in C_{-1,0}(0,\infty)$ and the Sonin condition holds [45,46]. Operator (4) with the kernel $M(x) = h_\alpha(x) = x^{\alpha-1}/\Gamma(\alpha)$, with $x > 0$ and $\alpha \in (0,1)$, is the well-known Riemann–Liouville fractional integral of the order α on the half-axis [4], p. 79.

In paper [55], the general fractional integral (GFI) on the finite interval $[a,b]$, with $-\infty < a < b < \infty$, is defined by as

$$\mathcal{J}^{(M)}_{[a,x]}[z]\, f(z) \;=\; \int_a^x M(x-z)\, f(z)\, dz, \tag{5}$$

where $f(x) \in C_{-1}(a,b]$, $M(x) \in C_{-1,0}(0, b-a)$ and the Sonin condition holds [55]. Note that

$$\mathcal{J}^{(M)}_{[a,b]}[z]\, f(z) \;=\; I^{b-a}_{(M)}[z]\, f(z+a). \tag{6}$$

In papers [66,69], the general fractional integral (GFI) on the finite interval $[a,b]$, with $0 \le a < b < \infty$, is defined as

$$I^{(M)}_{[a,b]}[z]\, f(z) \;:=\; I^{b}_{(M)}[z]\, f(z) \;-\; I^{a}_{(M)}[z]\, f(z) \;=\; \int_0^b M(b-z)\, f(z)\, dz \;-\; \int_0^a M(a-z)\, f(z)\, dz. \tag{7}$$

GFIs (5) and (7) with kernel $M(x) = h_\alpha(x) = x^{\alpha-1}/\Gamma(\alpha)$, with $x > 0$ and $\alpha \in (0,1)$, give the standard integral at the limit $\alpha \to 1-$.

STEP 3. It is obvious that the additivity property for operator (5) is violated, and we have the inequality

$$\mathcal{J}^{(M)}_{[a,b]}[z]\, f(z) \;+\; \mathcal{J}^{(M)}_{[b,c]}[z]\, f(z) \;\neq\; \mathcal{J}^{(M)}_{[a,c]}[z]\, f(z), \tag{8}$$

which holds for $b \in (a,c)$, where $f(x) \in C_{-1}(a,c]$ and $M(x) \in C_{-1,0}(0, c-a)$. Inequality (8) can be writted as

$$\mathcal{J}^{(M)}_{[b,c]}[z]\, f(z) \;\neq\; \mathcal{J}^{(M)}_{[a,c]}[z]\, f(z) \;-\; \mathcal{J}^{(M)}_{[a,b]}[z]\, f(z). \tag{9}$$

In the simple case $a = 0$, equality (9) has the form

$$\mathcal{J}^{(M)}_{[b,c]}[z]\, f(z) \;\neq\; I^{c}_{(M)}[z]\, f(z) \;-\; I^{b}_{(M)}[z]\, f(z), \tag{10}$$

where $0 < b < c < \infty$, $f(x) \in C_{-1}(0,\infty)$ and $M(x) \in C_{-1,0}(0,\infty)$.

Using the GFI operators (5) and (7), Equation (10) can be represented as

$$\mathcal{J}^{(M)}_{[a,b]}[z]\, f(z) \;\neq\; I^{(M)}_{[a,b]}[z]\, f(z). \tag{11}$$

Inequality (11), which describes the violation of the aditivity property for GFI, can be interpreted as an existence of two types of the different GF integral operators on the finite intervals.

As a result, the GFI operators (5) and (7) can be interpreted as different GF generalizations of the standard integral on the finite interval. GFI operator (7) can be interpreted as a GF generalization of the right-hand side of equality (3). GFI operator (5) is interpreted as a GF generalization of the left side of equality (3). Therefore, we should consider two non-equivalent forms of the GF integrals for the finite intervals. One of the main differences between these two operators is the violation and implementation of the additivity property. For GFIs (5), we have the inequality

$$\mathcal{J}^{(M)}_{[a,b]}[z]\, f(z) \;+\; \mathcal{J}^{(M)}_{[b,c]}[z]\, f(z) \;\neq\; \mathcal{J}^{(M)}_{[a,c]}[z]\, f(z) \tag{12}$$

that means the violation of the additivity property. For GFI operator (7), the additivity property is satisfied

$$I^{(M)}_{[a,b]}[z]\, f(z) \, + \, I^{(M)}_{[b,c]}[z]\, f(z) \, = \, I^{(M)}_{[a,c]}[z]\, f(z) \tag{13}$$

for all $b \in (a, c)$, where $0 < a < b < c < \infty$, $f(x) \in C_{-1}(0, \infty)$ and $M(x) \in C_{-1,0}(0, \infty)$.

In the limiting case, when $M(x) = h_\alpha(x)$ and $\alpha \to 1-$, we obtain

$$\lim_{\alpha \to 1-} \mathcal{J}^{(h_\alpha)}_{[a,b]}[x]\, f(x) \, = \, \lim_{\alpha \to 1-} I^{(h_\alpha)}_{[a,b]}[x]\, f(x) \, = \, \int_a^b f(x)\, dx. \tag{14}$$

The case of the additive GFIs on the finite interval $[a, b]$ with $a < 0$ of the real axis ($-\infty < a < b < \infty$) will not be discussed in this article to simplify the consideration. However, we point out that for this case the additive integral $I^{(M)}_{[a,b]}[z]$ can be defined (see Remark 4).

2.2. Why Is Additivity Needed for Applications?

Let us note an importance of the additivity properties in some mathematical theories, which are important to describe nonlocal processes and systems. We also note the importance of the additivity property in the corresponding branch of physics that deals with nonlocality in space and time.

(1) The additivity property is important for measure theory [111,112]. Let $F(x)$ be a nondecreasing and left continuous function defined on the real line. We can define the interval function

$$m(a, b] \, := \, F(b+) \, - \, F(a+). \tag{15}$$

For example, one can use

$$F(x) \, := \, \int_c^x M(x - z)\, f(z)\, dz \, = \, \mathcal{J}^{(M)}_{[c,x]}[z]\, f(z) \tag{16}$$

with $x > c$, or for $c = 0$,

$$F(x) \, := \, \int_0^x M(x - z)\, f(z)\, dz \, = \, I^x_{(M)}[z]\, f(z), \tag{17}$$

where $f(x) \geq 0$ and $M(x) \geq 0$ for all $x \in (c, \infty)$. It is easily verified that the interval function $m(a, b]$, which is defined in this way, is non-negative and additive. Using the function $m(a, b]$, we can construct a certain measure (see Chapter V in [111], p. 13, and Chapter V in [112], p. 279). The class of sets measurable with respect to this measure is closed under the operations of countable unions and intersections, and this measure is σ-additive [111]. The class of measurable sets will depend on the choice of the function $F(x)$.

(2) There is another mathematical theory, in which the additivity of the GF integrals is important. This is the theory of probability, which is directly related to the theory of measure and the property of additivity. Therefore, the additivity property is important to formulate a mathematically self-consistent nonlocal theory of probability [67,68,113] by using general fractional calculus. This problem of nonlocal probability theory is also directly correlated with the problems of formulations of nonlocal statistical mechanics [78,114–119].

(3) There is a third mathematical theory, in which the property of the additivity of the GF integrals on finite intervals is important. Such a theory is general fractional vector calculus [66], which is based on GFC. For example, if the GFI operator does not have the property of additivity, we cannot mathematically self-confidently define a linear GF integral along a continuous piecewise simple line. This is due to the fact that such a line GFI must be defined as the sum of line GFIs along simple lines. Note that the additive line GF integral along a piecewise simple line is proposed in [66]. The self-consistent definition can be proposed if the property of additivity is satisfied for GFI operators [66]. This problem

of nonlocal GF vector calculus is also directly correlated with the problems of nonlocal continuum mechanics [75], including the derivation of balance equations.

In general, the property of non-additivity can be considered (interpreted) from mathematical (M) and physical (P) points of view:

(M) From a mathematical point of view, a physical process should not depend on how we mentally divide the trajectory of the process into parts. In this case, non-additivity of the path-dependent quantity means that the definition of this quantity is incorrect and cannot be applied.

(P) From a physical point of view, if a process consists of two independent parts (subprocesses), then this separation of the process into two parts is not artificial. This separation reflects a qualitative property of these subprocesses that is the independence property of these parts. In this case, non-additivity means the need to distinguish between concepts of dependent and independent processes. A definition of the criteria for dependence and independence of processes should also be given.

Let us explain this physical point of view on a simple model example. If we use the GFIs $J^{(M)}_{[a,b]}$ and $\mathbf{J}_L$ to define nonlocal physical quantities that depend on the path, for example, work, amount of heat, and work of non-potential forces, then the corresponding quantities will be non-additive. Let $L[AC]$ be a continuous piecewise simple line that connects points A and C. One can consider the nonlocal work of non-potential forces that is defined by the equation

$$W(L[AC]) := \left(\mathbf{J}^{(M)}_{L[AC]}[x,y]\,\mathbf{F}(x,y)\right), \tag{18}$$

where a certain definition of non-additive of a line GFI is supposed to be existing. Then, wee can state that quantity (18) does not satisfy the additivity property

$$W(L[AC]) \neq W(L[AB]) + W(L[BC]) \tag{19}$$

for the arbitrary point $B \in L[A,C]$, where the line can be described as $L[A,C] := L[AB] \cup L[BC]$. This means that the work of non-potential forces on the path $L[A,C]$ is not equal to the sum of works on the paths $L[A,B]$ and $L[B,C]$.

If point B does not differ from other points of the line $L[A,C]$ from a physical point of view, then the dependence of the physical quantity $W(L[AC])$ on the mental division of the path $L[A,C]$, into two parts, $L[A,B]$ and $L[B,C]$, means that definition (18) is incorrect or the operator $\mathbf{J}^{(M)}_{L[AC]}[x,y]$ cannot be used to describe the physical process.

In physical theory, such a dependence of the full nonlocal process on the point $B \in L[A,C]$ is permissible only on condition that point B differs from other points of the line $L[A,C]$ from a physical point of view. For example, at point B, the nonlocal physical processes stop due to the damping of relaxation processes. In this case, point B can be considered as the starting point of the next independent process. In this case, the nonlocal work on path $L[AC]$ should be defined by the equation

$$W(L[AC]) := \left(\mathbf{J}^{(M)}_{L[AB]}[x,y]\,\mathbf{F}(x,y)\right) + \left(\mathbf{J}^{(M)}_{L[BC]}[x,y]\,\mathbf{F}(x,y)\right). \tag{20}$$

In this case, the non-stop nonlocal process $L[AC]$ and two independent non-stop nonlocal processes $L[AB]$ and $L[BC]$ with a stopping point $B \in L[AC]$ are not equivalent, and non-additivity (19) has a physical meaning.

Therefore, the non-additive line GFI along a piecewise simple line can be defined as a sum of the non-additive line GFIs along simple lines only on the condition that the processes, which are described by simple lines, are independent. Processes are called independent if the history of changes in the subsequent process does not depend on the history of the previous process.

For a visual explanation, let us consider some analogy between a piecewise linear integral and a multi-day trip by car with overnight stops. In this analogy, the processes described by simple lines are a trip by car within one day. The independence of these trips is that the traveler stays overnight (or for several days) to relax and restore strength.

3. Additive and Non-Additive General Fractional Operators

In this section, some basic elements of general fractional calculus (GFC), which are used in this paper, will be given. This section is based on GFC in the Luchko form proposed in papers [45,46] and papers [66,69]. GFC is a convenient mathematical tool for describing various types of nonlocalities in physics and other sciences [65–83].

3.1. Preliminary: Definitions of GFI and GFD

Let us define the set of kernel pairs that was proposed in [45].

Definition 1. *Let $M(x)$ and $K(x)$ be functions that satisfy the Sonin condition*

$$(M * K)(x) \, = \, \int_0^x M(x-x')\, K(x')\, dx' = 1 \tag{21}$$

for all $x \in (0,\infty)$, and the Luchko condition

$$M(x),\, K(x) \, \in \, C_{-1,0}(0,\infty), \tag{22}$$

where

$$C_{-1,0}(0,\infty) \, = \, \{f(x): \; f(x) \, = \, x^p\, g(x), \; x \, > \, 0, \; -1 \, < \, p \, < \, 0, \; g(x) \, \in \, C[0,\infty)\}. \tag{23}$$

Then, the set of pairs $(M(x),\, K(x))$ is called the Luchko set and is denoted by $\mathcal{L}$.

Remark 1. *Examples of kernel pairs $(M(x),\, K(x)) \in \mathcal{L}$ are proposed in Table 1 of [75], pp. 5–7, Table 1 of [76], p. 15, [78], p. 11, Table 1 of [69], pp. 21–22, and [68], p. 10. The physical dimensions of the kernels are $[M(x)] \, = \, [1]$ and $[K(x)] \, = \, [x]^{-1}$, where $\lambda \, > \, 0$, $[\lambda] \, = \, [x]^{-1}$ and $x \, > \, 0$.*

Note that one can consider the kernel pairs $\left(M_{new} \, = \, \lambda^{-1}\, K(x),\, K_{new} \, = \, \lambda\, M(x)\right)$, where $(M(x), K(x))$ are pairs of these tables.

Let us define operators of GFC on the half-axis $\mathbb{R}_+ \, = \, (0,\infty)$ proposed by Luchko in [45,46].

Definition 2. *Let $(M(x), K(x)) \in \mathcal{L}$ and $f(x) \in C_{-1}(0,\infty)$.*

Then, the general fractional integral (GFI) is defined as

$$I^x_{(M)}[x']f(x') \, = \, (M * f)(x) \, = \, \int_0^x dx'\, M(x-x')\, f(x'), \tag{24}$$

where

$$C_{-1}(0,\infty) \, = \, \{f(x): \; f(x) \, = \, x^p\, g(x), \; x \, > \, 0, \; p \, > \, -1, \; g(x) \in C[0,\infty)\}. \tag{25}$$

Definition 3. *Let $(M(x), K(x)) \in \mathcal{L}$ and $f(x) \in C^1_{-1}(0,\infty)$.*

Then, the general fractional derivative (GFD) of the Caputo type w is defined as

$$D^{x,*}_{(K)}[x']f(x') \, = \, \left(K * f^{(1)}\right)(x) \, = \, \int_0^x dx'\, K(x-x')\, f^{(1)}(x'), \tag{26}$$

where

$$C^1_{-1}(0,\infty) \, = \, \left\{f(x): \; f^{(1)}(x) \, \in \, C_{-1}[0,\infty)\right\}, \tag{27}$$

and $f^{(1)}(x) \, := \, df(x)/dx$.

Remark 2. *If* $(M(x), K(x)) \in \mathcal{L}$, *then* $(K(x), M(x)) \in \mathcal{L}$. *Therefore, both* $I^x_{(M)}[x']$, $D^{x,*}_{(K)}[x']$ *and* $I^x_{(K)}[x']$, $D^{x,*}_{(M)}[x']$ *can be used as GFIs and GFDs [46].*

Remark 3. *For the GFIs* $I^x_{(M)}$, *the semi-group property*

$$I^x_{(M_1)}[s]\, I^s_{(M_2)}[t]\, f(t) \,=\, I^x_{(M_1 * M_2)}[t]\, f(t) \tag{28}$$

is satisfied if the kernels $M_1(x)$ *and* $M_2(x)$ *belong to the space* $C_{-1,0}(0,\infty)$. *This equality is proved as the index law in [45], p. 8.*

In general, for the GFDs, we have the inequality

$$D^{x,*}_{(K_1)}[s]\, D^{s,*}_{(K_2)}[t]\, f(t) \,\neq\, D^{x,*}_{(K_1 * K_2)}[t]\, f(t), \tag{29}$$

if the kernels $K_1(x)$ *and* $K_2(x)$ *belong to the space* $C_{-1,0}(0,\infty)$.

The GFIs and GFDs are connected by the fundamental theorems of general fractional calculus (FT of GFC).

Theorem 1. *(First fundamental theorem of GFC). Let* $(M(x), K(x)) \in \mathcal{L}$.

Then, the equality

$$D^{x,*}_{(K)}[x']\, I^{x'}_{(M)}[x'']\, f(x'') \,=\, f(x) \tag{30}$$

holds for $f(x) \in C_{-1,(K)}(0,\infty)$, *where*

$$C_{-1,(K)}(0,\infty) \,:=\, \Big\{ f: \quad f(x) \,=\, I^x_{(K)}[x']g(x'), \quad g(x) \,\in\, C_{-1}(0,\infty) \Big\}.$$

Proof. This theorem is proved as Theorem 3 in [45], p. 9, (see also Theorem 1 in [46], p. 6). □

Theorem 2. *(Second fundamental theorem of GFC). Let* $(M(x), K(x)) \in \mathcal{L}$.

Then, the equality

$$I^x_{(M)}[x']\, D^{x',*}_{(K)}[x'']\, f(x'') \,=\, f(x) \,-\, f(0) \tag{31}$$

holds for $f(x) \in C^1_{-1}(0,\infty)$, *where* $C^1_{-1}(0,\infty) \,:=\, \Big\{ f(x): \quad f^{(1)}(x) \in C_{-1}(0,\infty) \Big\}$.

Proof. This theorem is proved as Theorem 4 in [45], p. 11, (see also Theorem 2 in [46], p. 7). □

3.2. Additive GF Operators on Interval $[a,b]$

Let us define the operators of GFC on the finite interval $[a,b]$, with $0 \leq a < b < \infty$, which are proposed in [66] to formulate general fractional vector calculus. These operators will be called the additive GFIs in contrast to the operators proposed by Luchko in [55,69] that will be called the non-additive GF operators.

Definition 4. *Let* $f(x) \in C_{-1}(0,\infty)$ *and* $(M(x), K(x)) \in \mathcal{L}$.

Then, the additive GFI on $[a,b]$, *with* $0 \leq a < x < b < \infty$, *is*

$$I^{(M)}_{[a,b]}[x]\, f(x) \,:=\, I^b_{(M)}[x]\, f(x) \,-\, I^a_{(M)}[x]\, f(x) \,=$$

$$\int_0^b dx\, M(b-x)\, f(x) \,-\, \int_0^a dx\, M(a-x)\, f(x) \tag{32}$$

for $a > 0$, *and*

$$I_{[a,b]}^{(M)}[x]\, f(x) \;:=\; I_{(M)}^{b}[x]\, f(x)$$

for $a = 0$, *i.e.,* $I_{[0,b]}^{(M)}[x]\, f(x) = I_{(M)}^{b}[x]\, f(x)$. *The additive GFI on* $[a,b]$ *is also called the GFI of the T-type.*

Remark 4. *The T-type GFI (32) is defined on the half-axis* $\mathbb{R}_+ = (0,\infty)$.

The T-type GFIs on the finite interval $[a,b]$ *of the real axis (* $-\infty < a < b < \infty$*) will not be discussed in this article to simplify the consideration. However, for this case, the additive (T-type) GFI can be defined in the following form.*

Let $f(z+c) \in C_{-1}(0,\infty)$, *where* $z > -c$ *(or* $f(x) \in C_{-1}(c,b]$*) and* $(M(x),\, K(x)) \in \mathcal{L}$. *Then, the additive GFI on* $[a,b]$, *with* $-\infty < a < b < \infty$, *is*

$$I_{[a,b]}^{(M),c}[z]\, f(z) \;:=\; I_{(M)}^{b-c}[z]\, f(z+c) \;-\; I_{(M)}^{a-c}[z]\, f(z+c) \;=$$

$$\int_c^b M(b-z)\, f(z)\, dz \;-\; \int_c^a M(a-z)\, f(z)\, dz, \tag{33}$$

where $c \in (-\infty,a]$, *and* $M(x) \in C_{-1,0}(0,\infty)$ *(or* $M(x) \in C_{-1}(0,b-c)$. *For* $c = 0$, *Equation (33) gives Equation (32), i.e.,*

$$I_{[a,b]}^{(M),0}[z]\, f(z) \;=\; I_{[a,b]}^{(M)}[z]\, f(z). \tag{34}$$

Definition 5. *Let function* $f(x) \in C_{-1}^{1}(0,\infty)$ *and* $(M(x),\, K(x)) \in \mathcal{L}$.

Then, the additive GFD on $[a,x]$, *with* $0 \le a < b < \infty$, *is*

$$D_{[a,b]}^{(K)}[x]\, f(x) \;:=\; D_{(K)}^{b,*}[x]\, f(x) \;-\; D_{(K)}^{a,*}[x]\, f(x) \;=$$

$$\int_0^b dx\, K(b-x)\, f^{(1)}(x) \;-\; \int_0^a dx\, K(a-x)\, f^{(1)}(x) \tag{35}$$

for $a > 0$, *and*

$$D_{[a,b]}^{(K)}[x]\, f(x) \;:=\; D_{(K)}^{b,*}[x]\, f(x),$$

for $a = 0$. *The additive GFD on* $[a,b]$ *is also called the GFD of the T-type.*

Remark 5. *The T-type GFDs on the finite interval* $[a,b]$ *of the real axis (* $-\infty < a < b < \infty$*) can be defined in the following form.*

Let $f^{(1)}(z+c) \in C_{-1}(0,\infty)$, *where* $x > c$, *(or* $f(x) \in C_{-1}^{1}(c,b]$*) and* $(M(x),\, K(x)) \in \mathcal{L}$. *Then, the additive GFD on* $[a,b]$, *with* $-\infty < a < b < \infty$, *is*

$$D_{[a,b]}^{(K)}[z]\, f(z) \;:=\; I_{(K)}^{b-c}[z]\, f^{(1)}(z+c) \;-\; I_{(K)}^{a-c}[z]\, f^{(1)}(z+c) \;=$$

$$\int_c^b K(b-z)\, f^{(1)}(z)\, dz \;-\; \int_c^a K(a-z)\, f^{(1)}(z)\, dz, \tag{36}$$

where $c \in (-\infty,a]$, *and* $K(x) \in C_{-1,0}(0,\infty)$ *(or* $K(x) \in C_{-1}(0,b-c)$. *For* $c = 0$, *Equation (36) gives Equation (35), i.e.,*

$$D_{[a,b]}^{(K),0}[z]\, f(z) \;=\; D_{[a,b]}^{(K)}[z]\, f(z). \tag{37}$$

Remark 6. *For arbitrary* $a, b \ge 0$, *the additive GF operator is defined as*

$${}_aI_b^{(M)}[x]\, f(x) \;:=\; \mathrm{sgn}(b-a)\, I_{\omega[a,b]}^{(M)}[x]\, f(x), \tag{38}$$

if $f(x) \in C_{-1}(0,\infty)$

$$ {}_aD_b^{(K)}[x]\, f(x) \;:=\; \mathrm{sgn}(b-a)\, D_{\omega[a,b]}^{(K)}[x]\, f(x), \tag{39}$$

if $f(x) \in C_{-1}^1(0,\infty)$, *where*

$$ \mathrm{sgn}(x) \;:=\; \begin{cases} -1 & (x<0), \\ \;\;1 & (x>0), \end{cases} \quad \omega[a,b] \;:=\; \begin{cases} [a,b] & (b>a), \\ [b,a] & (b<a), \end{cases} \tag{40}$$

and $(M(x),\, K(x)) \in \mathcal{L}$. *Note that the following property*

$$ {}_bI_a^{(M)}[x]\, f(x) \;:=\; -{}_aI_b^{(M)}[x]\, f(x), \tag{41}$$

$$ {}_bD_a^{(K)}[x]\, f(x) \;:=\; -{}_aD_b^{(K)}[x]\, f(x) \tag{42}$$

is satisfied for operators (38) and (39), which is similar to the standard property of the first-order integral

$$ \int_b^a f(x)\, dx \;=\; -\int_a^b f(x)\, dx. \tag{43}$$

One can use the notations

$$ \mathbb{R}_+^2 \;=\; \{(x,y): \quad x>0,\, y>0\},$$

$$ \mathbb{R}_{0,+}^2 \;=\; \{(x,y): \quad x \geq 0,\, y \geq 0\},$$

and $\mathbb{R}_+ = (0,\infty)$, $\mathbb{R}_{0,+} = [0,\infty)$.

Remark 7. *For the T-type GFIs on the finite interval, the semi-group property is violated and we have the inequality*

$$ I_{[a,x]}^{(M_1)}[s]\, I_{[a,s]}^{(M_2)}[t]\, f(t) \;\neq\; I_{[a,x]}^{(M_1*M_2)}[t]\, f(t), \tag{44}$$

if the kernels $M_1(x)$ *and* $M_2(x)$ *belong to the space* $C_{-1,0}(0,\infty)$. *This situation is similar to the violation of the semi-group property for the GFDs (29). It can be proved that*

$$ I_{[a,x]}^{(M_1)}[s]\, I_{[a,s]}^{(M_2)}[t]\, f(t) \;+\; I_{[a,x]}^{(M_1*M_2)}[t]\, f(t) \;-\; \Big(I_{[a,x]}^{(M_1)}[s]\, 1\Big)\, \Big(I_{(M_2)}^{a}[t]\, f(t)\Big). \tag{45}$$

For the T-type GFIs on the finite interval, the semi-group property is satisfied in the form

$$ I_{[a,x]}^{(M_1)}[s]\, I_{(M_2)}^{s}[t]\, f(t) \;=\; I_{[a,x]}^{(M_2)}[s]\, I_{(M_1)}^{s}[t]\, f(t) \;=\; I_{[a,x]}^{(M_1*M_2)}[t]\, f(t), \tag{46}$$

if the kernels $M_1(x)$ *and* $M_2(x)$ *belong to the space* $C_{-1,0}(0,\infty)$.

In general, for the T-type GFDs on the finite interval, we have the inequality

$$ D_{[a,x]}^{(K_1)}[s]\, D_{[a,s]}^{(K_2)}[t]\, f(t) \;\neq\; D_{[a,x]}^{(K_1*K_2)}[t]\, f(t), \tag{47}$$

if the kernels $M_1(x)$ *and* $M_2(x)$ *belong to the space* $C_{-1,0}(0,\infty)$.

Let us describe some properties of these GFIs and GFDs on the finite interval proposed in [66].

The additivity property of the proposed T-type GFIs and GFDs on $[a,b]$ is described by the following theorem.

Theorem 3. *Let $f(x)$ belongs to the space $C_{-1}(0,\infty)$ and $c > b > a \geq 0$. Then, the additivity property is satisfied*

$$I^{(M)}_{[a,b]}[x]\, f(x) \,+\, I^{(M)}_{[b,c]}[x]\, f(x) \,=\, I^{(M)}_{[a,c]}[x]\, f(x). \tag{48}$$

Let $f(x)$ belongs to the space $C^1_{-1}(0,\infty)$ and $c > b > a \geq 0$. Then, the additivity property is satisfied

$$D^{(K)}_{[a,b]}[x]\, f(x) \,+\, D^{(K)}_{[b,c]}[x]\, f(x) \,=\, D^{(K)}_{[a,c]}[x]\, f(x). \tag{49}$$

Proof. This theorem is proved as Theorem 3 in [66], p. 10. □

Note that the additivity property is also satisfied for the operators $I^{(M)}_{[a,b]}[z]$ and $D^{(K)}_{[a,b]}[z]$.

Let us formulate the first and second FT of GFC for the operator pair $I^{(M)}_{[a,x]}[s]$ and $D^{s,*}_{(M)}[x']$.

Theorem 4. *Let the kernel pair $(M(x), K(x))$ belong to the Luchko set $\mathcal{L}$.*

Then, the equality

$$D^{x,*}_{(K)}[s]\, I^{(M)}_{[a,s]}[x']\, f(x') \,=\, f(x)$$

holds for $f(x) \in C_{-1,(K)}(0,\infty)$.

Proof. This theorem is proved as Theorem 5 in [66], p. 11, where we use

$$D^{x,*}_{(K)}[s]\, 1 \,=\, 0.$$

□

Theorem 5. *Let $f(x)$ belongs to the space $C^1_{-1}(0,\infty)$, and the pair of kernels $(M(x), K(x))$ to the Luchko set $\mathcal{L}$.*

Then,

$$I^{(M)}_{[a,x]}[s]\, D^{s,*}_{(K)}[x']\, f(x') \,=\, f(x) \,-\, f(a), \tag{50}$$

where $x > a \geq 0$.

Proof. This theorem is proved as Theorem 4 in [66], pp. 10–11. □

Remark 8. *The fundamental theorems of GFC for the operator pair $D^{(K)}_{[a,x]}$ and $I^s_{(M)}[x']$ with $a > 0$ do not hold in the standard form and have a slightly different form.*

The first fundamental theorem of GFC for the GFD $D^{(K)}_{[a,x]}$ and GFI $I^x_{(M)}$ has the following form.

Let us assume that the conditions of the second FT of GFC for the operators $I^x_{(M)}$ and GFD $D^{x,*}_{(K)}$ are satisfied. Let us assume that $f(x) \in C_{-1}(0,\infty)$ and

$$I^x_{(M)}[x'] f(x') \,\in\, C^1_{-1}(0,\infty).$$

Then

$$D^{(K)}_{[a,x]}[s]\, I^s_{(M)}[x']\, f(x') \,=\, f(x) \,-\, f(a), \tag{51}$$

where $x > a \geq 0$.

This theorem is proved as Theorem 6 in [66], pp. 10–11.

If $a > 0$ and $f(a) \neq 0$, then Equation (51) gives

$$D^{(K)}_{[a,x]}[s]\, I^s_{(M)}[x']\, f(x') \,\neq\, f(x). \tag{52}$$

Therefore, the standard form of the first fundamental theorem of GFC does not hold for the operator $D^{(K)}_{[a,x]}$ *with* $a > 0$*, since* $I^{x}_{(M)}[x]\, 1 \neq 0$*. As a result, for the operator pair* $D^{(K)}_{[a,x]}$ *and* $I^{s}_{(M)}[x']$ *with* $a > 0$ *the first fundamental theorem of GFC has a non-standard form.*

3.3. Non-Additive GF Operators on Interval $[a, b]$

Let us define the GFIs and GFDs on the finite interval $[a, b]$, which are proposed in [55,69]. These operators will be called non-additive operators in contrast to the operators proposed in [66].

GFC on the finite interval $[a, b]$ is formulated for the Luchko set $\mathcal{L}_f$ of kernel pairs $(M(x),\, K(x))$ for $x \in \mathbb{R}_+$. Here the subscript "f" in $\mathcal{L}_f$ means the finiteness of the interval, in contrast to the Luchko set $\mathcal{L}$ on the infinite interval $\mathbb{R}_+$. The kernels of this set satisfy two conditions:

$$\int_0^x M(x - z)\, K(z)\, dz \;=\; \{1\} \;=\; \begin{cases} 1 & \text{if} x \in (0, b - a] \\ 0 & \text{if} x \notin (0, b - a] \end{cases}, \tag{53}$$

$$M(x),\, K(x) \;\in\; C_{-1}(0, b - a], \tag{54}$$

where $f(x) \in C_{-1}(0, b - a]$ if there is such a function $g(x) \in C_{-1}[0, b - a]$ that $f(x) = (x - a)^p\, g(x)$, with $p > -1$.

The left-sided GFI on $[a, b]$, with $-\infty < a < b < \infty$, is defined as

$$\mathcal{I}^{(M)}_{[a,b]}[x] f(x) \;=\; \left(\mathcal{I}^{(M)}_{a+} f\right)(b) \;=\; \int_a^b dx\, M(b - x)\, f(x) \tag{55}$$

for $f(x) \in C_{-1}(a, b]$. Function $f(x)$ belongs to the set $C_{-1}(a, b]$, if it can be represented as $f(x) = (x - a)^p\, g(x)$, where $g(x) \in C[a, b]$ and $p > -1$. The GFI (55) on $[a, b]$ is also called the GFI of the L-type.

Note that the GFI of the L-type can be expressed via the GFI of the T-type by the equation

$$\mathcal{I}^{(M)}_{[a,b]}[x] f(x) \;=\; I^{(M)}_{[0,b-a]}[z] f(z + a) \;=\; I^{b-a}_{(M)}[z] f(z + a), \tag{56}$$

where in the proof we use the change of variable $z = x - a$.

The left-sided GFDs of the Riemann–Liouville type on the finite interval are defined as

$$^{RL}\mathcal{D}^{(K)}_{[a,x]}[x'] f(x') \;=\; \left(^{RL}\mathcal{D}^{(K)}_{a+} f\right)(x) \;=\; \frac{d}{dx} \int_a^x dx'\, K(b - x')\, f(x'), \tag{57}$$

where $a < x < b$. The left-sided GFDs of the Caputo type are defined as

$$\mathcal{D}^{(K)}_{[a,b]}[x] f(x) \;=\; \left(\mathcal{D}^{(K)}_{a+} f\right)(b) \;=\; \int_a^b dx\, K(b - x)\, f^{(1)}(x), \tag{58}$$

where $a < x \leq b$ and $f(x) \in C^1_{-1}(a, b]$. The condition $f(x) \in C^1_{-1}(a, b)$ means that the first-order derivative of function $f(x)$ can be represented as $f^{(1)}(x) = (x - a)^p\, g_1(x)$, where $g_1(x) \in C(a, b]$ and $p > -1$, $f^{(1)}(x) = df(x)/dx$. The GFD (58) on $[a, b]$ is also called the GFD of the L-type.

Remark 9. *For the L-type GFIs on the finite interval, the semi-group property*

$$\mathcal{J}^{(M_1)}_{[a,x]}[s]\; \mathcal{J}^{(M_2)}_{[a,s]}[t]\; f(t) \;=\; \mathcal{J}^{(M_1 * M_2)}_{[a,x]}[t]\; f(t) \tag{59}$$

is satisfied if the kernels $M_1(x)$ *and* $M_2(x)$ *belong to the space* $C_{-1}(0, b - a]$*. This equality is proved in Proposition 3 of [55].*

In general, for the L-type GFDs on the finite interval, we have the inequality

$$\mathcal{D}^{(K_1)}_{[a,x]}[s]\,\mathcal{D}^{(K_2)}_{[a,s]}[t]\,f(t)\;\neq\;\mathcal{D}^{(K_1*K_2)}_{[a,x]}[t]\,f(t), \tag{60}$$

if the kernels $K_1(x)$ and $K_2(x)$ belong to the space $C_{-1}(0, b-a]$.

The properties of these GFIs and DFDs on the finite interval are proved in [55,69], including the fundamental theorems of GFC.

Remark 10. *Note that the additivity property of the L-type GFIs and GFDs on $[a,b]$, which are defined for GF operators (55) and (58), is violated*

$$\mathcal{I}^{(M)}_{[a,b]}[x]f(x)\;+\;\mathcal{I}^{(M)}_{[b,c]}[x]f(x)\;\neq\;\mathcal{I}^{(M)}_{[a,c]}[x]f(x), \tag{61}$$

$$\mathcal{D}^{(M)}_{[a,b]}[x]f(x)\;+\;\mathcal{D}^{(M)}_{[b,c]}[x]f(x)\;\neq\;\mathcal{D}^{(M)}_{[a,c]}[x]f(x), \tag{62}$$

where $a < b < c$. The additivity of these GF integrals (55) is violated.

Note that violation of additivity (61), (62) is an important difference between L-type operators on $[a,b]$ and T-type GF operators that are used in GF vector calculus [66], for which the additivity property (48) holds.

4. Additivity and Non-Additivity of Line GF Integrals

In this section, line general fractional integrals of the T-type, which are proposed in paper [66], are considered as a generalization of the standard line integrals. The GFIs of the L-type, which are proposed by Luchko in [45,46], are used to suggest line GFIs of the L-type.

4.1. Simple Line in $\mathbb{R}^2_{0,+}$

Let us define the concept of simple line in $\mathbb{R}^2_{0,+}$ of the XY plane.

Definition 6. *Let a line $L \subset \mathbb{R}^2_{0,+}$ be described by the equation*

$$y\;=\;y(x)\;\geq\;0,\quad x\;\in\;[a,b]\subset\mathbb{R}_{0,+}, \tag{63}$$

where $y(x) \in C^1(a,b)$. Then, the line L is called a Y-simple line on the XY plane.

Similarly, line $L \subset \mathbb{R}^2_{0,+}$ is called an X-simple line on the XY plane, if L can be described as

$$x\;=\;x(y)\;\geq\;0,\quad y\;\in\;[c,d]\subset\mathbb{R}_{0,+}, \tag{64}$$

where $x(y) \in C^1(c,d)$.

Definition 7. *The line L in $\mathbb{R}^2_{0,+}$ of the XY plane is called a simple line in $\mathbb{R}^2_{0,+}$ of the XY plane, if L is an X-simple and Y-simple line.*

The property of the simple line in $\mathbb{R}^2_{0,+}$ is described by the following well-known theorem.

Theorem 6. *Let $L \subset \mathbb{R}^2_{0,+}$ be a Y-simple line on the XY plane, where $y = y(x) \in C^1(a,b)$ and $y^{(1)}(x) > 0$ (or $y^{(1)}(x) < 0$) for all $x \in [a,b]$.*

Then, there is an inverse function $x = x(y) \in C^1(c,d)$, and the line L is an X-simple line with $y \in [c,d]$, and L is a simple line in $\mathbb{R}^2_{0,+}$ of the XY plane.

4.2. Definition of Line GFI of Vector Field

At the beginning, let us explain the difficulties associated with the correct definition of the line GF integrals, which is a generalization of the standard line integral. These difficulties are caused by the violation of the standard chain rule for the GFDs.

Let a simple line $L \subset \mathbb{R}^2_{0,+}$ in the XY plane be described as

$$y = y(x) \in C^1(a,b),$$

and

$$x = x(y) \in C^1(c,d),$$

where $(x,y) \in \mathbb{R}^2_{0,+}$.

The standard line integral of the second kind

$$\Big(\mathbf{I}^1_L, \mathbf{F}\Big) = \int_L (F_x(x,y)\,dx + F_y(x,y)\,dy)$$

for the vector field

$$\mathbf{F}(x,y) = \mathbf{e}_x\, F_x(x,y) + \mathbf{e}_y\, F_y(x,y),$$

can be written as

$$\Big(\mathbf{I}^1_L, \mathbf{F}\Big) = \int_a^b \Big(F_x(x,y(x))\,dx + F_y(x,y(x))\,y^{(1)}(x)\,dx\Big), \tag{65}$$

and

$$\Big(\mathbf{I}^1_L, \mathbf{F}\Big) = \int_c^d \Big(F_x(x(y),y)\,x^{(1)}(y)\,dy + F_y(x(y),y)\,dy\Big). \tag{66}$$

In this case, the line integrals (65) and (66) are equal, and we have the equality

$$\int_a^b \Big(F_x(x,y(x))\,dx + F_y(x,y(x))\,y^{(1)}(x)\,dx\Big) =$$

$$\int_c^d \Big(F_x(x(y),y)\,x^{(1)}(y)\,dy + F_y(x(y),y)\,dy\Big). \tag{67}$$

Equality (67) is based on the standard chain rule and the property

$$\int_c^d F_y(x(y),y)\,dy = \int_a^b F_y(x,y(x))\,y^{(1)}(x)\,dx, \tag{68}$$

$$\int_a^b F_x(x,y(x))\,dx = \int_c^d F_x(x(y),y)\,x^{(1)}(y)\,dy. \tag{69}$$

In GFC, the standard chain rule is violated. Therefore property (68), (69) does not hold for the GFIs of the T-type, and we have the inequality

$$I^{(M_2)}_{[c,d]}[y]\,F_y(x(y),y) \neq I^{(M_1)}_{[a,b]}[x]\,\Big(F_y(x,y(x))\,y^{(1)}(x)\Big)$$

$$I^{(M_1)}_{[b,a]}[x]\,F_y(x,y(x)) \neq I^{(M_2)}_{[c,d]}[y]\,\Big(F_y(x(y),y)\,x^{(1)}(y)\Big).$$

For example, these inequalities are caused by the inequalities that have the form

$$\int_0^d M_2(d-y)\,F_y(x(y),y)\,dy \neq \int_0^b M_1(b-x)\,F_y(x,y(x))\,y^{(1)}(x)\,dx.$$

However, the standard line integral of the first order over the simple line can be written as

$$\int_{L_1} \left(F_x(x,y)\,dx + F_y(x,y)\,dy\right) := \int_a^b F_x(x,y(x))\,dx + \int_c^d F_y(x(y),y)\,dy. \tag{70}$$

Representation (70) can be used to define a line GF integral [66]. In paper [66], the definition of the line GFI of the T-type in $\mathbb{R}^n_{0,+}$ is based on a generalization of Equation (70).

To simplify the consideration, at the beginning we will assume that the starting point, $A(a,c)$, and the ending point, $B(c,d)$, of the simple continuous line $L[AB]$ satisfy the conditions $0 \le a < b < \infty$ and $0 \le c < d < \infty$.

Let the simple lines $L[AB]$ be described as

$$L[AB] := \left\{(x,y): \quad x \in [a,b], \quad y = y(x) \in C^1(0,\infty)\right\}, \tag{71}$$

and

$$L[AB] := \left\{(x,y): \quad y \in [c,d], \quad x = x(y) \in C^1(0,\infty)\right\}, \tag{72}$$

where $y(a) = c$ and $y(b) = d$, such that $a < c$ and $b < d$.

Definition 8. *Let L be a simple line in $\mathbb{R}^2_{0,+}$ of the XY plane that can be represented as (71) and (72). Let vector field (68) satisfy the condition*

$$f(x) := F_x(x,y(x)) \in C_{-1}(0,\infty), \quad g(y) := F_y(x(y),y) \in C_{-1}(0,\infty). \tag{73}$$

Then, the T-type line GFI over the line L is defined as

$$\left(\mathbf{I}^{(M)}_L, \mathbf{F}\right) = I^{(M_1)}_{[a,b]}[x]\,F_x(x,y(x)) + I^{(M_2)}_{[c,d]}[y]\,F_y(x(y),y) =$$
$$\int_0^b M_1(b-x)\,F_x(x,y(x))\,dx - \int_0^a M_1(a-x)\,F_x(x,y(x))\,dx +$$
$$\int_0^d M_2(d-y)\,F_y(x(y),y)\,dy - \int_0^c M_2(c-y)\,F_y(x(y),y)\,dy, \tag{74}$$

where

$$(M_1(x),K_1(x)) \in \mathcal{L} \quad (M_2(y),K_2(y)) \in \mathcal{L}. \tag{75}$$

Remark 11. *Let us consider a simple line $L[A,C]$, where $A(a_1,b_1) \in \mathbb{R}^2_{0,+}$ $C(a_3,b_3) \in \mathbb{R}^2_{0,+}$. Then, for all points $B(a_2,b_2) \in L[A,C]$, with $a_1 < a_2 < a_3$ and $b_1 < b_2 < b_3$, we have the equality*

$$\left(\mathbf{I}^{(M)}_{L[A,C]}\,\mathbf{F}\right) = \left(\mathbf{I}^{(M)}_{L[A,B]}\,\mathbf{F}\right) + \left(\mathbf{I}^{(M)}_{L[B,C]}\,\mathbf{F}\right), \tag{76}$$

that holds since the additivity property

$$I^{(M_1)}_{[a_1,a_3]}[x]\,f(x) = I^{(M_1)}_{[a_1,a_2]}[x]\,f(x) + I^{(M_1)}_{[a_2,a_3]}[x]\,f(x), \tag{77}$$

$$I^{(M_2)}_{[b_1,b_3]}[y]\,g(y) = I^{(M_2)}_{[b_1,b_2]}[y]\,g(y) + I^{(M_2)}_{[b_2,b_3]}[y]\,g(y) \tag{78}$$

is satisfied. Therefore the T-type line GF integral is an additive operator.

Using the non-additive GFI on finite intervals, which are proposed in [55,69], we can propose a definition of a non-additive (L-type) line GFI by analogy with Definition 8.

Definition 9. *Let L be a simple line in $\mathbb{R}^2_{0,+}$ of the XY plane that can be represented as (71) and (72). Let vector field (68) satisfy the condition*

$$f(x) := F_x(x, y(x)) \in C_{-1}(a,b], \quad g(y) := F_y(x(y), y) \in C_{-1}(c,d]. \tag{79}$$

Then, the L-type line GFI over line L is defined as

$$\left(\mathbf{J}_L^{(M)}, \mathbf{F}\right) = \mathcal{J}^{(M_1)}_{[a,b]}[x]\, F_x(x,y(x)) + \mathcal{J}^{(M_2)}_{[c,d]}[y]\, F_y(x(y),y) =$$

$$\int_a^b M_1(b-x)\, F_x(x,y(x))\, dx + \int_c^d M_2(d-y)\, F_y(x(y),y)\, dy, \tag{80}$$

where

$$(M_1(x), K_1(x)) \in \mathcal{L}_f, \quad (M_2(y), K_2(y)) \in \mathcal{L}_f, \tag{81}$$

which includes the conditions $M_1(x) \in C_{-1}(0, b-a]$ and $M_2(y) \in C_{-1}(0, d-c]$.

It is necessary to make several important remarks about the line GF integral of the L-type (non-additive line GFI).

Remark 12. *In general, the additive and non-additive line integrals, which are defined by Definitions 8 and 9, do not coincide, and we have*

$$\left(\mathbf{I}_L^{(M)}, \mathbf{F}\right) \neq \left(\mathbf{J}_L^{(M)}, \mathbf{F}\right), \tag{82}$$

since

$$\int_0^b M_1(b-x)\, F_x(x,y(x))\, dx - \int_0^a M_1(a-x)\, F_x(x,y(x))\, dx \neq \int_a^b M_1(b-x)\, F_x(x,y(x))\, dx, \tag{83}$$

$$\int_0^d M_1(d-y)\, F_x(x(y),y)\, dy - \int_0^d M_1(c-y)\, F_x(x(y),y)\, dy \neq \int_c^d M_1(c-y)\, F_x(x(y),y)\, dy. \tag{84}$$

It should be noted that in the limiting case, when $M_1(x) = h_\alpha(x)$ and $M_2(y) = h_\alpha(y)$ at $\alpha \to 1-$, we obtain

$$\lim_{\alpha \to 1-} \left(\mathbf{I}_L^{(h_\alpha)}, \mathbf{F}\right) = \lim_{\alpha \to 1-} \left(\mathbf{J}_L^{(h_\alpha)}, \mathbf{F}\right), \tag{85}$$

since

$$\int_0^b F_x(x,y(x))\, dx - \int_0^a F_x(x,y(x))\, dx = \int_a^b F_x(x,y(x))\, dx, \tag{86}$$

and a similar equality for integrals along the Y-axis.

Remark 13. *Let us consider a simple line $L[A,C]$, where $A(a_1,b_1) \in \mathbb{R}^2_{0,+}$ $C(a_3,b_3) \in \mathbb{R}^2_{0,+}$. Then, for all points $B(a_2,b_2) \in L[A,C]$, with $a_1 < a_2 < a_3$ and $b_1 < b_2 < b_3$, we have the inequality*

$$\left(\mathbf{J}^{(M)}_{L[A,C]}, \mathbf{F}\right) \neq \left(\mathbf{J}^{(M)}_{L[A,B]}, \mathbf{F}\right) + \left(\mathbf{J}^{(M)}_{L[B,C]}, \mathbf{F}\right), \tag{87}$$

since the additivity property of the L-type GFIs (80) on finite intervals is violated (61), such that

$$\mathcal{J}^{(M_1)}_{[a_1,a_3]}[x]\, f(x) \neq \mathcal{J}^{(M_1)}_{[a_1,a_2]}[x]\, f(x) + \mathcal{J}^{(M_1)}_{[a_2,a_3]}[x]\, f(x), \tag{88}$$

$$\mathcal{J}^{(M_2)}_{[b_1,b_3]}[y]\, g(y) \neq \mathcal{J}^{(M_2)}_{[b_1,b_2]}[y]\, g(y) + \mathcal{J}^{(M_2)}_{[b_2,b_3]}[y]\, g(y), \tag{89}$$

in the general case.

4.3. Line GFI of T-Type for Piecewise Simple Lines

Let us define a line GFI, which consists of simple lines and lines parallel to the axes, according to paper [66].

Let $L \subset \mathbb{R}^2_{0,+}$ be a line that can be divided into several lines $L_k = L_k[A_k, A_{k+1}]$, $k = 1, \ldots, n$ that are simple lines or lines parallel to one of the axes:

$$L := \bigcup_{k=1}^{n} L_k = \bigcup_{k=1}^{n} L_k[A_k, A_{k+1}], \tag{90}$$

where the line $L_k = (A_k, A_{k+1})$ connects the points $A_k(a_k, c_k)$, and $A_{k+1}(a_{k+1}, c_{k+1})$, with $a_k, c_k \geq 0$ for all $k = 1, 2, \ldots, n$. Lines of this kind are called piecewise simple lines.

Let the simple lines $L_k = (A_k, A_{k+1})$ be described as

$$L_k := \Big\{(x, y): \quad x \in [a_k, b_k], \quad y = y_k(x) \in C^1(0, \infty)\Big\}, \tag{91}$$

where $b_k := a_{k+1}$ and $d_k := c_{k+1}$ and

$$L_k := \Big\{(x, y): \quad y \in [c_k, d_k], \quad x = x_k(y) \in C^1(0, \infty)\Big\}, \tag{92}$$

where $y_k(a_k) = c_k$ and $y_k(b_k) = d_k$.

Let the vector field

$$\mathbf{F} = \mathbf{F}(x, y) = \mathbf{e}_x F_x(x, y) + \mathbf{e}_y F_y(x, y) \tag{93}$$

satisfy the condition

$$F_x(x, y_k(x)),\ F_y(x_k(y), y) \in C_{-1}(0, \infty) \tag{94}$$

for all $k = 1, \ldots, n$, which is denoted as $\mathbf{F} \in \mathbb{F}_{-1}(L)$.

Then, for a piecewise simple line (90) with simple lines L_k ($k = 1, \ldots, n$), which can be represented by (91) and (92) in $\mathbb{R}^2_{0,+}$, and the vector field $\mathbf{F} \in \mathbb{F}_{-1}(L)$, the line GFI of the T-type for the piecewise simple line (90) is defined as

$$\Big(\mathbf{I}^{(M)}_{L}[x, y]\, \mathbf{F}(x, y)\Big) := \sum_{k=1}^{n} \Big(\mathbf{I}^{(M)}_{[A_k, A_{k+1}]}[x, y]\, \mathbf{F}(x, y)\Big),$$

where the line GFIs of the T-type for the simple lines $L_k = (A_k, A_{k+1})$ are defined as

$$\Big(\mathbf{I}^{(M)}_{[A_k, A_{k+1}]}[x, y]\, \mathbf{F}(x, y)\Big) := \operatorname{sgn}(x_{k+1} - x_k)\, I^{(M_1)}_{\omega[x_k, x_{k+1}]}[x]\, F_x(x, y_k(x)) +$$

$$\operatorname{sgn}(y_{k+1} - y_k)\, I^{(M_2)}_{\omega[y_k, y_{k+1}]}[y]\, F_y(x_k(y), y),$$

$$\operatorname{sgn}(x) := \begin{cases} -1 & (x < 0), \\ 0 & (x = 0), \\ 1 & (x > 0), \end{cases} \tag{95}$$

and

$$\omega[x_k, x_{k+1}] := \begin{cases} [x_k, x_{k+1}] & (x_{k+1} > x_k), \\ \{x_k\} & (x_{k+1} = x_k), \\ [x_{k+1}, x_k] & (x_{k+1} < x_k). \end{cases} \tag{96}$$

Here, the functions $y = y_k(x)$, $x = x_k(y)$, define the lines L_k that are simple or parallel to one of the axes.

The piecewise simple line (90), where $A_n(a_n, c_n) = A_1(a_1, c_1)$, is called the closed line. For closed lines, one can defined GF circulation [66].

4.4. Line GFI of L-Type for Piecewise Simple Lines

Let us describe the L-type line GF integral along a piecewise simple line, and we give possible interpretations of non-additivity.

Let the simple lines $L_k = (A_k, A_{k+1}) \subset \mathbb{R}^2$, which connect the points $A_k(a_k, c_k)$, and $A_{k+1}(a_{k+1}, c_{k+1})$ for all $k = 1, 2, \ldots, n$, be described as

$$L_k := \Big\{(x,y): \quad x \in \omega[a_k, b_k], \quad y = y_k(x) \in C^1(\omega(a_k, b_k))\Big\}, \tag{97}$$

and

$$L_k := \Big\{(x,y): \quad y \in \omega[c_k, d_k], \quad x = x_k(y) \in C^1(\omega([c_k, d_k]))\Big\}, \tag{98}$$

where $b_k := a_{k+1}$, $d_k := c_{k+1}$, $y_k(a_k) = c_k$, and $y_k(b_k) = d_k$. Here

$$\omega[a_k, b_k] := \begin{cases} [a_k, b_k] & (b_k > a_k), \\ \{a_k\} & (b_k = a_k), \\ [b_k, a_k] & (b_k < a_k). \end{cases} \tag{99}$$

The symbol $\omega(a_k, b_k)$ is defined similarly.

Let the vector field

$$\mathbf{F} = \mathbf{F}(x,y) = \mathbf{e}_x F_x(x,y) + \mathbf{e}_y F_y(x,y) \tag{100}$$

satisfy the condition

$$F_x(x, y_k(x)) \in C_{-1}(\omega[a_k, b_k]), \quad F_y(x_k(y), y) \in C_{-1}(\omega[c_k, d_k]) \tag{101}$$

for all $k = 1, \ldots, n$.

Then, the L-type line GFI of the vector field (100) along a piecewise simple line

$$L := \bigcup_{k=1}^{n} L_k = \bigcup_{k=1}^{n} L_k[A_k, A_{k+1}], \tag{102}$$

can be defined as

$$\Big(\mathbf{J}_L^{(M)}[x,y]\,\mathbf{F}(x,y)\Big) := \tag{103}$$

$$\sum_{k=1}^{n} \operatorname{sgn}(b_k - a_k)\, \mathcal{J}^{(M_1)}_{\omega[a_k,b_k]}[x] F_x(x, y_k(x)) + \operatorname{sgn}(d_k - c_k) \sum_{k=1}^{n-1} \mathcal{J}^{(M_2)}_{\omega[c_k,d_k]}[y] F_y(x_k(y), y),$$

where $\operatorname{sgn}(z)$ is defined by (95).

Remark 14. *It should be emphasized that the non-additivity property (61), (62) of the L-type GFIs does not allow us to give mathematically self-consistent GF vector calculus. For example, the non-additivity property (61), (62) does not allow us to formulate a self-consistent definition of the L-type line GF integral and L-type line GF gradient for piecewise simple lines. The self-consistency of the definition means that the mathematical quantity should not depend on the mental division of the line into segments. Therefore the non-additivity property can be interpreted as a sign of incorrect definition.*

The non-additivity property (61), (62) leads to the fact that the L-type line GF integral for the piecewise simple line L depends on the method of dividing this curved line into segments. For example, one can consider the case of two representations

$$L := \bigcup_{k=1}^{n} L_k[A_k, A_{k+1}] = \bigcup_{j=1}^{m} L_j[B_k, B_{k+1}], \tag{104}$$

where there is at least one finite non-zero segment L_k of the first representation, which is different from all segments of the second representation of line L, i.e., $L_k \neq L_j$ for all $j = 1, \ldots, m$. Then, for the L-type line GF integral, we have the inequality

$$\sum_{k=1}^{n} \left(\mathcal{I}_{L_k}^{(M)} \mathbf{F} \right) \neq \sum_{j=1}^{m} \left(\mathcal{I}_{L_j}^{(M)} \mathbf{F} \right) \tag{105}$$

in the general case.

Note the additivity property for the L-type line GF integrals can be satisfied for the piecewise simple line L, if the vector field, **F***, can represented as an L-type line GF gradient. For details, see Section 5.3 of this paper.*

Remark 15. *In general, the property of non-additivity can be considered (interpreted) from mathematical (M) and physical (P) points of view:*

(M) From a mathematical point of view, a physical process should not depend on how we mentally divide the trajectory of the process into parts. In this case, non-additivity of the path-dependent quantity means that the definition of the L-type quantity is incorrect and such a quantity cannot be applied.

(P) From a physical point of view, if a process consists of two independent parts (subprocesses), then this separation of the process into two parts is not artificial. This separation reflects a qualitative property of these subprocesses that is the independence property of these parts. In this case, non-additivity means that we should distinguish and define two types of processed concepts of dependent and independent processes.

As a result, it can be argued that, from a physical point of view, the L-type line GF integral along a piecewise line can be defined by Equation (103) only if the processes, which are described by simple lines L_k, are independent. Processes are called independent if the history of changes in the subsequent process does not depend on the history of the previous process.

In the framework of this approach, the L-type (non-additive) linear GFI should be considered as a sum L-type linear GFI along parts of this line if this line in the state space describes several successive independent processes.

As a result, the L-type line GFI for the line $L[AC] = L_1[AB] \bigcup L_2[BC]$, which describe a process, must take into account the dependence or independence of subprocesses $L_1[AB]$ and $L_2[BC]$. Processes are called independent if the history of changes in the subsequent process does not depend on the history of the previous process. For example, the self-consistent definition of the L-type line GFI should be given in the following form.

Definition 10. *Let a simple line L be represented as*

$$L[AC] = L_1[AB] \bigcup L_2[BC] \subset \mathbb{R}^2, \tag{106}$$

where $A = A(a_1, b_1)$, $C = C(a_3, b_3)$, and $B = B(a_2, b_2) \in L[A, C]$, with $-\infty < a_1 < a_2 < a_3 < \infty$ and $-\infty < b_1 < b_2 < b_3 < \infty$.

Then, the L-type line GFI is defined as

$$\left(\mathbf{J}_L^{(M)}, \mathbf{F} \right) = \begin{cases} \left(\mathbf{J}_{L_1 \bigcup L_2}^{(M)}, \mathbf{F} \right), & \text{if } L_1 \text{ and } L_2 \text{ describe dependent processes}, \\ \left(\mathbf{J}_{L_1}^{(M)}, \mathbf{F} \right) + \left(\mathbf{J}_{L_2}^{(M)}, \mathbf{F} \right), & \text{if } L_1 \text{ and } L_2 \text{ describe independent processes}, \end{cases} \tag{107}$$

where $(M_1(x), K_1(x)) \in \mathcal{L}_f$ and $(M_2(y), K_2(y)) \in \mathcal{L}_f$.

As a result, one can say that we have two different types of basic definitions of L-type line GFIs, which differ in the types of processes that they can describe.

Additional detailed discussion of the definition of L-type linear GFIs and the connections with the use of these operators will be continued in Section 6 of this paper.

5. Additivity by the GF Gradient Theorems

5.1. Line General Fractional Gradient in $\mathbb{R}^2_+$

Let us first consider the standard gradient theorem. Note that this theorem is proved by using the definition of the standard line integral and the standard chain rule.

Let us assume that a process be described by the continuous line L in the state space, $\mathbb{R}^2$, of parameters x and y. Let this line begin at point $A(x_1, y_1)$ and end at point $B(x_2, y_2)$ of the XY-flat. We will assume that line L is a XY-simple line, that is, X-simple and Y-simple (see Section 4.1 of this paper and [66]). This means that the continuous line $L[A, B]$ can be defined by two equivalent ways

$$L[A,B] \;=\; \Big\{(x,y) \in \mathbb{R}^2 : \quad y = y(x) \quad \text{for} \quad x \,\in\, [x_{min}, x_{max}]\Big\}, \tag{108}$$

$$L[A,B] \;=\; \Big\{(x,y) \in \mathbb{R}^2 : \quad x = x(y) \quad \text{for} \quad y \,\in\, [y_{min}, y_{max}]\Big\}, \tag{109}$$

where

$$x_{min} \,:=\, \min\{x_1, x_2\}, \quad x_{max} \,:=\, \max\{x_1, x_2\},$$

$$y_{min} \,:=\, \min\{y_1, y_2\}, \quad y_{max} \,:=\, \max\{y_1, y_2\}.$$

Since the line is simple, the derivatives of the functions $y = y(x)$ are not equal to zero and do not change sign on the interval $[x_{min}, x_{max}]$. By definition, the line integral of a vector field $\mathbf{F} \in C^1(\mathbb{R}^2)$ for the simple line L can be given by the equation

$$\int_L (\mathbf{F}, d\mathbf{r}) \;=\; \int_{x_1}^{x_2} F_x(x, y(x))\, dx \;+\; \int_{y_1}^{y_2} F_y(x(y), y)\, dy, \tag{110}$$

where $y_1 = y(x_1)$ and $y_3 = y(x_2)$.

For the vector field $\mathbf{F} = \operatorname{grad} U$, Equation (110) gives

$$\int_L (\operatorname{grad} U, d\mathbf{r}) \;=\; \int_{x_1}^{x_2} U^{(1)}_x(x, y(x))\, dx \;+\; \int_{y_1}^{y_2} U^{(1)}_y(x(y), y)\, dy \;=$$

$$\int_{x_1}^{x_2} \Big(U^{(1)}_x(x, y(x)) \;+\; U^{(1)}_y(x, y(x))\, y^{(1)}_x(x) \Big)\, dx, \tag{111}$$

where $U^{(1)}_x(x,y) \;:=\; \partial U(x,y)/\partial x$ and $U^{(1)}_y(x,y) \;:=\; \partial U(x,y)/\partial y$. Using the standard chain rule

$$\frac{dU(x, y(x))}{dx} \;=\; \left(\frac{\partial U(x,y)}{\partial x}\right)_{y=y(x)} \;+\; \left(\frac{\partial U(x,y)}{\partial y}\right)_{y=y(x)} \frac{dy(x)}{dx}, \tag{112}$$

Equation (111) takes the form

$$\int_L (\operatorname{grad} U, d\mathbf{r}) \;=\; \int_a^b \frac{dU(x, y(x))}{dx} \;=\; U(x_2, y(x_2)) \,-\, U(x_1, y(x_1)) \;=\; U(x_2, y_2) \,-\, U(x_1, y_1).$$

As a result, the following equality of the standard gradient theorem is satisfied

$$(\mathbf{I}_L, \operatorname{grad}\, U) \;=\; U(x_2, y_2) \,-\, U(x_1, y_1) \tag{113}$$

for differentiable function $U(x,y) \in C^1(\mathbb{R}^2)$, where

$$(\operatorname{grad}\, U)(x,y) \;:=\; \frac{\partial U(x,y)}{\partial x}\, \mathbf{e}_x \;+\; \frac{\partial U(x,y)}{\partial y}\, \mathbf{e}_y \tag{114}$$

is the standard gradient of the state function $U = U(x,y)$ in the space $\mathbb{R}^2$. In Equation (113), the line first-order integral $\mathbf{I}_L$ of the vector field

$$\mathbf{F}(x,y) = F_x(x,y)\,\mathbf{e}_x + F_y(x,y)\,\mathbf{e}_y \tag{115}$$

on the XY-simple line L is defined as

$$(\mathbf{I}_L, \mathbf{F}) = \int_{x_1}^{x_2} F_x(x, y(x))\,dx + \int_{y_1}^{y_2} F_y(x(y), y)\,dy. \tag{116}$$

In Equation (116), the following property of the standard integral can be applied

$$\int_{x_1}^{x_2} F_x(x, y(x))\,dx = \operatorname{sgn}(x_2 - x_1) \int_{x_{min}}^{x_{max}} F_x(x, y(x))\,dx, \tag{117}$$

$$\int_{y_1}^{y_2} F_y(x(y), y)\,dy. = \operatorname{sgn}(x_2 - x_1) \int_{y_{min}}^{y_{max}} F_y(x(y), y)\,dy. \tag{118}$$

If $\mathbf{F}(x,y) = (\text{grad}\ U)(x,y)$, then vector field (115) has the components

$$F_x(x,y) = \frac{\partial U(x,y)}{\partial x}, \quad F_y(x,y) = \frac{\partial U(x,y)}{\partial y}, \tag{119}$$

where $U(x,y) \in C^1(\mathbb{R}^2)$.

The standard gradient theorem has been generalized for the nonlocal case by using GFC in [66]. Let us give an explanation of the definition of the line general fractional gradient proposed in [66].

In fractional calculus and GFC, the standard chain rule (112) is violated. Because of this, there are significant restrictions on possible GF generalizations of the standard gradient theorem. In paper [66], it is proved that the GF generalization of the standard gradient theorem cannot be satisfied for a piecewise simple line if the GF gradient is defined for the region $W = \mathbb{R}^2_+$ as the regional GF gradient

$$\text{Grad}_W^{(K)} U = \mathbf{e}_x \left(\text{Grad}_W^{(K)} U\right)_x (x,y) + \mathbf{e}_y \left(\text{Grad}_W^{(K)} U\right)_y (x,y) =$$
$$\mathbf{e}_x\, D^{x,*}_{(K_1)}[x']\, U(x',y) + \mathbf{e}_y\, D^{y,*}_{(K_2)}[y']\, U(x,y'). \tag{120}$$

The GF gradient theorem can be proved only for the line GF gradients [66].

Let us define the line GF gradient for a simple line on the XY plane. In this definition, we use the fact that GFD can be represented as a sequential action of a first-order derivative and a general fractional integral:

$$D^{x,*}_{(K_1)}[x'] = I^{x}_{(K_1)}[x'] \frac{\partial}{\partial x'}, \quad D^{y,*}_{(K_2)}[y'] = I^{y}_{(K_2)}[y'] \frac{\partial}{\partial y'}. \tag{121}$$

Similarly, the line GF gradient can also be defined as a sequential action of first-order derivatives and general fractional integrals, if we take into account the substitution of a function describing a simple line.

Let us give a definition of the line GF gradient of the T-type.

Definition 11. *Let L be a simple line in $\mathbb{R}^2_{0,+}$ of the XY plane that is described by Equations (108) and (109) in $\mathbb{R}^2_+$.*

Let the condition $U(x,y) \in \mathbb{F}^1_{-1,L}(\mathbb{R}^2_+)$ be satisfied, which means that

$$U^{(1)}_x(x, y(x)) \in C_{-1}(0,\infty), \quad U^{(1)}_y(x(y), y) \in C_{-1}(0,\infty).$$

Then, the line general fractional gradient of the T-type for line L is defined as

$$\mathrm{Grad}_L^{(K)}\, U(x,y) \;=\; \mathbf{e}_x\, I^x_{(K_1)}[x']\, U^{(1)}_{x'}(x',y(x')) \;+\; \mathbf{e}_y\, I^y_{(K_2)}[y']\, U^{(1)}_{y'}(x(y'),y') \;=$$

$$\mathbf{e}_x\, I^x_{(K_1)}[x'] \left(\frac{\partial U(x',y)}{\partial x'}\right)_{y=y(x')} + \mathbf{e}_y\, I^y_{(K_2)}[y'] \left(\frac{\partial U(x,y')}{\partial y'}\right)_{x=x(y')}, \tag{122}$$

where $(M_x(x), K_x(x)) \in \mathcal{L}$ *and* $\left(M_y(y), K_y(y)\right) \in \mathcal{L}$.

Note that the regional GF gradient and the line GF gradient (122) are different operators

$$\mathrm{Grad}_L^{(K)}\, U(x,y) \neq \mathrm{Grad}_W^{(K)}\, U(x,y). \tag{123}$$

Remark 16. *Note that the T-type and L-type GFDs are not used in the definition of the linear GF gradient (122). Therefore, one can define a new form of the GFDs by the equation*

$$\left(\mathbb{D}_x^{(K)} F\right)(x) \;=\; I^x_{(K_x)}[x']\, U_x^{(1)}(x',y(x')), \tag{124}$$

$$\left(\mathbb{D}_y^{(K)} U\right)(y) \;=\; I^y_{(K_y)}[y']\, U_y^{(1)}(x(y'),y'), \tag{125}$$

where

$$U_x^{(1)}(x,y(x)) \;=\; \left(\frac{\partial U(x,y)}{\partial x}\right)_{y=y(x)}, \tag{126}$$

$$U_y^{(1)}(x(y),y) \;=\; \left(\frac{\partial U(x,y)}{\partial y}\right)_{x=x(y)}, \tag{127}$$

and

$$I^{(K)}_{[0,x]}[x']f(x') \;=\; \int_0^x K(x-x')\, f(x')dx' \;-\; \int_0^a K(a-x')\, f(x')dx'. \tag{128}$$

As a result, the T-type line GF gradient (122) is represented as

$$\left(\mathrm{Grad}_L^{(K)}\, U\right)(x,y) \;=\; \mathbf{e}_x \left(\mathbb{D}_x^{(K)} U\right)(x) \;+\; \mathbf{e}_y \left(\mathbb{D}_y^{(K)} U\right)(y), \tag{129}$$

where $\mathbb{D}_x^{(K)}$ *and* $\mathbb{D}_y^{(K)}$ *are defined by Equations (124) and (125).*

Note that the non-standard form of the GFDs (124) and (125) is caused by the fact that these derivatives are integro-differential operators along the simple line L and the fact that we define a line GF gradient and not a region GF gradient.

5.2. GF Gradient Theorem for T-Type Operators

We can use the definitions of the line GFI of the T-type (Definition 8) and line GF gradient of the T-type (Definition 11) to prove the following theorem.

Theorem 7 (GF gradient theorem for line GF gradient of T-type). *Let L be a simple line in* $\mathbb{R}^2_{0,+}$ *of the XY plane, which is described by Equations (108) and (109) in* $\mathbb{R}^2_+$ *and connects the points* $A(x_1,y_1)$ *and* $B(x_2,y_2)$*, and the scalar field* $U(x,y)$ *in* $\mathbb{F}^1_{-1,L}(\mathbb{R}^2_+)$.

Then, the equation

$$\left(\mathbf{I}_L^{(M)}, \mathrm{Grad}_L^{(K)}\, U\right) \;=\; U(x_2,y_2) \;-\; U(x_1,y_1) \tag{130}$$

holds, if $(M_x(x), K_x(x)) \in [\mathcal{L}]$ *and* $\left(M_y(y), K_y(y)\right) \in [\mathcal{L}]$.

Here the T-type line GFI is

$$\left(\mathbf{I}_L^{(M)}, \mathrm{Grad}_L^{(K)}\, U\right) := I^{(M_x)}_{[x_1,x_2]}[x] \left(\mathbb{D}_x^{(K)} U\right)(x) \;+\; I^{(M_y)}_{[y_1,y_2]}[y] \left(\mathbb{D}_y^{(K)} U\right)(y), \tag{131}$$

where

$$\left(\mathbb{D}_x^{(K)} U\right)(x) := I^x_{(K_x)}[x'] \, U_x^{(1)}(x', y(x')), \tag{132}$$

$$\left(\mathbb{D}_y^{(K)} U\right)(y) := I^y_{(K_y)}[y'] \, U_y^{(1)}(x(y'), y'))), \tag{133}$$

and

$$U_x^{(1)}(x,y) = \frac{\partial U(x,y)}{\partial x} \quad U_y^{(1)}(x,y) = \frac{\partial U(x,y)}{\partial y}.$$

Proof. Substitution of (132) and (133) into (131) gives

$$\left(\mathbf{I}_L^{(M)}, \mathrm{Grad}_L^{(K)} \, U\right) = I^{(M_x)}_{[x_1,x_2]}[x] \left(\mathbb{D}_x^{(K)} U\right)(x) + I^{(M_y)}_{[y_1,y_2]}[y] \left(\mathbb{D}_y^{(K)} U\right)(y),$$

$$I^{(M_1)}_{[x_1,x_2]}[x] \, I^x_{(K_1)}[x'] \, U_{x'}^{(1)}(x', y(x')) + I^{(M_2)}_{[y_1,y_2]}[y] \, I^{y,*}_{(K_2)}[y'] \, U_{y'}^{(1)}(x(y'), y'). \tag{134}$$

Using that $(M_x(x), K_x(x)) \in [\mathcal{L}]$ and $(M_y(y), K_y(y)) \in [\mathcal{L}]$, we obtain

$$I^{(M_1)}_{[x_1,x_2]}[x] \, I^x_{(K_1)}[x'] \, f(x') = (M_1 * (K_1 * f))(x_2) - (M_1 * (K_1 * f))(x_1) =$$

$$((M_1 * K_1) * f)(x_2) - ((M_1 * K_1) * f)(x_1) = (\{1\} * f)(x_2) - (\{1\} * f)(x_1) =$$

$$\int_0^{x_2} f(x)\, dx - \int_0^{x_1} f(x)\, dx = \int_{x_1}^{x_2} f(x)\, dx.$$

Similarly, we obtain the equation

$$I^{(M_2)}_{[y_2,y_1]}[y] \, I^y_{(K_2)}[y'] \, g(y') = (\{1\} * g)(y_2) - (\{1\} * g)(y_1) =$$

$$\int_0^{y_2} g(y)\, dy - \int_0^{y_1} g(y)\, dy = \int_{y_1}^{y_2} g(y)\, dy.$$

Therefore, we obtain

$$I^{(M_1)}_{[x_2,x_1]}[x] \, I^x_{(K_1)}[x'] \, U_{x'}^{(1)}(x', y(x')) + I^{(M_2)}_{[y_2,y_1]}[y] \, I^y_{(K_2)}[y'] \, U_{y'}^{(1)}(x(y'), y') =$$

$$\int_{x_1}^{x_2} \left(\frac{\partial U(x',y)}{\partial x'}\right)_{y=y(x')} dx' + \int_{y_1}^{y_2} \left(\frac{\partial U(x,y')}{\partial y'}\right)_{x=x(y')} dy' =$$

$$\int_{x_1}^{x_2} \left(\frac{\partial U(x',y)}{\partial x'}\right)_{y=y(x')} dx' + \int_{x_1}^{x_2} \left(\frac{\partial U(x',y)}{\partial y}\right)_{y=y(x'))} \frac{\partial y(x')}{\partial x'}\, dx' =$$

$$\int_{x_1}^{x_2} \frac{dU(x, y(x))}{dx}\, dx = U(x_2, y_2) - U(x_1, y_1),$$

where we use the standard gradient theorem. □

For the case

$$M_x(x) = h_\beta(x), \quad M_y(y) = h_\alpha(y),$$

where α, $\beta \in (0,1)$, in the limit α, $\beta \to 1-$, Equation (130) gives Equations (5) and (8) of standard vector calculus.

5.3. GF Gradient Theorem for L-Type Operators

Let us consider a nonlocal analog of the standard gradient theorem by using the non-additive GFIs and line GF integral of the L-type. In this case, we should give a generalization of the standard definition of the gradient (114) and a generalization of Equation (113) of the standard gradient theorem.

Let us give a definition of the line GF gradient of the L-type.

Definition 12. *Let $\Omega = (x_1, x_2] \times (y_1, y_2] \subset \mathbb{R}^2$ be a finite region*

$$\Omega := \{(x,y): \quad -\infty < x_1 < x \le x_2 < \infty, \quad -\infty < y_1 < y \le y_2 < \infty, \}. \tag{135}$$

Let L be a simple line $L \subset \Omega$, which is described by Equations (108) and (109), in finite region $\Omega \subset \mathbb{R}^2$, and $U(x,y) \in \mathbb{F}^1_{-1,L}(\Omega)$, which means

$$U^{(1)}_x(x, y(x)) \in C_{-1}(x_1, x_2], \quad U^{(1)}_y(x(y), y) \in C_{-1}(y_1, y_2]. \tag{136}$$

Then, the L-type line GF gradient for the line $L \subset \Omega$, is defined as

$$\left(\mathrm{Grad}^{(K),J}_L U\right)(x,y) = \mathbf{e}_x \, \mathcal{J}^{(K_x)}_{[x_1,x]}[x'] \, F_x(x', y(x')) + \mathbf{e}_y \, \mathcal{J}^{(K_y)}_{[y_1,y]}[y'] \, F_y(x(y'), y') =$$

$$\mathbf{e}_x \int_{x_1}^{x} K_x(x - x') \, F_x(y(x'), x') \, dx' + \mathbf{e}_y \int_{y_1}^{y} K_y(y - y') \, F_y(x(y'), y') \, dy', \tag{137}$$

where

$$F_x(x,y) = \frac{\partial U(x,y)}{\partial x}, \quad F_y(x,y) = \frac{\partial U(x,y)}{\partial y}, \tag{138}$$

and $(M_x(x), K_x(x)) \in \mathcal{L}_f$ and $\left(M_y(y), K_y(y)\right) \in \mathcal{L}_f$, which means

$$M_x(x),\, K_x(x) \in C_{-1}(0, x_2 - x_1], \quad M_y(y),\, K_y(y) \in C_{-1}(0, y_2 - y_1], \tag{139}$$

and the Sonin conditions on the finite intervals are satisfied for these kernel pairs.

Note that the L-type line GF gradient and the T-type line GF gradient are different operators

$$\mathrm{Grad}^{(K),J}_L \, U(x,y) \neq \mathrm{Grad}^{(K)}_L \, U(x,y), \tag{140}$$

in the general case.

Remark 17. *Note that the L-type GFDs on the finite interval, which are proposed in [55], are not used in the definition of the linear GF gradient (137). Therefore, we define a new class of L-type GFDs on the finite interval as*

$$\left(\mathbb{D}^{(K,J)}_x U\right)(x) = \mathcal{J}^{(K_x)}_{[x_1,x]}[x'] \, U^{(1)}_x(x', y(x')), \tag{141}$$

$$\left(\mathbb{D}^{(K,J)}_y U\right)(y) = \mathcal{J}^{(K_y)}_{[y_1,y]}[y'] \, U^{(1)}_y(x(y'), y'), \tag{142}$$

where

$$U^{(1)}_x(x, y(x)) = \left(\frac{\partial F(x,y)}{\partial x}\right)_{y=y(x)}, \tag{143}$$

$$U^{(1)}_y(x(y), y) = \left(\frac{\partial F(x,y)}{\partial y}\right)_{x=x(y)}, \tag{144}$$

and

$$\mathcal{J}^{(K)}_{[a,x]}[x'] \, f(x') = \int_a^x K(x - x') \, f(x') dx'. \tag{145}$$

Note that the GF operators $\mathbb{D}^{(K)}_x$ and $\mathbb{D}^{(K)}_y$ should not be confused with the operators $\mathbb{D}^{(K,J)}_x$ and $\mathbb{D}^{(K,J)}_y$ since these are different types of operators. In addition, note that $\mathbb{D}^{(K,J)}_x \neq \mathcal{D}^{(K)}_{[x_1,x]}$ and $\mathbb{D}^{(K,J)}_y \neq \mathcal{D}^{(K)}_{[y_1,y]}$.

As a result, the L-type line GF gradient can be represented as

$$\left(\mathrm{Grad}_L^{(K),J}\, U\right)(x,y) \;=\; \mathbf{e}_x \left(\mathbb{D}_x^{(K,J)} U\right)(x) \;+\; \mathbf{e}_y \left(\mathbb{D}_y^{(K,J)} U\right)(y), \tag{146}$$

where $\mathbb{D}_x^{(K,J)}$ *and* $\mathbb{D}_y^{(K,J)}$ *are defined by Equations (141) and (142).*

One can use the definitions of the L-type line GFI and line GF gradient to prove the following GF gradient theorem.

Theorem 8 (GF gradient theorem for line GF gradient of L-type). *Let $L[AB]$ be a simple line, which is described by Equations (108) and (109) in $\mathbb{R}^2$ of the XY plane and connects the points $A(x_1,y_1)$ and $B(x_2,y_2)$, and the scalar field $U(x,y)$ belongs to the set $\mathbb{F}^1_{-1,L}((x_1,x_2]\times(y_1,y_2])$.*
Then, the GF gradient theorem for the L-type operators is described by the equation

$$\left(\mathbf{J}_L^{(M)}, \mathrm{Grad}_L^{(K),J}\, U\right) \;=\; U(x_2,y_2) \;-\; U(x_1,y_1), \tag{147}$$

which holds if $(M_x(x),K_x(x)) \in \mathcal{L}_f$ and $(M_y(y),K_y(y)) \in \mathcal{L}_f$.
Here, the line GFI of the L-type is

$$\left(\mathbf{J}_L^{(M)}, \mathrm{Grad}_L^{(K),J}\, U\right) \;=\; \mathcal{J}^{(M_x)}_{[x_1,x_2]}[x] \left(\mathbb{D}_x^{(K,J)} U\right)(x) \;+\; \mathcal{J}^{(M_y)}_{[y_1,y_2]}[y] \left(\mathbb{D}_y^{(K,J)} U\right)(y) \;=$$
$$\int_{x_1}^{x_2} M_x(x_2-x) \left(\mathbb{D}_x^{(K,J)} U\right)(x)\, dx \;+\; \int_{y_1}^{y_2} M_y(y_2-y) \left(\mathbb{D}_y^{(K,J)} U\right)(y)\, dy, \tag{148}$$

where

$$\left(\mathbb{D}_x^{(K,J)} U\right)(x) \;=\; \mathcal{J}^{(K_x)}_{[x_1,x]}[x']\, U_x^{(1)}(x',y(x')), \quad \left(\mathbb{D}_y^{(K,J)} U\right)(y) \;=\; \mathcal{J}^{(K_y)}_{[y_1,y]}[y']\, U_y^{(1)}(x(y'),y'), \tag{149}$$

and

$$U_x^{(1)}(x,y) \;=\; \frac{\partial U(x,y)}{\partial x}, \quad U_y^{(1)}(x,y) \;=\; \frac{\partial U(x,y)}{\partial y}. \tag{150}$$

This theorem is proved similar to the proof of Theorem 7. In the proof, we should use the semi-group property of the L-type GFI on the finite interval

$$\mathcal{J}^{(M_1)}_{[a,x]}[s]\, \mathcal{J}^{(M_2)}_{[a,s]}[t]\, f(t) \;=\; \mathcal{J}^{(M_1*M_2)}_{[a,s]}[t]\, f(t). \tag{151}$$

For $(M_x(x),K_x(x)) \in \mathcal{L}_f$ and $(M_y(y),K_y(y)) \in \mathcal{L}_f$, we have

$$\mathcal{J}^{(M)}_{[a,b]}[s]\, \mathcal{J}^{(K)}_{[a,c]}[t]\, f(t) \;=\; \mathcal{J}^{(M*K)}_{[a,x]}[t]\, f(t) \;=\; \mathcal{J}^{(\{1\})}_{[a,c]}[t]\, f(t) \;=\; \int^{c} a f(t)\, dt. \tag{152}$$

For the case

$$M_x(x) \;=\; h_\alpha(x), \quad M_y(y) \;=\; h_\beta(y),$$

where α, $\beta \in (0,1)$, in the limit α, $\beta \to 1-$, Equation (148) give Equations (5) and (8) of standard vector calculus.

Remark 18. *In general, the property of additivity of L-type linear GFIs is not satisfied. For example, in Remark 13 we consider a simple line $L[A,C]$, which connects points $A(a_1,b_1) \in \mathbb{R}^2_{0,+}$ and $C(a_3,b_3) \in \mathbb{R}^2_{0,+}$. In general, for line GFIs of the L-type, we have the inequality*

$$\left(\mathbf{J}^{(M)}_{L[A,C]}, \mathbf{F}\right) \;\neq\; \left(\mathbf{J}^{(M)}_{L[A,B]}, \mathbf{F}\right) \;+\; \left(\mathbf{J}^{(M)}_{L[B,C]}, \mathbf{F}\right) \tag{153}$$

for all points $B(a_2, b_2) \in L[A, C]$, with $a_1 < a_2 < a_3$ and $b_1 < b_2 < b_3$ in general (See Remark 13).

If the vector field, $\mathbf{F}$, can be represented as a line GF gradient of the L-type

$$\mathbf{F}(x, y) = \left(\mathrm{Grad}_L^{(K),J} U\right)(x, y), \tag{154}$$

then we have the additivity property

$$\left(\mathbf{J}_{L[A,C]}^{(M)}, \mathbf{F}\right) = \left(\mathbf{J}_{L[A,B]}^{(M)}, \mathbf{F}\right) + \left(\mathbf{J}_{L[B,C]}^{(M)}, \mathbf{F}\right). \tag{155}$$

This statement directly follows from the GF gradient theorem for L-type operators in the form

$$\left(\mathbf{J}_{L[A,C]}^{(M)}, \mathbf{F}\right) = U(a_3, b_3) - U(a_1, b_1), \tag{156}$$

$$\left(\mathbf{J}_{L[A,B]}^{(M)}, \mathbf{F}\right) = U(a_2, b_2) - U(a_1, b_1), \tag{157}$$

$$\left(\mathbf{J}_{L[B,C]}^{(M)}, \mathbf{F}\right) = U(a_3, b_3) - U(a_2, b_1), \tag{158}$$

and the equality

$$U(a_2, b_2) - U(a_1, b_1) + U(a_3, b_3) - U(a_2, b_2) = U(a_3, b_3) - U(a_1, b_1) \tag{159}$$

for all points $B(a_2, b_2) \in L[A, C]$, with $a_1 < a_2 < a_3$ and $b_1 < b_2 < b_3$.

6. Nonlocality and Additivity Properties in Applications

6.1. Path Dependence and Nonlocality

Let us assume that states of a dynamical system are described by points of the set $\mathbb{R}^n$. The coordinates of these points are parameters that characterize this dynamical system. State space (or phase space) of a dynamical system is a space where the points represent possible states of this system. For example, the state (phase) space mechanical system is described by the set of all possible values of position and momentum variables. The evolution (dynamics) of the system is characterized by changes in the parameters of the system and it is mathematically described by some line (path) in the state space.

Quantities that do not depend on the history of the system (history of changes of the parameters) and are completely determined by its state are called state functions. Examples of such quantities are the potential energy, the internal energy, the entropy, and the Helmholtz free energy. Quantities that depend on the history of changes in the states of the system (the history of the system) are called path functions (path-dependent quantities). Examples of such quantities are the work of non-potential force, the thermodynamic work, and the amount of heat.

Mathematically, history is described by a continuous line (path) in the state space of the dynamical system. The dependence on history means dependence of quantity on the line (path) on the state space. The situation is similar to the dependence of the work of non-potential forces on the trajectory of the point mass in classical mechanics. Mathematically, path-dependent quantities (PDQs) are described by linear integrals along the line L on the space of states. The points of this line describe different states of the dynamical system.

For example, if the parameters of a dynamical system are x and y, then the path-dependent quantity, $Q(L)$, is described by the linear integral

$$Q(L) = \int_L P(x, y)\, dx. \tag{160}$$

Let us consider a simple line L on the space $\mathbb{R}^2$ of parameters x and y that is described by the equation $y = y(x)$ for $x \in [x_1, x_2]$. Then, path-dependent quantity, $Q(L)$, can be defined by the equation

$$Q(L) = \int_L P(x,y)\, dx = \int_{x_1}^{x_2} P(x, y(x))\, dx. \quad (161)$$

For example, if parameter x describes volume of gas in a vessel with a piston, parameter y describes gas temperature, and the function $P(x,y)$ is the gas pressure, then the path-dependent quantity, $Q(L)$, is the thermodynamic work of gas.

From Equation (161), it is clear that the dependence on the history is characterized by the identical influence of all previous states on the final state of the system, regardless of their distance along the line L from the final state. In other words, Equation (161) describes an influence of previous states on the current state that does not depend on their distances from the current state.

The first mathematical formulation of the fading principle for the influence of past states was proposed by Boltzmann [120–124] for a physical model of viscoelasticity. In the simplified form, this principle states that an increase in the time (parameter) interval leads to a decrease in the corresponding contribution. Then, this principle was used by Vito Volterra in 1928 and 1930 [125–129]. Then, this principle was considered in [29,130–134] and other works.

6.2. Path-Dependent Quantity of Nonlocal Processes

Processes are usually described by continuous lines $L \subset \mathbb{R}^n$ in the space of states $\mathbb{R}^n$. Note that the use of the length of the line as a measure of remoteness (or closeness) of the previous state from the final state is not quite adequate since parameters usually have different physical dimensions. For example, a volume $x_1 = V$ and a temperature $x_2 = T$ have different physical dimensions $[V] \neq [T]$.

Let us consider the process in which two parameters, x and y, are changed. One can define two types of the path-dependent quantity for nonlocal processes that are described by the single-valued function $y = y(x)$ at $x \in \left[x_i, x_f\right]$ on the state space of parameters x and y. To take into account nonlocality in the parameters on the state space, $\mathbb{R}^2$, of points (x,y), the path-dependent quantity for the nonlocal case can be described by the equations

$$Q_T(L) = I^{(M_x)}_{[x_i,x_f]}[x]\, P(x, y(x)) =$$

$$\int_0^{x_f} M_x\left(x_f - x\right) P(x, y(x))\, dx - \int_0^{x_i} M_x\left(x_f - x\right) P(x, y(x))\, dx, \quad (162)$$

$$Q_L(L) = \mathcal{J}^{(M_x)}_{[x_i,x_f]}[x]\, P(x, y(x)) = \int_{x_i}^{x_f} M_x\left(x_f - x\right) P(x, y(x))\, dx, \quad (163)$$

where is $x_f > x_i$, and $M_x\left(x_f - x\right)$ is the function of influence fading or weight function that describes the weight of the state remote from the final value by the size of $x_f - x$.

In the general case,

$$Q_L(L) \neq Q_T(L), \quad (164)$$

since

$$\int_{x_i}^{x_f} M_x\left(x_f - x\right) f(x)\, dx \neq \int_0^{x_f} M_x\left(x_f - x\right) f(x)\, dx - \int_0^{x_i} M_x(x_i - x)\, f(x)\, dx, \quad (165)$$

if $x_f > x_i > 0$ and $(M_x, K_x) \in \mathcal{L}$.

In general, parameter x can be increased and decreased in contrast to the time parameter. Therefore, in order to remove the restriction of $x_f > x_i$ in Equation (163), one can define the path-dependent quantity, $Q_L(L)$, by the equation

$$Q_L(L) = \operatorname{sgn}\Big(x_f - x_i\Big)\, \mathcal{J}^{(M_x)}_{[\max(x_i,x_f),\min(x_i,x_f)]}[x]\, P(x,y(x)) =$$

$$\operatorname{sgn}\Big(x_f - x_i\Big) \int_{\min(x_i,x_f)}^{\max(x_i,x_f)} M_x\Big(\max\Big(x_i,x_f\Big) - x\Big)\, P(x,y(x))\, dx, \tag{166}$$

where x_i is the initial value of x, and x_f is the final value of x. Here

$$\max\Big(x_i,x_f\Big) := \begin{cases} x_f, & x_f \geq x_i \\ x_i, & x_f < x_i, \end{cases} \tag{167}$$

$$\min\Big(x_i,x_f\Big) := \begin{cases} x_i, & x_f \leq x_i \\ x_f, & x_f > x_i. \end{cases} \tag{168}$$

The path-dependent quantity of the T-type $Q_T(L)$ for processes, which is described by equation $y = y(x)$ for $x \in [x_1,x_2]$ on the set of parameters (x,y), can be defined as

$$Q_T(L) = I^{(M_x)}_{[x_1,x_2]}[x]\, P(x,y(x)) =$$

$$\int_0^{x_2} M_x(x_2 - x)\, P(x,y(x))\, dx - \int_0^{x_1} M_x(x_1 - x)\, P(x,y(x))\, dx, \tag{169}$$

where $I^{(M_x)}_{[x_1,x_2]}[x]$ is the additive GF integral. Note that Equation (169) is also applicable in the case $x_2 < x_1$, since

$$I^{(M_x)}_{[x_1,x_2]}[x]\, P(x,y(x)) = -\, I^{(M_x)}_{[x_2,x_1]}[x]\, P(x,y(x)). \tag{170}$$

For the case $P(x,y(x)) = P(x)$, Equation (163) with $x_2 > x_1$ and (169) describe the one-dimensional case.

6.3. Properties of Path-Dependent Quantity for Nonlocal Processes

Let us describe the differences between these two types of the path-dependent quantity for nonlocal processes. One can state that the main difference between the two proposed definitions of the path-dependent quantity is non-additivity of $Q_L(L)$ in comparison with $Q_T(L)$ and the standard quantity, $Q(L)$. The non-additivity of the path-dependent quantity means that the quantity $Q_L(L)$ in the entire process, which is described by continuous line L, is not equal to the sum of quantities, $Q_L(L_k)$, of subprocesses

$$Q_L\left(\bigcup_{k=1}^{n} L_k\right) \neq \sum_{k=1}^{m} Q_L(L_k), \tag{171}$$

where

$$L := \bigcup_{k=1}^{m} L_k \tag{172}$$

is the piecewise simple line in the space of states. In the particular case $m = 2$, the L-type path-dependent quantity of the nonlocal process cannot be described as a sum of L-type quantities of two subprocesses

$$Q_L(L_{AB} \cup L_{BC}) \neq Q_L(L_{AB}) + Q_L(L_{BC}). \tag{173}$$

For an example of the power-law type of nonlocality, the kernel is $M_x(x) = h_\alpha(x)$, with the parameter $x > 0$, and inequality (173) is a consequence of the obvious inequality

$$(x_C - x_B)^\alpha + (x_B - x_A)^\alpha \neq (x_C - x_A)^\alpha \tag{174}$$

for $\alpha \in (0, 1)$ and $x_C > x_B > x_A$. Therefore, the quantity $Q_L(L)$ is a non-additive characteristic of nonlocal processes.

Note that, due to definition (166), the L-type quantities in the forward and inverse processes in the general case coincide in value but have opposite signs

$$Q_L(L_{AB}) + Q_L(L_{BA}) = 0. \tag{175}$$

This property of quantity $Q_L(L)$ is similar to the property of the standard path-dependent quantity, $Q(L)$.

In contrast to the L-type quantity, $Q_L(L)$, the T-type quantity, Q_T, is an additive quantity. The path-dependent quantity of the T-type in the entire process (172) is equal to the sum of the path-dependent quantities of subprocesses

$$Q_T\left(\bigcup_{k=1}^{n} L_k\right) = \sum_{k=1}^{n} Q_T(L_k), \tag{176}$$

where the line L is the piecewise simple line that is described by (172). In the particular case of $n = 2$, Equation (176) is

$$Q_T(L_{AB} \cup L_{BC}) = Q_T(L_{AB}) + Q_T(L_{BC}). \tag{177}$$

Therefore, quantity Q_T is an additive characteristic of nonlocal processes.

Quantity Q_T of the forward and inverse processes coincides in value

$$Q_T(L_{AB}) = -Q_T(L_{BA}). \tag{178}$$

We can see that the quantities $Q_L(L)$ and $Q_T(L)$ have the reversibility property (175), (178) similar to the property of the standard path-dependent quantity, $Q(L)$.

Note that additivity axiom for the path-dependent quantity with respect to the time parameter was proposed by Day in [135] for the physical concept of work. We expand this property on the T-type quantities of nonlocal theory. If we assume that the additivity axiom holds for the path-dependent quantity in nonlocal dynamics, then we should use the path-dependent quantities of the T-type only, and L-type quantities cannot be applied.

In general, the property of non-additivity can be considered (interpreted) from mathematical (M) and physical (P) points of view:

(M) From a mathematical point of view, a physical process should not depend on how we mentally divide the trajectory of the process into parts. In this case, non-additivity of the path-dependent quantity means that the definition of the L-type quantity is incorrect and cannot be used.

(P) From a physical point of view, if a process consists of two independent parts (subprocesses), then this separation of the process into two parts is not artificial. This separation reflects a qualitative property of these subprocesses that is the independence property of these parts. In this case, non-additivity means the need to distinguish between concepts of dependent and independent processes. Therefore, in order to use non-additive quantities, we should give a definition of the independence of the nonlocal processes.

Definition 13. *Two nonlocal processes are called independent if the history of the second (subsequent) process does not depend on the history of the first (previous) process.*

Processes, which are described by continuous lines L_{AB} and L_{BC}, are independent if the history of process L_{AB} does not affect process L_{BC}.

Remark 19. *The fact that the history of process L_{AB} does not affect process L_{BC} can be caused by the various physical reasons. Let us consider an example of such reasons, which is based on a possible source of nonlocality in the space of states. Nonlocality can be caused by non-standard properties of some relaxation processes. It can be assumed that relaxation has not completely disappeared in processes with nonlocality. In this case, the fading principle gives that that relaxation disappears more in more distant (old) states of the process. In states closer to the final state, relaxation fades less and the dependence on the history almost does not decrease. For such a source (interpretation) of nonlocality, the independence of the two processes means that enough time has passed between the end of the first process, L_{AB}, and the beginning of the second process, L_{BC}, so that all relaxation processes finish.*

Let two processes be described by piecewise simple lines L_{AB} and L_{BC}. Then, these processes can be considered as independent processes if at point B all relaxation processes are finished.

For independent processes L_{AB} and L_{BC}, the quantity $Q_L(L_{AB} \cup L_{BC})$, with $x_A < x_B < x_C$, cannot be described by the equation

$$Q_L(L_{AB} \cup L_{BC}) \;=\; \mathcal{J}^{(M)}_{[x_A,x_C]}[x]\, P(x, y(x)). \tag{179}$$

Equation (179) can be used to describe processes in which the path-dependent quantity, Q_L, depends on the entire history of process changes at interval $[x_A, x_C)$.

The path-dependent quantity, Q_L, of independent processes L_{AB} and L_{BC} should be described by the equation

$$Q_L(L_{AB} \cup L_{BC}) \;=\; \mathcal{J}^{(M)}_{[x_A,x_B]}[x]\, P(x, y(x)) \;+\; \mathcal{J}^{(M)}_{[x_B,x_C]}[x]\, P(x, y(x)), \tag{180}$$

where $x_A \;<\; x_B \;<\; x_C$.

As a result, the path-dependent quantity of the L-type must be described by Equations (179) and (180) depending on whether the subprocesses L_{AB} and L_{BC} are dependent or independent, respectively.

$$Q_L(L_{AB} \cup L_{BC}) \;=\; \begin{cases} \mathcal{J}^{(M)}_{[x_A,x_B]}[x]\, P(x, y(x)) \;+\; \mathcal{J}^{(M)}_{[x_B,x_C]}[x]\, P(x, y(x)), & \text{independent}, \\ \mathcal{J}^{(M)}_{[x_A,x_C]}[x]\, P(x, y(x)), & \text{dependent}, \end{cases} \tag{181}$$

where $x_A \;<\; x_B \;<\; x_C$.

For the piecewise simple line

$$L \;:=\; \bigcup_{k=1}^{n} L_k, \tag{182}$$

the L-type path-dependent quantity should be defined as

$$Q_L(L) \;=\; \sum_{k=1}^{n} Q_L(L_k) \tag{183}$$

only, if the simple lines L_k describe independent subprocesses. For example, the L-type path-dependent quantity for the nonlocal process by the equation

$$Q_L(L) \;=\; \sum_{k=1}^{n} \mathcal{J}^{(M)}_{[x_{A_k}, x_{A_{k+1}}]}[v_k]\, P_k(x_k, y(x_k)), \tag{184}$$

if $L_k \;=\; L_k\left[x_{A_k}, x_{A_{k+1}}\right]$ are independent subprocesses that are described by the equations $P \;=\; P_k(x_k, y(x_k))$.

We have the following property for the L-type path-dependent quantity, $Q_L(L)$, for nonlocal processes. If the nonlocal process, which is described by line L, cannot be divided into independent subprocesses, which are described by lines L_k, then the following inequality is satisfied

$$Q_L(L) \neq \sum_{k=1}^{n} Q_L(L_k), \tag{185}$$

despite the fact that the line itself can be represented mathematically in the form

$$L := \bigcup_{k=1}^{n} L_k. \tag{186}$$

If the nonlocal process line (186) is a sequence of processes independent from each other, which are described by lines L_k, then the equality

$$Q_L(L) = \sum_{k=1}^{n} Q_L(L_k) \tag{187}$$

is satisfied.

Remark 20. *Taking into account the described properties, various kinds of expressions of L-type and T-type path-dependent quantities can be used as different characteristics of nonlocal processes.*

If we assume the additivity axiom for the path-dependent quantities similar to the assumption of Day [135] for thermodynamics work, then we should use the T-type path-dependent quantity. For nonlocal processes with the additivity property of the path-dependent quantity, the L-type quantity, $Q_L(L)$, cannot be used in general cases.

Remark 21. *Replacing variables in standard theory is based on the use of the standard chain rule. For a nonlocal case and fractional calculus, the standard chain rule is violated and cannot be used. Therefore the nonlocal theory for one set of parameters cannot be transformed into a nonlocal theory for another set of parameters. By virtue of this, these two nonlocal theories should be formulated independently of each other, but on the basis of the same principles and using the same form. The nonlocalities on different sets of parameters are not equivalent to each other and the results depend on the choice of the parameter set.*

This situation is similar to the formulation of GF vector calculus in orthogonal curvilinear coordinates (OCC), which is proposed in paper [66]. The curvilinear coordinates (q_1, q_2, q_3) may be derived from a set of Cartesian coordinates (x, y, z) by using nonlinear coordinate transformations. The violation of the standard chain rule leads to the fact that the GF vector operators defined in different OCC (Cartesian, cylindrical, and spherical) are not related to each other by coordinate transformations. Therefore, the definitions of GF vector operators (for example, the linear GF gradient) should be formulated separately in different OCC. In this case, the most important requirement is that the GF gradient theorem and the general fractional Green, Stock, and Gauss theorems are satisfied [66].

7. Conclusions

General fractional calculus (GFC) is the theory of integral and integro-differential operators that satisfy analogs of the fundamental theorems of calculus. The operators, which satisfy the fundamental theorems of GFC, are called general fractional integrals (GFIs) and general fractional derivatives (GFDs). In this paper, we use the Luchko form of GFC, which is convenient for various applications.

There are two types of GF operators on a finite interval [55,66,69]. These types of GF operators are the L-type GFIs and GFDs, which are proposed in [55,69], and the T-type GFIs and GFDs, which are proposed in [66,69]. The main qualitative difference between these two types of GF operators is that the additivity property is satisfied for the T-type GF operators, and the additivity property is violated for the L-type operators.

In this paper, the additive and non-additive properties of GF operators on finite intervals and operators of GF vector calculus are discussed in detail. The additivity property is important for the application of GFC and in physics and other sciences. The violation and implementation of the additivity property leads to qualitative differences in the behavior of physical processes and systems. Understanding the mathematical aspects of violation and implementation of the additivity property with their physical interpretation allow us to build self-consistent adequate mathematical models of various processes and systems with nonlocality in space and time. The attention to the additive and non-additive properties of line GF integrals of T-type and L-type is due to the fact that the violation of the additivity gives rise to many problems when constructing mathematical models of nonlocal processes and systems in physics and other sciences. GF vector calculus has been proposed in [66] only for the T-type GF operators. In the 2021 paper [66], the line GF integrals, line GF gradient, and the GF gradient theorem were defined and proved only for the T-type operators. The T-type line GF integrals and the T-type line GF gradient operators are defined in [66]. The gradient theorem for these GF operators was also proved in [66]. The GF gradients are integro-differential operators that depend on the domain of integration in the space. Therefore, in three-dimensional space there are line, surface, and region GF gradient operators. Note that gradient theorem exists only for the line GF gradient, and gradient theorem does not hold for surface and region GF gradient operators [66]. The L-type operators have been proposed in 2023 papers.

In this 2024 paper, we propose definitions of the L-type line GF integrals and the L-type line GF gradient operators. For these L-type GF operators, we prove the GF gradient theorem. These operators are defined and this theorem is proved for the first time in this paper.

In general, the L-type line GFI along a simple line and piecewise simple line $L[A,B] \bigcup L[B,C]$ is not equal to the sum of the L-type line GFIs along simple lines $L[A,B]$ and $L[B,C]$. There exists two cases when the additivity property is satisfied for the L-type line GF integrals. The first case is when the L-type line GFI along the line $L[A,B] \bigcup L[B,C]$ is equal to the sum of the L-type line GFIs along lines $L[A,B]$ and $L[B,C]$, only on the condition that the processes, which are described by these lines, are independent. Processes are called independent if the history of changes in the subsequent process does not depend on the history of the previous process. The second case is, in this paper, we prove the additivity property holds for the L-type line GF integrals, if the conditions of the GF gradient theorems for the L-type operators are satisfied.

One can state that there exist two different operators that can be considered different tools that allow us to use GFC to describe various properties of nonlocal processes and systems.

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